First Edition

Precalculus

High School Edition

Julie Miller
Donna Gerken

Mc
Graw
Hill

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High School Edition

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PRECALCULUS, HIGH SCHOOL 1st EDITION

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Online-Only: Chapter 12 Preview of Calculus

The topics in Chapter 12 are outside the scope of most high school precalculus courses. The chapter is available online as part of the student eBook.

Preface

About the Authors: Julie Miller and Donna Gerken

Julie Miller is from Daytona State College, where she taught developmental and upper-level mathematics courses for 20 years. Prior to her work at DSC, she worked as a software engineer for General Electric in the area of flight and radar simulation. Julie earned a bachelor of science in applied mathematics from Union College in Schenectady, New York, and a master of science in mathematics from the University of Florida. In addition to this textbook, she has authored textbooks in developmental mathematics, trigonometry, and precalculus, as well as several short works of fiction and nonfiction for young readers.

"My father was a medical researcher, and I got hooked on math and science when I was young and would visit his laboratory. I remember doing simple calculations with him and using graph paper to plot data points for his experiments. He would then tell me what the peaks and features in the graph meant in the context of his experiment. I think that applications and hands-on experience made math come alive for me, and I'd like to see math come alive for my students."

Donna Gerken was a professor at Miami Dade College where she taught developmental courses, honors classes, and upper-level mathematics classes for decades. Throughout her career she has been actively involved with many projects at Miami Dade, including those on computer learning, curriculum design, and the use of technology in the classroom. Donna's bachelor of science in mathematics and master of science in mathematics are both from the University of Miami.

Dedications

In honor of Gus, Winston, Sine, Cosine, George, Pebbles, Lucy, Princess, Amber, Lady, Brownie, Aero, Chloe, Stanly, and Oliver for your past and present inspiration.

- Julie Miller & Donna Gerken

Acknowledgments

Paramount to the development of *Precalculus* was the invaluable feedback provided by instructors from around the country who reviewed the manuscript. In particular, we want to thank Alina Coronel for her amazing work in bringing our content into ALEKS, and Jennifer Blue for her steadfast scrutiny of our manuscript.

A special thanks to all of the event attendees who helped shape *Precalculus*. Focus groups and symposia were conducted to ensure the text was meeting the needs of students.

Preface

A Letter from the Author

Precalculus serves as a gateway course for many students interested in many disciplines. For some students, it is the entrance into the higher mathematics needed for careers in science, technology, engineering, and mathematics. For others, precalculus serves as the primary resource needed to understand the complexities of a modern world awash with statistics, investment strategies, financial planning, and even medical decisions. With the broad scope of this foundational course in mind, we worked to make the textbook, the digital tools, and the supplements as clear, relevant, and accessible as possible.

As part of our revision plan, we modernized content with new data sets and contemporary topics that are germane to all of today's students. Inside, you will find data and information relating to healthcare costs and ethics, income tax rates, the development of modern computers, extreme weather, and the work of modern scientists and inventors. We have also placed an emphasis on inclusion and diversity so that all students feel connected to the content in this book and can envision themselves in the many fields and situations in life that embrace mathematics.

In the text, we have expanded our exercise sets with an emphasis on mixed exercises that require multiple tools to complete their solutions. Likewise, we've added additional "challenge" exercises to promote more critical thinking and less rote repetition. Throughout this revision we were especially cognizant of the struggles for both teachers and students in adapting traditional content to the digital style of teaching that emerged during the pandemic.

We are excited for you, the student, to join an amazing journey in mathematics and hope it serves you well in your future.

Best to all,

Julie Miller

Donna Gerken

Preface

Key Features for Students

Clear, Precise Writing *Precalculus* is written for the diverse group of students taking this course. Julie Miller has written this text to use simple and accessible language. Through her friendly and engaging writing style, students are able to understand the material easily.

Fully Worked Examples The examples in the textbook are stepped-out in detail with thorough annotations at the right explaining each step. Following each example is a similar **Apply the Skills** exercise to engage students by practicing what they have just learned.

Modeling and Applications One of the most important tools to motivate our students is to make the mathematics they learn meaningful in their lives. The textbook is filled with robust applications and numerous opportunities for mathematical modeling for those instructors looking to incorporate these features into their course.

Building Confidence and Skills Throughout the text, students are encouraged to revisit key steps and tools while problem-solving. Specific, point-of-use call outs build student confidence in their abilities to solve complex problems.

- **Insights** boxes offer additional tips or observations on a concept or procedure
- **Check Your Work** boxes fend off common mistakes

Diverse Exercise Sets The exercises at the end of each lesson are graded, varied, and carefully organized to maximize student learning:

- **Prerequisite Review Exercises** begin the section-level exercises and ensure that students have the foundational skills to complete the homework sets successfully.
- **Concept Connections** prompt students to review the vocabulary and key concepts presented in the section.
- **Core Exercises** are presented next and are grouped by objective. These exercises are linked to examples in the text and direct students to similar problems whose solutions have been stepped-out in detail.
- **Mixed Exercises** do not refer to specific examples so that students can dip into their mathematical toolkit and decide on the best technique to use.
- **Write About It** exercises are designed to emphasize mathematical language by asking students to explain important concepts.
- **Technology Connections** require the use of a graphing utility and are found at the end of exercise sets. They can be easily skipped for those who do not encourage the use of calculators.

- **Expanding Your Skills Exercises** challenge and broaden students' understanding of the material.
- **Point of Interest** boxes feature interesting topics in mathematics from a diverse and inclusive set of contributors. These essays promote critical thinking, discussion, and research and often include follow-up questions and exercises.

End-of-Chapter Materials The textbook has the following end-of-chapter materials for students to review before test time:

- A **summary** with important equations, vocabulary, and key concepts. Detailed Chapter Summaries are available with the online resources.
- A **Chapter Test** reviewing the learning objectives of the full chapter.
- **Cumulative Review Exercises** that synthesize the learning objectives of the current chapter as well as all preceding chapters.

Key Features for Teachers

The accompanying teacher manual has been authored specifically for the high school classroom. This manual helps teachers to build inclusive, engaging lectures and classroom experiences, alongside providing key information about answers and pacing.

Pairing Examples and Exercises saves teachers time and helps to build lectures and presentations. Each fully worked example is paired with end-of-lesson Practice Exercises that mirror the example.

Guided Lecture Notes are keyed to the objectives in each section of the text. The notes step through the material with a series of questions and exercises that can be used in conjunction with lecture.

Classroom Activities, including online and in-class exercises, build collaboration and confidence in students. A diversity of exercises are available, including:

- **Wolfram Alpha activities** promote active learning in the classroom by using a powerful online resource.
- A **Group Activity** is available for each chapter to promote classroom discussion and collaboration.
- The **Problem Recognition Exercises** are available as worksheets for students to work on as individuals or in groups to help them determine appropriate methods of solution for related problem types.

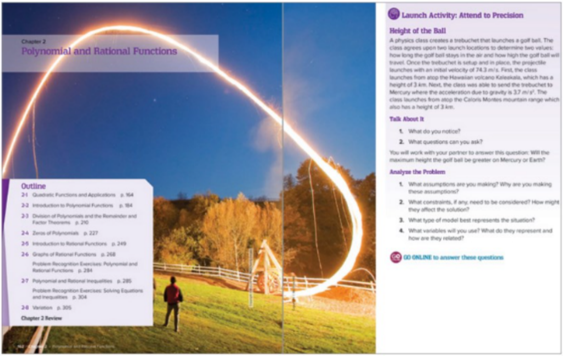
Guided Tour - Chapter and Lesson Openers

Exclusively for High School Precalculus

Precalculus is crafted specifically for the diverse group of students who will take this course as part of their high school math pathway. This text is organized around clear objectives, with abundant examples, and opportunities for students to develop mastery. Examples and concepts are connected to a wide range of applied topics certain to appeal to students with diverse backgrounds and interests, including business, chemistry, health, space travel, education, sports, travel, and more.

Stunning Visuals, Intuitive Layout

Making the study of advanced mathematics accessible and approachable for all students starts with its considerate, student-friendly design. Each chapter opens with a bright visual related to the topic of the **Launch Activity**. The chapter **Outline** provides a quick reference to what is covered in each chapter. The chapter openers are designed to be inviting, intriguing, and easy to navigate.



Chapter 2

Polynomial and Rational Functions

Outline

2-1 Quadratic Functions and Applications p. 164
2-2 Introduction to Polynomial Functions p. 184
2-3 Division of Polynomials and the Remainder and Factor Theorems p. 210
2-4 Zeros of Polynomials p. 227
2-5 Introduction to Rational Functions p. 249
2-6 Graphs of Rational Functions p. 268
Problem Recognition Exercises: Polynomial and Rational Functions p. 284
2-7 Polynomial and Rational Inequalities p. 285
Problem Recognition Exercises: Solving Equations and Inequalities p. 304
2-8 Variation p. 305
Chapter 2 Review

Launch Activity: Attend to Precision

Height of the Ball

A physics class creates a trebuchet that launches a golf ball. The class agrees upon two launch locations to determine two values: how long the golf ball stays in the air and how high the golf ball will travel. Once the trebuchet is setup and in place, the projectile launches with an initial velocity of 74.3 m/s. First, the class launches from atop the Hawaiian volcano Kaleakala, which has a height of 3 km. Next, the class was able to send the trebuchet to Mercury where the acceleration due to gravity is 3.7 m/s^2. The class launches from atop the Caloris Montes mountain range which also has a height of 3 km.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer this question: Will the maximum height the golf ball be greater on Mercury or Earth?

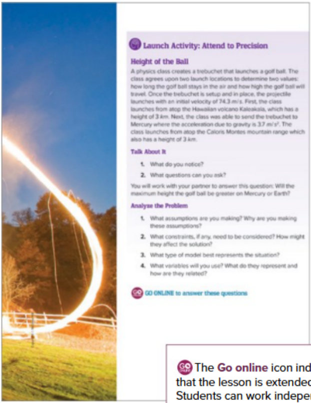
Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

GO ONLINE to answer these questions

Clear, Precise Narrative

At the core of *Precalculus* is the accessible narrative that invites students to engage with and grasp the content through clear, student-friendly instruction. The narrative is augmented with point-of-use features and elements designed to enhance student interest and understanding.



Launch Activity: Attend to Precision

Height of the Ball

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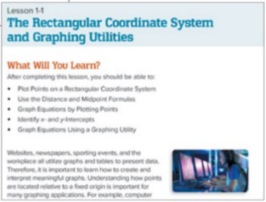
GO ONLINE to answer these questions

The **Launch Activity** helps students to develop their comfort and confidence with the mathematical practices including modeling, asking questions, and using mathematical reasoning.

The **Go online** icon indicates that the lesson is extended online. Students can work independently or in groups to answer the questions in the launch activity in their digital course.

Lessons open with **What Will You Learn?** sections, which summarize the main learning objectives of the lesson. They also connect the content of the lesson to the wider world, emphasizing the connection between precalculus and students' everyday lives.



Lesson 1-1

The Rectangular Coordinate System and Graphing Utilities

What Will You Learn?

After completing this lesson, you should be able to:

- Plot Points on a Rectangular Coordinate System
- Use the Distance and Midpoint Formulas
- Graph Equations by Plotting Points
- Identify *x*- and *y*-Intercepts
- Graph Equations Using a Graphing Utility

Websites, newspapers, sporting events, and the workplace all utilize graphs and tables to present data. Therefore, it is important to learn how to create and interpret meaningful graphs. Understanding how points are located relative to a fixed origin is important for many graphing applications. For example, computer

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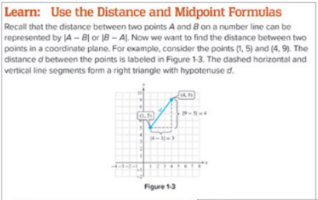
Guided Tour - Lesson Features

Guided Learning

Each chapter in *Precalculus* is divided into lessons, which follow a predictable pattern. This structure and repetition helps students build their foundational understanding of the concepts.



Learn: Use the Distance and Midpoint Formulas

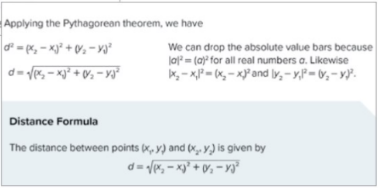
Recall that the distance between two points A and B on a number line can be represented by $|A - B|$ or $|B - A|$. Now we want to find the distance between two points in a coordinate plane. For example, consider the points (1, 5) and (4, 9). The distance d between the points is labeled in Figure 1-3. The dashed horizontal and vertical line segments form a right triangle with hypotenuse d.

Figure 1-3

The horizontal distance between the points is |4

The vertical distance between the points is |9 –

Chapter Lessons are divided into **Learn** sections, which focus on a particular topic or skill. Each Learn section starts with a Learn statement, and models a fully guided example for students to follow.



Applying the Pythagorean theorem, we have

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

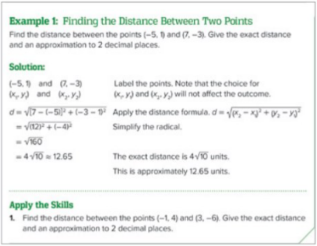
We can drop the absolute value bars because $|a|^2 = (a)^2$ for all real numbers a. Likewise $|x_2 - x_1|^2 = (x_2 - x_1)^2$ and $|y_2 - y_1|^2 = (y_2 - y_1)^2$.

Distance Formula

The distance between points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Important formulas and rules are highlighted in easy-to-find boxes for accessible student reference.



Example 1: Finding the Distance Between Two Points

Find the distance between the points (−5, 1) and (7, −3). Give the exact distance and an approximation to 2 decimal places.

Solution:

(−5, 1) and (7, −3)
(x_1, y_1) and (x_2, y_2)

Label the points. Note that the choice for (x_1, y_1) and (x_2, y_2) will not affect the outcome.

$d = \sqrt{[7 - (-5)]^2 + (-3 - 1)^2}$ Apply the distance formula. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(12)^2 + (-4)^2}$ Simplify the radical.

$= \sqrt{160}$

$= 4\sqrt{10} \approx 12.65$ The exact distance is $4\sqrt{10}$ units.

This is approximately 12.65 units.

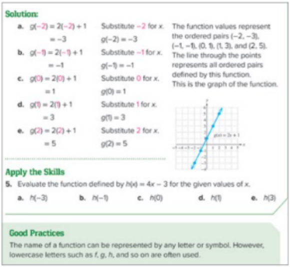
Apply the Skills

1. Find the distance between the points (−1, 4) and (3, −6). Give the exact distance and an approximation to 2 decimal places.

Examples are fully worked out with step-by-step explanations, illustrations, and detailed solutions. Students build confidence in their own problem-solving ability, and are then immediately able to apply the skills they've learned in the **Apply the Skills** exercise.

Meaningful Practice

In each lesson, students build toward independent problem solving. Scaffolded practice and timely reminders help to reinforce the learning while building their confidence. Several boxed reminders pop up that prompt students to double-check the assumptions they are making while problem-solving.



Solution:

a. $g(-2) = 2(-2) + 1$
$= -3$
Substitute -2 for x.
$g(-2) = -3$

b. $g(-1) = 2(-1) + 1$
$= -1$
Substitute -1 for x.
$g(-1) = -1$

c. $g(0) = 2(0) + 1$
$= 1$
Substitute 0 for x.
$g(0) = 1$

d. $g(1) = 2(1) + 1$
$= 3$
Substitute 1 for x.
$g(1) = 3$

e. $g(2) = 2(2) + 1$
$= 5$
Substitute 2 for x.
$g(2) = 5$

The function values represent the ordered pairs $(-2, -3)$, $(-1, -1)$, $(0, 1)$, $(1, 3)$, and $(2, 5)$. The line through the points represents all ordered pairs defined by this function. This is the graph of the function.

Apply the Skills

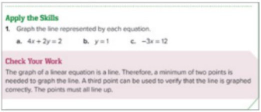
5. Evaluate the function defined by $h(x) = 4x - 3$ for the given values of x.

a. $h(-3)$ b. $h(-1)$ c. $h(0)$ d. $h(1)$ e. $h(3)$

Good Practices

The name of a function can be represented by any letter or symbol. However, lowercase letters such as f, g, h, and so on are often used.

Each example is followed by **Apply the Skills** exercises, which give students an immediate opportunity to practice the concepts they just learned in the example. **Good Practices** and **Check Your Work** boxes encourage students to slow down and refresh their problem-solving skills.



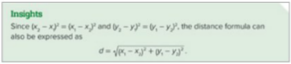
Apply the Skills

1. Graph the line represented by each equation.

a. $4x + 2y = 2$ b. $y = 1$ c. $-3x = 12$

Check Your Work

The graph of a linear equation is a line. Therefore, a minimum of two points is needed to graph the line. A third point can be used to verify that the line is graphed correctly. The points must all line up.

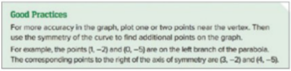


Insights

Since $(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$, the distance formula can also be expressed as

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Insights provide additional context or explanation, often offering an alternate formula or way of thinking about a problem.

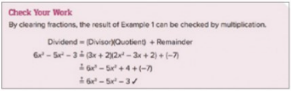


Good Practices

For more accuracy in the graph, plot one or two points near the vertex. Then use the symmetry of the curve to find additional points on the graph.

For example, the points $(1, -2)$ and $(0, -5)$ are on the left branch of the parabola. The corresponding points to the right of the axis of symmetry are $(3, -2)$ and $(4, -5)$.

Good Practices highlight proper notation and equation structure, to ensure consistency in problem solving.



Check Your Work

By clearing fractions, the result of Example 1 can be checked by multiplication.

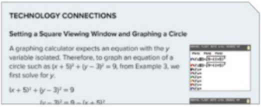
$$\begin{aligned} \text{Dividend} &= (\text{Divisor})(\text{Quotient}) + \text{Remainder} \\ 6x^3 - 5x^2 - 3 &\overset{?}{=} (3x + 2)(2x^2 - 3x + 2) + (-7) \\ &\overset{?}{=} 6x^3 - 5x^2 + 4 + (-7) \\ &\overset{?}{=} 6x^3 - 5x^2 - 3 \checkmark \end{aligned}$$

Check Your Work fends off common pitfalls by highlighting areas where mistakes are commonly made.

Guided Tour - Lesson Features

Real-World Connections

One of the most important tools to motivate students is to make the mathematics they learn meaningful to their lives. Numerous robust applications connect the abstract concepts to tangible events and circumstances for students.



TECHNOLOGY CONNECTIONS

Setting a Square Viewing Window and Graphing a Circle

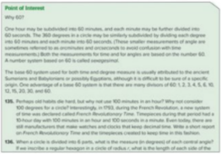
A graphing calculator expects an equation with the *y* variable isolated. Therefore, to graph an equation of a circle such as $(x+5)^2+(y-3)^2=9$, from Example 3, we first solve for *y*.

$(x+5)^2+(y-3)^2=9$

$(y-3)^2=9-(x+5)^2$

Technology Connections clearly explain how to solve different types of problems with the use of a graphing calculator. Students will apply this at the end of each lesson.

Point of Interest features interesting topics from diverse perspectives and a variety of applications. Perfect for partner activities or class discussions, these features provide students with the opportunity to see the relevance of mathematics to their everyday life.



Point of Interest

Why 60?

One hour may be subdivided into 60 minutes, and each minute may be further divided into 60 seconds. The 360 degrees in a circle may be similarly subdivided by dividing each degree into 60 minutes and each minute into 60 seconds. (These smaller measurements of angle are sometimes referred to as *arcminutes* and *arcseconds* to avoid confusion with time measurements.) Both the measurements for time and for angles are based on the number 60. A number system based on 60 is called *sexagesimal*.

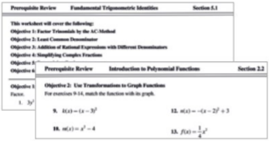
The base 60 system used for both time and degree measure is usually attributed to the ancient Sumerians and Babylonians or possibly Egyptians, although it is difficult to be sure of a specific origin. One advantage of a base 60 system is that there are many divisors of 60: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60.

135. Perhaps old habits die hard, but why not use 100 minutes in an hour? Why not consider 100 degrees for a circle? Interestingly, in 1793, during the French Revolution, a new system of time was declared called *French Revolutionary Time*. Timepieces during that period had a 10-hour day with 100 minutes in an hour and 100 seconds in a minute. Even today, there are still manufacturers that make watches and clocks that keep decimal time. Write a short report on *French Revolutionary Time* and the timepieces created to keep time in this fashion.

136. When a circle is divided into 6 parts, what is the measure (in degrees) of each central angle? If we inscribe a regular hexagon in a circle of radius *r*, what is the length of each side of the

Support for All Learners

Differentiated support addresses a range of abilities and proficiency levels, including students who need support with prerequisite skills and those who are ready for more challenging content.



Prerequisite Review **Fundamental Trigonometric Identities** **Section 5.1**

This worksheet will cover the following:
Objective 1: Factor Trinomials by the AC-Method
Objective 2: Least Common Denominator
Objective 3: Addition of Rational Expressions with Different Denominators
Objective 4: Simplifying Complex Fractions
Objective 5
Objective 6

Objective 1
Factor.
1. $3y^2$

Prerequisite Review **Introduction to Polynomial Functions** **Section 2.2**

Objective 2: Use Transformations to Graph Functions
For exercises 9-14, match the function with its graph.

9. $k(x)=(x-3)^3$

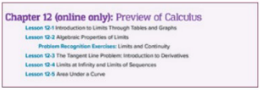
10. $m(x)=x^3-4$

12. $n(x)=-(x-2)^3+3$

13. $f(x)=\frac{1}{4}x^3$

The printable, interactive **Corequisite Workbook** is included in the digital resources. This workbook provides extra math practice on prerequisite content for each lesson in the textbook.

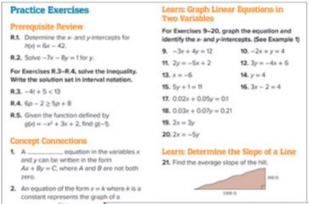
For accelerated learners, the online-only **Chapter 12** offers a preview of calculus.



Chapter 12 (online only): Preview of Calculus
Lesson 12-1 Introduction to Limits Through Tables and Graphs
Lesson 12-2 Algebraic Properties of Limits
Problem Recognition Exercises: Limits and Continuity
Lesson 12-3 The Tangent Line Problem: Introduction to Derivatives
Lesson 12-4 Limits at Infinity and Limits of Sequences
Lesson 12-5 Area Under a Curve

Abundant and Varied Lesson Assessments

Each lesson ends with a graded, varied, and carefully organized set of **Practice Exercises**. Several strands of problems are present to review the full breadth of knowledge covered in each lesson.



Practice Exercises

Prerequisite Review

R.1. Determine the x- and y-intercepts for $h(x) = 6x - 42$.

R.2. Solve $-7x - 8y = 1$ for y.

For Exercises R.3–R.4, solve the inequality. Write the solution set in interval notation.

R.3. $-4t + 5 < 13$

R.4. $6p - 2 \geq 5p + 8$

R.5. Given the function defined by $g(x) = -x^2 + 3x + 2$, find $g(-1)$.

Concept Connections

1. A ________ equation in the variables x and y can be written in the form $Ax + By = C$, where A and B are not both zero.
2. An equation of the form $x = k$ where k is a constant represents the graph of a ________ line.

Learn: Graph Linear Equations in Two Variables

For Exercises 9–20, graph the equation and identify the x- and y-intercepts. (See Example 1)

9. $-3x + 4y = 12$

10. $-2x + y = 4$

11. $2y = -5x + 2$

12. $3y = -4x + 6$

13. $x = -6$

14. $y = 4$

15. $5y + 1 = 11$

16. $3x - 2 = 4$

17. $0.02x + 0.05y = 0.1$

18. $0.03x + 0.07y = 0.21$

19. $2x = 3y$

20. $2x = -5y$

Learn: Determine the Slope of a Line

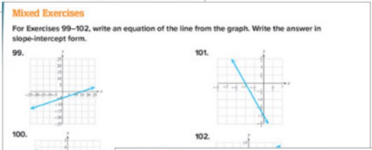
21. Find the average slope of the hill.

Prerequisite Review problems open each set of Practice Exercises.

Concept Connections ask students to review the vocabulary and key concepts presented in the section.

Each **Learn** title is reviewed independently with several practice exercises.

Mixed Exercises review topics across multiple learning objectives within a lesson.



Mixed Exercises

For Exercises 99–102, write an equation of the line from the graph. Write the answer in slope-intercept form.

99.

101.

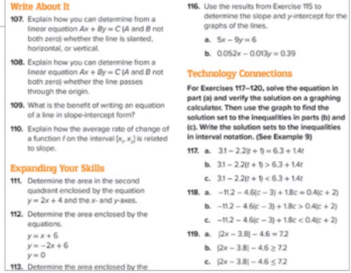
100.

102.

Write About It exercises emphasize mathematical language, giving students an opportunity to answer questions and describe key concepts in their own words.

Expanding Your Skills challenges and broadens students' understanding of the material.

Technology Connections are designed for use with students' graphing calculators.



Write About It

107. Explain how you can determine from a linear equation $Ax + By = C$ (A and B not both zero) whether the line is slanted, horizontal, or vertical.

108. Explain how you can determine from a linear equation $Ax + By = C$ (A and B not both zero) whether the line passes through the origin.

109. What is the benefit of writing an equation of a line in slope-intercept form?

110. Explain how the average rate of change of a function f on the interval $[x_1, x_2]$ is related to slope.

Expanding Your Skills

111. Determine the area in the second quadrant enclosed by the equation $y = 2x + 4$ and the x- and y-axes.

112. Determine the area enclosed by the equations.

$y = x + 6$
$y = -2x + 6$
$y = 0$

113. Determine the area enclosed by the

116. Use the results from Exercise 115 to determine the slope and y-intercept for the graphs of the lines.

- **a.** $5x - 9y = 6$
- **b.** $0.052x - 0.013y = 0.39$

Technology Connections

For Exercises 117–120, solve the equation in part (a) and verify the solution on a graphing calculator. Then use the graph to find the solution set to the inequalities in parts (b) and (c). Write the solution sets to the inequalities in interval notation. (See Example 9)

117. **a.** $3.1 - 2.2(t + 1) = 6.3 + 1.4t$
b. $3.1 - 2.2(t + 1) > 6.3 + 1.4t$
c. $3.1 - 2.2(t + 1) < 6.3 + 1.4t$

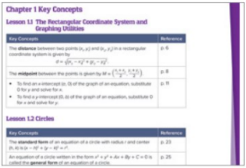
118. **a.** $-11.2 - 4.6(c - 3) + 1.8c = 0.4(c + 2)$
b. $-11.2 - 4.6(c - 3) + 1.8c > 0.4(c + 2)$
c. $-11.2 - 4.6(c - 3) + 1.8c < 0.4(c + 2)$

119. **a.** $|2x - 3.8| - 4.6 = 7.2$
b. $|2x - 3.8| - 4.6 \geq 7.2$
c. $|2x - 3.8| - 4.6 \leq 7.2$

Guided Tour - Chapter Review

Comprehensive Chapter Review

Each chapter ends with a **Chapter Review** including a summary with important equations, vocabulary, and key concepts. These elements review the full extent of the content covered in the chapter.



Chapter 1 Key Concepts

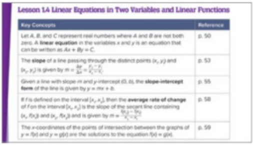
Lesson 1.1 The Rectangular Coordinate System and Graphing Utilities

Key Concepts	Reference
The **distance** between two points (x_1, y_1) and (x_2, y_2) in a rectangular coordinate system is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.	p. 6
The **midpoint** between the points is given by $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.	p. 8
• To find an *x*-intercept $(a, 0)$ of the graph of an equation, substitute 0 for *y* and solve for *x*. • To find a *y*-intercept $(0, b)$ of the graph of an equation, substitute 0 for *x* and solve for *y*.	p. 11

Lesson 1.2 Circles

Key Concepts	Reference
The **standard form** of an equation of a circle with radius *r* and center (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.	p. 23
An equation of a circle written in the form $x^2 + y^2 + Ax + By + C = 0$ is called the **general form** of an equation of a circle.	p. 25

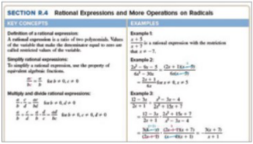
The Chapter Review is organized into **Key Concepts** tables. Each table summarizes the most important vocabulary words and formulas from each lesson.



Lesson 1.4 Linear Equations in Two Variables and Linear Functions

Key Concepts	Reference
Let *A*, *B*, and *C* represent real numbers where *A* and *B* are not both zero. A **linear equation** in the variables *x* and *y* is an equation that can be written as $Ax + By = C$.	p. 50
The **slope** of a line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.	p. 53
Given a line with slope *m* and *y*-intercept $(0, b)$, the **slope-intercept form** of the line is given by $y = mx + b$.	p. 55
If *f* is defined on the interval $[x_1, x_2]$, then the **average rate of change** of *f* on the interval $[x_1, x_2]$ is the slope of the secant line containing $(x_1, f(x_1))$ and $(x_2, f(x_2))$ and is given by $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.	p. 58
The *x*-coordinates of the points of intersection between the graphs of $y = f(x)$ and $y = g(x)$ are the solutions to the equation $f(x) = g(x)$.	p. 59

Each **key formula** or **vocabulary** word is tagged with a page reference, so that students can easily navigate back to the initial introduction of that concept for review.



SECTION R.4 Rational Expressions and More Operations on Radicals

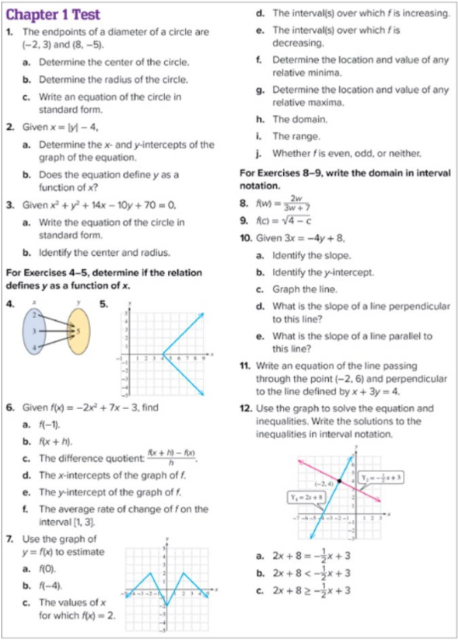
KEY CONCEPTS

EXAMPLES

Detailed **Chapter Summaries**, available with the digital resources, are excellent study tools.

Robust, Versioned Assessments

The **Chapter Test** assesses students on all the learning objectives covered in the chapter and can be used for self-study and review. Additional versioned **Lesson Quizzes** and **Chapter Tests** are available as part of the digital resources, alongside an expansive, editable question bank.



Chapter 1 Test

1. The endpoints of a diameter of a circle are $(-2, 3)$ and $(8, -5)$.

- **a.** Determine the center of the circle.
- **b.** Determine the radius of the circle.
- **c.** Write an equation of the circle in standard form.

2. Given $x = |y| - 4$,

- **a.** Determine the x- and y-intercepts of the graph of the equation.
- **b.** Does the equation define y as a function of x?

3. Given $x^2 + y^2 + 14x - 10y + 70 = 0$,

- **a.** Write the equation of the circle in standard form.
- **b.** Identify the center and radius.

For Exercises 4–5, determine if the relation defines y as a function of x.

4.

5.

6. Given $f(x) = -2x^2 + 7x - 3$, find

- **a.** $f(-1)$.
- **b.** $f(x + h)$.
- **c.** The difference quotient $\frac{f(x+h) - f(x)}{h}$.
- **d.** The x-intercepts of the graph of f.
- **e.** The y-intercept of the graph of f.
- **f.** The average rate of change of f on the interval $[1, 3]$.

7. Use the graph of $y = f(x)$ to estimate

- **a.** $f(0)$.
- **b.** $f(-4)$.
- **c.** The values of x for which $f(x) = 2$.
- **d.** The interval(s) over which f is increasing.
- **e.** The interval(s) over which f is decreasing.
- **f.** Determine the location and value of any relative minima.
- **g.** Determine the location and value of any relative maxima.
- **h.** The domain.
- **i.** The range.
- **j.** Whether f is even, odd, or neither.

For Exercises 8–9, write the domain in interval notation.

8. $f(w) = \frac{2w}{3w + 7}$

9. $f(c) = \sqrt{4 - c}$

10. Given $3x = -4y + 8$,

- **a.** Identify the slope.
- **b.** Identify the y-intercept.
- **c.** Graph the line.
- **d.** What is the slope of a line perpendicular to this line?
- **e.** What is the slope of a line parallel to this line?

11. Write an equation of the line passing through the point $(-2, 6)$ and perpendicular to the line defined by $x + 3y = 4$.

12. Use the graph to solve the equation and inequalities. Write the solutions to the inequalities in interval notation.

- **a.** $2x + 8 = -\frac{1}{2}x + 3$
- **b.** $2x + 8 < -\frac{1}{2}x + 3$
- **c.** $2x + 8 \geq -\frac{1}{2}x + 3$

Best in Class
Digital Resources

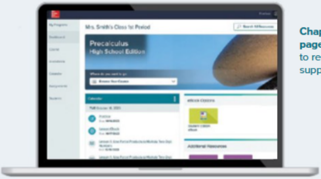
Precalculus **is enriched with multimedia content that enhance the teaching and learning experience both inside and outside of the classroom.**

Developed with the world's leading subject matter experts and organized by chapter level, the resources provide students with multiple opportunities to contextualize and apply their understanding. Teachers can save time, customize lessons, monitor student progress, and make data-driven decisions in the classroom with the flexible, easy-to-navigate instructional tools.

Student Assignments

Resources are organized at the chapter level. To enhance the core content, teachers can add assignments, activities, and instructional aids to any lesson. The chapter landing page gives students access to:

- assigned activities;
- customizable, auto-graded assessments;
- an interactive eBook;
- data sets, data projects, and classroom activities;
- and an interactive and printable Corequisite Workbook for additional math skills practice.



Chapter landing page links students to resources that support success.

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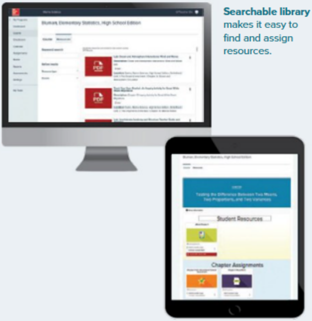
Access to course content on-the-go is easier and more efficient than ever before with the *McGraw Hill K12* Portal App.

Teacher Resources

Teachers have access to the interactive eBook, plus a wealth of customizable chapter resources and powerful gradebook tools. Resources include:

- a solutions manual with answers to the end-of-chapter questions in the student edition;
- actionable reporting features that track student progress with data-driven insights;
- customizable PowerPoint presentations, visual aids, and additional ideas for lecture enrichment;
- and customizable assignments and quiz banks that are automatically graded and populate easy-to-read reports.



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Adaptive Learning with ALEKS

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- assess a student's proficiency and knowledge with an "Initial Knowledge Check,"
- track which topics have been mastered,
- identify areas that need more study,
- and build student confidence with preparatory modules, video tutorials, and practice questions.

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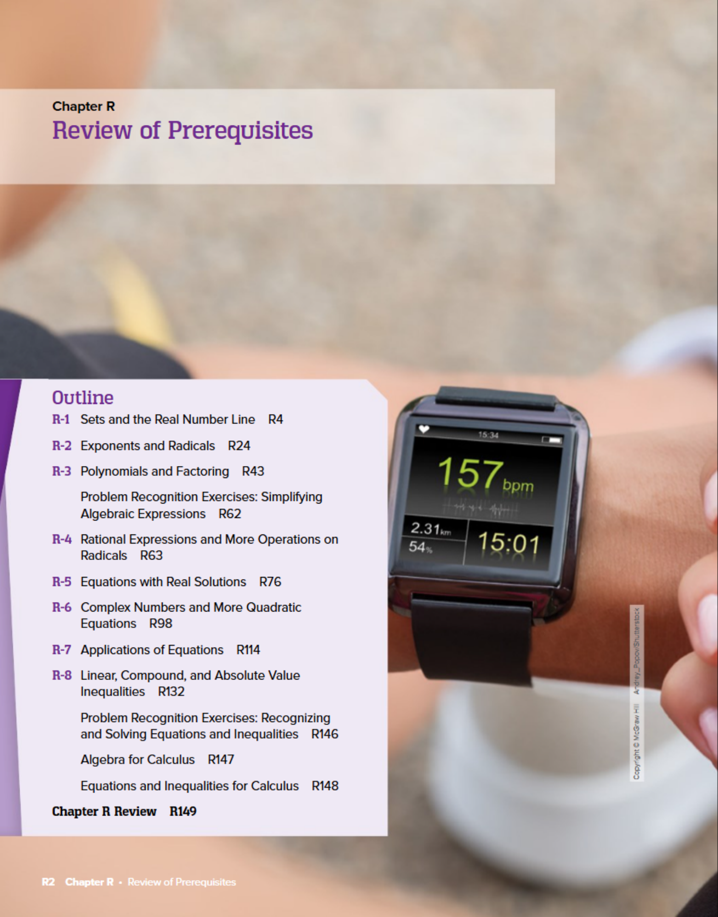

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15:3
Average Duration
06:45
Average Bed Tim
00:45 A
Jan
Feb
Mar
Apr
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Jun
Jul
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Bed Time: 00:15 AM
Wake Time: 07:45 AM
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Alarm Set: 08:00 AM
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Efficienc

Prepare: Mathematical Practices

Review and Build Skills

Athletes know that in order to optimize their performance they need to pace themselves and be mindful of their target heart rate. For example, a 25-year-old with a maximum heart rate of 195 beats per minute should strive for a target heart rate zone of between 98 and 166 beats per minute. This correlates to between 50% and 85% of the individual's maximum recommended heart rate (Source: American Heart Association, www.americanheart.org).

The mathematics involved in finding maximum recommended heart rate and an individual's target heart rate zone use a linear model relating age and resting heart rate. An introduction to modeling is presented here in Chapter R along with the standard order of operations used to carry out these calculations.

Chapter R reviews skills and concepts required for success in precalculus. Just as an athlete must first learn the basics of a sport and build endurance and speed, a student studying mathematics must focus on necessary basic skills to prepare for the challenge ahead. Preparation for algebra is comparable to an athlete preparing for a sporting event. Putting the time and effort into the basics here in Chapter R will be your foundation for success in later chapters.

Lesson R-1

Sets and the Real Number Line

What Will You Learn?

After completing this lesson, you should be able to:

- Identify Subsets of the Set of Real Numbers
- Use Inequality Symbols and Interval Notation
- Find the Union and Intersection of Sets
- Evaluate Absolute Value Expressions
- Use Absolute Value to Represent Distance
- Apply the Order of Operations
- Simplify Algebraic Expressions
- Write Algebraic Models

Learn: Identify Subsets of the Set of Real Numbers

A hybrid vehicle gets 48 mpg in city driving and 52 mpg on the highway. The formula $A = \frac{1}{48}c + \frac{1}{52}h$ gives the amount of gas A (in gal) for c miles of city driving and h miles of highway driving. In the formula, A, c, and h are called **variables**, and these represent values that are subject to change. The values $\frac{1}{48}$ and $\frac{1}{52}$ are called **constants** because their values do not change in the formula.

For a trip from Houston, Texas, to Dallas, Texas, a motorist travels 36 mi of city driving and 91 mi of highway driving. The amount of fuel used by this hybrid vehicle is given by

$$\begin{aligned} A &= \frac{1}{48}(36) + \frac{1}{52}(91) \\ &= 2.5 \text{ gal} \end{aligned}$$

The numbers used in day-to-day life such as those used to determine fuel consumption come from the set of real numbers, denoted by $\mathbb{R}$. A **set** is a collection of items called **elements**. The braces { and } are used to enclose the elements of a set. For example, {gold, silver, bronze} represents the set of medals awarded to the top three finishers in an Olympic event. A set that contains no elements is called the **empty set** (or **null set**) and is denoted by { } or $\varnothing$.

Mark Dierker/McGraw Hill

When referring to individual elements of a set, the symbol $\in$ means "is an element of," and the symbol $\notin$ means "is not an element of." For example,

$5 \in \{1, 3, 5, 7\}$ is read as "5 is an element of the set of elements 1, 3, 5, and 7."

$6 \notin \{1, 3, 5, 7\}$ is read as "6 is *not* an element of the set of elements 1, 3, 5, and 7."

A set can be defined in several ways. Listing the elements in a set within braces is called the **roster method**. Using the roster method, the set of the even numbers between 0 and 10 is represented by $\{2, 4, 6, 8\}$. Another method to define this set is by using **set-builder notation**. This uses a description of the elements of the set. For example,

$$\underbrace{\{}_{\text{The set of}}\underbrace{x}_{\text{all } x}\underbrace{|}_{\text{such that}}\underbrace{x \text{ is an even number between 0 and 10}}_{x \text{ is an even number between 0 and 10}}\}$$

In our study of college algebra, we will often refer to several important **subsets** (parts of) the set of real numbers (Table R-1).

TABLE R-1 Subsets of the Set of Real Numbers, $\mathbb{R}$

Set	Definition
Natural numbers, $\mathbb{N}$	$\{1, 2, 3, \ldots\}$
Whole numbers, $\mathbb{W}$	$\{0, 1, 2, 3, \ldots\}$
Integers, $\mathbb{Z}$	$\{\cdots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Rational numbers, $\mathbb{Q}$	$\left\{\frac{p}{q} \middle\| p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$ • Rational numbers can be expressed as a ratio of integers where the denominator is not zero. Examples: $-\frac{6}{11}$ (ratio of -6 and 11) and 9 (ratio of 9 and 1). • All terminating and repeating decimals are rational numbers. Examples: 0.71 (ratio of 71 and 100), $0.\overline{6} = 0.666\ldots$ (ratio of 2 and 3).
Irrational numbers, $\mathbb{H}$	Irrational numbers are real numbers that cannot be expressed as a ratio of integers. The decimal form of an irrational number is nonterminating and nonrepeating. Examples: π and $\sqrt{2}$

Insights

Notice that the first five letters of the word *rational* spell *ratio*. This will help you remember that a rational number is a *ratio* of integers.

The relationships among the subsets of real numbers defined in Table R-1 are shown in Figure R-1. In particular, notice that together the elements of the set of rational numbers and the set of irrational numbers make up the set of real numbers.

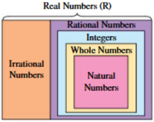


Figure R-1

Learn: Use Inequality Symbols and Interval Notation

All real numbers can be located on the real number line. We say that a is less than b (written symbolically as $a < b$) if a lies to the left of b on the number line. This is equivalent to saying that b is greater than a (written symbolically as $b > a$) because b lies to the right of a.

$a < b$ is equivalent to $b > a$

In Table R-2, we summarize other symbols used to compare two real numbers.

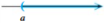
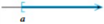
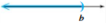
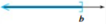
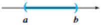
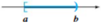
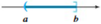
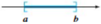
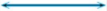
TABLE R-2 Summary of Inequality Symbols and Their Meanings

Inequality	Verbal Interpretation	Other Implied Meanings	Numerical Examples
$a < b$	a is less than b	b exceeds a b is greater than a	$5 < 7$
$a > b$	a is greater than b	a exceeds b b is less than a	$-3 > -6$
$a \leq b$	a is less than or equal to b	a is at most b a is no more than b	$4 \leq 5$ $5 \leq 5$
$a \geq b$	a is greater than or equal to b	a is no less than b a is at least b	$9 \geq 8$ $9 \geq 9$
$a = b$	a is equal to b	b is equal to a	$-4.3 = -4.3$
$a \neq b$	a is not equal to b	b is not equal to a	$-6 \neq -7$
$a \approx b$	a is approximately equal to b	b is approximately equal to a	$-12.99 \approx -13$

An interval on the real number line can be represented in set-builder notation or in interval notation. In Table R-3, observe that a parenthesis) or (indicates that an endpoint *is not* included in an interval. A bracket] or [indicates that an endpoint *is* included in the interval. The real number line extends infinitely far to the left and right. We use the symbols $-\infty$ and ∞ to denote the unbounded behavior to the left and right, respectively.

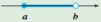
TABLE R-3 Summary of Interval Notation and Set-Builder Notation

Let a, b, and x represent real numbers.

Set-Builder Notation	Verbal Interpretation	Graph	Interval Notation
$\{x \mid x > a\}$	the set of real numbers greater than a		(a, ∞)
$\{x \mid x \geq a\}$	the set of real numbers greater than or equal to a		$[a, \infty)$
$\{x \mid x < b\}$	the set of real numbers less than b		$(-\infty, b)$
$\{x \mid x \leq b\}$	the set of real numbers less than or equal to b		$(-\infty, b]$
$\{x \mid a < x < b\}$	the set of real numbers between a and b		(a, b)
$\{x \mid a \leq x < b\}$	the set of real numbers greater than or equal to a and less than b		$[a, b)$
$\{x \mid a < x \leq b\}$	the set of real numbers greater than a and less than or equal to b		$(a, b]$
$\{x \mid a \leq x \leq b\}$	the set of real numbers between a and b, inclusive		$[a, b]$
$\{x \mid x \text{ is a real number}\}$ $\mathbb{R}$	the set of all real numbers		$(-\infty, \infty)$

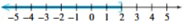
Insights

As an alternative to using parentheses and brackets to represent the endpoints of an interval, an open dot or closed dot may be used. For example, $\{x \mid a \leq x < b\}$ would be represented as follows.

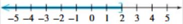
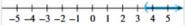
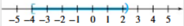


Example 1: Expressing Sets in Interval Notation and Set-Builder Notation

Complete the table.

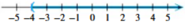
Graph	Interval Notation	Set-Builder Notation
		
	$\left(\frac{7}{2}, \infty\right)$	
		$\{y \mid -4 \leq y < 2.3\}$

Solution:

Graph	Interval Notation	Set-Builder Notation	Comments
	$(-\infty, 2]$	$\{x \mid x \le 2\}$	The bracket at 2 indicates that 2 is included in the set.
	$\left(\frac{7}{2}, \infty\right)$	$\left\{x \mid x > \frac{7}{2}\right\}$	The parenthesis at $\frac{7}{2} = 3.5$ indicates that $\frac{7}{2}$ is *not* included in the set.
	$[-4, 2.3)$	$\{y \mid -4 \le y < 2.3\}$	The set includes the real numbers between –4 and 2.3, including the endpoint –4.

Apply the Skills

1. a. Write the set represented by the graph in interval notation and set-builder notation.



b. Given the interval, $\left(-\infty, -\frac{4}{3}\right]$, graph the set and write the set-builder notation.

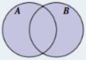
c. Given the set $\{x \mid 1.6 < x \le 5\}$, graph the set and write the interval notation.

Learn: Find the Union and Intersection of Sets

Two or more sets can be combined by the operations of union and intersection.

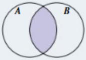
Union and Intersection of Sets

The **union** of sets A and B, denoted $A \cup B$, is the set of elements that belong to set A or to set B or to both sets A and B.



$A \cup B$
A union *B*
The elements in *A* or *B* or both

The **intersection** of sets A and B, denoted $A \cap B$, is the set of elements common to both set A and set B.



$A \cap B$
A intersection *B*
The elements common to *A* and *B*

In Examples 2 and 3, we practice finding the union and intersections of sets.

Example 2: Finding the Union and Intersection of Sets

Find the union or intersection of sets as indicated.

$A = \{-5, -3, -1, 1\}$ $\quad B = \{-5, 0, 5\}$ $\quad C = \{-4, -2, 0, 2, 4\}$

a. $A \cap B$ **b.** $A \cup B$ **c.** $A \cap C$

Solution:

a. $A \cap B = \{-5\}$

The only element common to both A and B is -5.
$A = \{-5, -3, -1, 1\}$, $B = \{-5, 0, 5\}$

b. $A \cup B = \{-5, -3, -1, 0, 1, 5\}$

The union of A and B consists of all elements from A along with all elements from B.

c. $A \cap C = \{\ \}$

Sets A and C have no common elements. Recall, from page R4, a set that contains no elements is called the empty set and is denoted by $\{\ \}$ or $\varnothing$.

Apply the Skills

2. From the sets A, B, and C defined in Example 2, find

a. $B \cap C$ **b.** $B \cup C$ **c.** $A \cup C$

Example 3: Finding the Union and Intersection of Sets

Find the union or intersection as indicated.

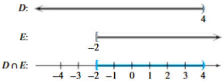
$D = \{x \mid x < 4\}$ $\quad E = \{x \mid x \geq -2\}$ $\quad F = \{x \mid x \leq -3\}$

a. $D \cap E$ **b.** $D \cup E$ **c.** $D \cup F$ **d.** $E \cap F$

Solution:

Graph each individual set and take the union or intersection.

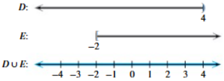
a.



The intersection is the region of overlap.

Set notation: $D \cap E = \{x \mid -2 \leq x < 4\}$
Interval notation: $[-2, 4)$

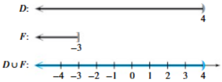
b.



The union contains the elements from D along with those from E.

Set notation: $D \cup E = \mathbb{R}$
Interval notation: $(-\infty, \infty)$

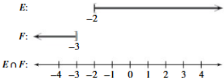
c.



Set F is contained within set D. The union is set D itself.

Set notation: $D \cup F = \{x | x < 4\}$
Interval notation: $(-\infty, 4)$

d.



There are no elements common to both sets E and F.

Set notation: $E \cap F = \{\,\}$

Apply the Skills

3. Given $X = \{x | x \geq -1\}$, $Y = \{x | x < 2\}$, and $Z = \{x | x < -4\}$, find the union or intersection of sets as indicated.

a. $X \cap Y$ **b.** $X \cup Y$ **c.** $Y \cap Z$ **d.** $X \cap Z$

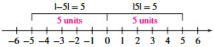
Learn: Evaluate Absolute Value Expressions

Every real number x has an opposite denoted by $-x$. For example, $-(4)$ is the opposite of 4 and simplifies to -4. Likewise, $-(-2.1)$ is the opposite of -2.1 and simplifies to 2.1.

The **absolute value** of a real number x, denoted by $|x|$, is the distance between x and zero on the number line. For example:

$|-5| = 5$ because -5 is 5 units from zero on the number line.

$|5| = 5$ because 5 is 5 units from zero on the number line.



Notice that if a number is negative, its absolute value is the opposite of the number. If a number is positive, its absolute value is the number itself.

Definition of Absolute Value

Let x be a real number. Then $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$

- If x is positive or zero, then $|x|$ is just x itself.

  For example: $|4| = 4$

  $|0| = 0$
- If x is negative, then $|x|$ is the opposite of x.

  For example: $|4| = -(-4) = 4$

Example 4: Removing Absolute Value Symbols

Use the definition of absolute value to rewrite each expression without absolute value bars.

a. $|\sqrt{3} - 3|$ **b.** $|3 - \sqrt{3}|$ **c.** $\dfrac{|x-4|}{x-4}$ for $x < 4$

Solution:

a. $|\sqrt{3} - 3| = -(\sqrt{3} - 3)$

$= -\sqrt{3} + 3$ or $3 - \sqrt{3}$

The value $\sqrt{3} \approx 1.73 < 3$, which implies that $\sqrt{3} - 3 < 0$. Since the expression inside the absolute value bars is negative, take the opposite.

b. $|3 - \sqrt{3}| = 3 - \sqrt{3}$

The value $\sqrt{3} \approx 1.73 < 3$, which implies that $3 - \sqrt{3} > 0$. Since the expression inside the absolute value bars is positive, the simplified form is the expression itself.

c. $\dfrac{|x-4|}{x-4}$ for $x < 4$

$= \dfrac{-(x-4)}{x-4}$

$= -1 \cdot \dfrac{x-4}{x-4}$

$= -1$

The condition $x < 4$ implies that $x - 4 < 0$. Since the expression inside the absolute value bars is negative, take the opposite.

Apply the Skills

4. Use the definition of absolute value to rewrite each expression without absolute value bars.

a. $|5 - \sqrt{7}|$ **b.** $|\sqrt{7} - 5|$ **c.** $\dfrac{x+6}{|x+6|}$ for $x > -6$

Insights

Calculator approximations can be used to show that $\sqrt{3} - 3 \approx -1.27$ is negative, and $3 - \sqrt{3} \approx 1.27$ is positive.

Learn: Use Absolute Value to Represent Distance

Absolute value is also used to denote distance between two points on a number line.

Distance Between Two Points on a Number Line

The distance between two points a and b on a number line is given by

$$|a - b| \quad \text{or} \quad |b - a|$$

That is, the distance between two points on a number line is the absolute value of their difference.

Example 5: Determining the Distance Between Two Points

Write an absolute value expression that represents the distance between 4 and –1 on the number line. Then simplify.

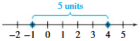
Solution:

$|4 - (-1)| = |5| = 5$

or

$|-1 - 4| = |-5| = 5$

The distance between 4 and –1 is represented by $|4 - (-1)|$ or by $|-1 - 4|$.



Apply the Skills

5. Write an absolute value expression that represents the distance between –9 and 2 on the number line. Then simplify.

Learn: Apply the Order of Operations

Repeated multiplication can be written by using exponential notation. For example, the product $5 \cdot 5 \cdot 5$ can be written as 5^3. In this case, 5 is called the **base** of the expression and 3 is the **exponent (or power)**. The exponent indicates how many times the base is used as a factor.

To find a square root of a nonnegative real number, we reverse the process to square a number. For example, a square root of 25 is a number that when squared equals 25. Both 5 and –5 are square roots of 25, because $5^2 = 25$ and $(-5)^2 = 25$. A radical sign $\sqrt{}$ is used to denote the principal square root of a number. The **principal square root** of a nonnegative real number is the square root that is greater than or equal to zero. Therefore, the principal square root of 25, denoted by $\sqrt{25}$, equals 5.

$$\sqrt{25} = 5 \text{ because } 5 \geq 0 \text{ and } 5^2 = 25$$

Insights

Note that the square of any real number is nonnegative. Therefore, there is no real number that is the square root of a negative number. For example, $\sqrt{-25}$ is not a real number because no real number when squared equals -25. *Note*: The value $\sqrt{-25}$ is an imaginary number and will be discussed in Lesson R-6.

The symbol $\sqrt[3]{}$ represents the cube root of a number. For example:

$$\sqrt[3]{64} = 4 \text{ because } 4^3 = 64$$

Many expressions involve multiple operations. In such a case, it is important to follow the order of operations.

Order of Operations

Step 1: Simplify expressions within parentheses and other grouping symbols. These include absolute value bars, fraction bars, and radicals. If nested grouping symbols are present, start with the innermost symbols.

Step 2: Evaluate expressions involving exponents.

Step 3: Perform multiplication or division in the order in which they occur from left to right.

Step 4: Perform addition or subtraction in the order in which they occur from left to right.

Example 6: Simplifying a Numerical Expression

Simplify. $7 - \{8 + 4[2 - (5 - \sqrt{64})^2]\}$

Solution:

$7 - \{8 + 4[2 - (5 - \sqrt{64})^2]\}$

$= 7 - \{8 + 4[2 - (5 - 8)^2]\}$	Simplify within inner parentheses, $\sqrt{64} = 8$.
$= 7 - \{8 + 4[2 - (-3)^2]\}$	Subtract within the inner parentheses.
$= 7 - [8 + 4(2 - 9)]$	Continue simplifying within the inner parentheses. Simplify $(-3)^2$ to get 9.
$= 7 - [8 + 4(-7)]$	Simplify $(2 - 9)$ to get -7.
$= 7 - (8 - 28)$	Multiply before adding or subtracting.
$= 7 - (-20)$	Simplify within parentheses.
$= 27$	

Apply the Skills

6. Simplify. $50 - \{2 - [\sqrt{121} + 3(-1 - 3)^2]\}$

When simplifying an expression, particular care must be taken with expressions involving division and zero.

Division Involving Zero

To investigate division involving zero, consider the expressions $\frac{5}{0}$, $\frac{0}{5}$, and $\frac{0}{0}$ and their related multiplicative forms.

1. Division by zero is undefined.
   For example: $\frac{5}{0} = n$ implies that $n \cdot 0 = 5$. No number n satisfies this requirement.
2. Zero divided by any nonzero number is zero.
   For example: $\frac{0}{5} = 0$ implies that $0 \cdot 5 = 0$, which is a true statement.
3. We say that $\frac{0}{0}$ is **indeterminate** (cannot be uniquely determined). This concept is investigated in detail in a first course in calculus.
   For example: $\frac{0}{0} = n$ implies that $n \cdot 0 = 0$. This is true for any number n. Therefore, the quotient cannot be uniquely determined.

Learn: Simplify Algebraic Expressions

An algebraic **term** is a product of factors that may include constants and variables. An algebraic **expression** is a single term or the sum of two or more terms. For example, the expression

$$-3xz^2 + \left(-\frac{4}{b}\right) + z\sqrt{x-y} + 5 \text{ has four terms.}$$

The terms $-3xz^2$, $-\frac{4}{b}$, and $z\sqrt{x-y}$ are **variable terms.** The term 5 is not subject to change and is called a **constant term**. The constant factor within each term is called the **coefficient** of the term. Writing the expression as $-3xz^2 + \left(-4 \cdot \frac{1}{b}\right) + 1z\sqrt{x-y} + 5$, we identify the coefficients of the four terms as -3, -4, 1, and 5, respectively.

The properties of real numbers summarized in Table R-4 are often helpful when working with algebraic expressions.

TABLE R-4 Properties of Real Numbers

Let a, b, and c represent real numbers or real-valued expressions.

Property	In Symbols and Words	Examples
Commutative property of addition	$a + b = b + a$ The order in which real numbers are added does not affect the sum.	1. $4 + (-7) = -7 + 4$ 2. $6 + w = w + 6$
Commutative property of multiplication	$a \cdot b = b \cdot a$ The order in which real numbers are multiplied does not affect the product.	1. $5 \cdot (-4) = -4 \cdot 5$ 2. $x \cdot 12 = 12x$
Associative property of addition	$(a + b) + c = a + (b + c)$ The order in which real numbers are grouped under addition does not affect the sum.	1. $(3 + 5) + 2 = 3 + (5 + 2)$ 2. $-9 + (2 + t) = (-9 + 2) + t$ $= -7 + t$
Associative property of multiplication	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$ The order in which real numbers are grouped under multiplication does not affect the product.	1. $(6 \cdot 7) \cdot 3 = 6 \cdot (7 \cdot 3)$ 2. $8 \cdot \left(\frac{1}{8} \cdot y\right) = \left(8 \cdot \frac{1}{8}\right) \cdot y$ $= 1y$
Identity property of addition	$a + 0 = a$ and $0 + a = a$ The number 0 is called the **identity element of addition** because any number plus 0 is the number itself.	1. $-5 + 0 = -5$ 2. $0 + \sqrt{z} = \sqrt{z}$
Identity property of multiplication	$a \cdot 1 = a$ and $1 \cdot a = a$ The number 1 is called the **identity element of multiplication** because any number times 1 is the number itself.	1. $\sqrt{2} \cdot 1 = \sqrt{2}$ 2. $1 \cdot (2w + 3) = 2w + 3$
Inverse property of addition	$a + (-a) = 0$ and $(-a) + a = 0$ For any real number a, the value $-a$ is called the **additive inverse of *a*** (also called the **opposite of *a***). The sum of any number and its additive inverse is the identity element for addition, 0.	1. $4\pi + (-4\pi) = 0$ 2. $-ab^2 + ab^2 = 0$
Inverse property of multiplication	$a \cdot \frac{1}{a} = 1$ and $\frac{1}{a} \cdot a = 1$ where $a \neq 0$ For any nonzero real number a, the value $\frac{1}{a}$ is called the **multiplicative inverse of *a*** (also called the **reciprocal of *a***). The product of any nonzero number and its multiplicative inverse is the identity element of multiplication, 1.	1. $-5 \cdot \left(-\frac{1}{5}\right) = 1$ 2. $\frac{1}{x^2} \cdot x^2 = 1$ for $x \neq 0$ *Note*: The number zero does not have a multiplicative inverse (reciprocal).
Distributive property of multiplication over addition	$a \cdot (b + c) = a \cdot b + a \cdot c$ The product of a number and a sum equals the sum of the products of the number and each term in the sum.	1. $4 \cdot (5 + x) = 4 \cdot 5 + 4 \cdot x$ $= 20 + 4x$ 2. $2(x + \sqrt{3}) = 2x + 2\sqrt{3}$

The distributive property is used to “clear” parentheses when a factor outside parentheses is multiplied by multiple terms inside parentheses. To simplify expressions, we often use the distributive property to clear parentheses and combine like terms.

Two terms are **like terms** if they have the same variables and the corresponding variables are raised to the same power. We can combine like terms by using the distributive property. For example,

$8x^2 + 6x^2 - x^2$	
$= 8x^2 + 6x^2 - 1x^2$	Note that the coefficient on the third term is –1.
$= (8 + 6 - 1)x^2$	Apply the distributive property.
$= 13x^2$	Simplify.

Insights

Although the distributive property is used to add and subtract like terms, it is tedious to write each step. Adding or subtracting like terms can also be done by combining the coefficients and leaving the variable factor unchanged.

$8x^2 + 6x^2 - 1x^2 = 13x^2$ This method will be used throughout the text.

In Example 7, we simplify an expression by applying the distributive property to “clear” parentheses and combine like terms.

Insights

After applying the distributive property, the original parentheses are removed. For this reason, we often call this process “*clearing parentheses*.”

Example 7: Clearing Parentheses and Combining Like Terms

Simplify. $5 - 2(4c - 8d) + 3(1 - d) + c$

Solution:

$5 - 2(4c - 8d) + 3(1 - d) + c$	Apply the distributive property to clear parentheses.
$= 5 - 8c + 16d + 3 - 3d + c$	Combine like terms.
$= 8 - 7c + 13d$	

Apply the Skills

7. Simplify. $12 - 3(5x - 2y) + 5(3 - x) - y$

Learn: Write Algebraic Models

An important skill in mathematics and science is to develop mathematical models. Example 8 offers practice writing algebraic expressions based on verbal statements.

Example 8: Writing an Algebraic Model

a. The maximum recommended heart rate M for adults is the difference of 220 and the person's age a. Write a model to represent an adult's maximum recommended heart rate in terms of age.

b. After eating at a restaurant, it is customary to leave a tip t for the server for at least 15% of the cost of the meal c. Write a model to represent the amount of the tip based on the cost of the meal.

Solution:

a.	$M = 220 - a$	The word "difference" implies subtraction in the order given.
b.	$t \geq 0.15c$	15% of c implies multiplication.

Apply the Skills

8. a. The sale price S on a lawn mower is the difference of the original price P and the amount of discount D. Write a model to represent the sale price.

b. The amount of simple interest I earned on a certain certificate of deposit is 4.65% of the amount of principal invested P. Write a model for the amount of interest.

Practice R-1

Practice Exercises

Concept Connections

1. A(n) ________ is a collection of items called elements.
2. $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is called the set of ________.
3. $\mathbb{R}$ is the notation used to denote the set of ________ ________.
4. A(n) ________ number is a real number that cannot be expressed as a ratio of two integers.
5. Write an absolute value expression to represent the distance between a and b on the real number line.
6. The ________ properties of addition and multiplication indicate that the order in which two real numbers are added or multiplied does not affect the sum or product.
7. The associative property of addition indicates that $a + (b + c) =$ ________, and the associative property of multiplication indicates that $a(bc) =$ ________.
8. The statement $a(b + c) = ab + ac$ represents the ________ property of multiplication over addition.

Learn: Identify Subsets of the Set of Real Numbers

For Exercises 9–10, write a complete sentence in words to represent the algebraic statement.

9. **a.** $3 \in \mathbb{N}$ **b.** $-3.1 \notin \mathbb{W}$

10. **a.** $\frac{2}{5} \in \mathbb{Q}$ **b.** $\pi \notin \mathbb{Q}$

For Exercises 11–12, determine whether the statement is true or false.

11. **a.** $-5 \in \mathbb{N}$ **b.** $-5 \in \mathbb{W}$

c. $-5 \in \mathbb{Z}$ **d.** $-5 \in \mathbb{Q}$

12. **a.** $\frac{1}{3} \in \mathbb{N}$ **b.** $\frac{1}{3} \in \mathbb{W}$

c. $\frac{1}{3} \in \mathbb{Z}$ **d.** $\frac{1}{3} \in \mathbb{Q}$

Learn: Use Inequality Symbols and Interval Notation

For Exercises 13–18, write each statement as an inequality.

13. a is at least 5.

14. b is at most -6.

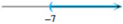
15. $3c$ is no more than 9.

16. $8d$ is no less than 16.

17. The quantity $(m + 4)$ exceeds 70.

18. The quantity $(n - 7)$ is approximately equal to 4.

For Exercises 19–24, write the interval notation and set-builder notation for each given graph. (See Example 1 on p. R7)

19.


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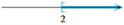

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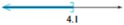
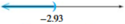
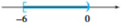
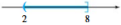


22.


23.


24.


For Exercises 25–30, graph the given set and write the corresponding interval notation. (See Example 1)

25. $\{x \mid x \le 6\}$ **26.** $\{x \mid x < -4\}$

27. $\left\{x \,\middle|\, -\frac{7}{6} < x \le \frac{1}{3}\right\}$ **28.** $\left\{x \,\middle|\, -\frac{4}{3} \le x < \frac{7}{4}\right\}$

29. $\{x \mid 4 < x\}$ **30.** $\{x \mid -3 \le x\}$

For Exercises 31–36, interval notation is given for several sets of real numbers. Graph the set and write the corresponding set-builder notation. (See Example 1 on p. R7)

31. $(-3, 7]$ **32.** $[-4, -1)$

33. $(-\infty, 6.7]$ **34.** $(-\infty, -3.2)$

35. $\left[-\frac{3}{5}, \infty\right)$ **36.** $\left(\frac{7}{8}, \infty\right)$

Learn: Find the Union and Intersection of Sets

For Exercises 37–40, refer to sets *A*, *B*, *C*, *X*, *Y*, and *Z* and find the union or intersection of sets as indicated. (See Example 2 on p. R9)

$A = \{0, 4, 8, 12\}$, $B = \{0, 3, 6, 9, 12\}$, $C = \{-2, 4, 8\}$

$X = \{1, 2, 3, 4, 5\}$, $Y = \{1, 2, 3\}$, $Z = \{6, 7, 8\}$

37. a. $A \cup B$ **b.** $A \cap B$
c. $A \cup C$ **d.** $A \cap C$
e. $B \cup C$ **f.** $B \cap C$

38. a. $X \cup Z$ **b.** $Y \cup Z$
c. $Y \cap Z$ **d.** $X \cup Y$
e. $X \cap Y$ **f.** $X \cap Z$

39. a. $A \cup X$ **b.** $A \cap Z$
c. $C \cap Y$ **d.** $B \cap Y$
e. $C \cup Z$ **f.** $A \cup Z$

40. a. $A \cap X$ **b.** $B \cup Z$
c. $C \cup Y$ **d.** $B \cap Z$
e. $C \cap Z$ **f.** $A \cup Y$

41. Refer to sets *C*, *D*, and *F* and find the union or intersection of sets as indicated. Write the answers in set notation. (**See Example 3 on p. R9**)

$C = \{x \mid x < 9\}$, $D = \{x \mid x \geq -1\}$, $F = \{x \mid x < -8\}$

a. $C \cup D$ **b.** $C \cap D$ **c.** $C \cup F$
d. $C \cap F$ **e.** $D \cup F$ **f.** $D \cap F$

42. Refer to sets *M*, *N*, and *P* and find the union or intersection of sets as indicated. Write the answers in set notation.

$M = \{y \mid y \geq -3\}$, $N = \{y \mid y \geq 5\}$, $P = \{y \mid y < 0\}$

a. $M \cup N$ **b.** $M \cap N$ **c.** $M \cup P$
d. $M \cap P$ **e.** $N \cup P$ **f.** $N \cap P$

For Exercises 43–46, find the union or intersection of the given intervals. Write the answers in interval notation.

43. a. $(-\infty, 4) \cup (-2, 1]$
b. $(-\infty, 4) \cap (-2, 1]$

44. a. $[0, 5) \cup [-1, \infty)$
b. $[0, 5) \cap [-1, \infty)$

45. a. $(-\infty, 5) \cup [3, \infty)$
b. $(-\infty, 5) \cap [3, \infty)$

46. a. $(-\infty, -1] \cup [-4, \infty)$
b. $(-\infty, -1] \cap [-4, \infty)$

Learn: Evaluate Absolute Value Expressions

For Exercises 47–58, simplify each expression by writing the expression without absolute value bars. (See Example 4 on p. R11)

47. $|-6|$ **48.** $|-4|$
49. $|0|$ **50.** $|1|$
51. $|\sqrt{17} - 5|$ **52.** $|\sqrt{6} - 6|$

53. a. $|\pi - 3|$
b. $|3 - \pi|$

54. a. $|m - 11|$ for $m \geq 11$
b. $|m - 11|$ for $m < 11$

55. a. $|x + 2|$ for $x \geq -2$
b. $|x + 2|$ for $x < -2$

56. a. $|t + 6|$ for $t < -6$
b. $|t + 6|$ for $t \geq -6$

57. a. $\frac{|z - 5|}{z - 5}$ for $z > 5$
b. $\frac{|z - 5|}{z - 5}$ for $z < 5$

58. a. $\frac{7 - x}{|7 - x|}$ for $x < 7$
b. $\frac{7 - x}{|7 - x|}$ for $z > 7$

Learn: Use Absolute Value to Represent Distance

For Exercises 59–64, write an absolute value expression to represent the distance between the two points on the number line. Then simplify without absolute value bars. (See Example 5 on p. R12)

59. 1 and 6 **60.** 2 and 9
61. 3 and -4 **62.** -8 and 2
63. 6 and 2π **64.** 3 and π

Learn: Apply the Order of Operations

For Exercises 65–78, simplify the expressions. (See Example 6 on p. R13)

65 a. 4^2 **b.** $(-4)^2$ **c.** -4^2

d. $\sqrt{4}$ **e.** $-\sqrt{4}$ **f.** $-\sqrt{4}$

66. a. 9^2 **b.** $(-9)^2$ **c.** -9^2

d. $\sqrt{9}$ **e.** $-\sqrt{9}$ **f.** $\sqrt{-9}$

67. a $\sqrt[3]{8}$ **b.** $\sqrt[3]{-8}$ **c.** $-\sqrt[3]{8}$

d. $\sqrt{100}$ **e.** $\sqrt{-100}$ **f.** $-\sqrt{100}$

68. a. $\sqrt[3]{27}$ **b.** $\sqrt[3]{-27}$ **c.** $-\sqrt[3]{27}$

d. $\sqrt{49}$ **e.** $\sqrt{-49}$ **f.** $-\sqrt{49}$

69. $20 - 12(36 \div 3^2 \div 2)$

70. $200 - 2^2(6 \div \frac{1}{2} \cdot 4)$

71. $6 - \{-12 + 3[(1 - 6)^2 - 18]\}$

72. $-5 - \{4 - 6[(2 - 8)^2 - 31]\}$

73. $-4 \cdot \left(\frac{2}{5} - \frac{7}{10}\right)^2$

74. $6 \cdot \left[\left(\frac{1}{3}\right)^2 - \left(\frac{1}{2}\right)^2\right]$

75. $9 - (6 + ||3 - 7| - 8|) \div \sqrt{25}$

76. $8 - 2(4 + ||2 - 5| - 5|) \div \sqrt{9}$

77. $\dfrac{|11 - 13| - 4 \cdot 2}{\sqrt{12^2 + 5^2} - 3 - 10}$

78. $\dfrac{(4 - 9)^2 + 2^2 - 3^2}{|-7 + 4| + (-12) \div 4}$

79. $\dfrac{\sqrt{25^2 - 24^2} - 14 \div 2}{20 + 4^2 - 8 \div 2}$

80. $\dfrac{5(2) - \sqrt{6^2 + 8^2}}{9 - 2^3 - 10 \cdot 6}$

Learn: Simplify Algebraic Expressions

For Exercises 81–86, combine like terms.

81. $-14w^3 - 3w^3 + w^3$

82. $12t^5 - t^5 - 6t^5$

83. $3.9x^3y - 2.2xy^3 + 5.1x^3y - 4.7xy^3$

84. $0.004m^4n - 0.005m^3n^2 - 0.01m^4n + 0.007m^3n^2$

85. $\frac{1}{3}c^7d + \frac{1}{2}cd^7 - \frac{2}{5}c^7d - 2cd^7$

86. $\frac{1}{10}yz^4 - \frac{3}{4}y^4z + yz^4 + \frac{3}{2}y^4z$

For Exercises 87–92, apply the distributive property.

87. $-(4x - \pi)$ **88.** $-\left(\frac{1}{2}k - \sqrt{7}\right)$

89. $-8(3x^2 + 2x - 1)$ **90.** $-6(-5y^2 - 3y + 4)$

91. $\frac{2}{3}(-6x^2y - 18yz^2 + 2z^3)$

92. $\frac{3}{4}(12p^5q - 8p^4q^2 - 6p^3q)$

For Exercises 93–100, simplify each expression. (See Example 7 on p. R16)

93. $2(4w + 8) + 7(2w - 4) + 12$

94. $3(2z - 4) + 8(z - 9) + 84$

95. $-(4u - 8v) - 3(7u - 2v) + 2v$

96. $-(10x - z) - 2(8x - 4z) - 3x$

97. $12 - 4[(8 - 2v) + 5(-3w - 4v)] - w$

98. $6 - 2[(9z + 6y) - 8(y - z)] - 11$

99. $2y^2 - [13 - \frac{2}{3}(6y^2 - 9) - 10] + 9$

100. $6 - [5t^2 - \frac{3}{4}(12 - 8t^2) + 5] + 11t^2$

Learn: Write Algebraic Models

101. Jorge is 1 yr younger than Caesar.

a. Write a model for Jorge's age *J* in terms of Caesar's age *C*.

b. Write a model for Caesar's age *C* in terms of Jorge's age *J*. (**See Example 8 on p. R17**)

102. For a recent NFL season, Patrick Mahomes had 12 more touchdown passes than Lamar Jackson.

a. Write a model for the number of touchdown passes *M* thrown by Mahomes in terms of the number thrown by Jackson *J*.

b. Write a model for the number of touchdown passes *J* thrown by Jackson in terms of the number thrown by Mahomes *M*.

103. At the end of the summer, a store discounts an outdoor grill for at least 25% of the original price. If the original price is *P*, write a model for the amount of the discount *D*. (**See Example 8 on p. R17**)

104. When Ms. Celano has excellent service at a restaurant, the amount she leaves for a tip t is at least 20% of the cost of the meal c. Write a model representing the amount of the tip.

105. Suppose that an object is dropped from a height h. Its velocity v at impact with the ground is given by the square root of twice the product of the acceleration due to gravity g and the height h. Write a model to represent the velocity of the object at impact.

106. The height of a sunflower plant can be determined by the time t in weeks after the seed has germinated. Write a model to represent the height h if the height is given by the product of 8 and the square root of t.

107. A power company charges one household \$0.16 per kilowatt-hour (kWh) and \$24.89 in monthly taxes.

- **a.** Write a formula for the monthly charge C for this household if it uses k kilowatt-hours.
- **b.** Compute the monthly charge if the household uses 1200 kWh.

108. A utility company charges a base rate for water usage of \$19.50 per month, plus \$4.58 for every additional 1000 gal of water used over a 2000-gal base.

- **a.** Write a formula for the monthly charge C for a household that uses n thousand gallons over the 2000-gal base.
- **b.** Compute the cost for a family that uses a total of 6000 gal of water for a given month.

109. The cost C (in \$) to rent an apartment is \$1100 per month, plus a \$500 nonrefundable security deposit, plus a \$200 nonrefundable deposit for each dog or cat.

- **a.** Write a formula for the total cost to rent an apartment for m months with n cats/dogs.
- **b.** Determine the cost to rent the apartment for 12 months, with 2 cats and 1 dog.

110. For a certain college, the cost C (in \$) for taking classes the first semester is \$620 per credit-hour, \$75 for each lab, plus a one-time admissions fee of \$100.

- **a.** Write a formula for the total cost to take n credit-hours and L labs the first semester.
- **b.** Determine the cost for the first semester if a student takes 12 credit-hours with 2 labs.

Mixed Exercises

For Exercises 111–128, simplify each expression.

111. **a.** $3(x-3)-5(2x+1)$

b. $-3(x-3)+5(2x+1)$

c. $3(x+3)-5(2x-1)$

112. **a.** $-4[2(x-1)+6x-3]$

b. $4-[2(x-1)+6x-3]$

c. $4-[2(x-1)+6(x-3)]$

113. $25-2[(7-3)^2 \div 4]+\sqrt{18-2}$

114. $\sqrt{29-2^2}+[8-3(6-2)]\div 4\cdot 5$

115. $\left(\frac{1}{2}\right)^2+\left(\frac{6-4}{5}\right)^2+\left(\frac{5+2}{10}\right)^2$

116. $\left(\frac{2^3}{2^3+1}\right)^2\div\left(\frac{8-(-2)}{3^2}\right)^2$

117. $3.1(2x+2)-4(1.2x-1)$

118. $4.5(5-y)+3(1.9y+1)$

119. $2\left[5\left(\frac{1}{2}a+3\right)-(a^2+a)+4\right]$

120. $-3\left[3\left(b-\frac{2}{3}\right)-2(b+4)-6b^2\right]$

121. $2.2\{4-8[6x-1.5(x+4)-6]+7.5x\}$

122. $-3.2-\{6.1y-4[9-(2y+2.5)]+7y\}$

123. $\frac{1}{8}(24n-16m)-\frac{2}{5}(3m-18n-2)+\frac{3}{2}$

124. $\frac{1}{5}(25a-20b)-\frac{4}{7}(21a-14b+2)+\frac{1}{7}$

125. $12-4[(8-2v)+5(-3w-4v)]-w$

126. $6-2[(9z+6y)-8(y-z)]-11$

127. $2y^2-\left[13-\frac{2}{3}(6y^2-9)-10\right]+9$

128. $6-\left[5t^2-\frac{3}{4}(12-8t^2)+5\right]+11t^2$

For Exercises 129–134, evaluate each expression for the given values of the variables.

129. $\frac{-b}{2a}$ for $a = -1, b = -6$

130. $\sqrt{b^2 - 4ac}$ for $a = 2, b = -6, c = 4$

131. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ for $x_1 = 2, x_2 = -1, y_1 = -4, y_2 = 1$

132. $\frac{y_2 - y_1}{x_2 - x_1}$ for $x_1 = -1.4, x_2 = 2, y_1 = 3.1, y_2 = -3.7$

133. $\frac{1}{3}\pi r^2 h$ for $\pi \approx \frac{22}{7}, r = 7, h = 6$

134. $\frac{4}{3}\pi r^3$ for $\pi \approx \frac{22}{7}, r = 3,$

135. Under selected conditions, a sports car gets 15 mpg in city driving and 25 mpg for highway driving. The model $G = \frac{1}{15}c + \frac{1}{25}h$ represents the amount of gasoline used (in gal) for c miles driven in the city and h miles driven on the highway. Determine the amount of gas required to drive 240 mi in the city and 500 mi on the highway.

136. Under selected conditions, a sedan gets 22 mpg in city driving and 32 mpg for highway driving. The model $G = \frac{1}{22}c + \frac{1}{32}h$ represents the amount of gasoline used (in gal) for c miles driven in the city and h miles driven on the highway. Determine the amount of gas required to drive 220 mi in the city and 512 mi on the highway.

Write About It

137. When is a parenthesis used when writing interval notation?

138. When is a bracket used when writing interval notation?

139. Explain the difference between the commutative property of addition and the associative property of addition.

140. Explain why 0 has no multiplicative inverse.

Expanding Your Skills

141. If $n > 0$, then $n - |n| =$ ____________.

142. If $n < 0$, then $n - |n| =$ ____________.

143. If $n > 0$, then $n + |n| =$ ____________.

144. If $n < 0$, then $n + |n| =$ ____________.

145. If $n > 0$, then $-|n| =$ ____________.

146. If $n < 0$, then $-|n| =$ ____________.

For Exercises 147–150, determine the sign of the expression. Assume that a, b, and c are real numbers and $a < 0$, $b > 0$, and $c < 0$.

147. $\frac{ab^2}{c^3}$

148. $\frac{a^2c}{b^4}$

149. $\frac{b(a + c)^3}{a^2}$

150. $\frac{(a + b)^2(b + c)^4}{b}$

For Exercises 151–154, write the set as a single interval.

151. $(-\infty, 2) \cap (-3, 4] \cap [1, 3]$

152. $(-\infty, 5) \cap (-1, \infty) \cap [0, 3)$

153. $[(-\infty, -2) \cup (4, \infty)] \cap [-5, 3)$

154. $[(-\infty, 6) \cup (10, \infty)] \cap [8, 12)$

Technology Connections

For Exercises 155–158, use a calculator to approximate the expression to 2 decimal places.

155. $5000\left(1 + \frac{0.06}{12}\right)^{(12)(5)}$

156. $8500\left(1 + \frac{0.05}{4}\right)^{(4)(30)}$

157. $\frac{-3 + 5\sqrt{2}}{7}$

158. $\frac{6 - 3\sqrt{5}}{4}$

Point of Interest

Many civilizations created symbols for numbers, but, in ancient mathematical texts, we do not find symbols for the operations +, −, ×, and ÷. Although Egyptians did have a hieroglyphic symbol that resembled a pair of walking legs to indicate addition or subtraction, the modern symbols of plus and minus appeared much later. The first appearance of the plus sign was in the 1300s; the minus sign appeared in the 15th century. The division symbol we now use appears to be a combination of a colon : used by William Oughtred and an Arab line symbol. William Oughtred's *Clavis Mathematicae* (1631) also introduced the cross symbol for multiplication.

For Exercises 159–162, insert each symbol +, −, ×, and ÷ one time to create a true statement. Multiple correct solutions are possible.

159. $2\square 2\square 2\square 2\square 2=-1$

160. $2\square 2\square 2\square 2\square 2=2$

161. $2\square 2\square 2\square 2\square 2=3$

162. $2\square 2\square 2\square 2\square 2=5$

Many numeric constants are also represented by specific letters or symbols. The irrational number *pi* was known as a ratio in antiquity, but it took many centuries before we saw the Greek letter π as a representation for a specific irrational number.

The Greek letter $\varphi\left(\varphi=\frac{1+\sqrt{5}}{2}\approx 1.618\right)$ is the designation for the value termed the *golden* ratio, *golden* mean, or *divine proportion*. Note that some symbols, such as the symbol for infinity, ∞, do not refer to an operation or a constant. The symbols ∞ and $-\infty$ refer more to *concepts* such as increasing or decreasing behavior without bound.

Further Reading

Mazur, Joseph. *Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers,* Princeton University Press, 2014.

Livio, Mario. *The Golden Ratio: The Story of Phi, The World's Most Astonishing Number,* Broadway Books, 2003.

Maor, Eli. *To Infinity and Beyond: A Cultural History of the Infinite,* Princeton University Press, 1991.

Go online for more practice problems.

Lesson R-2

Exponents and Radicals

What Will You Learn?

After completing this lesson, you should be able to:

- Apply Properties of Exponents
- Apply Scientific Notation
- Evaluate *n*th-Roots
- Simplify Expressions of the Forms $a^{1/n}$ and $a^{m/n}$
- Simplify Radicals
- Apply Operations on Radicals

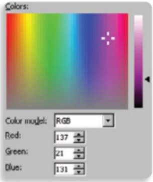
Learn: Apply Properties of Exponents

Recall that exponents are used to represent repeated multiplication. Applications of exponents appear in many fields of study, including computer science. Computer engineers define a *bit* as a fundamental unit of information having just two possible values. These values are represented by either 0 or 1. A *byte* is usually taken as 8 bits. Computer programmers know that there are 2^n possible values for an n-bit variable. So 1 byte has $2^8 = 256$ possible values.

Three bytes are often used to represent color on a computer screen. The intensity of each of the colors red, green, and blue ranges from 0 to 255 (a total of 256 possible values each). So the number of colors that can be represented by this system is

$$2^8 \cdot 2^8 \cdot 2^8 = (256)(256)(256) = 16{,}777{,}216$$

There are over 16 million possible colors available using this system. For example, the color given by red 137, green 21, blue 131, is a deep pink. See Figure R-2.



Color model: RGB
Red: 137
Green: 21
Blue: 131

Figure R-2

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The product $2^8 \cdot 2^8 \cdot 2^8$ can also be written out as follows.

$$(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)$$

$$= 2^{8+8+8} = 2^{24} = 16{,}777{,}216.$$

A similar approach can be used to show that $b^m b^n = b^{m+n}$ for natural numbers m and n.

We would like to extend this result to expressions where m and n are negative integers or zero. For example,

$\underline{b^0} \cdot b^4 = b^{0+4} = b^4$ For this to be true, the value b^0 must be 1.

$1 \cdot b^4 = b^4$

Now consider an example involving a negative exponent.

$\underline{b^{-4}} \cdot b^4 = b^{-4+4} = b^0 = 1$ For the product $b^{-4} \cdot b^4$ to be equal to 1, it follows

$\frac{1}{b^4} \cdot b^4 = 1$ that b^{-4} must be the reciprocal of b^4. That is, $b^{-4} = \frac{1}{b^4}$

These observations lead to two important definitions.

Definition of b^0 and b^{-n}

If b is a nonzero real number and n is a positive integer, then

- $b^0 = 1$ Examples: $(5000)^0 = 1$ and $(-3x)^0 = 1$ for $x \neq 0$
- $b^{-n} = \left(\frac{1}{b}\right)^n = \frac{1}{b^n}$ Examples: $4^{-1} = \left(\frac{1}{4}\right)^1 = \frac{1}{4}$ and $x^{-3} = \left(\frac{1}{x}\right)^3 = \frac{1}{x^3}$ for $x \neq 0$

The definitions given here have two important restrictions.

By definition, $b^0 = 1$ provided that $b \neq 0$. Therefore,

- The value of 0^0 is not defined here. The value of 0^0 is said to be indeterminate and is examined in calculus.

For a positive integer n, by definition, $b^{-n} = \frac{1}{b^n}$ provided that $b \neq 0$. Therefore,

- The value of 0^{-n} is not defined here.

All examples and exercises in the text will be given under the assumption that the variable expressions avoid these restrictions. For example, the expression x^0 will be stated with the implied restriction that $x \neq 0$.

Example 1: Simplifying Expressions with a Zero or Negative Exponent

Simplify.

a. $6y^0$ **b.** -5^0 **c.** 5^{-2} **d.** $\frac{1}{p^{-4}}$ **e.** $5x^{-8}y^2$

Solution:

a. $6y^0 = 6 \cdot y^0 = 6 \cdot 1 = 6$	The base for the exponent of 0 is y, not $6y$. The product of 6 and y^0 is $6 \cdot 1 = 6$.
b. $-5^0 = -1 \cdot 5^0 = -1 \cdot 1 = -1$	This is interpreted as the opposite of 5^0 or $-1 \cdot 5^0$.
c. $5^{-2} = \frac{1}{5^2}$ or $\frac{1}{25}$	Rewrite the expression using the reciprocal of 5 and change the exponent to positive 2.
d. $\frac{1}{p^{-4}} = \frac{1}{\frac{1}{p^4}} = 1 \cdot \frac{p^4}{1} = p^4$	Rewrite p^{-4} as $\frac{1}{p^4}$. Then divide fractions.
e. $5x^{-8}y^2 = 5 \cdot x^{-8} \cdot y^2$ $= 5 \cdot \frac{1}{x^8} \cdot y^2 = \frac{5y^2}{x^8}$	The exponent of -8 applies to x only, not to the factors of 5 or y.

Apply the Skills

1. Simplify.

a. $4z^0$ **b.** -4^0 **c.** 2^{-3} **d.** $\frac{1}{n^{-5}}$ **e.** $-3a^4b^{-9}$

The property $b^m b^n = b^{m+n}$ is one of several important properties of exponents that can be used to simplify an algebraic expression (Table R-5).

TABLE R-5 Properties of Exponents

Let a and b be real numbers and m and n be integers.

Property*	Example	Expanded Form
$b^m \cdot b^n = b^{m+n}$	$x^4 \cdot x^3 = x^{4+3} = x^7$	$x^4 \cdot x^3 = (x \cdot x \cdot x \cdot x)(x \cdot x \cdot x) = x^7$
$\frac{b^m}{b^n} = b^{m-n}$	$\frac{t^6}{t^2} = t^{6-2} = t^4$	$\frac{t^6}{t^2} = \frac{t \cdot t \cdot t \cdot t \cdot t \cdot t}{t \cdot t} = t^4$
$(b^m)^n = b^{m \cdot n}$	$(x^2)^3 = x^{2 \cdot 3} = x^6$	$(x^2)^3 = (x^2)(x^2)(x^2) = x^6$
$(ab)^m = a^m b^m$	$(4x)^3 = 4^3x^3$	$(4x)^3 = (4x)(4x)(4x)$ $= (4 \cdot 4 \cdot 4)(x \cdot x \cdot x)$ $= 4^3x^3$
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{2}{y}\right)^2 = \frac{2^2}{y^2}$	$\left(\frac{2}{y}\right)^2 = \left(\frac{2}{y}\right) \cdot \left(\frac{2}{y}\right) = \frac{2^2}{y^2}$

*The properties are stated under the assumption that the variables are restricted to avoid the expressions 0^0 and $\frac{1}{0}$. Expressions of the form 0^0 and $\frac{1}{0}$ are said to be indeterminate and are examined in calculus.

Example 2: Simplifying Expressions Containing Exponents

Simplify. Write the answers with positive exponents only.

a. $\left(\frac{14a^2b^7}{2a^5b}\right)^{-2}$ **b.** $(-3x)^{-4}(4x^{-2}y^3)^3$

Solution:

a. $\left(\frac{14a^2b^7}{2a^5b}\right)^{-2} = (7a^{2-5}b^{7-1})^{-2}$	Simplify within parentheses first. To divide common bases, subtract the exponents.
$= (7a^{-3}b^6)^{-2}$	
$= (7)^{-2}(a^{-3})^{-2}(b^6)^{-2}$	Raise each factor within parentheses to the −2 power.
$= 7^{-2}a^6b^{-12}$	
$= \frac{1}{7^2} \cdot a^6 \cdot \frac{1}{b^{12}}$	Simplify factors with negative exponents.
$= \frac{a^6}{49b^{12}}$	Simplify.
b. $(-3x) - 4(4x^{-2}y^3)^3$	
$= (-3)^{-4}x^{-4} \cdot 4^3(x^{-2})^3(y^3)^3$	Raise each factor inside parentheses to the power outside parentheses.
$= (-3)^{-4} \cdot 4^3 \cdot x^{-4} \cdot x^{-6} \cdot y^9$	Regroup factors.
$= (-3)^{-4} \cdot 4^3 \cdot x^{-10} \cdot y^9$	To multiply the factors of *x*, add the exponents.
$= \frac{4^3y^9}{(-3)^4x^{10}}$	Simplify factors with negative exponents.
$= \frac{64y^9}{81x^{10}}$	Simplify.

Apply the Skills

2. Simplify. Write the answers with positive exponents only.

a. $(3w)^{-3}(5wt^5)^2$ **b.** $\left(\frac{26c^5d^{-2}}{13cd^8}\right)^{-3}$

Learn: Apply Scientific Notation

In many applications of science, technology, and business we encounter very large or very small numbers. For example:

- San Francisco has some of the highest home prices in the United States. One of the most expensive homes recently sold for $45,000,000.
- The diameter of a capillary is measured as 0.000005 m.
- The mean surface temperature of the planet Saturn is −300°F.

Very large and very small numbers are sometimes cumbersome to write because they contain numerous zeros. Furthermore, it is difficult to determine the location of the decimal point when performing calculations with such numbers. For these reasons, scientists will often write numbers using scientific notation.

Scientific Notation

A number expressed in the form $a \times 10^n$, where $1 \le |a| < 10$ and n is an integer is said to be in **scientific notation.**

Examples:

Home Price	Capillary Size	Saturn Temperature
\$45,000,000	0.000005 m	–300 °F
$= \$4.5 \times 10{,}000{,}000$	$= 5.0 \times 0.000001$ m	$= -3 \times 100$ °F
$= \$4.5 \times 10^7$	$= 5.0 \times 10^{-6}$ m	$= -3 \times 10^2$ °F

To write a number in scientific notation, the number of positions that the decimal point must be moved determines the power of 10. Numbers 10 or greater require a positive exponent on 10. Numbers between 0 and 1 require a negative exponent on 10.

Example 3: Writing Numbers in Scientific Notation and Standard Decimal Notation

Write the numbers in scientific notation or standard decimal notation.

a. 0.0000002 **b.** 230 billion **c.** 1.36×10^7 **d.** 3.9×10^{-3}

Solution:

a. $0.\underbrace{0000002}_{\text{7 place positions}} = 2.0 \times 10^{-7}$

The number 0.0000002 is between 0 and 1. Use a negative power of 10.

b. 230 billion = 230,000,000,000

First write 230 billion in standard form.

$2\underbrace{30{,}000{,}000{,}000}_{\text{11 place positions}} = 2.3 \times 10^{11}$

The number 230 billion is greater than 10. Use a positive power of 10.

c. $1.36 \times 10^7 = 1\underbrace{3{,}600{,}000}_{\text{7 place positions}}$

10^7 is greater than 10. Move the decimal point 7 places to the right. Insert zeros to the right as needed.

d. $3.9 \times 10^{-3} = 0.\underbrace{003}_{\text{3 place positions}}9$

10^{-3} is between 0 and 1. Move the decimal point 3 places to the left. Insert zeros to the left as needed.

Apply the Skills

3. Write the numbers in scientific notation or standard decimal notation.

a. 0.0000035 **b.** 49,500 **c.** 5.86×10^2 **d.** 5.0×10^{-2}

Example 4 demonstrates the process to multiply numbers written in scientific notation.

Example 4: Performing Calculations with Scientific Notation

A light-year is the distance that light travels in 1 yr. If light travels at a speed of 6.7×10^8 mph, how far will it travel in 1 yr (8.76×10^3 hr)?

Solution:

Distance = (Rate)(Time)

$= (6.7 \times 10^8 \text{ mph})(8.76 \times 10^3 \text{ hr})$

$= (6.7)(8.76) \times (10^8)(10^3)$ mi — Regroup factors.

$= 58.692 \times 10^{11}$ mi — Multiply and add the powers of 10.

$= (5.8692 \times 10^1) \times 10^{11}$ mi — The number 58.692 is not between 1 and 10. Rewrite this as 5.8692×10^1.

$= 5.8692 \times 10^{12}$ mi — One light-year is approximately 5.87 trillion miles.

Apply the Skills

4. A satellite travels 1.72×10^4 mph. How far does it travel in 24 hr (2.4×10^1 hr)?

Learn: Evaluate *n*th-Roots

We now want to extend the definition of b^n to expressions in which the exponent, n, is a rational number.

First we need to understand the relationship between *n*th-powers and *n*th-roots. From Lesson R-1, we know that for $a \geq 0$, $\sqrt{a} = b$ if $b^2 = a$ and $b \geq 0$. Square roots are a special case of *n*th-roots.

Definition of an *n*th-Root

For a positive integer $n > 1$, the principal *n*th-root of a, denoted by $\sqrt[n]{a}$, is a number b such that

$$\sqrt[n]{a} = b \text{ means that } b^n = a.$$

If n is even, then we require that $a \geq 0$ and $b \geq 0$.

For the expression $\sqrt[n]{a}$, the symbol $\sqrt[n]{}$ is called a **radical sign,** the value a is called the **radicand,** and n is called the **index.**

Example 5: Simplifying *n*th-Roots

Simplify.

a. $\sqrt[5]{32}$ **b.** $\sqrt{\frac{49}{64}}$ **c.** $\sqrt[3]{-0.008}$ **d.** $\sqrt[4]{-1}$ **e.** $-\sqrt[4]{1}$

Solution:

a. $\sqrt[5]{32} = 2$ — $\sqrt[5]{32} = 2$ because $2^5 = 32$.

b. $\sqrt{\frac{49}{64}} = \frac{7}{8}$ — $\sqrt{\frac{49}{64}} = \frac{7}{8}$ because $\frac{7}{8} \geq 0$ and $\left(\frac{7}{8}\right)^2 = \frac{49}{64}$.

c. $\sqrt[3]{-0.008} = -0.2$ — $\sqrt[3]{-0.008} = -0.2$ because $(-0.2)^3 = -0.008$.

d. $\sqrt[4]{-1}$ is not a real number — $\sqrt[4]{-1}$ is not a real number because no real number when raised to the fourth power equals –1.

e. $-\sqrt[4]{1} = -1 \cdot \sqrt[4]{1}$ — $-\sqrt[4]{1}$ is interpreted as $-1\sqrt[4]{1}$. The factor of –1 is outside the radical.

$$= -1 \cdot 1$$
$$= -1$$

Apply the Skills

5. Simplify.

a. $\sqrt[3]{-125}$ **b.** $\sqrt{\frac{144}{121}}$ **c.** $\sqrt[5]{0.00001}$ **d.** $\sqrt[6]{-64}$ **e.** $-\sqrt[6]{64}$

Learn: Simplify Expressions of the Forms $a^{1/n}$ and $a^{m/n}$

Next, we want to define an expression of the form a^n, where n is a rational number. Furthermore, we want a definition for which the properties of integer exponents can be extended to rational exponents. For example, we want

$$(25^{1/2})^2 = 25^{(1/2)\cdot 2} = 25^1 = 25$$

$25^{1/2}$ must be a square root of 25, because when squared, it equals 25.

Definition of an $a^{1/n}$ and $a^{m/n}$

Let m and n be integers such that m/n is a rational number in lowest terms and $n > 1$. Then,

$$a^{1/n} = \sqrt[n]{a} \quad \text{and} \quad a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

If n is even, we require that $a \geq 0$.

The definition of $a^{m/n}$ indicates that $a^{m/n}$ can be written as a radical whose index is the denominator of the rational exponent. The order in which the nth-root and exponent m are performed within the radical does not affect the outcome. For example:

Take the 4th root first:
$$\begin{aligned} 16^{3/4} &= (\sqrt[4]{16})^3 \\ &= (2)^3 \\ &= 8 \end{aligned}$$

or Cube 16 first:
$$\begin{aligned} 16^{3/4} &= \sqrt[4]{16^3} \\ &= \sqrt[4]{4096} \\ &= 8 \end{aligned}$$

Example 6: Simplifying Expressions of the Form $a^{1/n}$ and $a^{m/n}$

Write the expressions using radical notation, and simplify if possible.

a. $25^{1/2}$ **b.** $\left(\frac{64}{27}\right)^{1/3}$ **c.** $-81^{1/4}$ **d.** $32^{3/5}$ **e.** $81^{-3/4}$

Solution:

a. $25^{1/2} = \sqrt{25} = 5$

b. $\left(\frac{64}{27}\right)^{1/3} = \sqrt[3]{\frac{64}{27}} = \frac{4}{3}$

c. $-81^{1/4}$ is undefined because $\sqrt[4]{-81}$ is not a real number.

d. $32^{3/5} = \sqrt[5]{32^3} = (\sqrt[5]{32})^3 = (2)^3 = 8$

e. $81^{-3/4} = \frac{1}{81^{3/4}}$ Rewrite the expression to remove the negative exponent.

$= \frac{1}{(\sqrt[4]{81})^3} = \frac{1}{(3)^3} = \frac{1}{27}$ Rewrite $81^{3/4}$ using radical notation.

Apply the Skills

6. Simplify the expressions if possible.

a. $36^{1/2}$ **b.** $\left(\frac{1}{125}\right)^{1/3}$ **c.** $(-9)^{1/2}$ **d.** $(-1)^{-4/3}$ **e.** $(-125)^{-2/3}$

Learn: Simplify Radicals

In Example 6, we simplified several expressions with rational exponents. Next, we want to simplify radical expressions. First consider expressions of the form $\sqrt[n]{a^n}$. The value of $\sqrt[n]{a^n}$ is not necessarily a. Since $\sqrt[n]{a}$ represents the principal nth-root of a, then $\sqrt[n]{a}$ must be nonnegative for even values of n. For example:

$$\sqrt{(5)^2} = 5 \quad \text{and} \quad \sqrt{(-5)^2} = |-5| = 5$$

The absolute value is needed here to guarantee a nonnegative result.

$$\sqrt[4]{(2)^4} = 2 \quad \text{and} \quad \sqrt[4]{(-2)^4} = |-2| = 2$$

In Table R-6, we generalize this result and give three other important properties of radicals.

TABLE R-6 Properties of Radicals

Let $n > 1$ be an integer and a and b be real numbers. The following properties are true provided that the given radicals are real numbers.

Property	**Examples**
1. If n is even, $\sqrt[n]{a^n} = \lvert a \rvert$	$\sqrt{x^2} = \lvert x \rvert$ $\sqrt[4]{x^8} = \lvert x^2 \rvert = x^2$
2. If n is odd, $\sqrt[n]{a^n} = a$.	$\sqrt[3]{x^3} = x$ $\sqrt[5]{(y+4)^5} = y + 4$
3. Product Property $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt[3]{7} \cdot \sqrt[3]{x} = \sqrt[3]{7x}$ $\sqrt{75} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$
4. Quotient Property $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$	$\dfrac{\sqrt{125}}{\sqrt{5}} = \sqrt{\dfrac{125}{5}} = \sqrt{25} = 5$
5. Nested Radical Property $\sqrt[m]{\sqrt[n]{a}} = \sqrt[m \cdot n]{a}$	$\sqrt[4]{\sqrt[3]{x}} = (x^{1/3})^{1/4} = x^{(1/3)(1/4)} = x^{1/12} = \sqrt[12]{x}$

Properties 1–5 follow from the definition of $a^{1/n}$ and the properties of exponents. We use these properties to simplify radical expressions and must address four specific criteria.

Simplified Form of a Radical

Suppose that the radicand of a radical is written as a product of prime factors. Then the radical is simplified if all of the following conditions are met.

1. The radicand has no factor other than 1 that is a perfect nth-power. This means that all exponents in the radicand must be less than the index.
2. No fractions may appear in the radicand.
3. No denominator of a fraction may contain a radical.
4. The exponents in the radicand may not all share a common factor with the index.

In Example 7, we simplify expressions that fail condition 1. Also notice that the expressions in Example 7 are assumed to have positive radicands. This eliminates the need to insert absolute value bars around the simplified form of $\sqrt[n]{a^n}$.

Example 7: Simplifying Radicals Using the Product Property

Simplify each expression. Assume that all variables represent positive real numbers.

a. $\sqrt[3]{c^5}$ **b.** $\sqrt{50}$ **c.** $\sqrt[4]{32x^9y^6}$

Solution:

a. $\sqrt[3]{c^5} = \sqrt[3]{c^3 \cdot c^2}$ Write the radicand as a product of a perfect cube and another factor.

$= \sqrt[3]{c^3} \cdot \sqrt[3]{c^2}$ Apply the product property of radicals.

$= c\sqrt[3]{c^2}$ Simplify $\sqrt[3]{c^3}$ as c.

b. $\sqrt{50} = \sqrt{5^2 \cdot 2}$ Factor the radicand. The radical is not simplified because the radicand has a perfect square.

$= \sqrt{5^2} \cdot \sqrt{2}$ Apply the product property of radicals.

$= 5\sqrt{2}$ Simplify.

c. $\sqrt[4]{32x^9y^6} = \sqrt[4]{2^5x^9y^6}$ Factor the radicand.

$= \sqrt[4]{(2^4x^8y^4)(2xy^2)}$ Write the radicand as the product of a perfect 4th power and another factor.

$= \sqrt[4]{(2^4x^8y^4)} \cdot \sqrt[4]{2xy^2}$ Apply the product property of radicals.

$= 2x^2y \cdot \sqrt[4]{2xy^2}$ Simplify the first radical.

Apply the Skills

7. Simplify each expression. Assume that all variables represent positive real numbers.

a. $\sqrt[4]{d^7}$ **b.** $\sqrt{45}$ **c.** $\sqrt[3]{54x^{13}y^8}$

Learn: Apply Operations on Radicals

In Example 8(a), we will use the product property of radicals to multiply two radical expressions. In Example 8(b), we will use the quotient property of radicals to divide radical expressions.

Example 8: Multiplying and Dividing Radical Expressions

Multiply or divide as indicated. Assume that *x* and *y* represent positive real numbers.

a. $\sqrt{6x} \cdot \sqrt{10x}$ **b.** $\dfrac{\sqrt[3]{3x^7y}}{\sqrt[3]{81xy^4}}$

Solution:

a. $\sqrt{6x} \cdot \sqrt{10x} = \sqrt{6x^2}$ The radicals have the same index. Apply the product property of radicals.

$= \sqrt{2^2 \cdot 3 \cdot 5 \cdot x^2}$ Factor the radicand.

$= \sqrt{(2x)^2 \cdot 3 \cdot 5}$ Write the radicand as the product of a perfect square and another factor.

$= \sqrt{(2x)^2} \cdot \sqrt{3 \cdot 5}$ Apply the product property of radicals.

$= 2x\sqrt{15}$ Simplify.

b. $\dfrac{\sqrt[3]{3x^7y}}{\sqrt[3]{81xy^4}} = \sqrt[3]{\dfrac{3x^7y}{81xy^4}}$ Apply the quotient property of radicals to write the expression as a single radical.

$= \sqrt[3]{\dfrac{x^6}{27y^3}}$ The numerator and denominator share common factors. Simplify the fraction.

$= \dfrac{x^2}{3y}$ Simplify.

Insights

In Example 8(b), the purpose of writing the quotient of two radicals as a single radical is to simplify the resulting fraction in the radicand.

Apply the Skills

8. Multiply or divide as indicated. Assume that *c*, *d*, and *y* represent positive real numbers.

a. $\sqrt{15y} \cdot \sqrt{21y}$ **b.** $\dfrac{\sqrt[3]{62c^2d^{10}}}{\sqrt[3]{5c^5d}}$

In Example 8(b), we removed the radical from the denominator of the fraction. This is called **rationalizing the denominator.**

We can use the distributive property to add or subtract radical expressions. However, the radicals must be like radicals. This means that the radicands must be the same and the indices must be the same. For example:

$3\sqrt{2x}$ and $-5\sqrt{2x}$ are like radicals.

$3\sqrt{2x}$ and $-5\sqrt[3]{2x}$ are not like radicals because the indices are different.

$3\sqrt{2x}$ and $-5\sqrt{2y}$ are not like radicals because the radicands are different.

Example 9: Adding and Subtracting Radicals

Add or subtract as indicated. Assume that all variables represent positive real numbers.

a. $5\sqrt[3]{7t^2} - 2\sqrt[3]{7t^2} + \sqrt[3]{7t^2}$ **b.** $x\sqrt{98x^3y} + 5\sqrt{18x^5y}$

Solution:

a. $5\sqrt[3]{7t^2} - 2\sqrt[3]{7t^2} + 1\sqrt[3]{7t^2}$ The radicals are like radicals. They have the same radicand and same index.

$= (5 - 2 + 1)\sqrt[3]{7t^2}$ Apply the distributive property.

$= 4\sqrt[3]{7t^2}$ Simplify.

b. $x\sqrt{98x^3y} + 5\sqrt{18x^5y}$	Each radical can be simplified. $x\sqrt{98x^3y} = x\sqrt{(7^2x^2)\cdot(2xy)} = 7x^2\sqrt{2xy}$ $5\sqrt{18x^5y} = 5\sqrt{(3^2x^4)(2xy)} = 15x^2\sqrt{2xy}$
$= 7x^2\sqrt{2xy} + 15x^2\sqrt{2xy}$	The terms are like terms.
$= (7 + 15)x^2\sqrt{2xy}$	Apply the distributive property.
$= 22x^2\sqrt{2xy}$	Simplify.

Apply the Skills

9. Add or subtract as indicated. Assume that all variables represent positive real numbers.

a. $-4\sqrt[3]{5w} + 9\sqrt[3]{5w} - 11\sqrt[3]{5w}$ **b.** $\sqrt{75cd^4} + 6d\sqrt{27cd^2}$

Practice R-2

Practice Exercises

Concept Connections

1. For a nonzero real number b, the value of $b^0 =$ ______________.
2. For a nonzero real number b, the value $b^{\square} = \frac{1}{b^n}$.
3. From the properties of exponents, $b^m b^n = b^{\square}$.
4. If $b \neq 0$, then $\frac{b^m}{b^n} = b^{\square}$.
5. From the properties of exponents, $(b^m)^n = b^{\square}$.
6. Given the expression $\sqrt[n]{a}$, the value a is called the ______________ and n is called the ______________.
7. The expression $a^{m/n}$ can be written in radical notation as ______________ provided that $\sqrt[n]{a}$ is a real number.
8. The expression $a^{1/n}$ can be written in radical notation as ________ provided that $\sqrt[n]{a}$ is a real number.

Learn: Apply Properties of Exponents

For Exercises 9–14, simplify each expression. (See Example 1 on p. R25)

9. **a.** 8^0 **b.** -8^0 **c.** $8x^0$ **d.** $(8x)^0$

10. **a.** 7^0 **b.** -7^0 **c.** $7y^0$ **d.** $(7y)^0$

11. **a.** 8^{-2} **b.** $8x^{-2}$ **c.** $(8x)^{-2}$ **d.** -8^{-2}

12. **a.** 7^{-2} **b.** $7y^{-2}$ **c.** $(7y)^{-2}$ **d.** -7^{-2}

13. **a.** $\frac{1}{q^{-2}}$ **b.** q^{-2} $\frac{1}{q^2}$
c. $5p^3q^{-2}$ **d.** $5p^{-3}q^2$

14. **a.** $\frac{1}{t^{-4}}$ **b.** t^{-4}
c. $11t^{-4}u^2$ **d.** $11t^4u^{-2}$

For Exercises 15–46, use the properties of exponents to simplify each expression. (See Example 2 on p. R27)

15. $x^7 \cdot x^6 \cdot x^{-2}$

16. $y^{-3} \cdot y^7 \cdot y^4$

17. $(-3c^2d^7)(4c^{-5}d)$

18. $(-7m^{-3}n^{-8})(3m^{-5}n)$

19. $\frac{y^{-3}y^6}{y^2}$

20. $\frac{z^{-8}z^{12}}{z^3}$

21. $\frac{18k^2p^9}{27k^5p^2}$

22. $\frac{10a^3b^{11}}{25a^7b^3}$

23. $\frac{2m^{-6}n^4}{6m^{-2}n^{-1}}$

24. $\frac{4p^{-7}q^5}{2p^{-2}q^{-2}}$

25. $(p^{-2})^7$

26. $(q^{-4})^2$

27. $\left(\frac{7a}{b}\right)^2$

28. $\left(\frac{pq}{4}\right)^4$

29. $(4x^2y^{-3})^2$

30. $(-3w^{-3}z^5)^2$

31. $\left(\frac{7k}{n^2}\right)^{-2}$

32. $\left(\frac{5w^5}{v}\right)^{-3}$

33. $\left(\frac{4x^3z^{-5}}{12y^{-2}}\right)^{-3}$

34. $\left(\frac{3z^2w^{-1}}{12p^{-4}}\right)^{-3}$

35. $(-2y)^{-3}(6y^{-2}z^8)^2$

36. $(-15z)^{-2}(5z^4w^{-6})^3$

37. $\left(\frac{1}{2}\right)^{-3} - \left(\frac{1}{3}\right)^{-3} + \left(\frac{1}{6}\right)^{-3}$

38. $\left(-\frac{1}{3}\right)^{-2} - \left(\frac{1}{4}\right)^{-2} + \left(\frac{1}{2}\right)^{-2}$

39. $\left(\frac{1}{4x^3z^{-5}}\right)^{-2}\left(\frac{8}{x^{-13}y^{14}}\right)^{-1}$

40. $\left(\frac{1}{3a^2b^{-4}}\right)^{-3}\left(\frac{3}{a^{-5}b^4}\right)^{-1}$

41. $\frac{(4vw^{-3}x^2)^2}{(2v^2w^3x^{-2})^4} \cdot (-v^3w^2x^{-4})^{-5}$

42. $\frac{(14v^2w^{-3}x^{-2})^3}{(21v^{-5}w^3x^{-2})^{-1}} \cdot (-7v^4w^{-1}x^{-4})^{-4}$

43. $(3x+5)^{14}(3x+5)^{-2}$

44. $(2y-7z)^{-4}(2y-7z)^{13}$

45. $2^{-2} + 2^{-1} + 2^0 + 2^1 + 2^2$

46. $3^{-2} + 3^{-1} + 3^0 + 3^1 + 3^2$

Learn: Apply Scientific Notation

For Exercises 47–52, write the numbers in scientific notation. (See Example 3 on p. R28)

47. a. 350,000 **b.** 0.000035
c. 3.5

48. a. 2710 **b.** 0.00271
c. 2.71

49. The speed of light is approximately 29,980,000,000 cm/sec.

50. The mean distance between the Earth and the Sun is approximately 149,000,000 km.

51. The size of an HIV particle is approximately 0.00001 cm.

52. One picosecond is 0.000000000001 sec.

For Exercises 53–58, write the number in standard decimal notation. (See Example 3 on p. R28)

53. a. 2.61×10^{-6} **b.** 2.61×10^{6}
c. 2.61×10^{0}

54. a. 3.52×10^{-2} **b.** 3.52×10^{2}
c. 3.52×10^{0}

55. A drop of water has approximately 1.67×10^{21} molecules of H_2O.

56. A computer with a 3-terabyte hard drive can store approximately 3.0×10^{12} bytes.

57. A typical red blood cell is 7.0×10^{-6} m.

58. The blue light used to read a laser disc has a wavelength of 4.7×10^{-7} m.

For Exercises 59–66, perform the indicated operation. Write the answer in scientific notation. (See Example 4 on p. R29)

59. $\dfrac{8.4 \times 10^{-6}}{2.1 \times 10^{-2}}$

60. $\dfrac{6.8 \times 10^{11}}{3.4 \times 10^{3}}$

61. $(6.2 \times 10^{11})(3 \times 10^{4})$

62. $(8.1 \times 10^{6})(2 \times 10^{5})$

63. $\dfrac{3.6 \times 10^{-14}}{5 \times 10^{5}}$

64. $\dfrac{3.68 \times 10^{-8}}{4 \times 10^{2}}$

65. $\dfrac{(6.2 \times 10^{5})(4.4 \times 10^{22})}{2.2 \times 10^{17}}$

66. $\dfrac{(3.8 \times 10^{4})(4.8 \times 10^{-2})}{2.5 \times 10^{-5}}$

67. Suyin has a personal music player with 80 gigabytes of memory (80 gigabytes is approximately 8×10^{10} bytes). If each song requires an average of 4 megabytes of memory (approximately 4×10^{6} bytes), how many songs can Suyin store on the device?

68. A math teacher has a personal web page with 60 gigabytes of memory (approximately 6×10^{10} bytes). Math videos are stored on the site for students to watch outside of class. If each video requires an average of 5 megabytes of memory (approximately 5×10^{6} bytes), how many videos can be stored?

69. A typical adult human body contains approximately 5 L of blood. If 1 μL (1 microliter) contains 5×10^{6} red blood cells, how many red blood cells does a typical adult have? (*Hint*: $1\text{ L} = 106\ \mu L$.)

70. The star Proxima Centauri is the closest star (other than the Sun) to the Earth. It is approximately 4.3 light-years away. If 1 light-year is approximately 5.9×10^{12} mi, how many miles is Proxima Centauri from the Earth?

Learn: Evaluate *n*th-Roots

For Exercises 71–78, simplify the expression. (See Example 5 on p. R30)

71. $\sqrt[4]{81}$

72. $\sqrt[3]{125}$

73. $\sqrt{\dfrac{4}{49}}$

74. $\sqrt{\dfrac{9}{121}}$

75. $\sqrt[4]{-81}$

76. $\sqrt[4]{-625}$

77. $\sqrt[3]{-\dfrac{1}{8}}$

78. $\sqrt[3]{-\dfrac{64}{125}}$

Learn: Simplify Expressions of the Forms $a^{1/n}$ and $a^{m/n}$

For Exercises 79–88, simplify each expression. (See Example 6 on p. R31)

79. a. $25^{1/2}$ **b.** $-25^{1/2}$ **c.** $-25^{1/2}$

80. a. $36^{1/2}$ **b.** $(-36)^{1/2}$ **c.** $-36^{1/2}$

81. a. $27^{1/2}$ **b.** $(-27)^{1/2}$ **c.** $-27^{1/2}$

82. a. $125^{1/3}$ **b.** $(-125)^{1/3}$ **c.** $-125^{1/3}$

83. a. $\left(\frac{121}{169}\right)^{1/2}$ **b.** $\left(\frac{121}{169}\right)^{-1/2}$

84. a. $\left(\frac{49}{144}\right)^{1/2}$ **b.** $\left(\frac{49}{144}\right)^{-1/2}$

85. a. $16^{3/4}$ **b.** $16^{-3/4}$ **c.** $-16^{3/4}$
d. $-16^{-3/4}$ **e.** $(-16)^{3/4}$ **f.** $(-16)^{-3/4}$

86. a. $81^{3/4}$ **b.** $81^{-3/4}$ **c.** $-81^{3/4}$
d. $-81^{-3/4}$ **e.** $(-81)^{-3/4}$ **f.** $(-81)^{-3/4}$

87. a. $64^{2/3}$ **b.** $64^{-2/3}$ **c.** $-64^{2/3}$
d. $-64^{-2/3}$ **e.** $(-64)^{2/3}$ **f.** $(-64)^{-2/3}$

88. a. $8^{2/3}$ **b.** $8^{-2/3}$ **c.** $-8^{2/3}$
d. $-8^{-2/3}$ **e.** $(-8)^{2/3}$ **f.** $(-8)^{-2/3}$

Learn: Simplify Radicals

89. a. For what values of t will the statement be true? $\sqrt{t^2} = t$

b. For what value of t will the statement be true? $\sqrt{t^2} = |t|$

90. a. For what values of c will the statement be true? $\sqrt[4]{(c+8)^4} = c + 8$

b. For what value of c will the statement be true? $\sqrt[4]{(c+8)^4} = |c+8|$

For Exercises 91–100, simplify each expression. Assume that all variable expressions represent positive real numbers. (See Example 7 on p. R32)

91. a. $\sqrt{c^7}$ **b.** $\sqrt[3]{c^7}$
c. $\sqrt[4]{c^7}$ **d.** $\sqrt[9]{c^7}$

92. a. $\sqrt{d^{11}}$ **b.** $\sqrt[3]{d^{11}}$
c. $\sqrt[4]{d^{11}}$ **d.** $\sqrt[12]{d^{11}}$

93. a. $\sqrt{24}$ **b.** $\sqrt[3]{24}$

94. a. $\sqrt{54}$ **b.** $\sqrt[3]{54}$

95. $\sqrt[3]{250x^2y^6z^{11}}$

96. $\sqrt[3]{40ab^{13}c^{17}}$

97. $\sqrt[4]{96p^{14}q^7}$

98. $\sqrt[4]{243m^{19}n^{10}}$

99. $\sqrt{84(y-2)^3}$

100. $\sqrt{18(w-6)^3}$

Learn: Apply Operations on Radicals

For Exercises 101–112, multiply or divide as indicated. Assume that all variable expressions represent positive real numbers. (See Example 8 on p. R33)

101. $\sqrt{10} \cdot \sqrt{14}$

102. $\sqrt{6} \cdot \sqrt{21}$

103. $\sqrt[3]{xy^2} \cdot \sqrt[3]{x^2y}$

104. $\sqrt[4]{a^3b} \cdot \sqrt[4]{ab^3}$

105. $\left(3\sqrt[4]{a^3}\right)\left(-5\sqrt[4]{a^3}\right)$

106. $\left(7\sqrt[6]{t^5}\right)\left(-2\sqrt[6]{t^5}\right)$

107. $\sqrt{\frac{p^7}{36}}$

108. $\sqrt{\frac{q^{11}}{4}}$

109. $4\sqrt[3]{\frac{w^3z^5}{8}}$

110. $8\sqrt[3]{\frac{c^6d^7}{64}}$

111. $\dfrac{\sqrt[3]{5x^2y}}{\sqrt[3]{625x^2y^4}}$

112. $\dfrac{\sqrt[3]{2m^2n^7}}{\sqrt[3]{16m^{14}n^4}}$

For Exercises 113–120, add or subtract as indicated. Assume that all variables represent positive real numbers. (See Example 9 on p. R34)

113. $3\sqrt[3]{2y^2} - 9\sqrt[3]{2y^2} + \sqrt[3]{2y^2}$

114. $8\sqrt[4]{3z^3} - \sqrt[4]{3z^3} + 2\sqrt[4]{3z^3}$

115. $\frac{1}{5}\sqrt{50} - \frac{7}{3}\sqrt{18} + \frac{5}{6}\sqrt{72}$

116. $\frac{2}{5}\sqrt{75} - \frac{2}{3}\sqrt{27} - \frac{1}{2}\sqrt{12}$

117. $-3x\sqrt[3]{16xy^4} + xy\sqrt[3]{54xy} - 5\sqrt[3]{250x^4y^4}$

118. $8\sqrt[4]{32a^5b^6} - 5b\sqrt[4]{2a^5b^2} - ab\sqrt[4]{162ab^2}$

119. $12\sqrt{2y} + 5y\sqrt{2y}$

120. $-8\sqrt{3w} + 3w\sqrt{3w}$

Mixed Exercises

For Exercises 121–141, simplify the expressions.

121. **a.** $\sqrt[3]{27y^9}$

b. $\sqrt{27y^9}$

122. **a.** $\sqrt{24}$

b. $\sqrt[3]{24}$

123. **a.** $\sqrt{54}$

b. $\sqrt[3]{54}$

124. **a.** $\sqrt{200y^6}$

b. $\sqrt[3]{200y^6}$

125. **a.** $\sqrt{32z^{15}}$

b. $\sqrt[3]{32z^{15}}$

126. **a.** $\sqrt{80}$

b. $\sqrt[3]{80}$

c. $\sqrt[4]{80}$

127. **a.** $\sqrt{48}$

b. $\sqrt[3]{48}$

c. $\sqrt[4]{48}$

128. **a.** $\sqrt{x^5y^6}$

b. $\sqrt[3]{x^5y^6}$

c. $\sqrt[4]{x^5y^6}$

129. **a.** $\sqrt{a^{10}b^9}$

b. $\sqrt[3]{a^{10}b^9}$

c. $\sqrt[4]{a^{10}b^9}$

130. **a.** $\sqrt[3]{32s^5t^6}$

b. $\sqrt[4]{32s^5t^6}$

c. $\sqrt[5]{32s^5t^6}$

131. **a.** $\sqrt[3]{96v^7w^{20}}$

b. $\sqrt[4]{96v^7w^{20}}$

c. $\sqrt[5]{96v^7w^{20}}$

132. **a.** $\sqrt{5}+\sqrt{5}$

b. $\sqrt{5}\cdot\sqrt{5}$

133. **a.** $\sqrt{10}+\sqrt{10}$

b. $\sqrt{10}\cdot\sqrt{10}$

134. **a.** $2\sqrt{6}-5\sqrt{6}$

b. $2\sqrt{6}\cdot 5\sqrt{6}$

135. **a.** $3\sqrt{7}-10\sqrt{7}$

b. $3\sqrt{7}\cdot 10\sqrt{7}$

136. **a.** $\sqrt{8}+\sqrt{2}$

b. $\sqrt{8}\cdot\sqrt{2}$

137. **a.** $\sqrt{12}+\sqrt{3}$

b. $\sqrt{12}\cdot\sqrt{3}$

138. **a.** $5\sqrt{18}-4\sqrt{8}$

b. $5\sqrt{18}\cdot 4\sqrt{8}$

139. **a.** $\sqrt{50}-\sqrt{72}$

b. $\sqrt{50}\cdot\sqrt{72}$

140. **a.** $4\sqrt[3]{24}+6\sqrt[3]{3}$

b. $4\sqrt[3]{24}\cdot 6\sqrt[3]{3}$

141. **a.** $2\sqrt[3]{54}-5\sqrt[3]{2}$

b. $2\sqrt[3]{54}\cdot 5\sqrt[3]{2}$

For Exercises 142–148, answer true or false. If an answer is false, explain why or give a counterexample. Assume that all variables represent positive real numbers.

142. $\sqrt{x}+\sqrt{y}=\sqrt{x+y}$

143. $\sqrt{x}+\sqrt{x}=2\sqrt{x}$

144. $5\sqrt[3]{x}+2\sqrt[3]{x}=7\sqrt[3]{x}$

145. $6\sqrt{x}+5\sqrt[3]{x}=11\sqrt{x}$

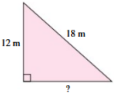
146. $\sqrt{y}+\sqrt{y}=\sqrt{2y}$

147. $\sqrt{c^2+d^2}=c+d$

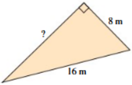
148. $2w\sqrt{5}+4w\sqrt{5}=6w^2\sqrt{5}$

For Exercises 149–150, use the Pythagorean theorem to determine the length of the missing side. Write the answer as a simplified radical.

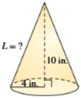
149.



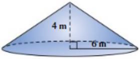
150.



151. The slant length L for a right circular cone is given by $L = \sqrt{r^2 + h^2}$, where r and h are the radius and height of the cone. Find the slant length of a cone with radius 4 in. and height 10 in. Determine the exact value and a decimal approximation to the nearest tenth of an inch.



152. The lateral surface area A of a right circular cone is given by $A = \pi r\sqrt{r^2 + h^2}$, where r and h are the radius and height of the cone. Determine the exact value (in terms of π) of the lateral surface area of a cone with radius 6 m and height 4 m. Then give a decimal approximation to the nearest square meter.



153. The depreciation rate for a car is given by $r = 1 - \left(\frac{S}{C}\right)^{1/n}$, where S is the value of the car after n years, and C is the initial cost. Determine the depreciation rate for a car that originally cost \$22,990 and was valued at \$11,500 after 4 yr. Round to the nearest tenth of a percent.

154. For a certain oven, the baking time t (in hours) for a turkey that weighs x pounds can be approximated by the model $t = 0.84x^{3/5}$. Determine the baking time for a 15-lb turkey. Round to 1 decimal place.

Write About It

155. Explain the difference between the expressions $6x^0$ and $(6x)^0$.

156. Explain why scientific notation is used.

157. Explain the similarity in simplifying the given expressions.

- **a.** $2x + 3x$
- **b.** $2\sqrt{x} + 3\sqrt{x}$
- **c.** $2\sqrt[3]{x} + 3\sqrt[3]{x}$

158. Explain why the given expressions cannot be simplified further.

- **a.** $2x + 3y$
- **b.** $2\sqrt{x} + 3\sqrt{y}$
- **c.** $2\sqrt[3]{x} + 3\sqrt{x}$

Expanding Your Skills

For Exercises 159–160, refer to the formula $F = \frac{Gm_1m_2}{d^2}$. This gives the gravitational force F (in Newtons, N) between two masses m_1 and m_2 (each measured in kg) that are a distance of d meters apart. In the formula, $G = 6.6726 \times 10^{-11}$ N-m²/kg².

159. Determine the gravitational force between the Earth (mass $= 5.98 \times 10^{24}$ kg) and Jupiter (mass $= 1.901 \times 10^{27}$ kg) if at one point in their orbits, the distance between them is 7.0×10^{11} m.

160. Determine the gravitational force between the Earth (mass $= 5.98 \times 10^{24}$ kg) and an 80-kg human standing at sea level. The mean radius of the Earth is approximately 6.371×10^{6} m.

For Exercises 161–164, without the assistance of a calculator, fill in the blank with the appropriate symbol <, >, or =.

161. a. 5^{15} _____ 5^{17} **b.** 5^{-15} _____ 5^{-17}

162. a. $\left(\frac{1}{5}\right)^{15}$ _____ $\left(\frac{1}{5}\right)^{17}$

b. $\left(\frac{1}{5}\right)^{-15}$ _____ $\left(\frac{1}{5}\right)^{-17}$

163. a. $(-1)^{86}$ _____ $(-1)^{87}$

b. $(1)^{86}$ _____ $(-1)^{87}$

164. a. $(-1)^{0}$ _____ -1^{41}

b. $(-1)^{42}$ _____ $(-1)^{0}$

For Exercises 165–170, write each expression as a single radical for positive values of the variable. (*Hint*: Write the radicals as expressions with rational exponents and simplify. Then convert back to radical form.)

165. $\sqrt[5]{x^3y^2} \cdot \sqrt[4]{x}$

166. $\sqrt[4]{a^2b} \cdot \sqrt[3]{ab^2}$

167. $\sqrt[6]{m\sqrt[3]{m^2}}$

168. $\sqrt[5]{y\sqrt[4]{y^2}}$

169. $\sqrt{x\sqrt{x\sqrt{x}}}$

170. $\sqrt[3]{y\sqrt[3]{y\sqrt[3]{y}}}$

For Exercises 171–172, evaluate the expression without the use of a calculator.

171. $\sqrt{\frac{8.0 \times 10^{12}}{2.0 \times 10^{4}}}$

172. $\sqrt{\frac{1.44 \times 10^{16}}{9.0 \times 10^{10}}}$

For Exercises 173–174, simplify the expression.

173. $\sqrt{\sqrt[3]{6 + \sqrt[4]{16}} + \sqrt{\sqrt{25} + \sqrt{16} + \sqrt{9}}}$

174. $\sqrt{\sqrt[4]{11 + \sqrt[3]{125}} + \sqrt{\sqrt{81} + \sqrt[3]{1000} + \sqrt{36} + \sqrt{25}}}$

Point of Interest

Mark E. Dean (1957-present) is an electrical engineer, computer scientist, and inventor who worked for IBM. One of his early inventions, a set of internal wires collectively called the 16-bit ISA bus, paved the way for widespread use of personal computers. Memory, disk drives, printers, and monitors could be plugged directly into a computer through the new ISA slot, therfore expanding the usability of the personal computer. Dean led the team that designed the original IBM PC, and many years later he worked on the development of the first gigahertz chip, a device that revolutionized the speed at which data can be processed.

175. a. The quantity 1 hertz (1 Hz) is defined as 1 cycle per second. For example, a hummingbird's wings flap roughly 80 times per second, or 80 Hz. In the metric system, the prefix "kilo" means 1000. So, 1 kilohertz (1 kHz) is equal to 1000 Hz. The prefix "mega" indicates 1 million and "giga" indicates 1 billion. Recreate and complete the table with the equivalent units in Hz in expanded form and in scientific notation.

Original Unit	Expanded Form in Hz	Scientific Notation in Hz
1 kilohertz (written as 1 kHz)	1000 Hz	1×10^3 Hz
1 megahertz (written as 1 MHz)		
1 gigahertz (written as 1 GHz)		

b. Based on the table in part (a), how many times faster is a computer processor that operates at 1 GHz than a computer that operates at 1 kHz?

c. In 1996, Dean became the first African American ever to be named an IBM fellow—the highest honor a scientist at IBM can achieve. Use an online resource to do further research on the distinguished career of Mark Dean, and write a short paragraph reporting your findings.

Go online for more practice problems.

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Lesson R-3

Polynomials and Factoring

What Will You Learn?

After completing this lesson, you should be able to:

- Add and Subtract Polynomials
- Multiply Polynomials
- Multiply Radical Expressions Involving Multiple Terms
- Factor by Grouping
- Factor Quadratic Trinomials
- Factor Binomials
- Apply a General Strategy to Factor Polynomials
- Factor Expressions Containing Negative and Rational Exponents

Learn: Add and Subtract Polynomials

The Environmental Protection Agency (EPA) is responsible for providing fuel economy data (gas mileage information) that is posted on the window stickers of new vehicles. Many variables contribute to fuel consumption, including the speed of the vehicle. For example, for one midsize sedan tested, the gas mileage G (in miles per gallon, mpg) can be approximated by

$$G = -0.008x^2 + 0.748x + 13.5,$$

where x is the speed of the vehicle in mph and $15 \leq x \leq 75$ mph.

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The expression on the right side of this equation is called a polynomial. A **polynomial** in the variable x is a finite sum of terms of the form ax^n. In each term, the coefficient, a, is a real number, and the exponent, n, is a whole number. The degree of ax^n is n.

The terms of the polynomial $-0.008x^2 + 0.748x + 13.5$ are written in **descending order** by degree. The term with highest degree is written first. This is called the **leading term**, and its coefficient is called the **leading coefficient**. The **degree of the polynomial** is the same as the degree of the leading term. Therefore, the polynomial $-0.008x^2 + 0.748x + 13.5$ is a degree 2 polynomial.

Insights

The number 0 can be written in infinitely many ways: $0x$, $0x^2$, $0x^3$, and so on. For this reason, the degree of 0 is undefined.

The preceding discussion is meant as an informal introduction to polynomials and associated key vocabulary. However, as your level of mathematical sophistication increases, you will encounter definitions written in a more concise mathematical language. Take a minute to read the formal definition of a polynomial.

Definition of a Polynomial in *x*

A **polynomial in the variable *x*** is an expression of the form:

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

The coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ are real numbers, where $a_n \neq 0$, and the exponents $n, n-1, n-2, \ldots, 0$ are whole numbers.

The term $a_n x^n$ is called the **leading term**, the coefficient a_n is the **leading coefficient**, and the exponent n is the **degree of the polynomial**.

In the preceding definition, subscript notation a_n (read as "a sub n"), a_{n-1} (read as "a sub $n-1$"), and so on, is used to denote the coefficients of the terms. Subscript notation is used rather than lettered variables such as a, b, c, and the like, when a large or undetermined number of terms is suggested.

A polynomial with one term is also called a **monomial**. A polynomial with two terms is called a **binomial**, and a polynomial with three terms is called a **trinomial**. To add or subtract polynomials, combine like terms. This is demonstrated in Example 1.

Insights

Some polynomials have more than one variable. In such a case, the degree of each term is the sum of the exponents on the variable factors. For example, the polynomial

$$-4x^4y^7 + 3xy^5$$

has two terms. The degree of the first term is 11 and the degree of the second term is 6.

Example 1: Adding and Subtracting Polynomials

Add or subtract as indicated, and simplify.

a. $(-4w^3 - 5w^2 + 6w + 3) + (8w^2 - 4w + 2)$

b. $(6.1a^2b + 2.9ab - 4.5b^2) - (2.6a^2b - 4.1ab + 2.1b^2)$

Solution:

a. $(-4w^3 - 5w^2 + 6w + 3) + (8w^2 - 4w + 2)$

$= -4w^3 - 5w^2 + 8w^2 + 6w - 4w + 3 + 2$ Group like terms.

$= -4w^3 + 3w^2 + 2w + 5$ Combine like terms.

b. $(6.1a^2b + 2.9ab - 4.5b^2) - (2.6a^2b - 4.1ab + 2.1b^2)$

$= (6.1a^2b + 2.9ab - 4.5b^2) + (-2.6a^2b + 4.1ab - 2.1b^2)$

$= 6.1a^2b - 2.6a^2b + 2.9ab + 4.1ab - 4.5b^2 - 2.1b^2$ Group like terms.

$= 3.5a^2b + 7ab - 6.6b^2$ Combine like terms.

Apply the Skills

1. Add or subtract as indicated, and simplify.

a. $(7t^5 - 3t^2 - 2t) + (2t^5 + 3t^2 - 5t - 4)$

b. $(0.08x^3y - 0.02x^2y - 0.1xy) - (0.05x^3y - 0.07x^2y + 0.02xy)$

Insights

Addition and subtraction of polynomials can also be done by aligning like terms in columns. The difference of polynomials from Example 1(b) is shown here.

$$\begin{array}{r} 6.1a^2b + 2.9ab - 4.5b^2 \\ -(2.6a^2b - 4.1ab + 2.1b^2) \\ \hline \end{array} \xrightarrow[\text{Add the opposite.}]{} \begin{array}{r} 6.1a^2b + 2.9ab - 4.5b^2 \\ +(-2.6a^2b + 4.1ab - 2.1b^2) \\ \hline 3.5a^2b + 7ab - 6.6b^2 \end{array}$$

Learn: Multiply Polynomials

To multiply polynomials we use the distributive property to multiply each term in the first polynomial by each term in the second polynomial.

Example 2: Multiplying Polynomials

Multiply and simplify.

$$(4x + 2)\left(x^2 - 6x + \frac{1}{2}\right)$$

Solution:

$(4x + 2)\left(x^2 - 6x + \frac{1}{2}\right)$ Multiply each term in the first polynomial by each term in the second.

$= 4x(x^2) + 4x(-6x) + 4x\left(\frac{1}{2}\right) + 2(x^2) + 2(-6x) + 2\left(\frac{1}{2}\right)$

$= 4x^3 - 24x^2 + 2x + 2x^2 - 12x + 1$ Simplify.

$= 4x^3 - 22x^2 - 10x + 1$ Combine like terms.

Insights

Multiplication of polynomials can also be performed vertically.

$$\begin{array}{rr} & x^2 - 6x + \frac{1}{2} \\ \times & 4x + 2 \\ & 2x^2 - 12x + 1 \\ + & 4x^3 - 24x^2 + 2x \\ \hline & 4x^3 - 22x^2 - 10x + 1 \end{array}$$

Apply the Skills

2. Multiply and simplify. $(3y - 6)\left(y^2 + 4y + \frac{2}{3}\right)$

Two expressions of the form $a - b$ and $a + b$ are called **conjugates**. The product of two conjugates results in a **difference of squares**.

$$\begin{aligned}(a - b)(a + b) &= a^2 + ab - ab - b^2 \\ &= a^2 - b^2 \qquad \text{(difference of squares)}\end{aligned}$$

Insights

The product of conjugates equals the square of the first term from the binomials, minus the square of the second term from the binomials.

An expression of the form $(a + b)^2$ or $(a - b)^2$ is called a **square of a binomial.** In expanded form, the product is a **perfect square trinomial**.

$$\begin{aligned}(a + b)^2 = (a + b)(a + b) &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2 \qquad \text{(perfect square trinomial)} \\ (a - b)^2 = (a - b)(a - b) &= a^2 - ab - ab + b^2 \\ &= a^2 - 2ab + b^2 \qquad \text{(perfect square trinomial)}\end{aligned}$$

Insights

The square of a binomial results in the square of the first term in the binomial, plus twice the product of terms in the binomial, plus the square of the second term in the binomial.

Special Case Products

Product of Conjugates: $(a+b)(a-b)=a^2-b^2$ (difference of squares)

Square of a Binomial: $(a+b)^2=a^2+2ab+b^2$ (perfect square trinomial)

$(a-b)^2=a^2-2ab+b^2$ (perfect square trinomial)

Example 3: Applying Special Case Products

a. Multiply. $(2t^2-9)(2t^2+9)$

b. Square the binomial. $(3x-7)^2$

Solution:

a. $(2t^2-9)(2t^2+9)$ This is a product of conjugates: $(a-b)(a+b)=a^2-b^2$.

$=(2t^2)^2-(9)^2$ The product is a difference of squares.

$=4t^4-81$ Simplify.

b. $(3x-7)^2$ This is the square of a binomial, $(a-b)^2$, where $a=3x$ and $b=7$.

$=(3x)^2-2(3x)(7)+(7)^2$ The product is $a^2-2ab+b^2$.

$=9x^2-42x+49$ Simplify.

Apply the Skills

3. **a.** Multiply. $(5w^3+2y)(5w^3-2y)$ **b.** Square the binomial. $(6p+5)^2$

Learn: Multiply Radical Expressions Involving Multiple Terms

The process used to multiply polynomials can be extended to algebraic expressions that are not polynomials. In Example 4 we multiply expressions containing two or more radical terms.

Example 4: Multiplying Radical Expressions with Multiple Terms

Multiply and simplify. Assume that all variable expressions represent positive real numbers.

a. $(3\sqrt{x}+5)(2\sqrt{x}-7)$ **b.** $(4\sqrt{5}+\sqrt{6})(4\sqrt{5}-\sqrt{6})$ **c.** $(3x+\sqrt{2})^2$

Solution:

a. $(3\sqrt{x} + 5)(2\sqrt{x} - 7)$

$= (3\sqrt{x})(2\sqrt{x}) + (3\sqrt{x})(-7) + (5)(2\sqrt{x}) + (5)(-7)$ Apply the distributive property.

$= 6\sqrt{x^2} - 21\sqrt{x} + 10\sqrt{x} - 35$

$= 6x - 11\sqrt{x} - 35$ Combine like terms.

b. $(4\sqrt{5} + \sqrt{6})(4\sqrt{5} - \sqrt{6})$ This is a product of conjugates: $(a + b)(a - b)$.

$= (4\sqrt{5})^2 - (\sqrt{6})^2$ The product is a difference of squares. Note that $(\sqrt{5})^2 = 5$ and $(\sqrt{6})^2 = 6$.

$= 16 \cdot 5 - 6$

$= 80 - 6$

$= 74$

c. $(3x + \sqrt{2})^2$ This is the square of a binomial, $(a + b)^2$, where $a = 3x$ and $b = \sqrt{2}$.

$= (3x)^2 - 2(3x)(\sqrt{2}) + (\sqrt{2})^2$ The product is $a^2 + 2ab + b^2$.

$= 9x^2 - 6\sqrt{2x} + 2$ Simplify.

Apply the Skills

4. Multiply and simplify. Assume that all variable expressions represent positive real numbers.

a. $(4\sqrt{t} + 10)(3\sqrt{t} - 8)$ **b.** $(3\sqrt{7} - \sqrt{5})(3\sqrt{7} + \sqrt{5})$ **c.** $(4y - \sqrt{3})^2$

Learn: Factor by Grouping

We now look at the multiplication of polynomials in reverse. The goal is to decompose a polynomial into a product of factors. This process is called **factoring**.

There are many techniques used to factor a polynomial, but the first step is always to factor out the greatest common factor (GCF). For example, each term in the polynomial $12x^5 + 18x^4 - 24x^3$ shares a common factor of x^3 and a greatest common numerical factor of 6. To factor out $6x^3$ from the polynomial, we have

$12x^5 + 18x^4 - 24x^3$ The GCF is $6x^3$.

$= 6x^3(2x^2) + 6x^3(3x) + 6x^3(-4)$ Write each term as a product of the GCF and another factor.

$= 6x^3(2x^2 + 3x - 4)$ Apply the distributive property.

Check Your Work

The factored form of any polynomial can be checked by multiplication.

$$6x^3(2x^2 + 3x - 4) = 12x^5 + 18x^4 - 24x^3 \quad ✓$$

Sometimes it is preferable to factor out a negative factor from a polynomial. For example, consider the polynomial $-4x^2 - 8x + 12$. If we factor out -4, then the leading coefficient of the remaining polynomial will be positive.

$-4x^2 - 8x + 12$	Write each term as a product of -4 and another factor.
$= (-4)(x^2) + (-4)(2x) + (-4)(-3)$	
$= -4(x^2 + 2x - 3)$	Apply the distributive property.

Insights

When a negative factor is factored out of a polynomial, the remaining terms in parentheses will have signs opposite to those in the original polynomial.

To factor a polynomial containing four terms, we often try factoring by grouping. This is demonstrated in Example 5.

Example 5: Factoring by Grouping

Factor by grouping.

$$2ax - 6ay - 5x + 15y$$

Solution:

$2ax - 6ay \mid - 5x + 15y$	Consider the first pair of terms and second pair of terms separately.
$= 2a(x - 3y) - 5(x - 3y)$	Factor out the GCF from the first two terms and from the second two terms.

Insights

After factoring out the GCF from each pair of terms, the binomial factors must match to be able to complete the process of factoring by grouping.

$= (x - 3y)(2a - 5)$	Factor out the common binomial factor $(x - 3y)$ from each term.

Check Your Work

$$(x - 3y)(2a - 5) = 2ax - 5x - 6ay + 15y \quad ✓$$

Apply the Skills

5. Factor by grouping: $7cd - c^2 + 14d - 2c$

Learn: Factor Quadratic Trinomials

Next we want to factor quadratic trinomials. These are trinomials of the form $ax^2 + bx + c$, where the coefficients a, b, and c are integers and $a \neq 0$. To understand the basis to factor a trinomial, first consider the product of two binomials.

$$(2x+3)(x+2) = \overset{\text{product of } 2x \text{ and } x}{\overset{\downarrow}{2x^2}} + \underbrace{4x + 3x}_{\text{sum of products of inner terms and outer terms}} + \overset{\text{product of 3 and 2}}{\overset{\downarrow}{6}}$$

To factor a trinomial, this process is reversed.

Example 6: Factoring a Quadratic Trinomial with Leading Coefficient 1

Factor. $x^2 - 8x + 12$

Solution:

factors of x^2

$x^2 - 8x + 12 = (\square x + \square)(\square x + \square)$ First fill in the blanks so that the product of first terms in the binomials is x^2. In this case, we have $1x$ and $1x$.

factors of 12

$x^2 - 8x + 12 = (1x + \square)(1x + \square)$ Fill in the remaining blanks with numbers whose product is 12 and whose sum is the middle term coefficient, -8.

$= (x - 2)(x - 6)$ The numbers are -2 and -6.

Check Your Work

$$(x-2)(x-6) = x^2 - 6x - 2x + 12$$
$$= x^2 - 8x + 12 \quad \checkmark$$

Apply the Skills

6. Factor. $y^2 - 9y + 14$

The trinomial in Example 6 has a leading coefficient of 1. This simplified the factorization process. For any trinomial with a leading coefficient of 1 ($x^2 + bx + c$), the constant terms in the binomial factors have a product of c and a sum equal to the middle term coefficient, b.

To factor a trinomial of the form $ax^2 + bx + c$, where $a \neq 1$, all possible factors of ax^2 must be tested with all factors of c until the correct middle term is found. This is demonstrated in Example 7.

Example 7: Factoring a Quadratic Trinomial

Factor. $10x^3 + 105x^2y - 55xy^2$

Solution:

$10x^3 + 105x^2y - 55xy^2$	Factor out the GCF, $5x$.
$= 5x(2x^2 + 21xy - 11y^2)$	
factors of $2x^2$ $= 5x(2x^2 + 21xy - 11y^2) = 5x(\square x + \square y)(\square x + \square y)$ factors of $-11y^2$	Fill in the blanks so that the product of first terms in the binomials is $2x^2$. We have $2x$ and x.
$= 5x(2x^2 + 21xy - 11y^2) = 5x(2x + \square)(x + \square)$	Fill in the remaining blanks with factors of $-11y^2$.
$(2x + 11y)(x - 1y) = 2x^2 + 9xy - 11y^2$	Incorrect.
$(2x + y)(x - 11y) = 2x^2 - 21xy - 11y^2$	Incorrect.
$(2x - 11y)(x + y) = 2x^2 - 9xy - 11y^2$	Incorrect.
$(2x - y)(x + 11y) = 2x^2 + 21xy - 11y^2$	Correct! ✓

The trinomial $10x^3 + 105x^2y - 55xy^2$ factors as $5x(2x - y)(x + 11y)$.

Insights

The order in which the product of factors is written does not affect the product. Therefore, $5x(2x - y)(x + 11y)$ equals $5x(x + 11y)(2x - y)$. This is guaranteed by the commutative property of multiplication.

Apply the Skills

7. Factor. $30m^3 - 200m^2n - 70mn^2$

The process to factor a trinomial can be challenging if the coefficients are large. In fact, we will learn other techniques that will come in handy in such cases. One technique we show here is to reverse the special case product formula for the square of a binomial. That is, given a perfect square trinomial, factor the trinomial as the square of a binomial.

Factored Form of a Perfect Square Trinomial

$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$	A perfect square trinomial factors as the square of a binomial.

Example 8: Factoring a Perfect Square Trinomial

Factor.

a. $4x^2 - 28x + 49$ **b.** $81c^4 + 90c^2d + 25d^2$

Solution:

a. $4x^2 - 28x + 49$

$= (2x)^2 - 2(2x)(7) + (7)^2$ The trinomial fits the pattern $a^2 - 2ab + b^2$, where $a = 2x$ and $b = 7$.

$= (2x - 7)^2$ Factor as $a^2 - 2ab + b^2 = (a - b)^2$.

b. $81c^4 + 90c^2d + 25d^2$

$= (9c^2)^2 + 2(9c^2)(5d) + (5d)^2$ The trinomial fits the pattern $a^2 + 2ab + b^2$, where $a = 9c^2$ and $b = 5d$.

$= (9c^2 + 5d)^2$ Factor as $a^2 + 2ab + b^2 = (a + b)^2$.

Apply the Skills

8. Factor.

a. $64y^2 + 16y + 1$ **b.** $9m^4 - 30m^2n + 25n^2$

Learn: Factor Binomials

Recall that the product of conjugates results in a difference of squares. Therefore, the factored form of a difference of squares is a product of conjugates.

Factored Form of a Difference of Squares

$$a^2 - b^2 = (a + b)(a - b)$$

Example 9: Factoring a Difference of Squares

Factor completely.

a. $p^2 - 100$ **b.** $32y^4 - 162$

Solution:

a. $p^2 - 100$

$= (p)^2 - (10)^2$ The binomial fits the pattern $a^2 - b^2$, where $a = p$ and $b = 10$.

$= (p + 10)(p - 10)$ Factor as $a^2 - b^2 = (a + b)(a - b)$.

b. $32y^4 - 162$

$= 2(16y^4 - 81)$	Factor out the GCF, 2.
$= 2[(4y^2)^2 - (9)^2]$	The binomial fits the pattern $a^2 - b^2$, where $a = 4y^2$ and $b = 9$.
$= 2(4y^2 + 9)(4y^2 - 9)$	Factor as $a^2 - b^2 = (a + b)(a - b)$.
$= 2(4y^2 + 9)[(2y)^2 - (3)^2]$	The factor $(4y^2 - 9)$ is also a difference of squares, with $a = 2y$ and $b = 3$.
$= 2(4y^2 + 9)(2y + 3)(2y - 3)$	The polynomial is now factored completely.

Apply the Skills

9. Factor completely.

a. $t^2 - 121$

b. $625z^5 - z$

A binomial can also be factored if it fits the pattern of a difference of cubes or a sum of cubes.

Factored Form of a Sum and Difference of Cubes

Sum of cubes:	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of cubes:	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Same sign **A**lways **P**ositive

Opposite signs

Insights

The factored form of a sum or difference of cubes is a binomial times a trinomial. To help remember the pattern for the signs, remember **SOAP**: **S**ame sign, **O**pposite signs, **A**lways **P**ositive.

Example 10: Factoring a Sum and Difference of Cubes

Factor completely.

a. $x^3 + 125$

b. $8m^6 - 27n^3$

Solution:

a. $x^3 + 125$

$= (x)^3 + (5)^3$	The binomial fits the pattern of a sum of cubes $a^3 + b^3$, where $a = x$ and $b = 5$.
$= (x + 5)[(x)^2 - (x)(5) + (5)^2]$	Factor as $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
$= (x + 5)(x^2 - 5x + 25)$	Simplify.

b. $8m^6 - 27n^3$

$= (2m^2)^3 - (3n)^3$	The binomial fits the pattern of a difference of cubes $a^3 - b^3$, where $a = 2m^2$ and $b = 3n$.
$= (2m^2 - 3n)[(2m^2)^2 + (2m^2)(3n) + (3n)^2]$	Factor as $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
$= (2m^2 - 3n)(4m^4 + 6m^2n + 9n^2)$	Simplify.

Apply the Skills

10. Factor completely.

a. $u^3 + 27$ **b.** $64v^3 - 125z^6$

Learn: Apply a General Strategy to Factor Polynomials

Factoring polynomials is a strategy game. The process requires that we identify the technique or techniques that best apply to a given polynomial. To do this we can follow the guidelines given in Table R-7.

TABLE R-7 Factoring Strategy

First Step	Number of Terms	Technique
Factor out the GCF	4 or more terms	Try factoring by grouping.
	3 terms	If possible, write the trinomial in the form $ax^2 + bx + c$. • If the trinomial is a perfect square trinomial, factor as $a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$ • Otherwise try factoring by the trial-and-error method.
	2 terms	• If the binomial is a difference of squares, factor as $a^2 - b^2 = (a + b)(a - b)$ • If the binomial is a sum of cubes, factor as $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ • If the binomial is a difference of cubes, factor as $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example 11: Factoring a 4-Term Polynomial

Factor completely. $x^5 - 9x^3 + 8x^2 - 72$

Solution:

$x^5 - 9x^3 + 8x^2 - 72$	Factor by grouping two terms with two terms.
$= x^3(x^2 - 9) + 8(x^2 - 9)$	Factor out the GCF from the first two terms. Factor out the GCF from the second two terms.
$= (x^2 - 9)(x^3 + 8)$	Factor out the common binomial factor $(x^2 - 9)$ from each term.
$= (x + 3)(x - 3)(x + 2)(x^2 - 2x + 4)$	Factor the first binomial as a difference of squares and the second binomial as a sum of cubes.

Apply the Skills

11. Factor completely. $x^5 - x^3 - 64x^2 + 64$

Example 12: Factoring by Grouping 1 Term with 3 Terms

Factor completely. $25 - x^2 - 4xy - 4y^2$

Solution:

This polynomial has 4 terms. However, the standard grouping method does not work even if we try rearranging terms.

$25 - x^2 - 4xy - 4y^2$	We might try grouping 1 term with 3 terms.
$= 25 - (\underbrace{x^2 + 4xy + 4y^2}_{\text{perfect square trinomial}})$	The reason is that after factoring out −1 from the last 3 terms, we have a perfect square trinomial, $a^2 + 2ab + b^2$ where $a = x$ and $b = 2y$.
$= 25 - (x + 2y)^2$	This results in a difference of squares $a^2 - b^2$ where $a = 5$ and $b = (x + 2y)$.
$= (5)^2 - (x + 2y)^2$	
$= [5 + (x + 2y)][5 - (x + 2y)]$	Factor as $a^2 - b^2 = (a + b)(a - b)$.
$= (5 + x + 2y)(5 - x - 2y)$	Simplify.

Apply the Skills

12. Factor completely. $36 - m^2 - 6mn - 9n^2$

In Example 12, it may be helpful to write the expression $25 - (x + 2y)^2$ by using a convenient substitution. If we let $u = x + 2y$, then the expression becomes $25 - u^2$. This is easier to recognize as a difference of squares.

$$\begin{aligned} 25 - u^2 &= (5 + u)(5 - u) & \\ &= [5 + (x + 2y)][5 - (x + 2y)] & \text{Back substitute.} \\ &= (5 + x + 2y)(5 - x - 2y) & \end{aligned}$$

In Example 13, we practice making an appropriate substitution to convert a cumbersome expression into one that is more easily recognizable and factorable.

Example 13: Factoring a Trinomial by Using Substitution

Factor completely. $(x^2 - 5)^2 + 2(x^2 - 5) - 24$

Solution:

$(x^2 - 5)^2 + 2(x^2 - 5) - 24$ — Notice that the first two terms in the polynomial share a common factor of $(x^2 - 5)$. Suppose that we let u represent this expression. That is, $u = x^2 - 5$.

$$\begin{aligned} &= u^2 + 2u - 24 & \\ &= (u + 6)(u - 4) & \text{Factor.} \\ &= [(x^2 - 5) + 6][(x^2 - 5) - 4] & \text{Back substitute. Replace } u \text{ by } (x^2 - 5). \\ &= (x^2 + 1)(x^2 - 9) & \text{Simplify.} \\ &= (x^2 + 1)(x + 3)(x - 3) & \text{Factor } x^2 - 9 \text{ as a difference of squares.} \end{aligned}$$

Apply the Skills

13. Factor completely. $(x^2 - 2)^2 + 5(x^2 - 2) - 14$

Some polynomials cannot be factored with the techniques learned so far. For example, no combination of binomial factors with integer coefficients will produce the trinomial $3x^2 + 9x + 5$.

$$\begin{aligned} (3x + 5)(x + 1) &= 3x^2 + 3x + 5x + 5 & \text{Incorrect. Wrong middle term.} \\ &= 3x^2 + 8x + 5 & \\ (3x + 1)(x + 5) &= 3x^2 + 15x + x + 5 & \text{Incorrect. Wrong middle term.} \\ &= 3x^2 + 16x + 5 & \end{aligned}$$

In such a case we say that the polynomial is a **prime polynomial**. Its only factors are 1 and itself.

Learn: Factor Expressions Containing Negative and Rational Exponents

We now revisit the process to factor out the greatest common factor. In some applications, it is necessary to factor out a variable factor with a negative integer exponent or a rational exponent. Before we demonstrate this in Example 14, take a minute to review a similar example with positive integer exponents.

$2x^6 + 5x^5 + 7x^4$	The GCF is x^4. This is x raised to the *smallest exponent* to which it appears in any term.
$\boxed{6-4}\ \boxed{5-4}\ \boxed{4-4}$ $= x^4(2x^2 + 5x^1 + x^0)$	The powers on the factors of x within parentheses are found by subtracting 4 from the original exponents.

Example 14: Factoring Out Negative and Rational Exponents

Factor. Write the answers with no negative exponents.

a. $2x^{-6} + 5x^{-5} + 7x^{-4}$ **b.** $x(2x+5)^{-1/2} + (2x+5)^{1/2}$

Solution:

a. $2x^{-6} + 5x^{-5} + 7x^{-4}$ $\boxed{-6-(-6)}\ \boxed{-5-(-6)}\ \boxed{-4-(-6)}$	The smallest exponent on x is -6. Factor out x^{-6}.
$= x^{-6}(2x^0 + 5x^1 + 7x^2)$ $= x^{-6}(2 + 5x + 7x^2)$	The powers on the factors of x within parentheses are found by subtracting -6 from the original exponents.
$= \dfrac{7x^2 + 5x + 2}{x^6}$	Simplify the negative exponent. $b^{-n} = \frac{1}{b^n}$
b. $x(2x+5)^{-1/2} + (2x+5)^{1/2}$	The smallest exponent on $(2x+5)$ is $-\frac{1}{2}$.
$\boxed{-1/2-(-1/2)}\ \boxed{1/2-(-1/2)}$	Factor out $(2x+5)^{-1/2}$.
$= (2x+5)^{-1/2}\,[x(2x+5)^0 + (2x+5)^1]$	The powers on the factors of $(2x+5)$ within parentheses are found by subtracting $-\frac{1}{2}$ from the original exponents.
$= (2x+5)^{-1/2}\,[x + (2x+5)]$	The expression $(2x+5)^0 = 1$, for $2x+5 \neq 0$.
$= (2x+5)^{-1/2}\,(3x+5)$	Simplify.
$= \dfrac{3x+5}{(2x+5)^{1/2}}$	Simplify the negative exponent.

Apply the Skills

14. Factor completely and write the answer with no negative exponents.

a. $11a^{-5} - 3a^{-4} + 2a^{-3}$ **b.** $x(4x+3)^{-3/4} + (4x+3)^{1/4}$

Practice R-3

Practice Exercises

Concept Connections

1. The __________ term of a polynomial is the term of highest degree.
2. The leading __________ of a polynomial is the numerical factor of the leading term.
3. A __________ a polynomial that has two terms, and a __________ is a polynomial with three terms.
4. The conjugate of $3 - \sqrt{x}$ is __________.
5. The binomial $a^3 + b^3$ is called a sum of __________ and factors as __________.
6. The binomial $a^3 - b^3$ is called a __________ of cubes and factors as __________.
7. The trinomial $a^2 + 2ab + b^2$ is a __________ square trinomial. Its factored form is __________.
8. The binomial $a^2 - b^2$ is called a difference of __________ and factors as __________.

Learn: Add and Subtract Polynomials

For Exercises 9–10, determine if the expression is a polynomial.

9. **a.** $4a^2 + 7b - 3$ **b.** $\frac{3}{4}x^2y$
c. $6x + \frac{7}{x} + 5$ **d.** $\sqrt{p^2 + 2p - 5}$

10. **a.** $3x^5 - 9x^2 + \frac{2}{x^3}$ **b.** $\sqrt{2}ab^4$
c. $3|y| + 2$ **d.** $-7x^3 - 4x^2 + 2x - 5$

For Exercises 11–14, add or subtract as indicated and simplify. (See Example 1 on p. R44)

11. $(-8p^7 - 4p^4 + 2p - 5) + (2p^7 + 6p^4 + p^2)$

12. $(-7w^5 + 3w^3 - 6) + (9w^5 - 5w^3 + 4w - 3)$

13. $(0.05c^3b + 0.02c^2b^2 - 0.09cb^3) - (-0.03c^3b + 0.08c^2b^2 - 0.1cb^3)$

14. $(0.004mn^5 - 0.001mn^4 + 0.05mn^3) - (0.003mn^5 + 0.007mn^4 - 0.07mn^3)$

Learn: Multiply Polynomials

For Exercises 15–22, multiply and simplify. (See Example 2 on p. R45)

15. $7m^2(2m^4 - 3m + 4)$

16. $8p^5(-2p^2 - 5p - 1)$

17. $(2x - 5)(x + 4)$

18. $(w + 7)(6w - 3)$

19. $(4u^2 - 5v^2)(2u^2 + 3v^2)$

20. $(2z^3 + 5u^2)(7z^3 - u^2)$

21. $(3y + 6)\left(\frac{1}{3}y^2 - 5y - 4\right)$

22. $(10v - 5)\left(\frac{1}{5}v^2 - 3v + 1\right)$

23. Write the expanded form for $(a + b)^2$.

24. Write the expanded form for $(a + b)(a - b)$.

For Exercises 25–38, perform the indicated operations and simplify. (See Example 3 on p. R47)

25. $(4x - 5)(4x + 5)$

26. $(3p - 2)(3p + 2)$

27. $(3w^2 - 7z)(3w^2 + 7z)$

28. $(9v^3 + 2u)(9v^3 - 2u)$

29. $\left(\frac{1}{5}c - \frac{2}{3}d^3\right)\left(\frac{1}{5}c + \frac{2}{3}d^3\right)$

30. $\left(\frac{1}{6}n - \frac{4}{5}p^4\right)\left(\frac{1}{6}n + \frac{4}{5}p^4\right)$

31. $(5m - 3)^2$

32. $(7v - 2)^2$

33. $(4t^2 + 3p^3)^2$

34. $(2a^2 + 11b^3)^2$

35. $(w + 4)^3$

36. $(p - 2)^3$

37. $[(u + v) - w][(u + v) + w]$

38. $[(c + d) - a][(c + d) + a]$

39. Suppose that x represents the smaller of two consecutive integers.

- **a.** Write a polynomial that represents the larger integer.
- **b.** Write a polynomial that represents the sum of the two integers. Then simplify.
- **c.** Write a polynomial that represents the product of the two integers. Then simplify.
- **d.** Write a polynomial that represents the sum of the squares of the two integers. Then simplify.

40. Suppose that x represents the larger of two consecutive odd integers.

a. Write a polynomial that represents the smaller integer.

b. Write a polynomial that represents the sum of the two integers. Then simplify.

c. Write a polynomial that represents the product of the two integers. Then simplify.

d. Write a polynomial that represents the difference of the squares of the two integers. Then simplify.

For Exercises 41–48, simplify each expression.

41. $(y + 7)^2 - 2(y - 3)^2$

42. $(x - 4)^2 - 6(x + 1)^2$

43. $(x^n + 3)(x^n - 7)$

44. $(y^n + 4)(y^n - 5)$

45. $(z^n + w^m)^2$

46. $(w^n - y^m)^2$

47. $(6x + 5)(6x - 5) - (6x + 5)^2$

48. $(2y - 7)(2y + 7) - (2y - 7)^2$

Learn: Multiply Radical Expressions Involving Multiple Terms

For Exercises 49–66, multiply and simplify. Assume that all variable expressions represent positive real numbers. (See Example 4 on p. R47)

49. $5\sqrt{2}(2\sqrt{2} + 6\sqrt{3} - 4)$

50. $5\sqrt{7}(3\sqrt{7} - 2\sqrt{5} + 3)$

51. $(2\sqrt{y} - 3)(4\sqrt{y} + 5)$

52. $(5\sqrt{z} - 4)(3\sqrt{z} + 6)$

53. $(2\sqrt{3} + \sqrt{7})(2\sqrt{3} - \sqrt{7})$

54. $(5\sqrt{2} - \sqrt{11})(5\sqrt{2} + \sqrt{11})$

55. $(4x\sqrt{y} - 2y\sqrt{x})(4x\sqrt{y} + 2y\sqrt{x})$

56. $(5u\sqrt{v} - 6v\sqrt{u})(5u\sqrt{v} + 6v\sqrt{u})$

57. $(6z - \sqrt{5})^2$

58. $(2v - \sqrt{17})^2$

59. $(\sqrt{x + 1} - 5)(\sqrt{x + 1} + 5)$

60. $(\sqrt{y + 2} - 4)(\sqrt{y + 2} + 4)$

61. $(\sqrt{x + 1} - 5)^2$

62. $(\sqrt{y + 2} - 4)^2$

63. $(\sqrt{5 + 2\sqrt{x}})(\sqrt{5 - 2\sqrt{x}})$

64. $(\sqrt{7 + 3\sqrt{z}})(\sqrt{7 - 3\sqrt{z}})$

65. $(\sqrt{x + y} - \sqrt{x - y})^2$

66. $(\sqrt{a + 2} - \sqrt{a - 2})^2$

Learn: Factor by Grouping

For Exercises 67–72, factor out the greatest common factor.

67. $15c^5 - 30c^4 + 5c^3$

68. $12m^4 + 15m^3 + 3m^2$

69. $21a^2b^5 - 14a^3b^4 + 35a^4b$

70. $36p^4q^7 + 18p^3q^5 - 27p^2q^6$

71. $5z(x - 6y) + 7(x - 6y)$

72. $4t(u + 8v) - 3(u + 8v)$

For Exercises 73–76, factor out the indicated common factor.

73. a. Factor out 3. $-6x^2 + 12x + 9$

b. Factor out –3. $-6x^2 + 12x + 9$

74. a. Factor out 5. $-15y^2 - 10y + 25$

b. Factor out –5. $-15y^2 - 10y + 25$

75. Factor out $-4x^2y$. $-12x^3y^2 - 8x^4y^3 + 4x^2y$

76. Factor out $-7a^3b$. $-14a^4b^3 + 21a^3b^2 - 7a^3b$

For Exercises 77–82, factor by grouping. (See Example 5 on p. R49)

77. $8ax + 18a + 20x + 45$

78. $6ty + 9y + 14t + 21$

79. $12x^3 - 9x^2 - 40x + 30$

80. $30z^3 - 35z^2 - 24z + 28$

81. $cd - 8d + 4c - 2d^2$

82. $7t - 6v^2 + tv - 42v$

Learn: Factor Quadratic Trinomials

For Exercises 83–94, factor the trinomials. (See Examples 6–8 on pp. R50–R52)

83. $p^2 + 2p - 63$

84. $w^2 + 5w - 66$

85. $2t^3 - 28t^2 + 80t$

86. $5u^4 - 40u^3 + 35u^2$

87. $7y^3z - 40y^2z^2 - 12yz^3$

88. $11a^3b + 18a^2b^2 - 8ab^3$

89. $t^2 - 18t + 81$

90. $p^2 + 8p + 16$

91. $50x^3 + 160x^2y + 128xy^2$

92. $48y^3 - 72y^2z + 27yz^2$

93. $4c^4 - 20c^2d^3 + 25d^6$

94. $9m^4 + 42m^2n^4 + 49n^8$

Learn: Factor Binomials

For Exercises 95–106, factor the binomials. (See Examples 9–10 on pp. R52–R53)

95. $9w^2 - 64$

96. $16t^2 - 49$

97. $200u^4 - 18v^6$

98. $75m^6 - 27n^4$

99. $625p^4 - 16$

100. $81z^4 - 1$

101. $y^3 + 64$

102. $u^3 + 343$

103. $c^4 - 27c$

104. $d^4 - 8d$

105. $8a^6 - 125b^9$

106. $27m^{12} - 64n^9$

Learn: Apply a General Strategy to Factor Polynomials

For Exercises 107–130, factor completely. (See Examples 11–13 on pp. R55–R56)

107. $30x^4 + 70x^3 - 120x^2 - 280x$

108. $4y^4 - 10y^3 - 36y^2 + 90y$

109. $a^2 - y^2 + 10y - 25$

110. $c^2 - z^2 + 8z - 16$

111. $(x^2 - 2)^2 - 3(x^2 - 2) - 28$

112. $(y^2 + 2)^2 + 5(y^2 + 2) - 24$

113. $(x^3 + 12)^2 - 16$

114. $(y^3 + 34)^2 - 49$

115. $(x + y)^3 + z^3$

116. $(a + 5)^3 - b^3$

117. $9m^2 + 42m(3n + 1) + 49(3n + 1)^2$

118. $4x^2 + 36x(7y - 1) + 81(7y - 1)^2$

119. $(c - 3)^2 - (2c - 5)^2$

120. $(d + 6)^2 - (4d - 3)^2$

121. $p^{11} - 64p^8 - p^3 + 64$

122. $t^7 + 27t^4 - t^3 - 27$

123. $m^6 + 26m^3 - 27$

124. $n^6 - 7n^3 - 8$

125. $16x^6z + 38x^3z - 54z$

126. $24y^7 + 21y^4 - 3y$

127. $x^2 - y^2 - x + y$

128. $a^2 - b^2 - a - b$

129. $a^2 + ac - 2c^2 - c + a$

130. $x^2 + 2xy - 3y^2 - y + x$

Learn: Factor Expressions Containing Negative and Rational Exponents

For Exercises 131–142, factor completely. Write the answers with positive exponents only. (See Example 14 on p. R57)

131. $2x^{-4} - 7x^{-3} + x^{-2}$

132. $5t^{-7} + 2t^{-6} - t^{-5}$

133. $y^{-2} - y^{-3} - 12y^{-4}$

134. $w^{-3} + 10w^{-4} + 9w^{-5}$

135. $2c^{7/4} + 4c^{3/4}$

136. $10y^{9/5} - 15y^{4/5}$

137. $\frac{8}{3}x^{1/3} + \frac{5}{3}x^{-2/3}$

138. $\frac{15}{2}x^{1/2} + \frac{3}{2}x^{-1/2}$

139. $5x(3x + 1)^{2/3} + (3x + 1)^{5/3}$

140. $7t(4t + 1)^{3/4} + (4t + 1)^{7/4}$

141. $x(3x + 2)^{-2/3} + (3x + 2)^{1/3}$

142. $x(5x - 8)^{-4/5} + (5x - 8)^{1/5}$

Mixed Exercises

For Exercises 143–146, perform the indicated operations.

143. $(y + 7)^2 - 2(y - 3)^2$

145. $(6x + 5)(6x - 5) - (6x + 5)^2$

144. $(x - 4)^2 - 6(x + 1)^2$

146. $(2y - 7)(2y + 7) - (2y - 7)^2$

For Exercises 147–154, factor completely.

147. **a.** $u^2 - 10u + 25$

b. $x^4 - 10x^2 + 25$

c. $(a + 1)^2 - 10(a + 1) + 25$

148. **a.** $u^2 + 12u + 36$

b. $y^4 + 12y^2 + 36$

c. $(b - 2)^2 + 12(b - 2) + 36$

149. $4(x - 3)^2 + 7(x - 3) + 3$

150. $3(y + 4)^2 + 5(y + 4) - 2$

151. $-20x^3 + 74x^2 - 60x$

152. $-24y^3 + 90y^2 - 75y$

153. $2(w^2 - 5)^2 + (w^2 - 5) - 15$

154. $5(t^2 + 3)^2 + 21(t^2 + 3) + 4$

For Exercises 155–164, factor completely. Write the answers with positive exponents only.

155. $2x^{-3} - 2x^{-4} - 24x^{-5}$

156. $3x^{-2} + 15x^{-3} - 72x^{-4}$

157. $(2x + 3)^{-1} + 4(2x + 3)^{-2} + 4(2x + 3)^{-3}$

158. $(x + 6)^{-1} + 6(x + 6)^{-2} + 9(x + 6)^{-3}$

159. $m^{5/2} - 5m^{3/2} - 36m^{1/2}$

160. $n^{8/3} - 8n^{5/3} - 20n^{2/3}$

161. $5x(3x + 1)^{2/3} + (3x + 1)^{5/3}$

162. $7t(4t + 1)^{3/4} + (4t + 1)^{7/4}$

163. $x(3x + 2)^{-2/3} + (3x + 2)^{1/3}$

164. $x(5x - 8)^{-4/5} + (5x - 8)^{1/5}$

Write About It

165. Explain the similarity in simplifying the given expressions.

a. $(3x + 2)(4x - 7)$

b. $(3\sqrt{x} + \sqrt{2})(4\sqrt{x} - \sqrt{7})$

166. Explain the similarity in simplifying the given expressions.

a. $(x + 3)(x - 3)$ **b.** $(\sqrt{x} + \sqrt{3})(\sqrt{x} - \sqrt{3})$

167. Why is the sum of squares $a^2 + b^2$ not factorable over the real numbers?

168. Explain the similarity in the process to factor out the GCF in the following two expressions.

$5x^4 + 4x^3$ and $5x^{-4} + 4x^{-3}$

Expanding Your Skills

For Exercises 169–178, factor out the greatest common factor. Assume that *n* is a positive integer, and write the answer with positive exponents only.

169. $x^{3n} + 2x^{2n} - 5x^n$ **170.** $4x^{5n} - 3x^{4n} + x^{3n}$

171. $14a^{-3n} - 6a^{-4n} + 2a^{-5n}$

172. $4t^{-4n} + 12t^{-5n} - 20t^{-6n}$

173. $2a^{3n} - x^{2n} - 15x^n$

174. $4x^{3n} - 12x^{2n} - 7x^n$

175. $4x^n - x^{-n}$ **176.** $x^n - 81x^{-n}$

177. $3x^{-2n} + x^{4-2n} + 4x^{2-2n}$

178. $10x^{-2n} + 3x^{4-2n} + 17x^{2-2n}$

We say that the expression $x^2 - 4$ is factorable over the integers as $(x + 2)(x - 2)$. Notice that the constant terms in the binomials are integers. The expression $x^2 - 3$ can be factored over the irrational numbers as $x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3})$. For Exercises 179–184, factor each expression over the irrational numbers.

179. $x^2 - 5$ **180.** $y^2 - 11$

181. $z^4 - 36$ **182.** $w^4 - 49$

183. $x^2 - 2\sqrt{5}x + 5$ **184.** $c^2 - 2\sqrt{3}c + 3$

For Exercises 185–188, determine if the statement is true or false. If a statement is false, explain why.

185. The sum of two polynomials each of degree 5 will be degree 5.

186. The sum of two polynomials each of degree 5 will be less than or equal to degree 5.

187. The product of two polynomials each of degree 4 will be degree 8.

188. The product of two polynomials each of degree 4 will be less than degree 8.

Problem Recognition Exercises

Simplifying Algebraic Expressions

Many expressions in algebra look similar, but the methods used to simplify the expressions may be different. For Exercises 1–14, simplify each expression. Assume that the variables are restricted so that each expression is defined.

1. **a.** $64^{1/2}$ **b.** $64^{1/3}$
c. $64^{2/3}$ **d.** 64^{-1}
e. $-64^{1/2}$ **f.** $(-64)^{1/2}$
g. $(-64)^{2/3}$ **h.** $(-64)^{-2/3}$

2. **a.** $(5ab^3)^2$ **b.** $(5a + b^3)^2$
c. $(5ab^3)^{-2}$ **d.** $(5a + b^3)^{-2}$

3. **a.** $(2x^4y)^2$ **b.** $(2x^4 - y)^2$
c. $(2x^4y)^{-2}$ **d.** $(2x^4 - y)^{-2}$

4. **a.** x^5x^3 **b.** $\frac{x^5}{x^3}$
c. $x^{-5}x^3$ **d.** $(x^5)^{-3}$

5. **a.** $\sqrt{x^8}$ **b.** $\sqrt[3]{x^8}$
c. $\sqrt[5]{x^8}$ **d.** $\sqrt[9]{x^8}$

6. **a.** $(3a + 4b^2) - (2a - b^2)$
b. $(3a + 4b^2)(2a - b^2)$

7. **a.** $(a^2 - b^2) + (a^2 + b^2)$
b. $(a^2 - b^2)(a^2 + b^2)$
c. $(a - b)^2 - (a + b)^2$
d. $(a - b)^2(a + b)^2$

8. **a.** $|a - b|$ for $a < b$
b. $|a - b|$ for $a > b$

9. **a.** $|x + 2|$ for $x > -2$
b. $|x + 2|$ for $x < -2$

10. **a.** $(\sqrt{x})^2 + (\sqrt{y})^2$
b. $\sqrt{x^2 + y^2}$
c. $(\sqrt{x} + \sqrt{y})^2$
d. $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$

11. **a.** $\sqrt[3]{2x} \cdot \sqrt[3]{2x}$
b. $\sqrt[3]{2x} + \sqrt[3]{2x}$

12. **a.** $\sqrt[4]{y} \cdot \sqrt[3]{y}$
b. $\sqrt[4]{y} + \sqrt[3]{y}$

13. **a.** $36 \div 12 \div 6 \div 2$
b. $36 \div (12 \div 6) \div 2$
c. $36 \div 12 \div (6 \div 2)$
d. $(36 \div 12) \div (6 \div 2)$

14. **a.** $\sqrt{6^2 + 8^2}$
b. $\sqrt{6^2} + \sqrt{8^2}$

Go online for more practice problems.

Lesson R-4

Rational Expressions and More Operations on Radicals

What Will You Learn?

After completing this lesson, you should be able to:

- Determine Restricted Values for a Rational Expression
- Simplify Rational Expressions
- Multiply and Divide Rational Expressions
- Add and Subtract Rational Expressions
- Simplify Complex Fractions
- Rationalize the Denominator of a Radical Expression

Learn: Determine Restricted Values for a Rational Expression

Suppose that an object that is originally 35°C is placed in a freezer. The temperature T (in °C) of the object t hours after being placed in the freezer can be approximated by the model

$$T = \frac{350}{t^2 + 3t + 10}.$$

For example, 2 hr after being placed in the freezer the temperature of the object is

$$T = \frac{350}{(2)^2 + 3(2) + 10} = 17.5°\text{C}.$$

The expression $\frac{350}{t^2 + 3t + 10}$ is called a rational expression. A **rational expression** is a ratio of two polynomials. Since a rational expression may have a variable in the denominator, we must be careful to exclude values of the variable that make the denominator zero.

Example 1: Determining Restricted Values for a Rational Expression

Determine the restrictions on the variable for each rational expression.

a. $\frac{x-3}{x+2}$ **b.** $\frac{x}{x^2 - 49}$ **c.** $\frac{4}{5x^2y}$

Solution:

a. $\frac{x-3}{x+2}$ $\boxed{x \neq -2}$

Division by zero is undefined.

For this expression $x \neq -2$. If -2 were substituted for x, the denominator would be zero. $-2 + 2 = 0$

b. $\frac{x}{x^2 - 49} = \frac{x}{(x+7)(x-7)}$ $\boxed{x \neq -7}$ $\boxed{x \neq 7}$

For this expression, $x \neq -7$ and $x \neq 7$. If x were -7, then $-7 + 7 = 0$. If x were 7, then $7 - 7 = 0$. In either case, the denominator would be zero.

c. $\frac{4}{5x^2y}$ $\boxed{x \neq 0}$ $\boxed{y \neq 0}$

For this expression, $x \neq 0$ and $y \neq 0$. If either x or y were zero, then the product would be zero.

Apply the Skills

1. Determine the restrictions on the variable.

a. $\frac{x+4}{x-3}$ **b.** $\frac{5}{c^2-16}$ **c.** $\frac{64}{7ab^3}$

Learn: Simplify Rational Expressions

A rational expression is simplified (in lowest terms) if the only factors shared by the numerator and denominator are 1 or –1. To simplify a rational expression, factor the numerator and denominator, then apply the property of equivalent algebraic fractions.

Equivalent Algebraic Fractions

If a, b, and c represent real-valued expressions, then

$$\frac{ac}{bc} = \frac{a}{b} \text{ for } b \neq 0 \text{ and } c \neq 0 \qquad \underline{\text{Example:}}\ \frac{10x}{15x} = \frac{2\cdot\overset{1}{\cancel{5\cdot x}}}{3\cdot\underset{1}{\cancel{5\cdot x}}} = \frac{2}{3}$$

Example 2: Simplifying Rational Expressions

Simplify.

a. $\frac{x^2-16}{x^2-x-12}$ **b.** $\frac{8+2\sqrt{7}}{4}$

Solution:

a. $\frac{x^2-16}{x^2-x-12} = \frac{(x+4)(x-4)}{(x+3)(x-4)}$

Factor the numerator and denominator. We have the restrictions that $x \neq -3$, $x \neq 4$.

$= \frac{(x+4)\overset{1}{\cancel{(x-4)}}}{(x+3)\underset{1}{\cancel{(x-4)}}}$

Divide out common factors that form a ratio of $\frac{1}{1}$.

$= \frac{(x+4)}{(x+3)}$ for $x \neq -3$, $x \neq 4$

The same restrictions for the original expression also apply to the simplified expression.

b. $\frac{8+2\sqrt{7}}{4} = \frac{2(4+\sqrt{7})}{2\cdot 2} = \frac{\overset{1}{\cancel{2}}(4+\sqrt{7})}{\underset{1}{\cancel{2}}\cdot 2} = \frac{4+\sqrt{7}}{2}$

Apply the Skills

2. Simplify.

a. $\frac{x^2 - 8x}{x^2 - 7x - 8}$

b. $\frac{3 + 9\sqrt{5}}{6}$

In Example 2(a), the expressions $\frac{x^2 - 16}{x^2 - x - 12}$ and $\frac{x + 4}{x + 3}$ are equal for all values of x for which *both* expressions are defined. This excludes the values $x = -3$ and $x = 4$.

The property of equivalent algebraic fractions tells us that we can divide out common factors that form a ratio of 1. We can also divide out factors that form a ratio of –1. For example:

$$\frac{4 - x}{x - 4} \xrightarrow{\text{Factor out } -1 \text{ from the numerator.}} \frac{-1(-4 + x)}{x - 4} = \frac{-1(x - 4)}{x - 4} = -1$$

Numerator and denominator are opposite polynomials. Their ratio is –1.

Insights

The expressions $4 - x$ and $x - 4$ are opposite polynomials (the signs of their terms are opposites). The ratio of two opposite factors is –1.

Learn: Multiply and Divide Rational Expressions

To multiply and divide rational expressions, use the following properties.

Multiplication and Division of Algebraic Fractions

Let a, b, c, and d be real-valued expressions.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \text{for } b \neq 0, d \neq 0$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \qquad \text{for } b \neq 0, c \neq 0, d \neq 0$$

To multiply or divide rational expressions, factor the numerator and denominator of each fraction completely. Then apply the multiplication and division properties for algebraic fractions.

Restriction Agreement

Operations on rational expressions are valid for all values of the variable for which the rational expressions are defined. In Examples 3–8 and in the exercises, we will perform operations on rational expressions without explicitly stating the restrictions. Instead, the restrictions on the variables will be implied.

Example 3: Dividing Rational Expressions

Divide. $\frac{x^3-8}{4-x^2} \div \frac{3x^2+6x+12}{x^2-x-6}$

Solution:

$$\frac{x^2-8}{4-x^2} \div \frac{3x^2+6x+12}{x^2-x-6}$$

$$= \frac{x^3-8}{4-x^2} \cdot \frac{x^2-x-6}{3x^2+6x+12}$$

Multiply the first fraction by the reciprocal of the second fraction.

$$= \frac{(x-2)(x^2+2x+4)}{(2-x)(2+x)} \cdot \frac{(x-3)(x+2)}{3(x^2+2x+4)}$$

Factor. Note that $x^3 - 8$ is a difference of cubes.

$$= \frac{\overset{(-1)}{\cancel{(x-2)}}\,\overset{1}{\cancel{(x^2+2x+4)}} \cdot (x-3)\overset{1}{\cancel{(x+2)}}}{\cancel{(2-x)}\cancel{(2+x)} \cdot 3\cancel{(x^2+2x+4)}}$$

$$= -\frac{x-3}{3}$$

Note that $(x-2)$ and $(2-x)$ are opposite polynomials, and their ratio is -1.

Check Your Work

For the expression $-\frac{x-3}{3}$ do not be tempted to "divide out" the 3 in the numerator with the 3 in the denominator. The 3's are *terms*, not factors. Only common *factors* can be divided out.

Apply the Skills

3. Divide. $\frac{y^3-27}{9-y^2} \div \frac{5y^2+15y+45}{y^2+8y+15}$

Learn: Add and Subtract Rational Expressions

Recall that fractions can be added or subtracted if they have a common denominator.

Addition and Subtraction of Algebraic Fractions

Let a, b, and c be real-valued expressions.

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b} \quad \text{for } b \neq 0$$

If two rational expressions have different denominators, then it is necessary to convert the expressions to equivalent expressions with the same denominator. We do this by applying the property of equivalent fractions.

Property of Equivalent Fractions

Let a, b, and c be real-valued expressions.

$\frac{a}{b} = \frac{ac}{bc}$, where $b \neq 0$ and $c \neq 0$ Example: $\frac{5}{x} = \frac{5 \cdot xy}{x \cdot xy} = \frac{5xy}{x^2y}$

When adding or subtracting numerical fractions or rational expressions, it is customary to use the least common denominator (LCD) of the original expressions and to follow these guidelines.

Adding and Subtracting Rational Expressions

Step 1: Factor the denominators and determine the LCD of all expressions. The LCD is the product of unique prime factors where each factor is raised to the greatest power to which it appears in any denominator.

Step 2: Write each expression as an equivalent expression with the LCD as its denominator.

Step 3: Add or subtract the numerators as indicated and write the result over the LCD.

Step 4: Simplify if possible.

Example 4: Adding Rational Expressions

Add the rational expressions and simplify the result. $\frac{7}{4a} + \frac{11}{10a^2}$

Solution:

$\frac{7}{4a} + \frac{11}{10a^2} = \frac{7}{2^2a} + \frac{11}{2 \cdot 5a^2}$	**Step 1:** Factor the denominators. The LCD is $2^2 \cdot 5 \cdot a^2$ or $20a^2$.
$= \frac{7 \cdot (5a)}{2^2a \cdot (5a)} + \frac{11 \cdot (2)}{2 \cdot 5a^2 \cdot (2)}$	**Step 2:** Multiply numerator and denominator of each expression by the factors missing from the denominators.
$= \frac{35a}{20a^2} + \frac{22}{20a^2}$	**Step 3:** Add the numerators and write the result over the common denominator.
$= \frac{35a + 22}{20a^2}$	**Step 4:** The expression is already simplified.

Apply the Skills

4. Add the rational expressions and simplify the result. $\frac{8}{9y^2} + \frac{1}{15y}$

Example 5: Subtracting Rational Expressions

Subtract the rational expressions and simplify the result.

$$\frac{3x+5}{x^2+4x+3}-\frac{x-5}{x^2+2x-3}$$

Solution:

$$\frac{3x+5}{x^2+4x+3}-\frac{x-5}{x^2+2x-3}$$

$$=\frac{3x+5}{(x+3)(x+1)}-\frac{x-5}{(x+3)(x-1)}$$

Factor the denominators. The LCD is $(x+3)(x+1)(x-1)$.

$$=\frac{(3x+5)(x-1)}{(x+3)(x+1)(x-1)}-\frac{(x-5)(x+1)}{(x+3)(x-1)(x+1)}$$

Multiply numerator and denominator of each expression by the factors missing from the denominators.

$$=\frac{(3x+5)(x-1)-(x-5)(x+1)}{(x+3)(x+1)(x-1)}$$

Subtract the numerators and write the result over the common denominator.

$$=\frac{3x^2+2x-5-(x^2-4x-5)}{(x+3)(x+1)(x-1)}$$

Check Your Work

It is very important to use parentheses around the second trinomial in the numerator. This will ensure that all terms that follow will be subtracted.

$$=\frac{3x^2+2x-5-x^2+4x+5}{(x+3)(x+1)(x-1)}$$

$$=\frac{2x^2+6x}{(x+3)(x+1)(x-1)}$$

$$=\frac{2x\overset{1}{\cancel{(x+3)}}}{\cancel{(x+3)}(x+1)(x-1)}$$

$$=\frac{2x}{(x+1)(x-1)}$$

Factor the numerator and denominator and simplify.

Apply the Skills

5. Subtract the rational expressions and simplify the result.

$$\frac{t}{t^2+5t+6}-\frac{2}{t^2+3t+2}$$

Learn: Simplify Complex Fractions

A **complex fraction** (also called a **compound fraction**) is an expression that contains one or more fractions in the numerator or denominator. We present two methods to simplify a complex fraction. Method I is an application of the order of operations.

Simplifying a Complex Fraction: Order of Operations (Method I)

Step 1: Add or subtract the fractions in the numerator to form a single fraction. Add or subtract the fractions in the denominator to form a single fraction.

Step 2: Divide the resulting expressions.

Step 3: Simplify if possible.

Example 6: Simplifying a Complex Fraction (Method I)

Simplify. $\dfrac{\frac{x}{4}-\frac{4}{x}}{\frac{1}{4}+\frac{1}{x}}$

Solution:

$$\frac{\frac{x}{4}-\frac{4}{x}}{\frac{1}{4}+\frac{1}{x}}=\frac{\frac{x\cdot x}{4\cdot x}-\frac{4\cdot 4}{x\cdot 4}}{\frac{1\cdot x}{4\cdot x}+\frac{1\cdot x}{x\cdot x}}=\frac{\frac{x^2-16}{4x}}{\frac{x+4}{4x}}$$

Step 1: Subtract the fractions in the numerator. Add the fractions in the denominator.

$$=\frac{x^2-16}{4x}\cdot\frac{4x}{x+4}$$

Step 2: Multiply the rational expression from the numerator by the reciprocal of the expression from the denominator.

$$=\frac{(x-4)\overset{1}{\cancel{(x+4)}}}{\cancel{4x}}\cdot\frac{\overset{1}{\cancel{4x}}}{\cancel{x+4}}$$

$$=x-4$$

Step 3: Simplify by factoring and dividing out common factors.

Apply the Skills

6. Simplify. $\dfrac{\frac{1}{7}+\frac{1}{y}}{\frac{y}{7}-\frac{7}{y}}$

In Example 7 we demonstrate another method (Method II) to simplify a complex fraction.

Simplifying a Complex Fraction: Multiply by the LCD (Method II)

Step 1: Multiply the numerator and denominator of the complex fraction by the LCD of all individual fractions.

Step 2: Apply the distributive property and simplify the numerator and denominator.

Step 3: Simplify the resulting expression if possible.

Example 7: Simplifying a Complex Fraction (Method II)

Simplify. $\dfrac{d^{-2}-c^{-2}}{d^{-1}-c^{-1}}$

Solution:

$$\frac{d^{-2}-c^{-2}}{d^{-1}-c^{-1}}=\frac{\frac{1}{d^2}-\frac{1}{c^2}}{\frac{1}{d}-\frac{1}{c}}$$

First write the expression with positive exponents.

$$=\frac{c^2d^2\cdot\left(\frac{1}{d^2}-\frac{1}{c^2}\right)}{c^2d^2\cdot\left(\frac{1}{d}-\frac{1}{c}\right)}$$

Step 1: Multiply numerator and denominator by the LCD of all four individual fractions: c^2d^2.

$$=\frac{\frac{c^2d^2}{1}\cdot\frac{1}{d^2}-\frac{c^2d^2}{1}\cdot\frac{1}{c^2}}{\frac{c^2d^2}{1}\cdot\frac{1}{d}-\frac{c^2d^2}{1}\cdot\frac{1}{c}}$$

Step 2: Apply the distributive property.

$$=\frac{c^2-d^2}{c^2d-cd^2}$$

$$=\frac{\cancel{(c-d)}(c+d)}{cd\cancel{(c-d)}}=\frac{c+d}{cd}$$

Step 3: Simplify by factoring and dividing out common factors.

Apply the Skills

7. Simplify. $\dfrac{4-6x^{-1}}{2x^{-1}-3x^{-2}}$

Example 8: Simplifying a Complex Fraction (Method II)

Simplify. $\dfrac{\frac{2}{1+h}-2}{h}$

Solution:

$$\frac{\frac{2}{1+h}-2}{h}=\frac{\frac{2}{1+h}-\frac{2}{1}}{h}=\frac{(1+h)\cdot\left(\frac{2}{1+h}-\frac{2}{1}\right)}{(1+h)\cdot(h)}$$

Step 1: Multiply numerator and denominator by the LCD, which is $(1+h)$.

$$=\frac{\frac{(1+h)}{1}\cdot\left(\frac{2}{1+h}\right)-\frac{(1+h)}{1}\cdot\left(\frac{2}{1}\right)}{(1+h)\cdot(h)}$$

Step 2: Apply the distributive property.

$$=\frac{2-2(1+h)}{h(1+h)}$$

$$=\frac{2-2-2h}{h(1+h)}=\frac{-2\cancel{h}}{\cancel{h}(1+h)}=\frac{-2}{1+h}\quad\text{or}\quad-\frac{2}{1+h}$$

Step 3: Simplify.

Apply the Skills

8. Simplify. $\dfrac{\frac{5}{1+h}-5}{h}$

Insights

The expression given in Example 8 is a pattern we see in a first-semester calculus course.

Learn: Rationalize the Denominator of a Radical Expression

The same principle that applies to simplifying rational expressions also applies to simplifying algebraic fractions. For example, $\frac{5}{\sqrt{x}}$ is an algebraic fraction, but not a rational expression because the denominator is not a polynomial.

From Lesson R.2, we outlined the criteria for a radical expression to be simplified. Conditions 2 and 3 are stated here.

- No fraction may appear in the radicand.
- No denominator of a fraction may contain a radical.

In Example 9, we use the property of equivalent fractions to remove a radical from the denominator of a fraction. This is called **rationalizing the denominator.**

Example 9: Rationalizing the Denominator

Simplify. Assume that x is a positive real number.

a. $\frac{5}{\sqrt{x}}$

b. $\frac{4}{\sqrt{7}-\sqrt{5}}$

Solution:

a. $\frac{5}{\sqrt{x}} = \frac{5 \cdot \sqrt{x}}{\sqrt{x} \cdot \sqrt{x}}$

Multiply numerator and denominator by $\sqrt{x}$ so that the radicand in the denominator is a perfect square.

$= \frac{5\sqrt{x}}{\sqrt{x^2}}$

Apply the product property of radicals.

$= \frac{5\sqrt{x}}{x}$

Simplify the radical in the denominator.

b. $\frac{4}{\sqrt{7}-\sqrt{5}} = \frac{4 \cdot (\sqrt{7}+\sqrt{5})}{(\sqrt{7}-\sqrt{5}) \cdot (\sqrt{7}+\sqrt{5})}$

Multiply numerator and denominator by the conjugate of the denominator.

$= \frac{4(\sqrt{7}+\sqrt{5})}{(\sqrt{7})^2-(\sqrt{5})^2}$

Recall that $(a-b)(a+b) = a^2 - b^2$.

$= \frac{4(\sqrt{7}+\sqrt{5})}{7-5}$

Simplify the radicals in the denominator.

$= \frac{\overset{2}{\cancel{4}}(\sqrt{7}+\sqrt{5})}{\cancel{2}}$

Simplify the fraction.

$= 2(\sqrt{7}+\sqrt{5})$ or $2\sqrt{7}+2\sqrt{5}$

Insights

Keep the numerator in factored form until the denominator is simplified completely. By so doing, it will be easier to identify common factors in the numerator and denominator.

Apply the Skills

9. Simplify.

a. $\frac{\sqrt{5}}{\sqrt{7}}$

b. $\frac{12}{\sqrt{13}-\sqrt{10}}$

In Example 9(b), we multiplied the numerator and denominator by the conjugate of the denominator. The rationale is that product $(a+b)(a-b)$ results in a difference of squares $a^2 - b^2$. If either a or b has a square root factor, then the product will simplify to an expression without square roots.

Practice R-4

Practice Exercises

Concept Connections

1. A ______ expression is a ratio of two polynomials.
2. The restricted values of the variable for a rational expression are those that make the denominator equal to ______.
3. The expression $\frac{5(x+2)}{(x+2)(x-1)}$ equals $\frac{5}{x-1}$ provided that $x \neq$ ______ and $x \neq$ ______.
4. The ratio of a polynomial and its opposite equals ______.
5. A ______ fraction is an expression that contains one or more fractions in the numerator or denominator.
6. The process to remove a radical from the denominator of a fraction is called ______ the denominator.

Learn: Determine Restricted Values for a Rational Expression

For Exercises 7–14, determine the restrictions on the variable. (See Example 1 on p. R63)

7. $\frac{x-4}{x+7}$
8. $\frac{y-1}{y+10}$
9. $\frac{a}{a^2-81}$
10. $\frac{t}{t^2-16}$
11. $\frac{a}{a^2+81}$
12. $\frac{t}{t^2+16}$
13. $\frac{6c}{7a^3b^2}$
14. $\frac{11z}{8x^5y}$

Learn: Simplify Rational Expressions

15. Determine which expressions are equal to $-\frac{5}{x-3}$.
    a. $\frac{-5}{x-3}$
    b. $\frac{5}{3-x}$
    c. $-\frac{5}{3-x}$
    d. $\frac{-5}{3-x}$
16. Determine which expressions are equal to $\frac{-2}{a+b}$.
    a. $\frac{-2}{a-b}$
    b. $-\frac{2}{a+b}$
    c. $\frac{2}{-a-b}$
    d. $\frac{2}{a-b}$

For Exercises 17–26, simplify the expression and state the restrictions on the variable. (See Example 2 on p. R64)

17. $\frac{x^2-9}{x^2-4x-21}$
18. $\frac{y^2-64}{y^2-7y-8}$
19. $\frac{12a^2bc}{3ab^5}$
20. $-\frac{15tu^5v}{3t^3u}$
21. $\frac{10-5\sqrt{6}}{15}$
22. $\frac{12+4\sqrt{3}}{8}$
23. $\frac{2y^2-16y}{64-y^2}$
24. $\frac{81-t^2}{7t^2-63t}$
25. $\frac{4b-4a}{ax-xb-2a+2b}$
26. $\frac{2z-2y}{xy-xz+3y-3z}$

Learn: Multiply and Divide Rational Expressions

For Exercises 27–34, multiply or divide as indicated. The restrictions on the variables are implied. (See Example 3 on p. R66)

27. $\frac{3a^5b^7}{a-5b} \cdot \frac{2a-10b}{12a^4b^{10}}$
28. $\frac{8x-3y}{x^3y^4} \cdot \frac{6xy^8}{24x-9y}$
29. $\frac{c^2-d^2}{cd^{11}} \div \frac{8c^2+4cd-4d^2}{8c^2d^{10}}$
30. $\frac{m^{11}n^2}{m^2-n^2} \div \frac{18m^9n^5}{9m^2+6mn-15n^2}$
31. $\frac{2a^2b-ab^2}{8b^2+ab} \cdot \frac{a^2+16ab+64b^2}{2a^2+15ab-8b^2}$
32. $\frac{2c^2-2cd}{3c^2d+2c^3} \cdot \frac{4c^2+12cd+9d^2}{2c^2+cd-3d^2}$
33. $\frac{x^3-64}{16x-x^3} \div \frac{2x^2+8x+32}{x^2+2x-8}$
34. $\frac{3y^2+21y+147}{25y-y^3} \div \frac{y^3-343}{y^2-12y+35}$

Learn: Add and Subtract Rational Expressions

For Exercises 35–40, identify the least common denominator for each pair of expressions.

35. $\frac{7}{6x^5yz^4}$ and $\frac{3}{20xy^2z^3}$
36. $\frac{12}{35b^4cd^3}$ and $\frac{8}{25b^2c^3d}$
37. $\frac{2t+1}{(3t+4)^3(t-2)}$ and $\frac{4}{t(3t+4)^2(t-2)}$
38. $\frac{5y-7}{y(2y-5)(y+6)^4}$ and $\frac{6}{(2y-5)^3(y+6)^2}$
39. $\frac{x+3}{x^2+20x+100}$ and $\frac{3}{2x^2+20x}$
40. $\frac{z-4}{4z^2-20z+25}$ and $\frac{5}{12z^2-30z}$

For Exercises 41–56, add or subtract as indicated. (See Examples 4–5 on pp. R67–R68)

41. $\frac{m^2}{m+3}+\frac{6m+9}{m+3}$

42. $\frac{n^2}{n+5}+\frac{7n+10}{n+5}$

43. $\frac{2}{9c}+\frac{7}{15c^3}$

44. $\frac{6}{25x}+\frac{7}{10x^4}$

45. $\frac{9}{2x^2y^4}-\frac{11}{xy^5}$

46. $\frac{-2}{3m^3n}-\frac{5}{m^2n^4}$

47. $\frac{2}{x+3}-\frac{7}{x}$

48. $\frac{4}{m-2}-\frac{3}{m}$

49. $\frac{1}{x^2+xy}-\frac{2}{x^2-y^2}$

50. $\frac{4}{4a^2-b^2}-\frac{1}{2a^2-ab}$

51. $\frac{5}{y}+\frac{2}{y+1}-\frac{6}{y^2}$

52. $\frac{5}{t^2}+\frac{4}{t+1}-\frac{3}{t}$

53. $\frac{3w}{w-4}+\frac{2w+4}{4-w}$

54. $\frac{2x-1}{x-7}+-\frac{x+6}{7-x}$

55. $\frac{4}{x^2+6x+5}-\frac{3}{x^2+7x+10}$

56. $\frac{3}{x^2-4x-5}-\frac{2}{x^2-6x+5}$

Learn: Simplify Complex Fractions

For Exercises 57–68, simplify the complex fraction. (See Examples 6–8 pp. R69–R70)

57. $\frac{\frac{1}{27x}+\frac{1}{9}}{\frac{1}{3}+\frac{1}{9x}}$

58. $\frac{\frac{1}{8x}+\frac{1}{4}}{\frac{1}{2}+\frac{1}{4x}}$

59. $\frac{\frac{x}{6}-\frac{5x+14}{6x}}{\frac{1}{6}-\frac{7}{6x}}$

60. $\frac{\frac{x}{3}-\frac{2x+3}{3x}}{\frac{1}{3}+\frac{1}{3x}}$

61. $\frac{2a^{-1}-b^{-1}}{4a^{-2}-b^{-2}}$

62. $\frac{3u^{-1}-v^{-1}}{9u^{-2}-v^{-2}}$

63. $\frac{\frac{3}{1+h}-3}{h}$

64. $\frac{\frac{4}{1+h}-4}{h}$

65. $\frac{\frac{7}{x+h}-\frac{7}{x}}{h}$

66. $\frac{\frac{8}{x+h}-\frac{8}{x}}{h}$

67. $\frac{\frac{3}{x-1}-\frac{1}{x+1}}{\frac{6}{x^2-1}}$

68. $\frac{\frac{1}{x+1}}{\frac{-5}{x^2-3x-4}+\frac{1}{x-4}}$

Learn: Rationalize the Denominator of a Radical Expression

For Exercises 69–84, simplify the expression. Assume that the variable expressions represent positive real numbers. (See Example 9 on p. R71)

69. $\frac{4}{\sqrt{y}}$

70. $\frac{7}{\sqrt{z}}$

71. $\frac{4}{\sqrt[3]{y}}$

72. $\frac{7}{\sqrt[4]{z}}$

73. $\frac{\sqrt{12}}{\sqrt{x+1}}$

74. $\frac{\sqrt{50}}{\sqrt{x-2}}$

75. $\frac{8}{\sqrt{15}-\sqrt{11}}$

76. $\frac{12}{\sqrt{6}-\sqrt{2}}$

77. $\frac{x-5}{\sqrt{x}+\sqrt{5}}$

78. $\frac{y-3}{\sqrt{y}+\sqrt{3}}$

79. $\frac{2\sqrt{10}+3\sqrt{15}}{4\sqrt{10}+2\sqrt{5}}$

80. $\frac{3\sqrt{3}+\sqrt{6}}{5\sqrt{3}-2\sqrt{6}}$

81. $\frac{7}{\sqrt{3x}}+\frac{\sqrt{3x}}{x}$

82. $\frac{4}{\sqrt{11y}}+\frac{\sqrt{11y}}{y}$

83. $\frac{5}{w\sqrt{7}}-\frac{\sqrt{7}}{w}$

84. $\frac{13}{t\sqrt{2}}-\frac{\sqrt{2}}{t}$

Mixed Exercises

85. The average round trip speed S (in mph) of a vehicle traveling a distance of d miles each way is given by

$$s=\frac{2d}{\frac{d}{r_1}+\frac{d}{r_2}}.$$

In this formula, r_1 is the average speed going one way, and r_2 is the average speed on the return trip.

a. Simplify the complex fraction.

b. If a plane flies 400 mph from Orlando to Albuquerque and 460 mph on the way back, compute the average speed of the round trip. Round to 1 decimal place.

86. The formula $R=\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$ gives the total electrical resistance R (in ohms, Ω) when two resistors of resistance R_1 and R_2 are connected in parallel.

a. Simplify the complex fraction.

b. Find the total resistance when $R_1 = 12\,\Omega$ and $R_2 = 20\,\Omega$.

87. The concentration C (in ng/mL) of a drug in the bloodstream t hours after ingestion is modeled by $C=\frac{600t}{t^3+125}$.

a. Determine the concentration at 1 hr, 12 hr, 24 hr, and 48 hr. Round to 1 decimal place.

b. What appears to be the limiting concentration for large values of t?

88. An object that is originally 35°C is placed in a freezer. The temperature T (in °C) of the object can be approximated by the model $T=\frac{350}{t^2+3t+10}$, where t is the time in hours after the object is placed in the freezer.

a. Determine the temperature at 2 hr, 4 hr, 12 hr, and 24 hr. Round to 1 decimal place.

b. What appears to be the limiting temperature for large values of t?

For Exercises 89–102, simplify the expression.

89. $\dfrac{2x^3y}{x^3y + 3xy} \cdot \dfrac{x^2 + 6x + 9}{2x + 6} \div 5xy^4$

90. $\dfrac{2y^2 + 20y + 50}{12 - 4y} \cdot \dfrac{y - 3}{y^2 + 12y + 35} \div (3y + 15)$

91. $\left(\dfrac{4}{2t + 1} - \dfrac{t}{2t^2 + 17t + 8}\right)(t + 8)$

92. $\left(\dfrac{2m}{6m + 3} - \dfrac{1}{m + 4}\right)(2m + 1)$

93. $\dfrac{n - 2}{n - 4} + \dfrac{2n^2 - 15n + 12}{n^2 - 16} - \dfrac{2n - 5}{n + 4}$

94. $\dfrac{c^2 + 13c + 18}{c^2 - 9} + \dfrac{c + 1}{c + 3} - \dfrac{c + 8}{c - 3}$

95. $\dfrac{1 - a^{-1} - 6a^{-2}}{1 - 4a^{-1} + 3a^{-2}}$

96. $\dfrac{1 + t^{-1} - 12t^{-2}}{1 - 4t^{-1} + 3t^{-2}}$

97. $\dfrac{34}{2\sqrt{5} - \sqrt{3}}$

98. $\dfrac{13}{2\sqrt{7} + \sqrt{2}}$

99. $\dfrac{8 - \sqrt{48}}{6}$

100. $\dfrac{10 - \sqrt{50}}{15}$

101. $\dfrac{t + 6}{1 + \frac{2}{t}} - t - 4$

102. $\dfrac{m - 4}{1 - \frac{2}{m}} - m + 2$

For Exercises 103–110, write the expression as a single term, factored completely. Do not rationalize the denominator.

103. $1 - x^{-2} - 2x^{-3}$

104. $1 - 8x^{-5} + 30x^{-7}$

105. $\dfrac{3}{2\sqrt{3x}} + \sqrt{3x}$

106. $\dfrac{2}{\sqrt{x}} + \sqrt{x}$

107. $\dfrac{\sqrt{x^2 + 1} + \frac{x^2}{\sqrt{x^2 + 1}}}{x^2 + 1}$

108. $\dfrac{\frac{x^2}{\sqrt{x^2 + 9}} - \sqrt{x^2 + 9}}{x^2}$

109. $2\sqrt{4x^2 + 9} + \dfrac{8x^2}{\sqrt{4x^2 + 9}}$

110. $3\sqrt{9x^2 + 1} + \dfrac{27x}{\sqrt{9x^2 + 1}}$

Write About It

111. Explain why the expression $\frac{x}{x - y}$ is not defined for $x = y$.

112. Is the statement $\frac{3(x - 4)}{(x + 2)(x - 4)} = \frac{3}{x + 2}$ true for all values of x? Explain why or why not.

Expanding Your Skills

113. The numbers 1, 2, 4, 5, 10, and 20 are natural numbers that are factors of 20. There are other factors of 20 within the set of rational numbers and the set of irrational numbers. For example:

a. Show that $\frac{14}{3}$ and $\frac{30}{7}$ are factors of 20 over the set of rational numbers.

b. Show that $(5 - \sqrt{5})$ and $(5 + \sqrt{5})$ are factors of 20 over the set of irrational numbers.

114. a. Show that $\frac{15}{2}$ and $\frac{4}{5}$ are factors of 6 over the set of rational numbers.

b. Show that $(3 - \sqrt{3})$ and $(3 + \sqrt{3})$ are factors of 6 over the set of irrational numbers.

For Exercises 115–122, simplify the expression.

115. $\dfrac{w^{3n+1} - w^{3n}z}{w^{n+2} - w^n z^2}$

116. $\dfrac{2a^{4n+1} + a^{4n}y}{a^n y^2 - 4a^{n+2}}$

117. $\dfrac{x^{2n+1} - x^{2n}y}{x^{n+3} - x^n y^3}$

118. $\dfrac{8z^{n+3} + z^n w^3}{2z^{3n+1} + z^{3n}w}$

119. $\dfrac{\sqrt{5}}{\sqrt[3]{2}}$

120. $\dfrac{\sqrt{7}}{\sqrt[4]{3}}$

121. $\dfrac{a - b}{\sqrt[3]{a} - \sqrt[3]{b}}$ (*Hint*: Factor the numerator as a difference of cubes over the set of irrational numbers.)

122. $\dfrac{x + y}{\sqrt[3]{x} + \sqrt[3]{y}}$

For Exercises 123–130, rationalize the numerator by multiplying numerator and denominator by the conjugate of the *numerator*.

123. $\dfrac{\sqrt{3} + 6}{2}$

124. $\dfrac{\sqrt{7} - 2}{5}$

125. $\dfrac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}$

126. $\dfrac{\sqrt{p} + \sqrt{q}}{\sqrt{p} - \sqrt{q}}$

127. $\dfrac{\sqrt{4 + h} - 2}{h}$

128. $\dfrac{\sqrt{x + h} - \sqrt{x}}{h}$

129. $\dfrac{\sqrt{4 + 5h} - 2}{h}$

130. $\dfrac{\sqrt{9 + 4h} - 3}{h}$

Go online for more practice problems.

Lesson R-5

Equations with Real Solutions

What Will You Learn?

After completing this lesson, you should be able to:

- Solve Linear Equations in One Variable
- Solve Quadratic Equations with Real Solutions
- Solve Rational Equations
- Solve Absolute Value Equations
- Solve Radical Equations
- Solve Equations for a Specified Variable

Learn: Solve Linear Equations in One Variable

In this lesson we solve a variety of equations of different types beginning with linear equations in one variable.

Definition of a Linear Equation in One Variable

A **linear equation in one variable** is an equation that can be written in the form $ax + b = 0$, where a and b are real numbers, $a \neq 0$, and x is the variable.

A linear equation in one variable is also called a **first-degree equation** because the degree of the variable term must be exactly 1.

A **solution** to an equation is a value of the variable that makes the equation a true statement. The set of all solutions to an equation is called the **solution set** of the equation. **Equivalent equations** have the same solution set. To solve a linear equation in one variable, we form simpler, equivalent equations until we obtain an equation whose solution is obvious. The properties used to produce equivalent equations include the addition and multiplication properties of equality.

Properties of Equality

Let a, b, and c be real-valued expressions.

- **Addition property of equality**

  $a = b$ is equivalent to $a + c = b + c$.

  Example:

$$x + 4 = 9$$
$$x + 4 + (-4) = 9 + (-4)$$
$$x = 5$$

- **Multiplication property of equality**

  $a = b$ is equivalent to $ac = bc$ ($c \neq 0$).

  Example:

$$4x = 24$$
$$\frac{1}{4} \cdot 4x = \frac{1}{4} \cdot 24$$
$$x = 6$$

Insights

Given $x + 4 = 9$, we can add the opposite of 4 to both sides or subtract 4 from both sides. Likewise, given $4x = 24$, we can multiply both sides by $\frac{1}{4}$ or divide both sides by 4.

To solve a linear equation in one variable, isolate the variable by following these guidelines.

Solving a Linear Equation in One Variable

Step 1: Simplify both sides of the equation.

- Use the distributive property to clear parentheses.
- Combine like terms.
- Consider clearing fractions or decimals by multiplying both sides of the equation by the least common denominator (LCD) of all terms.

Step 2: Use the addition property of equality to collect the variable terms on one side of the equation and the constant terms on the other side.

Step 3: Use the multiplication property of equality to make the coefficient of the variable term equal to 1.

Step 4: Check the potential solution in the original equation.

Step 5: Write the solution set.

Example 1: Solving a Linear Equation

Solve. $-3(w-4)+5=10-(w+1)$

Solution:

$-3(w-4)+5=10-(w+1)$	
$-3w+12+5=10-w-1$	Apply the distributive property.
$-3w+17=9-w$	Combine like terms.
$-3w+w+17=9-w+w$	Add w to both sides of the equation.
$-2w+17=9$	Combine like terms.
$-2w+17-17=9-17$	Subtract 17 from both sides.
$-2w=-8$	Combine like terms.
$\frac{-2w}{-2}=\frac{-8}{-2}$	Divide both sides by -2 to obtain a coefficient of 1 for w.
$w=4$	

Check Your Work

$$-3(w-4)+5=10-(w+1)$$

$$-3[(4)-4]+5\overset{?}{=}10-[(4)+1]$$

$$-3(0)+5\overset{?}{=}10-(5)$$

$$5\overset{?}{=}5 \checkmark \text{ true}$$

The solution set is {4}.

Apply the Skills

1. Solve. $5(v-4)-2=2(v+7)-3$

The equation in Example 1 has exactly one solution. This equation is an example of a conditional equation. A **conditional equation** is true for some values of the variable and false for other values.

An equation that is true for all values of the variable for which the expressions in an equation are defined is called an **identity.** An equation that is false for all values of the variable is called a **contradiction.** Example 2 presents each of these three types of equations.

Example 2: Identifying Conditional Equations, Contradictions, and Identities

Identify each equation as a conditional equation, a contradiction, or an identity. Then give the solution set.

a. $3(2x-1)=2(3x-2)$ **b.** $3(2x-1)=2(3x-2)+1$ **c.** $3(2x-1)=5x-4$

Solution:

a. $3(2x-1)=2(3x-2)$

$6x-3=6x-4$ Apply the distributive property.

$-3=-4$ Subtract $6x$ from both sides. Contradiction

This equation is a contradiction.

The solution set is the empty set { }.

Insights No real number substituted for x will make $-3=-4$.

b. $3(2x-1)=2(3x-2)+1$

$6x-3=6x-4+1$ Apply the distributive property.

$6x-3=6x-3$ Combine like terms.

$0=0$ Subtract $6x$ from both sides. Add 3 to both sides.

This equation is an identity.

The solution set is the set of all real numbers, $\mathbb{R}$.

Insights

The statement $0=0$ is true regardless of the value of x. All real numbers are solutions.

c. $3(2x-1)=5x-4$ Apply the distributive property.

$6x-3=5x-4$ Subtract $5x$ from both sides. Add 3 to both sides.

$x=-1$

This is a conditional equation. Conditional equation. The statement is true only under the condition that $x=-1$.

The solution set is $\{-1\}$.

Apply the Skills

2. Identify each equation as a conditional equation, a contradiction, or an identity. Then give the solution set.

a. $4x+1-x=6x-2$ **b.** $2(-5x-1)=2x-12x+6$ **c.** $2(3x-1)=6(x+1)-8$

Learn: Solve Quadratic Equations with Real Solutions

A linear equation in one variable is an equation of the form $ax + b = 0$, where $a \neq 0$. A linear equation is a first-degree equation. We now turn our attention to a quadratic equation. This is identified as a second-degree equation.

Definition of a Quadratic Equation

Let a, b, and c represent real numbers, where $a \neq 0$. A **quadratic equation** in the variable x is an equation of the form

$$ax^2 + bx + c = 0.$$

To solve a quadratic equation, we make use of the zero product property.

Zero Product Property

If $mn = 0$, then $m = 0$ or $n = 0$.

To solve a quadratic equation using the zero product property, set one side of the equation equal to zero and factor the other side.

Example 3: Applying the Zero Product Property

Solve the equation. $2x(2x - 7) = -12$

Solution:

$$\begin{aligned} 2x(2x-7) &= -12 \\ 4x^2 - 14x &= -12 && \text{Apply the distributive property.} \\ 4x^2 - 14x + 12 &= 0 && \text{Set one side to zero.} \\ 2(2x^2 - 7x + 6) &= 0 && \text{Factor.} \\ 2(x-2)(2x-3) &= 0 \end{aligned}$$

$$\begin{array}{llllll} \underset{\substack{\uparrow \\ \text{contradiction}}}{2 = 0} & \text{or} & x - 2 = 0 & \text{or} & 2x - 3 = 0 & \text{Set each factor equal to zero.} \\ & & x = 2 & & x = \frac{3}{2} & \text{Both solutions check.} \end{array}$$

The solution set is $\left\{2, \frac{3}{2}\right\}$.

Apply the Skills

3. Solve the equation. $10(3x^2 - 13x) = -40$

The zero product property can be used to solve equations of the form $x^2 = k$.

$$x^2 = k$$

$$x^2 - k = 0$$

$$(x - \sqrt{k})(x + \sqrt{k}) = 0$$

$$x - \sqrt{k} = 0 \quad \text{or} \quad x + \sqrt{k} = 0$$

$$x = \sqrt{k} \qquad\qquad x = -\sqrt{k}$$

The solution set is $\{\pm\sqrt{k}\}$.

The solutions can also be written as $x = \pm\sqrt{k}$. This is read as "x equals plus or minus the square root of k."

This result is formalized as the square root property.

Square Root Property

If $x^2 = k$, then $x = \pm\sqrt{k}$.

The solution set is $\{\sqrt{k}, -\sqrt{k}\}$ or more concisely $\{\pm\sqrt{k}\}$.

To apply the square root property to solve a quadratic equation, first isolate the square term on one side and the constant term on the other side.

Example 4: Applying the Square Root Property

Solve the equations by using the square root property.

a. $x^2 = 64$ **b.** $(w + 3)^2 = 8$

Solution:

a. $x^2 = 64$

$x = \pm\sqrt{64}$ Apply the square root property.

$x = \pm 8$ Both solutions check in the original equation.

The solution set is $\{\pm 8\}$.

b. $(w + 3)^2 = 8$

$w + 3 = \pm\sqrt{8}$ Apply the square root property.

$w = -3 \pm \sqrt{8}$ Subtract 3 from both sides to isolate w.

$w = -3 \pm 2\sqrt{2}$ Simplify the radical. $\sqrt{8} = \sqrt{2^3} = 2\sqrt{2}$

The solution set is $\{-3 \pm 2\sqrt{2}\}$. Both solutions check.

Apply the Skills

4. Solve the equations by using the square root property.

a. $a^2 = 49$ **b.** $(t + 4)^2 = 24$

In Example 4(b), the left side of the equation is the square of a binomial and the right side is a constant. We can manipulate a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ to write it as the square of a binomial equal to a constant. First look at the relationship between a perfect square trinomial and its factored form.

Perfect Square Trinomial	Factored Form
$x^2 + 10x + 25 \longrightarrow$	$(x + 5)^2$
$t^2 - 6t + 9 \longrightarrow$	$(t - 3)^2$
$p^2 - 14p + 49 \longrightarrow$	$(p - 7)^2$

For a perfect square trinomial with a leading coefficient of 1, the constant term is the square of one-half the linear term coefficient. For example:

$$\begin{array}{c} t^2 - 6t + 9 \\ \left[\frac{1}{2}(-6)\right]^2 \end{array}$$

In general, an expression of the form $x^2 + bx + n$ is a perfect square trinomial if $n = \left(\frac{1}{2}b\right)^2$. The process to create a perfect square trinomial is called **completing the square.**

We can solve a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ by completing the square and then applying the square root property.

Solving a Quadratic Equation $ax^2 + bx + c = 0$ by Completing the Square and Applying the Square Root Property

Step 1: Divide both sides by a to make the leading coefficient 1.

Step 2: Isolate the variable terms on one side of the equation.

Step 3: Complete the square.

- Add the square of one-half the linear term coefficient to both sides.
- Factor the resulting perfect square trinomial.

Step 4: Apply the square root property and solve for x.

Example 5: Completing the Square and Applying the Square Root Property

Solve the equation by completing the square and applying the square root property.

$$2x^2 + 20x - 6 = 0$$

Solution:

$2x^2+20x-6=0$	The equation is in the form $ax^2+bx+c=0$.
$\frac{2x^2}{2}+\frac{20x}{2}-\frac{6}{2}=\frac{0}{2}$	**Step 1:** Divide both sides by the leading coefficient 2.
$x^2+10x-3=0$	
$x^2+10x=3$	**Step 2:** Add 3 to both sides to isolate the variable terms.
$x^2+10x+25=3+25$	**Step 3:** Add $\left[\frac{1}{2}(10)\right]^2=[5]^2=25$ to both sides.
$(x+5)^2=28$	Factor.
$x+5=\pm\sqrt{28}$	**Step 4:** Apply the square root property and solve for x.
$x=-5\pm 2\sqrt{7}$	Both solutions check in the original equation.
$\{-5\pm 2\sqrt{7}\}$	Write the solution set.

Apply the Skills

5. Solve the equation by completing the square and applying the square root property. $3x^2-24x-6=0$

If we solve a general quadratic equation $ax^2+bx+c=0$ $(a\neq 0)$ by completing the square and using the square root property, the result is a formula that gives the solutions for x in terms of a, b, and c.

$ax^2+bc+c=0$	Begin with a quadratic equation in standard form with $a>0$.
$\frac{ax^2}{a}+\frac{bx}{a}+\frac{c}{a}=\frac{0}{a}$	Divide by the leading coefficient.
$x^2+\frac{b}{a}x+\frac{c}{a}=0$	
$x^2+\frac{b}{a}x=-\frac{c}{a}$	Isolate the terms containing x.
$x^2+\frac{b}{a}x+\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2=\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2-\frac{c}{a}$	Add the square of $\frac{1}{2}$ the linear term coefficient to both sides of the equation.
$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$	Factor the left side as a perfect square.
$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$	Combine fractions on the right side by finding a common denominator.
$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$	Apply the square root property
$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$	Simplify the denominator.
$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$	Subtract $\frac{b}{2a}$ from both sides.
$=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Combine fractions.

The result is called the quadratic formula.

The Quadratic Formula

For a quadratic equation of the form $ax^2 + bx + c = 0$ ($a \neq 0$), the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 6: Using the Quadratic Formula

Solve the equation by applying the quadratic formula. $x(x - 6) = 3$

Solution:

$x(x-6)=3$ $x^2-6x-3=0$	Write the equation in the form $ax^2+bx+c=0$.
$a=1$, $b=-6$, $c=-3$	Identify the values of a, b, and c.
$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-3)}}{2(1)}$	Apply the quadratic formula. $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$=\frac{6\pm\sqrt{48}}{2}$	Simplify.
$=\frac{6\pm4\sqrt{3}}{2}$	Simplify the radical. $\sqrt{48}=\sqrt{2^4\cdot3}=2^2\sqrt{3}=4\sqrt{3}$
$=\frac{\cancel{2}(3\pm2\sqrt{3})}{\cancel{2}}$	Factor the numerator.
$=3\pm2\sqrt{3}$	Simplify the fraction.
The solution set is $\{3\pm2\sqrt{3}\}$.	The solutions both check in the original equation.

Insights

When applying the quadratic formula, note that a, b, and c are constants. The variable is x.

Apply the Skills

6. Solve the equation by applying the quadratic formula.

$$x(x - 8) = 3$$

Insights

If a quadratic equation has fractional or decimal coefficients, we have the option of clearing fractions or decimals to create integer coefficients. This makes the application of the quadratic formula easier.

$$\frac{3}{10}x^2-\frac{2}{5}x-\frac{1}{2}=0$$
The LCD among all terms in the equation is 10.

$$10\left(\frac{3}{10}x^2-\frac{2}{5}x-\frac{1}{2}\right)=10(0)$$
Multiply both sides by 10 to clear fractions.

$$3x^2-4x-5=0$$
Apply the quadratic formula with $a=3$, $b=-4$, and $c=-5$.

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(3)(-5)}}{2(3)}=\frac{4\pm\sqrt{76}}{6}=\frac{4\pm2\sqrt{19}}{6}=\frac{\overset{1}{\cancel{2}}(2\pm\sqrt{19})}{\underset{3}{\cancel{6}}}=\frac{2\pm\sqrt{19}}{3}$$

Learn: Solve Rational Equations

A rational equation is an equation in which each term is a rational expression. For example $\frac{12}{x}=\frac{6}{2x}+3$ is a rational equation. One technique to solve a rational equation is to multiply both sides by the least common denominator of all the terms. This is demonstrated in Example 7.

Example 7: Solving a Rational Equation

Solve the equation and check the solution. $\frac{12}{x}=\frac{6}{2x}+3$

Solution:

$$\frac{12}{x}=\frac{6}{2x}+3$$
Restrict x so that $x\neq0$.

$$2x\left(\frac{12}{x}\right)=2x\left(\frac{6}{2x}+\frac{3}{1}\right)$$
Clear fractions by multiplying both sides by the LCD, $2x$. Since $x\neq0$, this will produce an equivalent equation.

$$\frac{2x}{1}\left(\frac{12}{x}\right)=\frac{\cancel{2x}}{1}\left(\frac{6}{\cancel{2x}}\right)+\frac{2x}{1}\left(\frac{3}{1}\right)$$
Apply the distributive property.

$$24=6+6x$$
Simplify.

$$18=6x$$
Subtract 6 from both sides.

$$3=x$$

Check Your Work

$$\frac{12}{x}\overset{?}{=}\frac{6}{2x}+\frac{3}{1}$$
$$\frac{12}{(3)}\overset{?}{=}\frac{6}{2(3)}+\frac{3}{1}$$
$$4\overset{?}{=}1+3\ \checkmark\ \text{true}$$

The solution set is $\{3\}$.

Apply the Skills

7. Solve the equation and check the solution. $\frac{15}{y}=\frac{21}{3y}+2$

In Example 7, notice that two of the expressions in the equation are undefined for $x = 0$ because the denominator would be zero. Thus, when we solve a rational equation, we must restrict the values of the variable to avoid division by zero. This is demonstrated in Example 8.

Example 8: Solving a Rational Equation

Solve. $\frac{2x}{x-4} - \frac{3}{x+2} = \frac{x^2+14}{x^2-2x-8}$

Solution:

$$\frac{2x}{x-4} - \frac{3}{x+2} = \frac{x^2+14}{x^2-2x-8}$$

Factor the denominators.

$$\frac{2x}{x-4} - \frac{3}{x+2} = \frac{x^2+14}{(x-4)(x+2)}$$

The variable x has the restrictions that $x \neq 4$ and $x \neq -2$.

$$(x-4)(x+2)\left(\frac{2x}{x-4} - \frac{3}{x+2}\right) = (x-4)(x+2)\left[\frac{x^2+14}{(x-4)(x+2)}\right]$$

Multiply both sides by the LCD to clear fractions.

$$2x(x+2) - 3(x-4) = x^2 + 14$$

The resulting equation is quadratic.

$$2x^2 + 4x - 3x + 12 = x^2 + 14$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

Apply the zero product property.

$$\cancel{x=-2} \text{ or } x = 1$$

The value -2 is not a solution because it is a restricted value. It does not check.

Check Your Work

Check: $x = -2$

$$\frac{2(-2)}{(-2)-4} - \underbrace{\frac{3}{(-2)+2}}_{\text{undefined}} \stackrel{?}{=} \underbrace{\frac{(-2)^2+14}{(-2)^2-2(-2)-8}}_{\text{undefined}}$$

Check: $x = 1$

$$\frac{2(1)}{(1)-4} - \frac{3}{(1)+2} \stackrel{?}{=} \frac{(1)^2+14}{(1)^2-2(1)-8}$$

$$\frac{2}{-3} - 1 \stackrel{?}{=} \frac{15}{-9}$$

$$-\frac{5}{3} \stackrel{?}{=} -\frac{5}{3} \checkmark \text{ true}$$

The solution set is {1}. The value -2 does not check.

Apply the Skills

8. Solve. $\frac{3x}{x-5} = \frac{2}{x+1} + \frac{2x^2+40}{x^2-4x-5}$

Learn: Solve Absolute Value Equations

We now turn our attention to equations involving absolute value expressions. For example, given $|x| = 5$, the solution set is $\{-5, 5\}$. The equation can also be written as $|x - 0| = 5$, implying that the solutions are the values of x that are 5 units from 0 on the number line (Figure R-3).

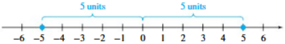


Figure R-3

Given a nonnegative real number k, the generic absolute value equation $|u| = k$ can be solved directly from the definition of absolute value. Recall that

$$|u| = \begin{cases} u \text{ if } u \geq 0 \\ -u \text{ if } u < 0 \end{cases}$$

Thus, $|u| = k$ means that $u = k$ or $-u = k$. Solving for u, we have $u = k$ or $u = -k$.

This and three other properties summarized in Table R-8 follow directly from the definition of absolute value.

TABLE R-8 Absolute Value Equations

Let k represent a positive real number.

	Absolute Value Equation Equivalency
1.	$|u| = k$ is equivalent to $u = k$ or $u = -k$.
2.	$|u| = 0$ is equivalent to $u = 0$.
3.	$|u| = -k$ has no solution.
4.	$|u| = |w|$ is equivalent to $u = w$ or $u = -w$.

To solve an absolute value equation, first isolate the absolute value. Then solve the equation by rewriting the equation in its equivalent form given in Table R-8.

Example 9: Solving Absolute Value Equations

Solve the equations. **a.** $2|3 - 2t| = 6$ **b.** $2 = |7w - 3| + 8$

Solution:

a. $2|3 - 2t| = 6$ Divide by 2 to isolate the absolute value.

$|3 - 2t| = 3$ The equation is in the form $|u| = k$, where $u = 3 - 2t$ and $k > 0$.

$3 - 2t = 3$ or $3 - 2t = -3$ Rewrite the equation in the form $u = k$ or $u = -k$.

$-2t = 0$ or $-2t = -6$

$t = 0$ $t = 3$

Check Your Work

Check: $t = 0$

$2|3 - 2|0 \stackrel{?}{=} 6$

$2|3| = 6$ ✓

Check: $t = 3$

$2|3 - 2(3)| \stackrel{?}{=} 6$

$2|3 - 6| \stackrel{?}{=} 6$

$2|-3| = 6$ ✓

The solution set is $\{3, 0\}$.

b. $2 = \lvert 7w - 3\rvert + 8$	Subtract 8 from both sides to isolate the absolute value.
$-6 = \lvert 7w - 3\rvert$	The equation is in the form $\lvert u\rvert = k$, where $u = 7w - 3$ and $k < 0$. By definition, an absolute value cannot be negative. There is no solution.
The solution set is { }.	

Apply the Skills

9. Solve the equations.

a. $5\lvert 2 - 4t\rvert = 50$ **b.** $5 = \lvert 6c - 7\rvert + 9$

In Example 10, we solve equations involving two absolute values by writing $\lvert u\rvert = \lvert w\rvert$ in the equivalent form $u = w$ or $u = -w$.

Example 10: Solving Equations with Two Absolute Values

Solve the equations. **a.** $\lvert 2x - 5\rvert = \lvert x + 1\rvert$ **b.** $\lvert 6 - x\rvert = \lvert x - 6\rvert$

Solution:

a. $\lvert 2x - 5\rvert = \lvert x + 1\rvert$	The equation is in the form $\lvert u\rvert = \lvert w\rvert$.
$2x - 5 = x + 1$ or $2x - 5 = -(x + 1)$	Rewrite the equation in the form $u = w$ or $u = -w$.
$x - 5 = 1$ or $2x - 5 = -x - 1$	Solve each individual equation.
$x = 6$ or $3x = 4$	
$x = \frac{4}{3}$	Both solutions check in the original equation.
The solution set is $\left\{6, \frac{4}{3}\right\}$.	

b. $\lvert 6 - x\rvert = \lvert x - 6\rvert$	The equation is in the form $\lvert u\rvert = \lvert w\rvert$.
$6 - x = x - 6$ or $6 - x = -(x - 6)$	Rewrite the equation in the form $u = w$ or $u = -w$.
$-2x = -12$ or $6 - x = -x + 6$	Solve each individual equation.
$x = 6$ or $6 = 6$ (indentity)	The solution to this equation is all real numbers (including 6).
The solution set is $\mathbb{R}$.	

Apply the Skills

10. Solve the equations.

a. $\lvert 3x - 4\rvert = \lvert 2x + 1\rvert$ **b.** $\lvert 4 + x\rvert = \lvert -4 - x\rvert$

Learn: Solve Radical Equations

An equation with one or more radicals containing a variable $\left(\text{such as } \sqrt[3]{x} = 5\right)$ is called a **radical equation.** We can eliminate the radical by raising both sides of the equation to a power equal to the index of the radical.

$\sqrt[3]{x} = 5$

$(\sqrt[3]{x})^3 = (5)^3$ The index is 3. Therefore, raise both sides to the third power.

$x = 125$

By raising each side of a radical equation to a power equal to the index, a new equation is produced. However, some (or all) of the solutions to the new equation may *not* be solutions to the original equation. These are called **extraneous solutions.** For this reason, it is necessary to check all potential solutions in the original equation. For example, consider the equation $\sqrt{x} = -10$. By inspection, this equation has no solution because the principal square root of x must be nonnegative. However, if we square both sides of the equation, it appears as though a solution exists:

$\sqrt{x} = -10$ Solution set: { }

$(\sqrt{x})^2 = (-10)^2$ Square both sides.

$x = 100$ The value 100 does not check in the original equation. Therefore, 100 is an extraneous solution.

Solving a Radical Equation

Step 1: Isolate the radical. If an equation has more than one radical, choose one of the radicals to isolate.

Step 2: Raise each side of the equation to a power equal to the index of the radical.

Step 3: Solve the resulting equation. If the equation still has a radical, repeat steps 1 and 2.

***Step 4:** Check the potential solutions in the original equation and write the solution set.

*In solving radical equations, extraneous solutions potentially arise when both sides of the equation are raised to an even power. Therefore, an equation with only odd-indexed roots will not have extraneous solutions. However, it is still recommended that all potential solutions be checked.

Example 11: Solving a Radical Equation

Solve. $\sqrt{x+10}-4=x$

Solution:

$\sqrt{x+10}-4=x$	
$\sqrt{x+10}=x+4$	Isolate the radical.
$(\sqrt{x+10})^2=(x+4)^2$	The index is 2. Therefore, raise both sides to the second power.

Check Your Work

When raising both sides of an equation to a power, be sure to enclose both sides of the equation in parentheses.

$x+10=x^2+8x+16$	The resulting equation is quadratic.
$0=x^2+7x+6$	Set one side equal to zero.
$0=(x+6)(x+1)$	Factor.
$x=-6$ or $x=-1$	Apply the zero product property.

Both sides of the equation were raised to an even power. Therefore, it is necessary to check the potential solutions.

Check Your Work

Check: $x=-6$	Check: $x=-1$
$\sqrt{x+10}-4=x$	$\sqrt{x+10}-4=x$
$\sqrt{(-6)+10}-4\overset{?}{=}(-6)$	$\sqrt{(-1)+10}-4\overset{?}{=}(-1)$
$\sqrt{4}-4\overset{?}{=}-6$	$\sqrt{9}-4\overset{?}{=}-1$
$2-4\overset{?}{=}-6$	$3-4\overset{?}{=}-1$
$-2\overset{?}{=}-6$ false	$-1\overset{?}{=}-1$ ✓ true

The solution set is $\{-1\}$. The value -6 does not check.

Apply the Skills

11. Solve the equation. $\sqrt{t+7}=t-5$

In Example 12, we solve the equation $\sqrt{m-1}-\sqrt{3m+1}=-2$. The first step is to isolate one of the radicals. However, the presence of the constant term, -2, makes it impossible to isolate both radicals simultaneously. As a result, it is necessary to square both sides of the equation twice.

Example 12: Solving an Equation Containing Two Radicals

Solve. $\sqrt{m-1} - \sqrt{3m+1} = -2$

Solution:

$\sqrt{m-1} - \sqrt{3m+1} = -2$	
$\sqrt{m-1} = \sqrt{3m+1} - 2$	Isolate one of the radicals.
$(\sqrt{m-1})^2 = (\sqrt{3m+1} - 2)^2$	The index is 2. Therefore, raise both sides to the second power.
$m - 1 = 3m + 1 - 4\sqrt{3m+1} + 4$	
$m - 1 = 3m + 5 - 4\sqrt{3m+1}$	
$4\sqrt{3m+1} = 2m + 6$	Isolate the remaining radical.
$2\sqrt{3m+1} = m + 3$	Divide both sides by 2 to simplify.
$(2\sqrt{3m+1})^2 = (m+3)^2$	The resulting equation has another radical. Isolate the radical, and square both sides again.
$4(3m+1) = m^2 + 6m + 9$	
$12m + 4 = m^2 + 6m + 9$	The resulting equation is quadratic.
$0 = m^2 - 6m + 5$	
$0 = (m-5)(m-1)$	Apply the zero product property.
$m = 5 \quad \text{or} \quad m = 1$	Both sides of the equation were raised to an even power. Check both potential solutions.

Check Your Work

Check: $m = 5$

$$\sqrt{m-1} - \sqrt{3m+1} = -2$$
$$\sqrt{(5)-1} - \sqrt{3(5)+1} \overset{?}{=} -2$$
$$\sqrt{4} - \sqrt{16} \overset{?}{=} -2$$
$$2 - 4 \overset{?}{=} -2 \checkmark \text{ true}$$

Check: $m = 1$

$$\sqrt{m-1} - \sqrt{3m+1} = -2$$
$$\sqrt{(1)-1} - \sqrt{3(1)+1} \overset{?}{=} -2$$
$$\sqrt{0} - \sqrt{4} \overset{?}{=} -2$$
$$0 - 2 \overset{?}{=} -2 \checkmark \text{ true}$$

Both solutions check. The solution set is $\{1, 5\}$.

Good Practices

Use caution when squaring the two-term expression on the right in Example 12:

$$\begin{aligned}(\sqrt{3m+1} - 2)^2 &= (\sqrt{3m+1})^2 - 2(\sqrt{3m+1})(2) + (2)^2 \\ &= 3m + 1 - 4\sqrt{3m+1} + 4 \\ &= 3m + 5 - 4\sqrt{3m+1}\end{aligned}$$

Apply the Skills

12. Solve. $1 + \sqrt{n+4} = \sqrt{3n+1}$

In Example 13, we solve an equation containing rational exponents.

Example 13: Solving an Equation Containing Rational

Solve the equation. $2(x+1)^{2/3}=8$

Solution:

$$2(x+1)^{2/3}=8$$

$$2\sqrt[3]{(x+1)^2}=8$$ Write the expression on the left in radical notation.

$$\sqrt[3]{(x+1)^2}=4$$ Divide by 2 to isolate the radical.

$$\left[\sqrt[3]{(x+1)^2}\right]^3=(4)^3$$ Raise both sides to a power equal to the index of the radical.

$$(x+1)^2=64$$

$$x+1=\pm\sqrt{64}$$ Apply the square root property.

$$x=-1\pm 8$$

$$x=-1-8=-9 \quad \text{or} \quad x=-1+8=7$$

Check Your Work

Check: $x=-9$	Check: $x=7$
$(-9+1)^{2/3}\overset{?}{=}4$	$(7+1)^{2/3}\overset{?}{=}4$
$(-8)^{2/3}=4$ ✓	$(8)^{2/3}=4$ ✓

The solution set is $\{-9, 7\}$.

Apply the Skills

13. Solve the equation. $2(x-4)^{3/4}=54$

Learn: Solve Equations for a Specified Variable

The techniques presented in this lesson can also be used to manipulate an equation with multiple variables. In particular, we are often interested in solving an equation for a specified variable in terms of the other variables in the equation.

Example 14: Solving an Equation for a Specified Variable

Solve for r. $V=\frac{1}{3}\pi r^2 h \quad (r>0)$

Solution:

$$V = \frac{1}{3}\pi r^2 h$$

This equation is quadratic in the variable r. The strategy in this example is to isolate r^2 and then apply the square root property.

$$3(V) = 3\left(\frac{1}{3}\pi r^2 h\right)$$

Multiply both sides by 3 to clear fractions.

$$3V = \pi r^2 h$$

$$\frac{3V}{\pi h} = \frac{\pi r^2 h}{\pi h}$$

Divide both sides by πh to isolate r^2.

$$\frac{3V}{\pi h} = r^2$$

$$r = \sqrt{\frac{3V}{\pi h}} \quad \text{or} \quad r = \frac{\sqrt{3V\pi h}}{\pi h}$$

Apply the square root property. Since $r > 0$, we take the positive square root only.

Insights

The equation $V = \frac{1}{3}\pi r^2 h$ is linear in the variables V and h, and quadratic in the variable r.

The formula $V = \frac{1}{3}\pi r^2 h$ gives the volume of a right circular cone with radius r. Therefore, $r > 0$.

Apply the Skills

14. Solve for v. $E = \frac{1}{2}mv^2 \quad (v > 0)$

Example 15: Solving an Equation for a Specified Variable

Solve for t. $mt^2 + nt = z$

Solution:

This equation is quadratic in the variable t. The strategy is to write the polynomial in descending order by powers of t. Then since there are two t terms with different exponents, we cannot isolate t directly. Instead we apply the quadratic formula.

$$mt^2 + nt = z$$

$$mt^2 + nt - z = 0$$

Write the polynomial in descending order by t.

$$a = m,\ b = n,\ c = -z$$

Identify the coefficients of each term.

$$t = \frac{-(n) \pm \sqrt{(n)^2 - 4(m)(-z)}}{2m}$$

Apply the quadratic formula.

$$t = \frac{-n \pm \sqrt{n^2 + 4mz}}{2m}$$

Simplify.

Check Your Work

In the equation $mt^2 + nt - z = 0$, t is the variable, and m, n, and z are the coefficients.

Apply the Skills

15. Solve for p. $cp^2 - dp = k$

Practice R-5

Practice Exercises

Concept Connections

1. A ___________ equation is a second-degree equation of the form $ax^2 + bx + c = 0$ where $a \neq 0$.

2. A _____________ equation is a first-degree equation of the form $ax + b = 0$ where $a \neq 0$.

3. A _____________ equation is one that is true for some values of the variable and false for others.

4. An _____________ is an equation that is true for all values of the variable for which the expressions in the equation are defined.

5. A _____________ is an equation that is false for all values of the variable.

6. The square root property indicates that if $x^2 = k$, then $x =$ _____________.

7. Given $ax^2 + bx + c = 0$ $(a \neq 0)$, write the quadratic formula.

8. A _____________ equation is an equation in which each term contains a rational expression.

Learn: Solve Linear Equations in One Variable

For Exercises 9–20, solve the equation. (See Example 1 on p. R78)

9. $4 = 7 - 3(4t + 1)$

10. $11 = 7 - 2(5p - 2)$

11. $-6(v - 2) + 3 = 9 - (v + 4)$

12. $-5(u - 4) + 2 = 11 - (u - 3)$

13. $0.05y + 0.02(6000 - y) = 270$

14. $0.06x + 0.04(10{,}000 - x) = 520$

15. $2(5x - 6) = 4[x - 3(x - 10)]$

16. $4(y - 3) = 3[y + 2(y - 2)]$

17. $\frac{1}{2}w - \frac{3}{4} = \frac{2}{3}w + 2$

18. $\frac{2}{5}p - \frac{3}{10} = \frac{7}{15}p - 1$

19. $\frac{n+3}{4} - \frac{n-2}{5} = \frac{n+1}{10} - 1$

20. $\frac{t-2}{3} - \frac{t+7}{5} = \frac{t-4}{10} + 2$

21. In the mid-nineteenth century, explorers used the boiling point of water to estimate altitude. The boiling temperature of water T (in °F) can be approximated by the model $T = -1.83a + 212$, where a is the altitude in thousands of feet.

- **a.** Determine the temperature at which water boils at an altitude of 4000 ft. Round to the nearest degree.
- **b.** Two campers hiking in Colorado boil water for tea. If the water boils at 193°F, approximate the altitude of the campers. Give the result to the nearest hundred feet.

22. For a recent year, the cost C (in \$) for in-state tuition and fees for x credit-hours at a public college was given by $C = 219.95x + 94$.

- **a.** Determine the cost to take 9 credit-hours.
- **b.** If a student spent \$3173.30 for in-state tuition and fees, how many credit-hours did the student take?

For Exercises 23–28, identify the equation as a conditional equation, a contradiction, or an identity. Then give the solution set. (See Example 2 on p. R79)

23. $2x - 3 = 4(x - 1) - 1 - 2x$

24. $4(3 - 5n) + 1 = -4n - 8 - 16n$

25. $-(6 - 2w) = 4(w + 1) - 2w - 10$

26. $-5 + 3x = 3(x - 1) - 2$

27. $\frac{1}{2}x + 3 = \frac{1}{4}x + 1$

28. $\frac{2}{3}y - 5 = \frac{1}{6}y - 4$

Learn: Solve Quadratic Equations with Real Solutions

For Exercises 29–36, solve by applying the zero product property. (See Example 3 on p. R80)

29. $n^2 + 5n = 24$ **30.** $y^2 = 18 - 7y$

31. $8t(t + 3) = 2t - 5$ **32.** $6m(m + 4) = m - 15$

33. $3x^2 = 12x$

34. $z^2 = 25z$

35. $(m + 4)(m - 5) = -8$

36. $(n + 2)(n - 4) = 27$

For Exercises 37–42, solve by using the square root property. (See Example 4 on p. R81)

37. $x^2 = 81$ **38.** $w^2 = 121$

39. $5y^2 - 35 = 0$ **40.** $6v^2 - 30 = 0$

41. $(k + 2)^2 = 28$ **42.** $3(z + 11)^2 - 10 = 110$

For Exercises 43–48, determine the value of *n* that makes the polynomial a perfect square trinomial. Then factor as the square of a binomial.

43. $x^2 + 14x + n$ **44.** $y^2 + 22y + n$

45. $w^2 - 3w + n$ **46.** $v^2 - 11v + n$

47. $m^2 + \frac{2}{9}m + n$ **48.** $k^2 + \frac{2}{5}k + n$

For Exercises 49–54, solve by completing the square and applying the square root property. (See Example 5 on p. R82)

49. $y^2 + 22y - 4 = 0$

50. $x^2 + 14x - 3 = 0$

51. $2x(x - 3) = 4 + x$

52. $5c(c - 2) = 6 + 3c$

53. $-4y^2 - 12y + 5 = 0$

54. $-2x^2 - 14x + 5 = 0$

For Exercises 55–64, solve by using the quadratic formula. (See Example 6 on p. R84)

55. $x^2 - 3x - 7 = 0$

56. $x^2 - 5x - 9 = 0$

57. $(6x + 5)(x - 3) = -2x(7x + 5) + x - 12$

58. $(5c + 7)(2c - 3) = -2c(c + 15) - 35$

59. $\frac{1}{2}x^2 - \frac{2}{7} = \frac{5}{14}x$

60. $\frac{1}{3}x^2 - \frac{7}{6} = \frac{3}{2}x$

61. $0.4y^2 = 2y - 2.5$

62. $0.09n^2 = 0.42n - 0.49$

63. $m^2 + 4m = -2$

64. $n^2 + 8n = -3$

Learn: Solve Rational Equations

For Exercises 65–66, determine the restrictions on *x*.

65. $\frac{3}{x - 5} + \frac{2}{x + 4} = \frac{5}{7}$

66. $\frac{2}{x + 1} - \frac{5}{x - 7} = \frac{2}{3}$

For Exercises 67–84, solve the equation. (See Examples 7–8 on pp. R85–R86)

67. $\frac{1}{2} - \frac{7}{2y} = \frac{5}{y}$

68. $\frac{1}{3} - \frac{4}{3t} = \frac{7}{t}$

69. $\frac{w + 3}{4w} + 1 = \frac{w - 5}{w}$

70. $\frac{x + 2}{6x} + 1 = \frac{x - 7}{x}$

71. $\frac{c}{c - 3} = \frac{3}{c - 3} - \frac{3}{4}$

72. $\frac{7}{d - 7} - \frac{7}{8} = \frac{d}{d - 7}$

73. $\frac{2}{x - 5} - \frac{1}{x + 5} = \frac{11}{x^2 - 25}$

74. $\frac{2}{c + 3} - \frac{1}{c - 3} = \frac{10}{c^2 - 9}$

75. $\frac{-14}{x^2 - x - 12} - \frac{1}{x - 4} = \frac{2}{x + 3}$

76. $\frac{2}{x^2 + 5x + 6} - \frac{2}{x + 2} = \frac{1}{x + 3}$

77. $\frac{5}{x^2 - x - 2} - \frac{2}{x^2 - 4} = \frac{4}{x^2 + 3x + 2}$

78. $\frac{4}{x^2 + 2x - 8} - \frac{1}{x^2 - 16} = \frac{2}{x^2 + 6x + 8}$

79. $\frac{3x}{x + 2} - \frac{5}{x - 4} = \frac{2x^2 - 14x}{x^2 - 2x - 8}$

80. $\frac{4c}{c - 5} - \frac{1}{c + 1} = \frac{3c^2 + 3}{c^2 - 4c - 5}$

81. $\frac{m}{2m + 1} + 1 = \frac{2}{m - 3}$

82. $7 + \frac{20}{z} = \frac{3}{z^2}$

83. $\frac{18}{m^2 - 3m} + 2 = \frac{6}{m - 3}$

84. $\frac{48}{m^2 - 4m} + 3 = \frac{12}{m - 4}$

Learn: Solve Absolute Value Equations

For Exercises 85–102, solve the equations. (See Examples 9 and 10 on pp. R87–R88)

85. **a.** $|p| = 6$
  b. $|p| = 0$
  c. $|p| = -6$

86. **a.** $|w| = 2$
  b. $|w| = 0$
  c. $|w| = -2$

87. a. $|x-3|=4$

b. $|x-3|=0$

c. $|x-3|=-7$

88. a. $|m+1|=5$

b. $|m+1|=0$

c. $|m+1|=-1$

89. $2|3x-4|+7=9$

90. $4|2t+7|+2=22$

91. $-3=-|c-7|+1$

92. $-4=-|z+8|-3$

93. $2=8+|11y+4|$

94 $6=7+|9z-3|$

95. $|3y+5|=|y+1|$

96. $|2a-3|=|a+2|$

97. $\left|\frac{1}{4}w\right|=|4w|$

98. $|3z|=\left|\frac{1}{3}z\right|$

99. $|x+4|=|x-7|$

100. $|k-3|=|k+3|$

101. $|2p-1|=|1-2p|$

102. $|4d-3|=|3-4d|$

Learn: Solve Radical Equations

For Exercises 103–122, solve the equation. (See Examples 11–13 on pp. R90–R92)

103. $1=-2+\sqrt{2x+7}$

104. $6=9+\sqrt{5-3x}$

105. $\sqrt{m+18}+2=m$

106. $\sqrt{2n+29}+3=n$

107. $-4\sqrt[3]{2x-5}+6=10$

108. $-3\sqrt[5]{4x-1}+2=8$

109. $\sqrt{8-p}-\sqrt{p+5}=1$

110. $\sqrt{d+4}-\sqrt{6+2d}=-1$

111. $3-\sqrt{y+3}=\sqrt{2-y}$

112. $\sqrt{k-2}=\sqrt{2k+3}-2$

113. $2(x+5)^{2/3}=18$

114. $3(x-6)^{2/3}=48$

115. $(3x+1)^{3/2}+2=66$

116. $(2x-1)^{3/2}-3=122$

117. $m^{3/4}=5$

118. $n^{5/6}=3$

119. $2p^{4/5}=\frac{1}{8}$

120. $5t^{2/3}=\frac{1}{5}$

121. $(2v+7)^{1/3}-(v-3)^{1/3}=0$

122. $(5u-6)^{1/5}-(3u+1)^{1/5}=0$

Learn: Solve Equations for a Specified Variable

For Exercises 123–152, solve for the specified variable. (See Examples 14–15 on pp. R92–R93)

123. $A=lw$ for l

124. $E=IR$ for R

125. $P=a+b+c$ for c

126. $W=K-T$ for K

127. $7x+2y=8$ for y

128. $3x+5y=15$ for y

129. $5x-4y=2$ for y

130. $7x-2y=5$ for y

131. $S=\frac{n}{2}(a+d)$ for d

132. $S=\frac{n}{2}[2a+(n-1)d]$ for a

133. $6=4x+tx$ for x

134. $8=3x+kx$ for x

135. $6x+ay=bx+5$ for x

136. $3x+2y=cx+d$ for x

137. $A=\pi r^2$ for $r>0$

138. $V=\pi r^2h$ for $r>0$

139. $a^2+b^2=c^2$ for $a>0$

140. $a^2+b^2=c^2$ for $c>0$

141. $kw^2-cw=r$ for w

142. $dy^2+my=p$ for y

143. $s=v_0t+\frac{1}{2}at^2$ for t

144. $S=2\pi rh+\pi r^2h$ for r

145. $\frac{1}{f}=\frac{1}{p}+\frac{1}{q}$ for p

146. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ for R_3

147. $16 + \sqrt{x^2 - y^2} = z$ for x

148. $4 + \sqrt{x^2 + y^2} = z$ for y

149. $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$ for T_1

150. $\frac{t_1}{s_1v_1} = \frac{t_2}{s_2v_2}$ for v_2

151. $T = 2\pi\sqrt{\frac{L}{g}}$ for g

152. $t = \sqrt{\frac{2s}{g}}$ for s

Mixed Exercises

For Exercises 153–176, solve the equation.

153. $-3x(2x - 1)(x + 6)^2 = 0$

154. $5y(3 - y)(4y + 1)^2 = 0$

155. $98t^3 - 49t^2 - 8t + 4 = 0$

156. $2m^3 + 3m^2 = 9(2m + 3)$

157. $\frac{1}{x-3} + 1 = \frac{x}{x+1}$

158. $\frac{1}{x+4} = \frac{x}{x-2} - 1$

159. $\frac{3}{4}(x + 1) - \frac{5}{6} = 2x - \frac{1}{2}$

160. $\frac{3}{4}x - \frac{1}{3} = \frac{2}{3}\left(x - \frac{1}{4}\right)$

161. $(x - 3)^2 - (x + 2)^2 = 0$

162. $(2x + 3)^2 - (2x + 7)^2 = 0$

163. $0 = 2x^3 + 5x^2 - 18x - 45$

164. $0 = 3y^3 + y^2 - 48y - 16$

165. $x^3 + 7x^2 = 4(x + 7)$

166. $2m^3 + 3m^2 = 9(2m + 3)$

167. $4 - \frac{x}{6-x} = \frac{72}{x^2-36}$

168. $\frac{x}{x+4} + \frac{32}{16-x^2} = 3$

169. $\frac{y}{y+3} + \frac{2}{y^2+3y} = \frac{6}{y}$

170. $\frac{-8}{t^2-6t} + \frac{t}{t-6} = \frac{1}{t}$

171. $\sqrt{6m+7} - \sqrt{3m+3} = 1$

172. $\sqrt{5w+1} - \sqrt{3w} = 1$

173. $5(x - 2) = \sqrt{12 - x} + 4(x - 1)$

174. $8x - 13 = \sqrt{x+7} + 7(x - 2)$

175. **a.** $|3x| = |-3x|$ **b.** $|3 + x| = |-3 + x|$
c. $|3x| = |-3 + x|$ **d.** $|3 + x| = -|3 - x|$

176. **a.** $|x + 2| - |x - 8| = 0$
b. $|x + 2| + |x - 8| = 0$
c. $|x + 2| - |x + 8| = 0$
d. $|x + 8| - |2x| = 0$

Write About It

177. Explain why the value 5 is not a solution to the equation $\frac{x}{x-5} + \frac{1}{5} = \frac{5}{x-5}$.

178. Explain why the value 2 is not the only solution to the equation $2x + 4 = 2(x - 3) + 10$.

Expanding Your Skills

For Exercises 179–180, solve for the indicated variable.

179. $x^2 - xy - 2y^2 = 0$ for x

180. $3a^2 + 2ab - b^2 = 0$ for a

For Exercises 181–184, solve for *x*. Then indicate the restrictions on the constants *a* and *b* to have a valid solution.

181. $(x - 3a)(x + 2) = (x - 8a)(x - 1)$

182. $(2x - 1)(2x + 1) = (x - 3a)(4x + 5a)$

183. $a^2x - 2a^2 + 3ab = b^2x + b^2$

184. $b^2x - 6ab - a^2 = a^2x + 5b^2$

For Exercises 185–190, write an equation with integer coefficients and the variable *x* that has the given solution set. [*Hint*: Apply the zero product property in reverse. For example, to build an equation whose solution set is $\left\{2, -\frac{5}{2}\right\}$ we have $(x - 2)(2x + 5) = 0$, or simply $2x^2 + x - 10 = 0$.]

185. $\{4, -2\}$

186. $\{7, -1\}$

187. $\left\{\frac{2}{3}, \frac{1}{4}\right\}$

188. $\left\{\frac{3}{5}, \frac{1}{7}\right\}$

189. $\{\sqrt{5}, -\sqrt{5}\}$

190. $\{\sqrt{2}, -\sqrt{2}\}$

Go online for more practice problems.

Lesson R-6

Numbers and More Quadratic Equations

What Will You Learn?

After completing this lesson, you should be able to:

- Simplify Imaginary Numbers
- Perform Operations on Complex Numbers
- Solve Quadratic Equations over the Set of Complex Numbers
- Use the Discriminant
- Solve Equations in Quadratic Form

Learn: Simplify Imaginary Numbers

In our study of algebra so far, we have worked exclusively with real numbers. However, as we encounter new types of equations, we need to look outside the set of real numbers to find solutions. For example, the equation $x^2 = 1$ has two solutions: 1 and –1. But what about the equation $x^2 = -1$? There is no real number x for which $x^2 = -1$. For this reason, mathematicians defined a new number i such that $i^2 = -1$. The number i is called an *imaginary number* and is used to represent $\sqrt{-1}$. Furthermore, the square root of any negative real number is an imaginary number that can be expressed in terms of i.

The Imaginary Number *i*

- $i = \sqrt{-1}$ and $i^2 = -1$
- If b is a positive real number, then $\sqrt{-b} = i\sqrt{b}$.

Example 1: Writing Imaginary Numbers in Terms of *i*

Write each expression in terms of i.

a. $\sqrt{-25}$ **b.** $\sqrt{-12}$ **c.** $\sqrt{-13}$

Solution:

a. $\sqrt{-25} = i\sqrt{25} = 5i$

b. $\sqrt{-12} = i\sqrt{12} = i \cdot 2\sqrt{3}$
$= 2i\sqrt{3}$ or $2\sqrt{3}i$

The value $i \cdot 2\sqrt{3}$ can be written as $2i\sqrt{3}$ or as $2\sqrt{3}i$. Note, however, that the factor i is written *outside* the radical.

c. $\sqrt{-13} = i\sqrt{13}$ or $\sqrt{13}i$

Apply the Skills

1. Write each expression in terms of i.

a. $\sqrt{-81}$ **b.** $\sqrt{-50}$ **c.** $\sqrt{-11}$

It is important to note that the multiplication and division properties of radicals can be used only if the radicals represent real-valued expressions.

$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ provided that the roots represent real numbers.

$\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ provided that the roots represent real numbers.

For this reason, we write the radical expressions in terms of *i* first, before applying the multiplication or division property of radicals. For example,

$$\begin{aligned} \sqrt{-9} \cdot \sqrt{-25} &= i\sqrt{9} \cdot i\sqrt{25} && \text{Write each radical in terms of } i \text{ first, } \textit{before} \text{ multiplying.} \\ &= 3i \cdot 5i && \text{Simplify the radicals.} \\ &= 15i^2 && \text{Multiply.} \\ &= 15(-1) && \text{By definition, } i^2 = -1. \\ &= -15 && \end{aligned}$$

$$\frac{\sqrt{-50}}{\sqrt{-2}} = \frac{i\sqrt{50}}{i\sqrt{2}} = \frac{\sqrt{50}}{\sqrt{2}} = \sqrt{\frac{50}{2}} = \sqrt{25} = 5$$

By definition, $i^2 = -1$, but what about other powers of *i*? Consider the following pattern.

$$\left.\begin{array}{ll} i^1 = i & i^1 = i \\ i^2 = -1 & i^2 = -1 \\ i^3 = i^2 \cdot i = (-1)i = -i & i^3 = -i \\ i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 & i^4 = 1 \end{array}\right\} \text{Pattern: } i, -1, i, 1$$

$$\left.\begin{array}{ll} i^5 = i^4 \cdot i(1) = i & i^5 = i \\ i^6 = i^4 \cdot i^2 = (1)(-1) = -1 & i^6 = -1 \\ i^7 = i^4 \cdot i^2 \cdot i = (1)(-1)i = -i & i^7 = -i \\ i^8 = i^4 \cdot i^4 = (1)(1) = 1 & i^8 = 1 \end{array}\right\} \text{Pattern repeats: } i, -1, -i, 1, \ldots$$

Insights

Notice that even powers of *i* simplify to 1 or –1.

- If the exponent is a multiple of 4, then the expression equals 1.
- If the exponent is even but *not* a multiple of 4, then the expression equals –1.

Notice that the fourth powers of *i* (i^4, i^8, i^{12}, ...) equal the real number 1. For other powers of *i*, we can write the expression as a product of a fourth power of *i* and a factor of *i*, i^2, or i^3, which equals *i*, –1, or –*i*, respectively.

Example 2: Simplifying Powers of *i*

Simplify.

a. i^{48} **b.** i^{23} **c.** i^{50} **d.** i^{-19}

Solution:

a. $i^{48} = (i^4)^{12} = (1)^{12} = 1$

The exponent 48 is a multiple of 4. Thus, i^{48} is equal to 1.

b. $i^{23} = i^{20} \cdot i^3$

$= (1) \cdot i^3 = -i$

Write i^{23} as a product of the largest fourth power of *i* and a remaining factor.

c. $i^{50} = i^{48} \cdot i^2$

$= (1)(-1) = -1$

d. $i^{-19} = i^{-20} \cdot i^1$

$= (i^4)^{-5} \cdot i = (1) \cdot i = i$

Apply the Skills

2. Simplify.

a. i^{13} **b.** i^{103} **c.** i^{64} **d.** i^{-30}

Insights

To simplify i^n, divide the exponent, *n*, by 4. The remainder is the exponent of the remaining factor of *i* once the fourth power of *i* has been extracted.

Example: $i^{50} = i^{48} \cdot i^2 = (1)i^2$

$$\begin{array}{r} 12 \\ 4\overline{)50} \\ \underline{48} \\ 2 \end{array}$$

So $i^{50} = (1) \cdot i^2 = -1$

Learn: Perform Operations on Complex Numbers

We now define a new set of numbers that includes the real numbers and imaginary numbers. This is called the set of complex numbers.

Complex Numbers

Given real numbers a and b, a number written in the form $a + bi$ is called a **complex number**. The value a is called the **real part** of the complex number and the value b is called the **imaginary part**.

$$5 - 7i = \overset{\text{Real part: 5}}{5} + \overset{\text{Imaginary part: }-7}{(-7)}i$$

Notes

- If $b = 0$, then $a + bi$ equals the real number a. This tells us that all real numbers are complex numbers.

  Example: $4 + 0i$ is generally written as the real number 4.

- If $a = 0$ and $b \neq 0$, then $a + bi$ equals bi, which we say is **pure imaginary**.

  Example: The number $0 + 8i$ is a pure imaginary number and is generally written as simply $8i$.

A complex number written in the form $a + bi$ is said to be in **standard form**. That being said, we sometimes write $a - bi$ in place of $a + (-b)i$. Furthermore, a number such as $5 + \sqrt{3}i$ is sometimes written as $5 + i\sqrt{3}$ to emphasize that the factor of i is not under the radical.

To add or subtract complex numbers, add or subtract their real parts, and add or subtract their imaginary parts. That is,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Example 3: Adding and Subtracting Complex Numbers

Perform the indicated operations. Write the answer in the form $a + bi$.

$$(-2 - 4i) + (5 + 2i) - (3 - 6i)$$

Solution:

$$\begin{aligned}&(-2 - 4i) + (5 + 2i) - (3 - 6i)\\ &\quad = (-2 + 5 - 3) + [-4 + 2 - (-6)]i\\ &\quad = 0 + 4i\end{aligned}$$

Combine the real parts and combine the imaginary parts.

Write the result in the form $a + bi$.

Apply the Skills

3. Perform the indicated operations. Write the answer in the form $a + bi$.

$$(8 - 3i) - (2 + 4i) + (5 + 7i)$$

In Examples 4 and 5, we multiply complex numbers using a process similar to multiplying polynomials.

Example 4: Multiplying Complex Numbers

Multiply. Write the result in the form $a + bi$. $(-2 + 6i)(4 - 3i)$

Solution:

$(-2 + 6i)(4 - 3i)$ Apply the distributive property

$= -2(4) + (-2)(-3i) + 6i(4) + 6i(-3i)$

$= -8 + 6i + 24i - 18i^2$

$= -8 + 30i - 18(-1)$ Recall that $i^2 = -1$.

$= -8 + 30i + 18$

$= 10 + 30i$ Write the result in the form $a + bi$.

Apply the Skills

4. Multiply. Write the result in the form $a + bi$.

$$(-5 + 4i)(3 - i)$$

In Example 5, we make use of the special case products:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2 \quad \text{and} \quad (a + b)(a - b) = a^2 - b^2$$

Example 5: Evaluating Special Products with Complex Numbers

Multiply. Write the results in the form $a + bi$.

a. $(3 + 4i)^2$ **b.** $(5 + 2i)(5 - 2i)$

Solution:

a. $(3 + 4i)^2 = (3)^2 + 2(3)(4i) + (4i)^2$ Apply the property $(a + b)^2 = a^2 + 2ab + b^2$.

$= 9 + 24i + 16i^2$

$= 9 + 24i + 16(-1)$

$= 9 + 24i - 16$

$= -7 + 24i$ Write the result in the form $a + bi$.

b. $(5 + 2i)(5 - 2i) = (5)^2 - (2i)^2$ Apply the property $(a + b)(a - b) = a^2 - b^2$.

$= 25 - 4i^2$

$= 25 - 4(-1)$

$= 25 + 4$

$= 29$ or $29 + 0i$ Write the result in the form $a + bi$.

Apply the Skills

5. Multiply. Write the result in the form $a + bi$.

a. $(4 - 7i)^2$ **b.** $(10 - 3i)(10 + 3i)$

In Lesson R.3 we noted that the expressions $(a + b)$ and $(a - b)$ are conjugates. Similarly, the expressions $(a + bi)$ and $(a - bi)$ are called **complex conjugates**. Furthermore, as illustrated in Example 5(b), the product of complex conjugates is a real number.

$$\begin{aligned}(a + bi)(a - bi) &= (a)^2 - (bi)^2\\ &= a^2 - b^2i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2\end{aligned}$$

Product of Complex Conjugates

If a and b are real numbers, then $(a + bi)(a - bi) = a^2 + b^2$.

In Example 6, we demonstrate division of complex numbers such as $\frac{8 + 2i}{3 - 5i}$. The goal is to make the denominator a real number so that the quotient can be written in standard form $a + bi$. This can be accomplished by multiplying the denominator by its complex conjugate. Of course, this means that we must also multiply the numerator by the same quantity.

Example 6: Dividing Complex Numbers

Divide. Write the result in the form $a + bi$. $\frac{8 + 2i}{3 - 5i}$

Solution:

$$\frac{8 + 2i}{3 - 5i} = \frac{(8 + 2i) \cdot (3 + 5i)}{(3 - 5i) \cdot (3 + 5i)}$$

Multiply numerator and denominator by the conjugate of the denominator.

$$= \frac{24 + 40i + 6i + 10i^2}{(3)^2 + (5)^2}$$

Apply the distributive property in the numerator.
Multiply conjugates in the denominator.

$$= \frac{24 + 46i + 10(-1)}{9 + 25}$$

Replace i^2 by -1.

$$= \frac{14 + 46i}{34}$$

$$= \frac{14}{34} + \frac{46}{34}i = \frac{7}{17} + \frac{23}{17}i$$

Write the result in the form $a + bi$.

Apply the Skills

6. Divide. Write the result in the form $a + bi$.

$$\frac{5 + 6i}{2 - 7i}$$

Learn: Solve Quadratic Equations over the Set of Complex Numbers

In Examples 7–9, we solve quadratic equations over the set of complex numbers. This means that in addition to having real solutions to quadratic equations, we now consider solutions of the form $a + bi$.

Example 7: Solving a Quadratic Equation with Imaginary Solutions

Solve the equation by applying the square root property. $2y^2 + 36 = 0$

Solution:

$2y^2 + 36 = 0$	
$2y^2 = -36$	Isolate the square term.
$y^2 = -18$	Write the equation in the form $y^2 = k$.
$y^2 = \pm\sqrt{-18}$	Apply the square root property.
$y = \pm 3i\sqrt{2}$	Simplify the radical. $\sqrt{-18} = i\sqrt{18} = i\sqrt{3^2 \cdot 2} = 3i\sqrt{2}$
The solution set is $\{\pm 3i\sqrt{2}\}$.	Both solutions check in the original equation.

Insights

The solutions to the equation in Example 7 are written concisely as $\pm 3i\sqrt{2}$. Do not forget that this actually represents two solutions:

$$y = 3i\sqrt{2} \quad \text{and} \quad y = -3i\sqrt{2}$$

Apply the Skills

7. Solve the equation by applying the square root property.

$$2c^2 + 80 = 0$$

Example 8: Using the Quadratic Formula

Solve the equation by applying the quadratic formula. $\frac{3}{10}x^2 - \frac{2}{5}x + \frac{7}{10} = 0$

Solution:

$\frac{3}{10}x^2 - \frac{2}{5}x + \frac{7}{10} = 0$	The equation is in the form $ax^2 + bx + c = 0$.
$10\left(\frac{3}{10}x^2 - \frac{2}{5}x + \frac{7}{10}\right) = 10(0)$	Multiply by 10 to clear fractions.
$3x^2 - 4x + 7 = 0$	

$a = 3,\ b = -4,\ c = 7$	Identify the values of *a*, *b*, and *c*.
$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(7)}}{2(3)}$	Apply the quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$= \frac{4 \pm \sqrt{-68}}{6}$	Simplify.
$= \frac{4 \pm 2i\sqrt{17}}{6}$	Simplify the radical.
$= \frac{\cancel{2}(2 \pm i\sqrt{17})}{\cancel{2} \cdot 3}$	Factor the numerator and denominator.
$= \frac{2 \pm i\sqrt{17}}{3}$	Simplify the fraction.
$= \frac{2}{3} \pm \frac{\sqrt{17}}{3}i$	Write the solutions in standard form, $a + bi$.
The solution set is $\left\{\frac{2}{3} \pm \frac{\sqrt{17}}{3}i\right\}$.	The solutions both check in the original equation.

Apply the Skills

8. Solve the equation by applying the quadratic formula.

$$\frac{5}{12}x^2 - \frac{1}{2}x + \frac{1}{4} = 0$$

Learn: Use the Discriminant

The solutions to a quadratic equation are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The radicand, $b^2 - 4ac$, is called the *discriminant*. The value of the discriminant tells us the number and type of solutions to the equation. We examine three different cases.

Using the Discriminant to Determine the Number and Type of Solutions to a Quadratic Equation $ax^2 + bx + c = 0$

Given a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$, the quantity $b^2 - 4ac$ is called the **discriminant**.

- $b^2 - 4ac < 0$: The quadratic equation has 2 nonreal solutions.

  Example: $2x^2 - 3x + 5 = 0$

  Discriminant: $b^2 - 4ac = (-3)^2 - 4(2)(5) = -31$

  Result of quadratic formula: $x = \frac{3 \pm \sqrt{-31}}{4}$

- $b^2 - 4ac = 0$: The quadratic equation has 1 real solution.

  Example: $x^2 + 6x + 9 = 0$

  Discriminant: $b^2 - 4ac = (6)^2 - 4(1)(9) = 0$

  Result of quadratic formula: $x = \frac{-6 \pm \sqrt{0}}{2} = -3$

- $b^2 - 4ac > 0$: The quadratic equation has 2 real solutions.

  Example: $2x^2 + 7x - 1 = 0$

  Discriminant: $b^2 - 4ac = (7)^2 - 4(2)(-1) = 57$

  Result of quadratic formula: $x = \frac{-7 \pm \sqrt{57}}{4}$

Example 9: Using the Discriminant

Use the discriminant to determine the number and type of solutions for each equation.

a. $5x^2 - 3x + 1 = 0$ **b.** $2x^2 = 3 - 6x$ **c.** $4x^2 + 12x = -9$

Solution:

Equation	$b^2 - 4ac$	**Solution Type and Number**
a. $5x^2 - 3x + 1 = 0$	$(-3)^2 - 4(5)(1) = -11$	Because $-11 < 0$, there are two nonreal solutions.
b. $2x^2 = 3 - 6x$ $2x^2 + 6x - 3 = 0$	$(6)^2 - 4(2)(-3) = 60$	Because $60 > 0$, there are two real solutions.
c. $4x^2 + 12x = -9$ $4x^2 + 12x + 9 = 0$	$(12)^2 - 4(4)(9) = 0$	Because the discriminant is 0, there is one real solution.

Apply the Skills

9. Use the discriminant to determine the number and type of solutions for each equation.

a. $2x^2 - 4x + 5 = 0$ **b.** $25x^2 = 10x - 1$ **c.** $x^2 + 10x = -9$

Learn: Solve Equations in Quadratic Form

We have learned to solve quadratic equations by applying the quadratic formula or by completing the square and applying the square root property. This is particularly important because many other equations are quadratic in form. That is, with a simple substitution, these equations can be expressed as quadratic equations in a new variable. For example:

Equation in Quadratic Form		**New Equation**
$\left(2 + \frac{3}{x}\right)^2 - \left(2 + \frac{3}{x}\right) - 12 = 0$	$\xrightarrow{\text{Let } u = 2 + \frac{3}{x}}$	$u^2 - u - 12 = 0$
$2w^{2/3} - 3w^{1/3} - 20 = 0$	$\xrightarrow{\text{Let } u = w^{1/3}}$	$2u^2 - 3u - 20 = 0$

The equations on the right are quadratic and easily solved. Then using back substitution, we can solve for the original variable.

Insights

For an equation written in quadratic form, notice that the expression for u is taken to be the variable expression from the middle term.

Example 10: Solving an Equation in Quadratic Form

Solve. $(2x^2 - 3)^2 + 36(2x^2 - 3) + 35 = 0$

Solution:

$(2x^2 - 3)^2 + 36(2x^2 - 3) + 35 = 0$ The equation is in quadratic form.

$u^2 + 36u + 35 = 0$ Let $u = 2x^2 - 3$.

$(u + 35)(u + 1) = 0$ Set one side equal to zero and factor the other side.

$u = -35$ or $u = -1$ Apply the zero product property.

$2x^2 - 3 = -35$ or $2x^2 - 3 = -1$ Back substitute. Replace u by $2x^2 - 3$.

$2x^2 = -32$ or $2x^2 = 2$

$x^2 = -16$ or $x^2 = 1$ Isolate the square term.

$x = \pm 4i$ or $x = \pm 1$ Apply the square root property.

The solution set is $\{\pm 4i, \pm 1\}$. The solutions all check in the original equation.

Apply the Skills

10. Solve. $(x^2 - 6)^2 + 33(x^2 - 6) + 62 = 0$

Example 11: Solving an Equation in Quadratic Form

Solve. $2w^{2/3} = 3w^{1/3} + 20$

Solution:

$2w^{2/3} = 3w^{1/3} + 20$ Set one side equal to zero, and write the expression on the left in descending order.

$2w^{2/3} - 3w^{1/3} - 20 = 0$

$2(w^{1/3})^2 - 3(w^{1/3}) - 20 = 0$ The equation is in quadratic form.

$2u^2 - 3u - 20 = 0$ Let $u = w^{1/3}$.

$(2u + 5)(u - 4) = 0$ Factor.

$u = -\frac{5}{2}$ or $u = 4$ Apply the zero product property.

$w^{1/3} = -\frac{5}{2}$ or $w^{1/3} = 4$ Back substitute. Replace u by $w^{1/3}$.

$(w^{1/3})^3 = \left(-\frac{5}{2}\right)^3$ or $(w^{1/3})^3 = (4)^3$ Cube both sides.

$w = -\frac{125}{8}$ or $w = 64$ Both solutions check in the original equation.

The solution set is $\left\{-\frac{125}{8}, 64\right\}$.

Insights

Consider the equation from Example 11:

$$2w^{2/3} - 3w^{1/3} - 20 = 0$$

As an alternative to using substitution, the expression on the left can be factored directly.

$$(2w^{1/3} + 5)(w^{1/3} - 4) = 0$$

Applying the zero product property results in the same solutions.

Apply the Skills

11. Solve. $2t^{2/3} = 15 - 7t^{1/3}$

Practice R-6

Practice Exercises

Concept Connections

1. The imaginary number i is defined so that $i = \sqrt{-1}$ and $i^2 =$ __________.

2. For a positive real number b, the value $\sqrt{-b} =$ __________.

3. Given a complex number $a + bi$, the value of a is called the __________ part and the value of b is called the __________ part.

4. Given a complex number $a + bi$, the expression $a - bi$ is called the complex __________.

5. The equation $m^{2/3} + 10m^{1/3} + 9 = 0$ is said to be in __________ form, because making the substitution $u =$ __________ results in a new equation that is quadratic.

6. Consider the equation $(4x^2 + 1)^2 + 4(4x^2 + 1) + 4 = 0$. If the substitution $u =$ __________ is made, then the equation becomes $u^2 + 4u + 4 = 0$.

Learn: Simplify Imaginary Numbers

For Exercises 7–18, write each expression in terms of *i* and simplify. (See Example 1 on p. R98)

7. $\sqrt{-121}$

8. $\sqrt{-100}$

9. $\sqrt{-98}$

10. $\sqrt{-63}$

11. $\sqrt{-4}\sqrt{-9}$

12. $\sqrt{-1}\sqrt{-36}$

13. $\sqrt{-10}\sqrt{-5}$

14. $\sqrt{-6}\sqrt{-15}$

15. $\dfrac{\sqrt{-98}}{\sqrt{-2}}$ **16.** $\dfrac{\sqrt{-45}}{\sqrt{-5}}$

17. $\dfrac{\sqrt{-63}}{\sqrt{7}}$ **18.** $\dfrac{\sqrt{-80}}{\sqrt{5}}$

For Exercises 19–26, simplify each expression and write the result in standard form, *a* + *bi*.

19. $\dfrac{8 + 3i}{14}$

20. $\dfrac{4 + 5i}{6}$

21. $\dfrac{-4 - 6i}{-2}$

22. $\dfrac{9 - 15i}{-3}$

23. $\dfrac{-18 + \sqrt{-48}}{4}$

24. $\dfrac{-20 + \sqrt{-50}}{-10}$

25. $\dfrac{14 - \sqrt{-98}}{-7}$

26. $\dfrac{-10 + \sqrt{-125}}{5}$

Learn: Perform Operations on Complex Numbers

For Exercises 27–30, simplify the powers of *i*. (See Example 2 on p. R100)

27. a. i^{20} **b.** i^{29}

c. i^{50} **d.** i^{-41}

28. a. i^{32} **b.** i^{47}

c. i^{66} **d.** i^{-27}

29. a. i^{37} **b.** i^{-37}

c. i^{82} **d.** i^{-82}

30. a. i^{103} **b.** i^{-103}

c. i^{52} **d.** i^{-52}

For Exercises 31–56, perform the indicated operations. Write the answers in standard form, *a* + *bi*. (See Examples 3–6 on pp. R101–R103)

31. $(2 - 7i) + (8 - 3i)$

32. $(6 - 10i) + (8 + 4i)$

33. $(15 + 21i) - (18 - 40i)$

34. $(250 + 100i) - (80 + 25i)$

35. $(2.3 + 4i) - (8.1 - 2.7i) + (4.6 - 6.7i)$

36. $(0.05 - 0.03i) + (-0.12 + 0.08i) - (0.07 + 0.05i)$

37. $2i(5 + i)$

38. $4i(6 + 5i)$

39. $(3 - 6i)(10 + i)$

40. $(2 - 5i)(8 + 2i)$

41. $(3 - 7i)^2$

42. $(10 - 3i)^2$

43. $(3 - \sqrt{-5})(4 + \sqrt{-5})$

44. $(2 + \sqrt{-7})(10 + \sqrt{-7})$

45. $(2 - i)^2 + (2 + i)^2$

46. $(3 - 2i)^2 + (3 + 2i)^2$

47. $(10 - 4i)(10 + 4i)$

48. $(3 - 9i)(3 + 9i)$

49. $(\sqrt{2} + \sqrt{3}i)(\sqrt{2} - \sqrt{3}i)$

50. $(\sqrt{5} + \sqrt{7}i)(\sqrt{5} - \sqrt{7}i)$

51. $\frac{6 + 2i}{3 - i}$

52. $\frac{5 + i}{4 - i}$

53. $\frac{8 - 5i}{13 + 2i}$

54. $\frac{10 - 3i}{11 + 4i}$

55. $\frac{5}{13i}$

56. $\frac{6}{7i}$

Learn: Solve Quadratic Equations over the Set of Complex Numbers

For Exercises 57–72, solve the equations. (See Examples 7–8 on p. R104)

57. $4u^2 + 64 = 0$

58. $8p^2 + 72 = 0$

59. $\left(t - \frac{1}{2}\right)^2 = -\frac{17}{4}$

60. $\left(a - \frac{1}{3}\right)^2 = -\frac{47}{9}$

61. $t^2 - 8t = -24$

62. $p^2 - 24p = -156$

63. $4z^2 + 24z = -160$

64. $2m^2 + 20m = -70$

65. $y^2 = -4y - 6$

66. $z^2 = -8z - 19$

67. $t(t - 6) = -10$

68. $m(m + 10) = -34$

69. $-\frac{7}{6}c + \frac{1}{2} = -\frac{5}{6}c^2$

70. $-\frac{5}{2}d + 1 = -3d^2$

71. $9x^2 + 49 = 0$

72. $121x^2 + 4 = 0$

Learn: Use the Discriminant

For Exercises 73–80, (a) evaluate the discriminant and (b) determine the number and type of solutions to each equation. (See Example 9 on p. R106)

73. $3x^2 - 4x + 6 = 0$

74. $5x^2 - 2x + 4 = 0$

75. $-2w^2 + 8w = 3$

76. $-6d^2 + 9d = 2$

77. $3x(x - 4) = x - 4$

78. $2x(x - 2) = x + 3$

79. $-1.4m + 0.1 = -4.9m^2$

80. $3.6n + 0.4 = -8.1n^2$

Learn: Solve Equations in Quadratic Form

For Exercises 81–100, make an appropriate substitution and solve the equation. (See Examples 10–11 on p. R107)

81. $(2x + 5)^2 - 7(2x + 5) - 30 = 0$

82. $(3x - 7)^2 - 6(3x - 7) - 16 = 0$

83. $(x^2 + 2x)^2 - 18(x^2 + 2x) = -45$

84. $(x^2 + 3x)^2 - 14(x^2 + 3x) = -40$

85. $(x^2 + 2)^2 + (x^2 + 2) - 42 = 0$

86. $(y^2 - 3)^2 - 9(y^2 - 3) - 52 = 0$

87. $\left(m - \frac{10}{m}\right)^2 - 6\left(m - \frac{10}{m}\right) - 27 = 0$

88. $\left(x + \frac{6}{x}\right)^2 - 12\left(x + \frac{6}{x}\right) + 35 = 0$

89. $\left(2 + \frac{3}{t}\right)^2 - \left(2 + \frac{3}{t}\right) = 12$

90. $\left(\frac{5}{y} + 3\right)^2 + 6\left(\frac{5}{y} + 3\right) = -8$

91. $5c^{2/5} - 11c^{1/5} + 2 = 0$

92. $3d^{2/3} - d^{1/3} - 4 = 0$

93. $y^{1/2} - y^{1/4} - 6 = 0$

94. $n^{1/2} + 6n^{1/4} - 16 = 0$

95. $9y^{-4} - 10y^{-2} + 1 = 0$

96. $100x^{-4} - 29x^{-2} + 1 = 0$

97. $4t - 25\sqrt{t} = 0$

98. $9m - 16\sqrt{m} = 0$

99. $30k^{-2} - 23k^{-1} + 2 = 0$

100. $3q^{-2} + 16q^{-1} + 5 = 0$

Mixed Exercises

For Exercises 101–110, perform the indicated operations.

101. **a.** $5i + 2(-4 + 3i)$ **b.** $5i(-4 + 3i)$
c. $\frac{-4 + 3i}{5i}$ **d.** $(-4 + 3i)(-4 - 3i)$

102. **a.** $\frac{-7i}{2 - 4i}$ **b.** $(-7i)^3$
c. $(-7i)(2 - 4i)$ **d.** $-7i - 3(2 - 4i)$

103. **a.** $3i - 2(4 - 8i)$ **b.** $3i - (4 - 8i)^2$
c. $\frac{3i - 2}{4 - 8i}$ **d.** $(3i - 2)(4 - 8i)$

104. **a.** $(2 - 3i)(4 + 5i)$ **b.** $(2 - 3i) - (4 + 5i)$
c. $\frac{2 - 3i}{4 + 5i}$ **d.** $(2 - 3i)^2$

105. **a.** $(2 + \sqrt{3}i)(2 - \sqrt{3}i)$
b. $(2 + \sqrt{3}i) + (2 - \sqrt{3}i)$
c. $\frac{2 + \sqrt{3}i}{\sqrt{3}i}$ **d.** $(2 + \sqrt{3}i)^2$

106. **a.** $(\sqrt{2} - 5i)^2$
b. $(\sqrt{2} - 5i) - (3\sqrt{2} + i)$
c. $(\sqrt{2} - 5i)(3\sqrt{2} + i)$
d. $\frac{3\sqrt{2}}{\sqrt{2} - 5i}$

107. **a.** $\frac{i^{22} - i}{-1 + i}$ **b.** $(i^{22} - i)(-1 + i)$

108. **a.** $\frac{3i^{15} - i}{1 - i}$ **b.** $(3i^{15} - i)(1 - i)$

109. $(1 + i)^{10}$ (*Hint*: rewrite as $[(1 + i)^2]^5$.)

110. $(-1 + i)^8$

For Exercises 111–116, solve the equations over the set of complex numbers.

111. **a.** $x^4 + 5x^2 - 36 = 0$
b. $x + 5x^{1/2} - 36 = 0$
c. $x^{1/2} + 5x^{1/4} - 36 = 0$
d. $x^{-4} + 5x^{-2} - 36 = 0$

112. **a.** $x^4 - 7x^2 - 8 = 0$
b. $x^{1/2} - 7x^{1/4} - 8 = 0$
c. $x^6 - 7x^3 - 8 = 0$
d. $x^{2/3} - 7x^{1/3} - 8 = 0$

113. $9x^{-2} - x^2 + 8 = 0$

114. $4x^2 + 3x^{-2} - 13 = 0$

115. $32x^{-3} - 10x^{-1} + \frac{x}{2} = 0$

116. $\frac{x}{3} + x^{-1} - 18x^{-3} = 0$

For Exercises 117–120, verify by substitution that the given values of *x* are solutions to the given equation.

117. $x^2 + 25 = 0$
a. $x = 5i$
b. $x = -5i$

118. $x^2 + 49 = 0$
a. $x = 7i$
b. $x = -7i$

119. $x^2 - 4x + 7 = 0$
a. $x = 2 + i\sqrt{3}$
b. $x = 2 - i\sqrt{3}$

120. $x^2 - 6x + 11 = 0$
a. $x = 3 + i\sqrt{2}$
b. $x = 3 - i\sqrt{2}$

121. Prove that $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

122. Prove that $(a + bi)^2 = (a^2 - b^2) + (2ab)i$.

For Exercises 123–124, solve the equation in two ways.

a. Solve as a radical equation by first isolating the radical.

b. Solve by writing the equation in quadratic form and using an appropriate substitution.

123. $y + 4\sqrt{y} = 21$

124. $w - 3\sqrt{w} = 10$

Write About It

125. Explain the flaw in the following logic.

$$\sqrt{-9} \cdot \sqrt{-4} = \sqrt{(-9)(-4)} = \sqrt{36} = 6$$

126. Discuss the difference between the products $(a + b)(a - b)$ and $(a + bi)(a - bi)$.

127. Give an example of a complex number that is its own conjugate.

128. Give an example of two complex numbers that are not real numbers, but whose product is a real number.

129. Given a quadratic equation, what is the discriminant and what information does it provide about the given quadratic equation?

130. Explain how to determine if an equation is in quadratic form.

Expanding Your Skills

For Exercises 131–136, factor the expressions over the set of complex numbers. For assistance, consider these examples.

- In Lesson R.3 we saw that some expressions factor over the set of integers. For example: $x^2 - 4 = (x + 2)(x - 2)$.
- Some expressions factor over the set of irrational numbers. For example: $x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$.
- To factor an expression such as $x^2 + 4$, we need to factor over the set of complex numbers. For example, verify that $x^2 + 4 = (x + 2i)(x - 2i)$.

131. a. $x^2 - 9$

b. $x^2 + 9$

132. a. $x^2 - 100$

b. $x^2 + 100$

133. a. $x^2 - 64$

b. $x^2 + 64$

134. a. $x^2 - 25$

b. $x^2 + 25$

135. a. $x^2 - 3$

b. $x^2 + 3$

136. a. $x^2 - 11$

b. $x^2 + 11$

For Exercises 137–140, write an equation with integer coefficients and the variable *x* that has the given solution set. [*Hint*: Apply the zero product property in reverse. For example, to build an equation whose solution set is $\left\{2, -\frac{5}{2}\right\}$ we have $(x - 2)(2x + 5) = 0$, or simply $2x^2 + x - 10 = 0$.]

137. $\{2i, -2i\}$

138. $\{9i, -9i\}$

139. $\{1 \pm 2i\}$

140. $\{2 \pm 9i\}$

The solutions to the equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$. For Exercises 141–144, prove the given statements.

141. Prove that $x_1 + x_2 = -\frac{b}{a}$.

142. Prove that $x_1x_2 = \frac{c}{a}$.

143. Prove that $x_1 = \frac{-2c}{b + \sqrt{b^2 - 4ac}}$.

144. Prove that $x_2 = \frac{-2c}{b - \sqrt{b^2 - 4ac}}$.

The variable *z* is often used to denote a complex number, and $\bar{z}$ is used to denote its conjugate. If $z = a + bi$, simplify the expressions in Exercises 145–148.

145. $z \cdot z$

146. $z^2 - \bar{z}^2$

147. $\frac{z + z}{2}$

148. $\frac{z - z}{i}$

For Exercises 149–150, let *z* represent the complex number $a + bi$, and let $\bar{z}$ be the conjugate. Solve for *z*. (*Hint*: Solve for *a* and *b* by equating the real parts and equating the imaginary parts.)

149. $z - 2\bar{z} = 5$

150. $2z + 3\bar{z} = i$

For Exercises 151–158, solve for *z*.

151. $z^2 - 2iz - 5 = 0$

152. $z^2 - 6iz - 10 = 0$

153. $z^2 - 2iz - 1 = 0$

154. $z^2 + 6iz - 9 = 0$

155. $2z^2 + 7iz + 15 = 0$

156. $3z^2 - 4iz + 4 = 0$

157. $2iz^2 - z + i = 0$

158. $iz^2 - 2z + 3i = 0$

Point of Interest

A series of unfortunate names? In the long history of mathematics, number systems have been expanded to accommodate meaningful solutions to equations. But the unfortunate connotation of their names may suggest a reluctance by early mathematicians to accept these new concepts.

- Negative numbers are not *unpleasant* or *disagreeable*.
- Irrational numbers are not *illogical* or *absurd*.
- Imaginary numbers are not "fake."

All these sets of numbers were found necessary to render solutions to equations.

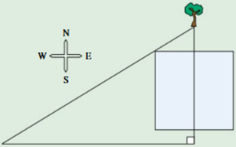
For example, the earliest written appearance of negative numbers occurred in 200 BCE in the text *The Nine Chapters on the Mathematical Art* (Jiǔzhāng Suànshù) and were used to solve systems of equations. In the seventh century, the Indian mathematician and astronomer Brahmagupta used negative numbers in the solutions to equations from the third-century Greek mathematician Diophantus. The Egyptian mathematician Abu Kamil Shuja (c.850–c.930) is considered to be the first to use irrational numbers as solutions (and coefficients) to equations. We can thank the seventeenth century mathematician and philosopher René Descartes for the term *imaginary* numbers, although many other mathematicians had previously worked with negative numbers under a square root.

159. This exercise is taken from Chapter 9 of the Chinese text *Nine Chapters* concerning right triangles.

There is a walled village in the shape of a square of unknown dimensions. At the center of both the north and south walls of the village is a gate. Walking 20 paces northward from the north gate, one finds a tree. If one walks 14 paces southward from the south gate, then turns west and walks 1775 paces, the tree just becomes visible.

a. Label the diagram with the given information. Let *x* represent the length of a side of the village.

b. Find the dimensions of the village. (*Hint*: Use similar triangles.)



Further Reading

Lu Yan, Du Shiran. *Chinese Mathematics: A Concise History*, Oxford University Press, 1987.

Joseph, George Gheverghese. *The Crest of the Peacock: Non-European Roots of Mathematics*, Princeton University Press (3rd ed.), 2010.

Go online for more practice problems.

Lesson R-7

Applications of Equations

What Will You Learn?

After completing this lesson, you should be able to:

- Solve Applications Involving Simple Interest
- Solve Applications Involving Mixtures
- Solve Applications Involving Uniform Motion
- Solve Applications Involving Rate of Work Done
- Solve Applications of Quadratic Equations

Learn: Solve Applications Involving Simple Interest

In Examples 1–6, we use linear and rational equations to model physical situations to solve applications. While there is no magic formula to apply to all word problems, we do offer the following guidelines to help you organize the given information and to form a useful model.

Problem-Solving Strategy

1. Read the problem carefully. Determine what the problem is asking for, and assign variables to the unknown quantities.
2. Make an appropriate figure or table if applicable. Label the given information and variables in the figure or table.
3. Write an equation that represents the verbal model. The equation may be a known formula or one that you create that is unique to the problem.
4. Solve the equation from step 3.
5. Interpret the solution to the equation and check that it is reasonable in the context of the problem.

Simple interest I for a loan or an investment is based on the principal P (amount of money invested or borrowed), the annual interest rate r, and the time of the loan t in years. The relationship among the variables is given by $I = Prt$.

For example, if \$5000 is invested at 4% simple interest for 18 months (1.5 yr), then the amount of simple interest earned is

$$\begin{aligned} I &= Prt \\ I &= (\$5000)(0.04)(1.5) \\ &= \$300 \end{aligned}$$

The formula for simple interest is used in Example 1.

Example 1: Solving an Application Involving Simple Interest

Imani invested a total of \$8000. Part of the money was invested for 2 yr in a stock fund that earned the equivalent of 6.5% simple interest. The remaining money was put in an 18-month certificate of deposit (CD) that earned 0.8% simple interest. If the total interest from both investments was \$804, determine the amount invested in each account.

Solution:

We can assign a variable to *either* the amount invested in the stock fund or the amount invested in the CD.

Let x represent the principal invested in the stock fund.

Then, $(8000 - x)$ is the remaining amount in the CD.

The interest from each account is computed from the formula $I = Prt$. Consider organizing this information in a table.

	Stock Fund (6.5% yield)	CD (0.8% yield)	Total
Principal (\$)	x	$8000 - x$	\$8000
Interest (\$)	$x(0.065)(2)$	$(8000 - x)(0.008)(1.5)$	\$804

Check Your Work

The CD was invested for 18 months. Be sure to convert to years.

$$18 \text{ months} = 1.5 \text{ yr}$$

To build an equation, note that

$$\begin{pmatrix}\text{Interest from}\\ \text{stock fund}\end{pmatrix} + \begin{pmatrix}\text{Interest}\\ \text{from CD}\end{pmatrix} = \begin{pmatrix}\text{Total}\\ \text{interest}\end{pmatrix}$$

$$\begin{aligned}
x(0.065)(2) + (8000 - x)(0.008)(1.5) &= 804 &&\text{Second row of table}\\
0.13x + 0.012(8000 - x) &= 804 &&\text{Simplify.}\\
0.13x + 96 - 0.012x &= 804 &&\text{Apply the distributive property.}\\
0.118x + 96 &= 804 &&\text{Combine like terms.}\\
0.118x &= 708 &&\text{Subtract 96 from both sides.}\\
x &= 6000
\end{aligned}$$

The amount invested in the stock fund is x: \$6000.

The amount invested in the CD is $\$8000 - x = \$8000 - \$6000 = \$2000$.

Check Your Work

Check that the answer is reasonable.

Amount of interest from the stock fund: ($6000)(0.065)(2) = $780

Amount of interest from the CD: ($2000)(0.008)(1.5) = $24

Total interest: $780 + $24 = $804 ✓

Apply the Skills

1. Ladonna borrowed a total of $10,000. Part of the money was borrowed from a lending institution that charged 5.5% simple interest. The rest of the money was borrowed from a friend to whom Ladonna paid 2.5% simple interest. Ladonna paid the friend back after 9 months (0.75 yr) and paid the lending institution after 2 yr. If the total amount Ladonna paid in interest was $735, how much was borrowed from each source?

Learn: Solve Applications Involving Mixtures

In Example 1, we "mixed" money between two different investments. We had to find the correct distribution of principal between two accounts to produce the given amount of interest. Example 2 presents a similar type of application that involves mixing different concentrations of a bleach solution to produce a third mixture of a given concentration.

For example, household bleach contains 6% sodium hypochlorite (active ingredient). This means that the remaining 94% of liquid is some other mixing agent such as water. Therefore, given 200 cL of household bleach, 6% would be pure sodium hypochlorite, and 94% would be some other mixing agent.

$$\text{Pure sodium hypochlorite} = (0.06)(200\text{ cL}) = 12\text{ cL}$$

$$\text{Other mixing agent} = (0.94)(200\text{ cL}) = 188\text{ cL}$$

To find the amount of pure sodium hypochlorite, we multiplied the concentration rate by the amount of solution.

Example 2: Solving an Application Involving Mixtures

Household bleach contains 6% sodium hypochlorite. How much household bleach should be combined with 70 L of a weaker 1% hypochlorite solution to form a solution that is 2.5% sodium hypochlorite?

Solution:

Let x represent the amount of 6% sodium hypochlorite solution (in liters).

70 L is the amount of 1% sodium hypochlorite solution.

Therefore, $x + 70$ is the amount of the resulting mixture (2.5% solution).

The amount of pure sodium hypochlorite in each mixture is found by multiplying the concentration rate by the amount of solution.

	6% Solution	1% Solution	2.5% Solution
Amount of solution (L)	x	70	$x + 70$
Pure sodium hypochlorite (L)	$0.06x$	$0.01(70)$	$0.025(x + 70)$

To build an equation, note that

$$\begin{pmatrix}\text{Amount of sodium}\\ \text{hypochlorite in}\\ \text{6\% solution}\end{pmatrix} + \begin{pmatrix}\text{Amount of sodium}\\ \text{hypochlorite in}\\ \text{1\% solution}\end{pmatrix} = \begin{pmatrix}\text{Amount of sodium}\\ \text{hypochlorite in}\\ \text{2.5\% solution}\end{pmatrix}$$

$$\begin{aligned} 0.06x + 0.01(70) &= 0.025(x + 70) && \text{Second row in the table} \\ 0.06x + 0.7 &= 0.025x + 1.75 && \text{Solve the equation.} \\ 0.035x &= 1.05 \\ x &= 30 \end{aligned}$$

The amount of household bleach (6% sodium hypochlorite solution) needed is 30 L.

Check Your Work

Check that the answer is reasonable. The total amount of the resulting solution is 30 L + 70 L, which is 100 L.

Amount of sodium hypochlorite in 6% solution: $0.06(30\text{ L}) = 1.8\text{ L}$

Amount of sodium hypochlorite in 1% solution: $\underline{0.01(70\text{ L}) = 0.7\text{ L}}$

Amount of sodium hypochlorite in 2.5% solution: $0.025(100\text{ L}) = 2.5\text{ L}$ ✓

Apply the Skills

2. How much 4% acid solution should be mixed with 200 mL of a 12% acid solution to make a 9% acid solution?

Learn: Solve Applications Involving Uniform Motion

Example 3 involves uniform motion. Recall that the distance that an object travels is given by

$$d = rt \qquad \text{Distance} = (\text{Rate})(\text{Time})$$

It follows that $$t = \frac{d}{r} \qquad \text{Time} = \frac{\text{Distance}}{\text{Rate}}$$

Example 3: Solving an Application Involving Uniform Motion

Alina takes a boat 6 mi downstream with a 1.5-mph current. The return trip against the current takes 1 hr longer. Find the speed of the boat in still water (in the absence of current).

Solution:

Let b represent the speed of the boat in still water.

Assign a variable to the unknown quantity.

	Distance (mi)	Rate (mph)	Time (hr)
With current	6	$b + 1.5$	$\frac{6}{b + 1.5}$
Against current	6	$b - 1.5$	$\frac{6}{b - 1.5}$

Organize the given information in a figure or chart.

Build an equation.

$$\left(\begin{array}{c}\text{Time of trip}\\ \text{against current}\end{array}\right) - \left(\begin{array}{c}\text{Time of trip}\\ \text{with current}\end{array}\right) = 1 \longleftarrow$$

The return trip against the current takes longer. The difference in time between the return trip and the original trip is 1 hr.

$$\frac{6}{b - 1.5} - \frac{6}{b + 1.5} = 1$$

The restrictions on b are $b \neq 1.5$ and $b \neq -1.5$.

$$(b - 1.5)(b + 1.5)\left(\frac{6}{b - 1.5} - \frac{6}{b + 1.5}\right) = (b - 1.5)(b + 1.5)\,(1)$$

$$6(b + 1.5) - 6(b - 1.5) = (b - 1.5)(b + 1.5)$$

Apply the distributive property.

$$6b + 9 - 6b + 9 = b^2 - 2.25$$

$$20.25 = b^2$$

$$b = \pm\sqrt{20.25}$$

Apply the square root property.

$$b = 4.5$$

Reject the negative solution because b represents the speed of the boat.

The speed of the boat in still water is 4.5 mph.

Apply the Skills

3. A fishing boat can travel 60 km with a 2.5-km/hr current in 2 hr less time than it can travel 60 km against the current. Determine the speed of the fishing boat in still water.

Learn: Solve Applications Involving Rate of Work Done

In Example 4, we investigate an application involving the combined rates at which two individuals work to complete a task. From this information, we can determine the amount of time required for the individuals to complete a task by working together.

Example 4: Solving an Application Involving "Work" Rates

At a mail-order company, Derrick can process 100 orders in 4 hr. Miguela can process 100 orders in 3 hr.

a. How long would it take them to process 100 orders if they worked together?

b. How long would it take them to process 1400 orders if they worked together?

Solution:

a. Let *t* represent the amount of time required to process 100 orders working together.

One method to approach this problem is to add the rates of speed at which each person works.

$$\begin{pmatrix}\text{Derrick's}\\ \text{speed}\end{pmatrix} + \begin{pmatrix}\text{Miguela's}\\ \text{speed}\end{pmatrix} = \begin{pmatrix}\text{Speed working}\\ \text{together}\end{pmatrix}$$

Build an equation.

$$\frac{1\text{ job}}{4\text{ hr}} + \frac{1\text{ job}}{3\text{ hr}} = \frac{1\textit{ job}}{t\text{ hr}}$$

1 job = 100 orders.

$$\frac{1}{4} + \frac{1}{3} = \frac{1}{t}$$

$$12t\left(\frac{1}{4} + \frac{1}{3}\right) = 12t\left(\frac{1}{t}\right)$$

Multiply both sides by the LCD, 12*t*.

$$3t + 4t = 12$$

Apply the distributive property.

$$7t = 12$$

$$t = \frac{12}{7} \quad \text{or} \quad 1\frac{5}{7}$$

Derrick and Miguela can process 100 orders in $1\frac{5}{7}$ hr working together.

b. The time required to process 1400 orders is 14 times as long as the time to process 100 orders. $\left(\frac{12}{7}\text{hr}\right)(14) = 24$ hr.

Apply the Skills

4. Sheldon and Penny were awarded a contract to paint 16 offices in the new math building at a university. Once all the preparation work is complete, Sheldon can paint an office in 30 min and Penny can paint an office in 45 min.

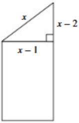
a. How long would it take them to paint one office working together?

b. How long would it take them to paint all 16 offices?

Learn: Solve Applications of Quadratic Equations

In Example 5, we use the Pythagorean theorem and a quadratic equation to find the lengths of the sides of a right triangle.

Example 5: Solving an Application Involving the Pythagorean Theorem



A window is in the shape of a rectangle with an adjacent right triangle above (see figure). The length of one leg of the right triangle is 2 ft less than the length of the hypotenuse. The length of the other leg is 1 ft less than the length of the hypotenuse. Find the lengths of the sides.

Solution:

Let x represent the length of the hypotenuse.

$x-1$ represents the length of the longer leg.

$x-2$ represents the length of the shorter leg.

Use the Pythagorean theorem to relate the lengths of the sides.

$a^2+b^2=c^2$	Pythagorean theorem
$(x-1)^2+(x-2)^2=(x)^2$	Substitute $x-1$, $x-2$, and x for the lengths of the sides.
$x^2-2x+1+x^2-4x+4=x^2$	
$x^2-6x+5=0$	Set one side of the equation equal to zero.
$(x-1)(x-5)=0$	Factor.
$x-1=0$ or $x-5=0$	Apply the zero product property.
$\cancel{x=1}$ or $x=5$	Reject $x=1$ because if x were 1, the lengths of the legs would be 0 ft and –1 ft, which is impossible.

The hypotenuse is 5 ft.

The length of the longer leg is given by $x-1$: 5 ft – 1 ft = 4 ft.

The length of the shorter leg is given by $x-2$: 5 ft – 2 ft = 3 ft.

Check Your Work

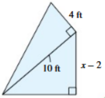
In Example 5, be sure to square the binomials correctly. Recall that $(a-b)^2=a^2-2ab+b^2$. Therefore,

$$(x-1)^2=x^2-2x+1$$

$$(x-2)^2=x^2-4x+4$$

Apply the Skills

5. A sail on a sailboat is in the shape of two adjacent right triangles. The hypotenuse of the lower triangle is 10 ft, and one leg is 2 ft shorter than the other leg. Find the lengths of the legs of the lower triangle.



In the study of physical science, a common model used to represent the vertical position of an object moving vertically under the influence of gravity is given in the following box.

Vertical Position of an Object

Suppose that an object has an initial vertical position of s_0 and initial velocity v_0 straight upward. The vertical position s of the object is given by

$$s = -\frac{1}{2}gt^2 + v_0t + s_0,$$

where

g = the acceleration due to gravity

t = the time of travel

v_0 = the initial velocity

On Earth, $g = 32$ ft/sec² or $g = 9.8$ m/sec².

s_0 = the initial vertical position

s = the vertical position of the object at time t

For example, suppose that a child tosses a ball straight upward from a height of 1.5 ft, with an initial velocity of 48 ft/sec.

The initial height is $s_0 = 1.5$ ft.

The initial velocity is $v_0 = 48$ ft/sec.

The acceleration due to gravity is $g = 32$ ft/sec².

The vertical position of the ball (in feet) is given by

$$\begin{aligned} s &= -\frac{1}{2}gt^2 + v_0t + s_0 \\ s &= -\frac{1}{2}(32)t^2 + (48)t + (1.5) \\ &= -16t^2 + 48t + 1.5 \end{aligned}$$

Insights

The value of g is chosen to be consistent with the units for position and velocity. In this case, the initial height is given in ft. The initial velocity is given in ft/sec. Therefore, we choose g in ft/sec² rather than m/sec².

Example 6: Analyzing an Object Moving Vertically

A model rocket is shot straight upward from a launch pad of 1 m above ground level with an initial velocity of 24 m/sec.

a. Write a model to express the height of the rocket s (in meters) above ground level.

b. Find the time(s) at which the rocket is at a height of 20 m. Round to 1 decimal place.

c. Find the time(s) at which the rocket is at a height of 40 m.

Solution:

a. $s = -\frac{1}{2}gt^2 + v_0t + s_0$

$s = -\frac{1}{2}(9.8)t^2 + (24)t + (1)$

$= -4.9t^2 + 24t + 1$

In this example,

$s_0 = 1$ m

$v_0 = 24$ m/sec

$g = 9.8$ m/sec^2

Insights

Choose $g = 9.8$ m/sec^2 because the height is given in meters and velocity is given in meters per second.

b. $20 = -4.9t^2 + 24t + 1$ Substitute 20 for *s*.

$4.9t^2 - 24t + 19 = 0$ Set one side equal to zero.

$t = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(4.9)(19)}}{2(4.9)}$ Apply the quadratic formula.

$$\begin{tikzcd} & t = \frac{24 + \sqrt{203.6}}{9.8} \approx 3.9 \\ t = \frac{24 \pm \sqrt{203.6}}{9.8} \arrow[ur] \arrow[dr] & \\ & t = \frac{24 - \sqrt{203.6}}{9.8} \approx 1.0 \end{tikzcd}$$

The rocket will be at a height of 20 m at 1 sec and 3.9 sec after launch.

c. $40 = -4.9t^2 + 24t + 1$ Substitute 40 for *s*.

$4.9t^2 - 24t + 39 = 0$

$t = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(4.9)(39)}}{2(4.9)}$ Apply the quadratic formula.

$t = \frac{24 \pm \sqrt{-188.4}}{9.8}$ The solutions are not real numbers.

There is no real number *t* for which the height of the rocket will be 40 m. The rocket will not reach a height of 40 m.

Apply the Skills

6. A fireworks mortar is launched straight upward from a pool deck 2 m off the ground at an initial velocity of 40 m/sec.

a. Write a model to express the height of the mortar *s* (in meters) above ground level.

b. Find the time(s) at which the mortar is at a height of 60 m. Round to 1 decimal place.

c. Find the time(s) at which the rocket is at a height of 100 m.

Practice R-7

Practice Exercises

Concept Connections

1. If \$6000 is borrowed at 7.5% simple interest for 2 yr, then the amount of interest is ____________________.

2. Suppose that 8% of a solution is fertilizer by volume and the remaining 92% is water. How much fertilizer is in a 2-L bucket of solution?

3. If $d = rt$, then $t = \frac{\square}{\square}$

4. If $d = rt$, then $r = \frac{\square}{\square}$

Learn: Solve Applications Involving Simple Interest

5. Rocco borrowed a total of \$5000 from two student loans. One loan charged 3% simple interest and the other charged 2.5% simple interest, both payable after graduation. If the interest owed after 1 yr was \$132.50, determine the amount of principal for each loan. (**See Example 1 on p. R115**)

6. Laura borrowed a total of \$22,000 from two different banks to start a business. One bank charged the equivalent of 4% simple interest, and the other charged 5.5% interest. If the total interest after 1 yr was \$910, determine the amount borrowed from each bank.

7. Fernando invested money in a 3-yr CD (certificate of deposit) that returned the equivalent of 1.4% simple interest. He invested \$2000 less in an 18-month CD that had a 0.9% return. If the total amount of interest from these investments was \$222.75, determine how much was invested in each CD.

8. Ebony bought a 5-yr Treasury note that paid the equivalent of 2.8% simple interest. She invested \$5000 more in a 10-yr bond earning 3.6% than she did in the Treasury note. If the total amount of interest from these investments was \$5300, determine the amount of principal for each investment.

Learn: Solve Applications Involving Mixtures

9. Ethanol fuel mixtures have "E" numbers that indicate the percentage of ethanol in the mixture by volume. For example, E10 is a mixture of 10% ethanol and 90% gasoline. How much E5 should be mixed with 5000 gal of E10 to make an E9 mixture? (**See Example 2 on p. R116**)

10. A nurse mixes 60 cc of a 50% saline solution with a 10% saline solution to produce a 25% saline solution. How much of the 10% solution should be used?

11. The density and strength of concrete are determined by the ratio of cement and aggregate (aggregate is sand, gravel, or crushed stone). Suppose that a contractor has 480 ft³ of a dry concrete mixture that is 70% sand by volume. How much pure sand must be added to form a new mixture that is 75% sand by volume?

12. Antifreeze is a compound added to water to reduce the freezing point of a mixture. In extreme cold (less than –35°F), one car manufacturer recommends that a mixture of 65% antifreeze be used. How much 50% antifreeze solution should be drained from a 4-gal tank and replaced with pure antifreeze to produce a 65% antifreeze mixture?

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Learn: Solve Applications Involving Uniform Motion

13. Jesse takes a 3-day kayak trip and travels 72 km south from Everglades City to a camp area in Everglades National Park. The trip to the camp area with a 2-km/hr current takes 9 hr less time than the return trip against the current. Find the speed that Jesse travels in still water. (**See Example 3 on p. R118**)

14. A plane travels 800 mi from Dallas, Texas, to Atlanta, Georgia, with a prevailing west wind of 40 mph. The return trip against the wind takes $\frac{1}{2}$ hr longer. Find the average speed of the plane in still air.

15. Jean runs 6 mi and then rides 24 mi on her bicycle in a biathlon. She rides 8 mph faster than she runs. If the total time for her to complete the race is 2.25 hr, determine her average speed running and her average speed riding her bicycle.

16. Carmen drives between Miami, Florida, and West Palm Beach, Florida. She drives 50 mi in clear weather and then encounters a thunderstorm for the last 15 mi. She drives 20 mph slower through the thunderstorm than she does in clear weather. If the total time for the trip is 1.5 hr, determine her average speed in nice weather and her average speed driving in the thunderstorm.

17. Two passengers leave the airport at Kansas City, Missouri. One flies to Los Angeles, California, in 3.4 hr and the other flies in the opposite direction to New York City in 2.4 hr. With prevailing westerly winds, the speed of the plane to New York City is 60 mph faster than the speed of the plane to Los Angeles. If the total distance traveled by both planes is 2464 mi, determine the average speed of each plane.

18. Two planes leave from Atlanta, Georgia. One makes a 5.2-hr flight to Seattle, Washington, and the other makes a 2.5-hr flight to Boston, Massachusetts. The plane to Boston averages 44 mph slower than the plane to Seattle. If the total distance traveled by both planes is 3124 mi, determine the average speed of each plane.

19. Darren drives to school in rush hour traffic and averages 32 mph. He returns home in mid-afternoon when there is less traffic and averages 48 mph. What is the distance between his home and school if the total traveling time is 1 hr 15 min?

20. Fatima competes in a biathlon by running and bicycling around a large loop through a city. She runs the loop one time and bicycles the loop five times. She can run 8 mph and she can ride 16 mph. If the total time it takes her to complete the race is 1 hr 45 min, determine the distance of the loop.

Learn: Solve Applications Involving Rate of Work Done

21. Natalie can run around a $\frac{1}{4}$-mi track in 66 sec, and Matteo can run around the track in 60 sec. If the runners start at the same point on the track and run in opposite directions, how long will it take the runners to cover $\frac{1}{4}$mi? (**See Example 4 on p. R119**)

22. One custodian can vacuum the library of a school in 40 min. It takes a different custodian 1 hr to vacuum the library. How long would it take them if they worked together?

23. One pump can fill a pool in 10 hr. Working with a second slower pump, the two pumps together can fill the pool in 6 hr. How fast can the second pump fill the pool by itself?

24. George and Amal can mow their yard together with two lawn mowers in 30 min. When Amal works alone, it takes 50 min. How long would it take George to mow the lawn working alone?

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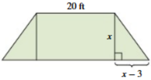
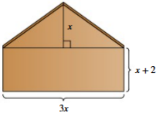
Learn: Solve Applications of Quadratic Equations

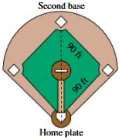
25. A patio is configured from a rectangle with two right triangles of equal size attached at the two ends. The length of the rectangle is 20 ft. The base of the right triangle is 3 ft less than the height of the triangle. If the total area of the patio is 348 ft², determine the base and height of the triangular portions.



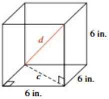
26. The front face of a house is in the shape of a rectangle with a Queen post roof truss above. The length of the rectangular region is 3 times the height of the truss. The height of the rectangle is 2 ft more than the height of the truss. If the total area of the front face of the house is 336 ft², determine the length and width of the rectangular region.



27. A baseball diamond is in the shape of a square with 90-ft sides. How far is it from home plate to second base? Give the exact value and give an approximation to the nearest tenth of a foot.



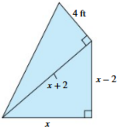
28. The figure shown is a cube with 6-in. sides. Find the exact length of the diagonal through the interior of the cube d by following these steps.



a. Apply the Pythagorean theorem using the sides on the base of the cube to find the length of diagonal c.

b. Apply the Pythagorean theorem using c and the height of the cube as the legs of the right triangle through the interior of the cube.

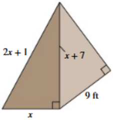
29. The sail on a sailboat is in the shape of two adjacent right triangles. In the lower triangle, the shorter leg is 2 ft less than the longer leg. The hypotenuse is 2 ft more than the longer leg. (**See Example 5 on p. R120**)



a. Find the lengths of the sides of the lower triangle.

b. Find the total sail area.

30. A portion of a roof truss is given in the figure. The triangle on the left is configured such that the longer leg is 7 ft longer than the shorter leg, and the hypotenuse is 1 ft more than twice the shorter leg.



a. Find the lengths of the sides of the triangle on the left.

b. Find the lengths of the sides of the triangle on the right.

31. The display area on a cell phone has a 3.5-in. diagonal.

a. If the aspect ratio of length to width is 1.5 to 1, determine the length and width of the display area. Round the values to the nearest hundredth of an inch.

b. If the phone has 326 pixels per inch, approximate the dimensions in pixels.

32. The display area on a computer has a 15-in. diagonal. If the aspect ratio of length to width is 1.6 to 1, determine the length and width of the display area. Round the values to the nearest hundredth of an inch.

For Exercises 33–36, use the model $s = -\frac{1}{2}gt^2 + v_0 t + s_0$ with $g = 32$ ft/sec² or $g = 9.8$ m/sec². (See Example 6 on p. R121)

33. NBA basketball legend Michael Jordan had a 48-in. vertical leap. Suppose that Michael jumped from ground level with an initial velocity of 16 ft/sec.

a. Write a model to express Michael's height (in ft) above ground level *t* seconds after leaving the ground.

b. Use the model from part (a) to determine how long it would take Michael to reach his maximum height of 48 in. (4 ft).

34. At the time of this printing, the highest vertical leap on record is 60 in., held by Kadour Ziani. For this record-setting jump, Kadour left the ground with an initial velocity of $8\sqrt{5}$ ft/sec.

a. Write a model to express Kadour's height (in ft) above ground level *t* seconds after leaving the ground.

b. Use the model from part (a) to determine how long it would take Kadour to reach his maximum height of 60 in. (5 ft). Round to the nearest hundredth of a second.

35. A bad punter on a football team kicks a football approximately straight upward with an initial velocity of 75 ft/sec.

a. If the ball leaves his foot from a height of 4 ft, write an equation for the vertical height *s* (in ft) of the ball *t* seconds after being kicked.

b. Find the time(s) at which the ball is at a height of 80 ft. Round to 1 decimal place.

36. In a classic *Seinfeld* episode, Jerry tosses a loaf of bread (a marble rye) straight upward to his friend George who is leaning out of a third-story window.

a. If the loaf of bread leaves Jerry's hand at a height of 1 m with an initial velocity of 18 m/sec, write an equation for the vertical position of the bread *s* (in meters) *t* seconds after release.

b. How long will it take the bread to reach George if he catches the bread on the way up at a height of 16 m? Round to the nearest tenth of a second.

Mixed Exercises

37. Suppose that 40 deer are introduced in a protected wilderness area. The population of the herd *P* can be approximated by $P = \frac{40 + 20x}{1 + 0.05x}$, where *x* is the time in years since introducing the deer. Determine the time required for the deer population to reach 200.

38. Starting from rest, an automobile's velocity *v* (in ft/sec) is given by $v = \frac{180t}{2t + 10}$, where *t* is the time in seconds after the car begins forward motion. Determine the time required for the car to reach a speed of 60 ft/sec (≈ 41 mph).

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39. Dominique's SUV gets 22 mpg in the city and 30 mpg on the highway. The amount of gas used A (in gal) is given by $A = \frac{1}{22}c + \frac{1}{30}h$, where c is the number of city miles driven and h is the number of highway miles driven. If Dominique used 7 gal of gas on a trip where 165 mi was on the highway, how many city miles were driven?

40. Dexter's truck gets 32 mpg on the highway and 24 mpg in the city. The amount of gas used A (in gal) is given by $A = \frac{1}{24}c + \frac{1}{32}h$, where c is the number of city miles driven and h is the number of highway miles driven. If Dexter used 9 gal of gas on a trip where 60 mi was in the city, how many highway miles were driven?

41. At a construction site, cement, sand, and gravel are mixed to make concrete. The ratio of cement to sand to gravel is 1 to 2.4 to 3.6. If a 150-lb bag of sand is used, how much cement and gravel must be used?

42. The property tax on a $180,000 house is $1296. At this rate, what is the property tax on a house that is $240,000?

43. In addition to measuring a person's individual HDL and LDL cholesterol levels, doctors also compute the ratio of total cholesterol to HDL cholesterol. Doctors recommend that the ratio of total cholesterol to HDL cholesterol be kept under 4. Suppose that the ratio of a patient's total cholesterol to HDL is 3.4. If the HDL is 60 mg/dL, determine the patient's LDL level and total cholesterol. (Assume that total cholesterol is the sum of the LDL and HDL levels.)

44. An administrator at a health clinic takes inventory of PPE (personal protective equipment). The clinic has 27 more packages of N95 masks than packages of N99 masks. (An N95 mask filters at least 95% of airborne particles, and an N99 mask filters at least 99% of airborne particles.) This is a 4 to 1 ratio of boxes of N95 masks to boxes of N99 masks. How many boxes of each type of mask does the clinic have?

45. When studying wildlife populations, biologists sometimes use a technique called "mark-recapture." For example, a researcher captured and tagged 30 deer in a wildlife management area. Several months later, the researcher observed a new sample of 80 deer and determined that 5 were tagged. What is the total number of deer in the population?

46. To estimate the number of bass in a lake, a biologist catches and tags 24 bass. Several weeks later, the biologist catches a new sample of 40 bass and finds that 4 are tagged. How many bass are in the lake?

47. Seismographs can record two types of wave energy (P waves and S waves) that travel through the Earth after an earthquake. Traveling through granite, P waves travel approximately 5 km/sec and S waves travel approximately 3 km/sec. If a geologist working at a seismic station measures a time difference of 40 sec between an earthquake's P waves and S waves, how far from the epicenter of the earthquake is the station?

48. Suppose that a shallow earthquake occurs in which the P waves travel 8 km/sec and the S waves travel 4.8 km/sec. If a seismologist measures a time difference of 20 sec between the arrival of the P waves and the S waves, how far is the seismologist from the epicenter of the earthquake?

49. Henri needs to have a toilet repaired in his house. The cost of the new plumbing fixtures is $110 and labor is $60/hr.

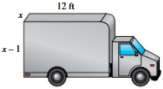
a. Write a model that represents the cost of the repair C (in $) in terms of the number of hours of labor x.

b. After how many hours of labor would the cost of the repair job equal the cost of a new toilet of $350?

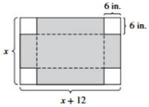
50. After a hurricane, repairs to a roof will cost \$2400 for materials and \$80/hr in labor.

- **a.** Write a model that represents the cost of the repair C (in \$) in terms of the number of hours of labor x.
- **b.** If an estimate for a new roof is \$5520, after how many hours of labor would the cost to repair the roof equal the cost of a new roof?

51. On moving day, Guyton needs to rent a truck. The length of the cargo space is 12 ft, and the height is 1 ft less than the width. The brochure indicates that the truck can hold 504 ft³. What are the dimensions of the cargo space? Assume that the cargo space is in the shape of a rectangular solid. (**See Example 1**)



52. Lorene plans to make several open-topped boxes in which to carry plants. The boxes are made from rectangular sheets of cardboard from which Lorene cuts out 6-in. squares from each corner. The length of the original piece of cardboard is 12 in. more than the width. If the volume of the box is 1728 in.³, determine the dimensions of the original piece of cardboard.



53. The population P of a culture of *Pseudomonas aeruginosa* bacteria is given by $P = -1718t^2 + 82{,}000t + 10{,}000$, where t is the time in hours since the culture was started. Determine the time(s) at which the population was 600,000. Round to the nearest hour.

54. The distance d (in ft) required to stop a car that was traveling at speed v (in mph) before the brakes were applied depends on the amount of friction between the tires and the road and the driver's reaction time. After an accident, a legal team hired an engineering firm to collect data for the stretch of road where the accident occurred. Based on the data, the stopping distance is given by $d = 0.05v^2 + 2.2v$.

- **a.** Determine the distance required to stop a car going 50 mph.
- **b.** Up to what speed (to the nearest mph) could a motorist be traveling and still have adequate stopping distance to avoid hitting a deer 330 ft away?

Write About It

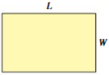
55. Is it possible for the measures of the angles in a triangle to be represented by three consecutive odd integers? Explain.

56. A bank customer wants to change a \$100 bill into an equal number of \$20 bills, \$10 bills, and \$5 bills. Is this possible? Explain.

Gaja Snover/Alamy Stock Photo

Expanding Your Skills

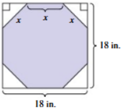
57. A **golden rectangle** is a rectangle in which the ratio of its length to its width is equal to the ratio of the sum of its length and width to its length: $\frac{L}{W} = \frac{L+W}{L}$ (values of L and W that meet this condition are said to be in the **golden ratio**).



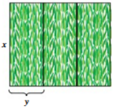
a. Suppose that a golden rectangle has a width of 1 unit. Solve the equation to find the exact value for the length. Then give a decimal approximation to 2 decimal places.

b. To create a golden rectangle with a width of 9 ft, what should be the length? Round to 1 decimal place.

58. An artist has been commissioned to make a stained glass window in the shape of a regular octagon. The octagon must fit inside an 18-in. square space. Determine the length of each side of the octagon. Round to the nearest hundredth of an inch.



59. A farmer has 160 yd of fencing material and wants to enclose three rectangular pens. Suppose that x represents the length of each pen and y represents the width as shown in the figure.



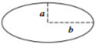
a. Assuming that the farmer uses all 160 yd of fencing, write an expression for y in terms of x.

b. Write an expression in terms of x for the area of one individual pen.

c. If the farmer wants to design the structure so that each pen encloses 250 yd², determine the dimensions of each pen.

60. At noon, a ship leaves a harbor and sails south at 10 knots. Two hours later, a second ship leaves the harbor and sails east at 15 knots. When will the ships be 100 nautical miles apart? Round to the nearest minute.

61. A gardener plans to install decorative tile around an oval-shaped pond that is twice as long as it is wide. The formula $\rho = 2\pi\sqrt{\frac{a^2+b^2}{2}}$ gives an approximation for the perimeter ρ of the pond, where a and b represent half the width and half the length of the oval, respectively.



If the gardener has enough tile for a perimeter of 30 ft,

a. Find the exact values of a and b.

b. Approximate the length and width of the pond to the nearest foot.

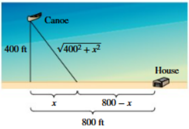
62. A catering company fills plastic cones with small gifts for the guests at a wedding reception. Each cone measures 12 in. across at the circular opening.

If each cone has a surface area of 190 in.^{2},

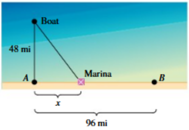
a. Find the exact height of the cone. Use the fact that the lateral surface area A of a right circular cone is given by the formula $A = \pi R\sqrt{R^2 + h^2}$, where R is the radius of the opening and h is the height of the cone.

b. Approximate the height of the cone to the nearest inch.

63. Bruny is in a canoe on a lake 400 ft from the closest point on a straight shoreline. Her house is 800 ft up the road along the shoreline. She can row 2.5 ft/sec and she can walk 5 ft/sec. If the total time it takes for her to get home is 5 min (300 sec), determine the point along the shoreline at which she landed her canoe.



64. A boat is in the ocean 48 mi from point A, the closest point along a straight shoreline. The captain plans to dock the boat at a marina x miles farther up the coast, and then drive along the coast to point B, 96 mi from point A. The average speed of the boat is 20 mph, and the average speed of the car is 60 mph. If the total trip takes 4 hr, determine the distance x along the shoreline.



Point of Interest

The relationship $d = rt$ is a familiar formula indicating that distance equals rate times time, or equivalently that $t = \frac{d}{r}$. However, suppose that a spaceship travels to a distant planet and then returns to Earth. Einstein's theory of special relativity indicates that $t = \frac{d}{r}$ only represents the trip's duration for an observer on Earth. For a person on the spaceship, the time will be shorter by a factor of $\sqrt{1 - \frac{r^2}{c^2}}$, where r is the speed of the spaceship and c is the speed of light.

For example, suppose that a spaceship travels to a planet 10 light-years away (a light-year is the distance that light travels in 1 yr) and then returns. The round trip is 20 light-years. Further suppose that the spaceship travels at half the speed of light, that is, $r = 0.5c$.

To an observer on Earth, the elapsed time of travel (in years) is

$$t_E = \frac{d}{r} = \frac{20}{0.5} = 40 \text{ yr}$$

To an observer on the spaceship, the elapsed time (in years) is

$$t_s = \frac{d}{r}\sqrt{1 - \frac{r^2}{c^2}} = \frac{20}{0.5}\sqrt{1 - \frac{(0.5c)^2}{c^2}} \approx 34.6 \text{ yr}$$

The unit of measurement in each case is years. So, the observer on Earth perceives the time of travel to be 40 yr, whereas to an observer on the spaceship the time of travel is only 34.6 yr.

65. a. NASA's Parker Solar Probe (launched in 2018) has a top speed of approximately 192 km/sec. If the speed of light, c, is approximately 300,000 km/sec, represent the top speed of the probe in terms of c.

b. One of the nearest stars outside our solar system is Alpha Centauri at approximately 4.3 light-years. If a spaceship travels at the top speed of the Parker Solar Probe, to an observer on Earth, how long (in years) would the journey to Alpha Centauri take? Do not round.

c. For an observer on the spaceship, how many years would appear to have passed? Do not round.

d. What is the difference in the perceived time of the journey between an observer on Earth versus an observer on the spaceship? Round to the nearest tenth of a day.

In 1994, Mexican theoretical physicist Miguel Alcubierre published a paper in *Classical and Quantum Gravity* that showed the mathematical possibility for space to be *warped* to achieve faster than light travel while still staying within the constraints of general relativity. Using this theoretical drive, the spaceship would create a "wave" in space-time by compressing space-time in front of the ship and expanding space-time behind. The ship itself would exist inside a "bubble" of space-time and surf the wave. However, this theoretical solution required material that had "negative mass." More recently (2021), another approach was proposed by Erik Lentz using *solitons* that would require dense matter, but not negative energy.

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Alcubierre, Miguel. "The Warp Drive: Hyper-Fast Travel Within General Relativity." *Classical and Quantum Gravity,* 1994.

Lentz, Erik. "Breaking the Warp Barrier: Hyper-Fast Solitons in Einstein-Maxwell-Plasma Theory." *Classical and Quantum Gravity,* 2021.

Further Reading

Chang, Kenneth. "Where's Our Warp Drive to the Stars?" *New York Times,* 11 November 2018.

Yirka, Bob. "A Potential Model for a Real, Physical Warp Drive." Phys.org, 4 March 2021. https://phys.org/news/2021-03-potential-real-physical-warp.html

Lesson R-8

Linear, Compound, and Absolute Value Inequalities

What Will You Learn?

After completing this lesson, you should be able to:

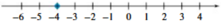
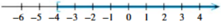
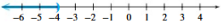
- Solve Linear Inequalities in One Variable
- Solve Compound Linear Inequalities
- Solve Absolute Value Inequalities
- Solve Applications of Inequalities

Learn: Solve Linear Inequalities in One Variable

Shanice wants to earn an “A” in her College Algebra course and knows that the average of her tests and assignments must be at least 90. She has five test grades of 96, 84, 80, 98, and 88. She also has a score of 100 for online homework, and this carries the same weight as a test grade. She still needs to take the final exam, and the final is weighted as two test grades. To determine the scores on the final exam that would result in an average of 90 or more, Shanice would solve the following inequality (see Example 9):

$$\frac{96 + 84 + 80 + 98 + 88 + 100 + 2x}{8} \geq 90, \text{ where } x \text{ is Shanice's score on the final.}$$

A linear equation in one variable is an equation that can be written as $ax + b = 0$, where a and b are real numbers and $a \neq 0$. A **linear inequality** is any relationship of the form $ax + b < 0$, $ax + b \leq 0$, $ax + b > 0$, or $ax + b \geq 0$. The solution set to a linear equation consists of a single element that can be represented by a point on the number line. The solution set to a linear inequality contains an infinite number of elements and can be expressed in set-builder notation or in interval notation.

Equation/ Inequality	Solution Set	Graph
$x + 4 = 0$	$\{-4\}$	
$x + 4 \geq 0$	$\{x \mid x \geq -4\}$ or $[-4, \infty)$	
$x + 4 < 0$	$\{x \mid x < -4\}$ or $(-\infty, -4)$	

To solve a linear inequality in one variable, we use the following properties of inequality.

Properties of Inequality

Let a, b, and c represent real numbers.

1. If $x < a$, then $a > x$.
2. If $a < b$ and $b < c$, then $a < c$.
3. If $a < b$ and $c < d$, then $a + c < b + d$.
4. If $a < b$, then $a + c < b + c$ and $a - c < b - c$.
5. If c is *positive* and $a < b$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
6. If c is *negative* and $a < b$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

These statements are also true expressed with the symbols $>$, $\leq$, and $\geq$.

Property 6 indicates that if both sides of an inequality are multiplied or divided by a negative number, then the direction of the inequality sign must be reversed.

Example 1: Solving a Linear Inequality

Solve the inequality. Graph the solution set and write the solution set in set-builder notation and in interval notation.

$$-6x + 4 < 34$$

Solution:

$-6x + 4 < 34$

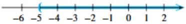
$-6x + 4 - 4 < 34 - 4$ Subtract 4 from both sides.

$-6x < 30$

$\frac{-6x}{-6} > \frac{30}{-6}$ Divide both sides by -6. Reverse the inequality sign.

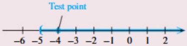
$x > -5$

The solution set is $\{x \mid x > -5\}$.
Interval notation: $(-5, \infty)$



Check Your Work

In Example 1, the solution set to the inequality $-6x + 4 < 34$ is $\{x \mid x > -5\}$. This means that all numbers greater than -5 make the inequality a true statement. You can check by taking an arbitrary test point from the interval $(-5, \infty)$. For example, the value $x = -4$ makes the original inequality true.



Check: $x = -4$

$-6(-4) + 4 \overset{?}{<} 34$

$24 + 4 \overset{?}{<} 34$ ✓ true

Apply the Skills

1. Solve the inequality. Graph the solution set and write the solution set in set-builder notation and in interval notation. $-5t - 6 \geq 24$

Example 2: Solving a Linear Inequality Containing Fractions

Solve the inequality. Graph the solution set and write the solution set in set-builder notation and in interval notation.

$$\frac{x+1}{3} - \frac{2x-4}{6} \leq -\frac{x}{2}$$

Solution:

$$\frac{x+1}{3} - \frac{2x-4}{6} \leq -\frac{x}{2}$$

$$6\left(\frac{x+1}{3} - \frac{2x-4}{6}\right) \leq 6\left(-\frac{x}{2}\right)$$

Multiply both sides by the LCD of 6 to clear fractions.

$$2(x+1) - (2x-4) \leq -3x$$

Apply the distributive property.

$$2x + 2 - 2x + 4 \leq -3x$$

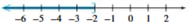
$$6 \leq -3x$$

$$\frac{6}{-3} \geq \frac{-3x}{-3}$$

Divide both sides by −3. Since −3 is a negative number, reverse the inequality sign.

$$-2 \geq x \quad \text{or} \quad x \leq -2$$

The solution set is $\{x \mid x \leq -2\}$.
Interval notation: $(-\infty, -2]$



Apply the Skills

2. Solve the inequality. Graph the solution set and write the solution set in set-builder notation and in interval notation.

$$\frac{m-4}{2} - \frac{3m+4}{10} > -\frac{3m}{5}$$

Learn: Solve Compound Linear Inequalities

In Examples 3–5, we solve **compound inequalities.** These are statements with two or more inequalities joined by the word “and” or the word “or.” For example, suppose that x represents the glucose level measured from a fasting blood sugar test.

- The normal glucose range is given by $x \geq 70$ mg/dL and $x \leq 100$ mg/dL.
- An abnormal glucose level is given by $x < 70$ mg/dL or $x > 100$ mg/dL.

To find the solution sets for compound inequalities, follow these guidelines.

Solving a Compound Inequality

Step 1 To solve a compound inequality, first solve the individual inequalities.

Step 2
- If two inequalities are joined by the word "and," the solutions are the values of the variable that simultaneously satisfy each inequality. That is, we take the *intersection* of the individual solution sets.
- If two inequalities are joined by the word "or," the solutions are the values of the variable that satisfy either inequality. Therefore, we take the *union* of the individual solution sets.

Example 3: Solving a Compound Inequality "Or"

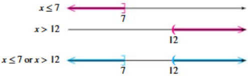
Solve. $x - 2 \leq 5$ or $\frac{1}{2}x > 6$

Solution:

$x - 2 \leq 5$ or $\frac{1}{2}x > 6$

$x \leq 7$ or $x > 12$

First solve the individual inequalities.
Then take the *union* of the individual solution sets.



The solution set is $\{x \mid x \leq 7 \text{ or } x > 12\}$.
Interval notation: $(-\infty, 7] \cup (12, \infty)$

Apply the Skills

3. Solve. $\frac{1}{4}y < -1$ or $3 + y \geq 5$

In Example 4, we solve a compound inequality in which the individual inequalities are joined by the word "and." In this case, we take the intersection of the individual solution sets.

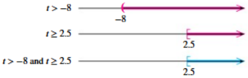
Example 4: Solving a Compound Inequality "And"

Solve. $-\frac{1}{4}t < 2$ and $0.52t \geq 1.3$

Solution:

$-\frac{1}{4}t < 2$ and $0.52t \geq 1.3$ First solve the individual inequalities. Multiply both sides of the first inequality by –4 (reverse the inequality sign). Divide the second inequality by 0.52.

$t > -8$ and $t \geq 2.5$ Take the *intersection* of the individual solution sets. The intervals overlap for values of t greater than or equal to 2.5.



The solution set is $\{t | t \geq 2.5\}$. Interval notation: $[2.5, \infty)$

Apply the Skills

4. Solve. $0.36w \leq 0.54$ and $-\frac{1}{2}w > 3$

Sometimes a compound inequality joined by the word "and" is written as a three-part inequality. For example:

$5 < -2x + 7$ and $-2x + 7 \leq 11$ In this example, two simultaneous conditions are imposed on the quantity $-2x + 7$.

$5 < -2x + 7 \leq 11$

To solve a three-part inequality, the goal is to isolate x in the middle region. This is demonstrated in Example 5.

Example 5: Solving a Three-Part Compound Inequality

Solve. $5 < -2x + 7 \leq 11$

Solution:

$$5 < -2x + 7 \leq 11$$

$$5 - 7 < -2x + 7 - 7 \leq 11 - 7$$

Subtract 7 from all three parts of the inequality.

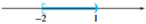
$$-2 < -2x \leq 4$$

$$\frac{-2}{-2} > \frac{-2x}{-2} \geq \frac{4}{-2}$$

Divide all three parts by –2.

$$1 > x \geq -2 \text{ or equivalently } -2 \leq x < 1.$$

The solution set is $\{x | -2 \leq x < 1\}$.
Interval notation: $[-2, 1)$

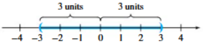
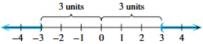


Apply the Skills

5. Solve. $-16 \le -3y - 4 < 2$

Learn: Solve Absolute Value Inequalities

We now investigate the solutions to absolute value inequalities. For example:

Inequality	**Graph**	**Solution Set**
$\|x\| < 3$		$\{x\|-3 < x < 3\}$
$\|x\| > 3$		$\{x\|x < -3 \text{ or } x > 3\}$

Insights

The solution set to $|x| < 3$ is the set of real numbers within 3 units of zero on the number line.

The solution set to $|x| > 3$ is the set of real numbers more than 3 units from zero on the number line.

We can generalize these observations with the following properties involving absolute value inequalities.

Properties Involving Absolute Value Inequalities

For a real number $k > 0$,

1. $|u| < k$ is equivalent to $-k < u < k$. (1)

2. $|u| > k$ is equivalent to $u < -k$ or $u > k$. (2)

Note: The statements also hold true for the inequality symbols $\le$ and $\ge$, respectively.

Properties (1) and (2) follow directly from the definition: $|u| = \begin{cases} u & \text{if } u \ge 0 \\ -u & \text{if } u < 0 \end{cases}$

By definition, $|u| < k$ is equivalent to

$$u < k \quad \text{and} \quad -u < k$$
$$u < k \quad \text{and} \quad u > -k$$
$$-k < u < k \qquad (1)$$

By definition, $|u| > k$ is equivalent to

$$u > k \quad \text{or} \quad -u > k$$
$$u > k \quad \text{or} \quad u < -k$$
$$u < -k \quad \text{or} \quad u > k \qquad (2)$$

Example 6: Solving an Absolute Value Inequality

Solve the inequality and write the solution set in interval notation.

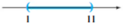
$$2|6-m|-3<7$$

Solution:

$2\lvert 6-m\rvert-3<7$	First isolate the absolute value. Add 3 and divide by 2.
$\lvert 6-m\rvert<5$	The inequality is in the form $\lvert u\rvert<k$, where $u=6-m$.
$-5<6-m<5$	Write the equivalent compound inequality $-k<u<k$.
$-11<-m<-1$	Subtract 6 from all three parts.
$\frac{-11}{-1}>\frac{-m}{-1}>\frac{-1}{-1}$	Divide by -1 and reverse the inequality signs.

$11>m>1$ or equivalently $1<m<11$.

The solution set is $\{m|1<m<11\}$.
Interval notation: $(1, 11)$



Apply the Skills

6. Solve the inequality and write the solution set in interval notation.

$$3|5-x|+2\le 14$$

Example 7: Solving an Absolute Value Inequality

Solve the inequality and write the solution set in interval notation.

$$-4\ge -2|3x+1|$$

Solution:

$-4\ge-2\lvert 3x+1\rvert$	First isolate the absolute value.
$\frac{-4}{-2}\le\frac{-2\lvert 3x+1\rvert}{-2}$	Divide both sides by -2 and reverse the inequality sign.
$2\le\lvert 3x+1\rvert$	
$\lvert 3x+1\rvert\ge 2$	Write the absolute value on the left. Notice that the direction of the inequality sign is also changed. The inequality is now in the form $\lvert u\rvert\ge k$, where $u=3x+1$.
$3x+1\le-2$ or $3x+1\ge 2$	Write the equivalent form $u\le-k$ or $u\ge k$.
$x\le-1$ or $x\ge\frac{1}{3}$	Take the union of the solution sets of the individual inequalities.

The solution set is $\left\{x\middle|x\le-1 \text{ or } x\ge\frac{1}{3}\right\}$.

Interval notation: $(-\infty,-1]\cup\left[\frac{1}{3},\infty\right)$

Apply the Skills

7. Solve the inequality and write the solution set in interval notation.

$$-18 > -3|2y - 4|$$

An absolute value equation such as $|7x - 3| = -6$ has no solution because an absolute value cannot be equal to a negative number. We must also exercise caution when an absolute value is compared to a negative number or zero within an inequality. This is demonstrated in Example 8.

Example 8: Solving Absolute Value Inequalities with Special Case Solution Sets

Solve the inequality and write the solution set in interval notation where appropriate.

a. $|x + 2| < -4$

b. $|x + 2| \geq -4$

c. $|x - 5| \leq 0$

d. $|x - 5| > 0$

Solution:

a. $|x + 2| < -4$

The solution set is { }.

By definition an absolute value is greater than or equal to zero. Therefore, the absolute value of an expression cannot be less than zero or any negative number. This inequality has no solution.

b. $|x + 2| \geq -4$

The solution set is $\mathbb{R}$.

Interval notation: $(-\infty, \infty)$

An absolute value of any real number is greater than or equal to zero. Therefore, it is also greater than every negative number. This inequality is true for all real numbers, x.

c. $|x - 5| \leq 0$

$|x - 5| = 0$

$x - 5 = 0$

$x = 5$

The absolute value of x minus 5 cannot be less than zero, but it can be *equal* to zero.

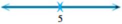
The solution set is $\{5\}$.

d. $|x - 5| > 0$

$|x - 5| > 0$ for all values of x except 5. When $x = 5$, we have $|5 - 5| = 0$, and this is not greater than zero. The solution set is all real numbers excluding 5.

The solution set is $\{x \mid x < 5 \text{ or } x > 5\}$.

Interval notation: $(-\infty, 5) \cup (5, \infty)$



Apply the Skills

8. Solve the inequality and write the solution set in interval notation where appropriate.

a. $|x - 3| < -2$

b. $|x - 3| > -2$

c. $|x + 1| \leq 0$

d. $|x + 1| > 0$

Learn: Solve Applications of Inequalities

In Example 9, we use a linear inequality to solve an application.

Example 9: Using a Linear Inequality in an Application of Grades

Shanice has test scores of 96, 84, 80, 98, and 88. Her score for online homework is 100 and is weighted as one test grade. Shanice still needs to take the final exam, which counts as two test grades. What score does she need on the final exam to have an average of at least 90? (This is the minimum average to earn an "A" in the class.)

Solution:

Let x represent the grade needed on the final exam.

$\begin{pmatrix}\text{Average of}\\ \text{all scores}\end{pmatrix} \geq 90$	To earn an "A," Shanice's average must be at least 90.
$\dfrac{96 + 84 + 80 + 98 + 88 + 100 + 2x}{8} \geq 90$	Take the sum of all grades. Divide by a total of eight grades.
$\dfrac{546 + 2x}{8} \geq 90$	
$8\left(\dfrac{546 + 2x}{8}\right) \geq 8(90)$	Multiply by 8 to clear fractions.
$546 + 2x \geq 720$	
$2x \geq 174$	Subtract 546 from both sides.
$x \geq 87$	Divide by 2.

Shanice must earn a score of at least 87 to earn an "A" in the class.

Apply the Skills

9. For a recent year, the monthly snowfall (in inches) for Chicago, Illinois, for November, December, January, and February was 2, 8.4, 11.2, and 7.9, respectively. How much snow would be necessary in March for Chicago to exceed its monthly average snowfall of 7.28 in. for these five months?

Practice R-8

Practice Exercises

Concept Connections

1. The multiplication and division properties of inequality indicate that if both sides of an inequality are multiplied or divided by a negative real number, the direction of the ________ sign must be reversed.
2. If a compound inequality consists of two inequalities joined by the word "and," the solution set is the ________ of the solution sets of the individual inequalities.
3. The compound inequality $a < x$ and $x < b$ can be written as the three-part inequality ________.
4. If a compound inequality consists of two inequalities joined by the word "or," the solution set is the ________ of the solution sets of the individual inequalities.
5. If k is a positive real number, then the inequality $|x| < k$ is equivalent to ________ $< x <$ ________.
6. If k is a positive real number, then the inequality $|x| > k$ is equivalent to $x <$ ________ or x ________ k.
7. If k is a positive real number, then the solution set to the inequality $|x| > -k$ is ________.
8. If k is a positive real number, then the solution set to the inequality $|x| < -k$ is ________.

Learn: Solve Linear Inequalities in One Variable

For Exercises 9–26, solve the inequality. Graph the solution set, and write the solution set in set-builder notation and interval notation. (See Examples 1–2, pp. R133–R134)

9. $-2x - 5 > 17$
10. $-8t + 1 < 17$
11. $-3 \le -\frac{4}{3}w + 1$
12. $8 \ge -\frac{5}{2}y - 2$
13. $-1.2 + 0.6a \le 0.4a + 0.5$
14. $-0.7 + 0.3x \le 0.9x - 0.4$
15. $-5 > 6(c - 4) + 7$
16. $-14 < 3(m - 7) + 7$
17. $\frac{4 + x}{2} - \frac{x - 3}{5} < -\frac{x}{10}$
18. $\frac{y + 3}{4} - \frac{3y + 1}{6} > -\frac{1}{12}$
19. $\frac{1}{3}(x + 4) - \frac{5}{6}(x - 3) \ge \frac{1}{2}x + 1$
20. $\frac{1}{2}(t - 6) - \frac{4}{3}(t + 2) \ge -\frac{3}{4}t - 2$
21. $5(7 - x) + 2x < 6x - 2 - 9x$
22. $2(3x + 1) - 4x > 2(x + 8) - 5$
23. $5 - 3[2 - 4(x - 2)] \ge 6\{2 - [4 - (x - 3)]\}$
24. $8 - [6 - 10(x - 1)] \ge 2\{1 - 3[2 - (x + 4)]\}$
25. $4 - 3k > -2(k + 3) - k$
26. $2x - 9 < 6(x - 1) - 4x$

Learn: Solve Compound Linear Inequalities

For Exercises 27–34, solve the compound inequality. Graph the solution set, and write the solution set in interval notation. (See Examples 3–4 on pp. R135–R136)

27. a. $x < 4$ and $x \ge -2$
    b. $x < 4$ or $x \ge -2$
28. a. $y \le -2$ and $y > -5$
    b. $y \le -2$ or $y > -5$
29. a. $m + 1 \le 6$ or $\frac{1}{3}m < -2$
    b. $m + 1 \le 6$ and $\frac{1}{3}m < -2$
30. a. $n - 6 > 1$ or $\frac{3}{4}n \ge 6$
    b. $n - 6 > 1$ and $\frac{3}{4}n \ge 6$
31. a. $-\frac{2}{3}y > -12$ and $2.08 \ge 0.65y$
    b. $-\frac{2}{3}y > -12$ or $2.08 \ge 0.65y$
32. a. $-\frac{4}{5}m < 8$ and $0.85 \le 0.34m$
    b. $-\frac{4}{5}m < 8$ or $0.85 \le 0.34m$

33. a. $3(x-2)+2 \le x-8$ or
$4(x+1)+2 > -2x+4$

b. $3(x-2)+2 \le x-8$ and
$4(x+1)+2 > -2x+4$

34. a. $5(t-4)+2 > 3(t+1)-3$ or
$2t-6 > 3(t-4)-2$

b. $5(t-4)+2 > 3(t+1)-3$ and
$2t-6 > 3(t-4)-2$

35. Write $-2.8 < y \le 15$ as two separate inequalities joined by "and."

36. Write $-\frac{1}{2} \le z < 2.4$ as two separate inequalities joined by "and."

For Exercises 37–42, solve the compound inequality. Graph the solution set, and write the solution set in interval notation. (See Example 5 on p. R136)

37. $-3 < -2x+1 \le 9$

38. $-6 \le -3x+9 < 0$

39. $1 \le \frac{5x-4}{2} < 3$

40. $-2 \le \frac{4x-1}{3} \le 5$

41. $-2 \le \frac{-2x+1}{-3} \le 4$

42. $-4 < \frac{-5x-2}{-2} < 4$

Learn: Solve Absolute Value Inequalities

For Exercises 43–46, solve the equation or inequality. Write the solution set to each inequality in interval notation.

43. a. $|x| = 7$

b. $|x| < 7$

c. $|x| > 7$

44. a. $|y| = 8$

b. $|y| < 8$

c. $|y| > 8$

45. a. $|a+9|+2 = 6$

b. $|a+9|+2 \le 6$

c. $|a+9|+2 \ge 6$

46. a. $|b+1|-4 = 1$

b. $|b+1|-4 \le 1$

c. $|b+1|-4 \ge 1$

For Exercises 47–60, solve the inequality, and write the solution set in interval notation if possible. (See Examples 6–7 on p. R138)

47. $3|4-x|-2 < 16$

48. $2|7-y|+1 < 17$

49. $2|x+3|-4 \ge 6$

50. $5|x+1|-9 \ge -4$

51. $|4w-5|+6 \le 2$

52. $|2x+7|+5 < 1$

53. $|5-p|+13 > 6$

54. $|12-7x|+5 \ge 4$

55. $-11 \le 5-|2p+4|$

56. $-18 \le 6-|3z+3|$

57. $10 < |-5c-4|+2$

58. $15 < |-2d-3|+6$

59. $\left|\frac{y+3}{6}\right| < 2$

60. $\left|\frac{m-4}{2}\right| < 14$

For Exercises 61–68, write the solution set. (See Example 8 on p. R139)

61. a. $|x| = -9$

b. $|x| < -9$

c. $|x| > -9$

62. a. $|y| = -2$

b. $|y| < -2$

c. $|y| > -2$

63. a. $18 = 4-|y+7|$

b. $18 \le 4-|y+7|$

c. $18 \ge 4-|y+7|$

64. a. $15 = 2-|p-3|$

b. $15 \le 2-|p-3|$

c. $15 \ge 2-|p-3|$

65. a. $|z| = 0$

b. $|z| < 0$

c. $|z| \leq 0$

d. $|z| > 0$

e. $|z| \geq 0$

66. a. $|2w| = 0$

b. $|2w| < 0$

c. $|2w| \leq 0$

d. $|2w| > 0$

e. $|2w| \geq 0$

67. a. $|k + 4| = 0$

b. $|k + 4| < 0$

c. $|k + 4| \leq 0$

d. $|k + 4| > 0$

e. $|k + 4| \geq 0$

68. a. $|c - 3| = 0$

b. $|c - 3| < 0$

c. $|c - 3| \leq 0$

d. $|c - 3| > 0$

e. $|c - 3| \geq 0$

Learn: Solve Applications of Inequalities

69. Marilee wants to earn an "A" in a class and needs an overall average of at least 92. Her test grades are 88, 92, 100, and 80. The average of her quizzes is 90 and counts as one test grade. The final exam counts as 2.5 test grades. What score on the final exam would result in Marilee's overall average being 92 or greater? (**See Example 9 on p. R140**)

70. A 10-yr-old competes in gymnastics. For several competitions she received the following "All-Around" scores: 36, 36.9, 37.1, and 37.4. Her coach recommends that gymnasts whose "All-Around" scores average at least 37 move up to the next level. What "All-Around" scores in the next competition would result in the child being eligible to move up?

71. A student earns scores of 78, 82, 90, 80, and 75 on five chapter tests for a certain class and a grade of 85 on the class project. The overall average for the course is computed as follows: the average of the five chapter tests makes up 60% of the course grade; the project accounts for 10% of the grade; and the final exam accounts for 30%. What scores can the student earn on the final exam to earn a "B" in the course if the cut-off for a "B" is an overall score greater than or equal to 80, but less than 90? Assume that 100 is the highest score that can be earned on the final exam and that only whole-number scores are given.

72. Trent earns scores of 66, 84, and 72 on three chapter tests for a certain class. His homework grade is 60, and his grade for a class project is 85. The overall average for the course is computed as follows: the average of the three chapter tests makes up 50% of the course grade; homework accounts for 20% of the grade; the project accounts for 10%; and the final exam accounts for 20%. What scores can Trent earn on the final exam to pass the course if he needs a "C" or better? A "C" or better requires an overall score of 70 or better, and 100 is the highest score that can be earned on the final exam. Assume that only whole-number scores are given.

73. A car travels 50 mph and passes a truck traveling 40 mph. How long will it take the car to be more than 16 mi ahead?

74. A rectangular garden is to be constructed so that the width is 100 ft. What are the possible values for the length of the garden if at most 800 ft of fencing is to be used?

75. For a certain bowling league, a bowler's handicap is found by taking 90% of the difference between 220 and the bowler's average score in league play. Determine the average scores that would produce a handicap of 72 or less. Also assume that a negative handicap is not possible in this league.

76. Body temperature is usually maintained between 36.5°C and 37.5°C, inclusive. Determine the corresponding range of temperatures in Fahrenheit. Use the relationship between degrees Celsius *C* and degrees Fahrenheit *F*:

$$C = \frac{5}{9}(F - 32).$$

77. Donovan has offers for two sales jobs. Job A pays a base salary of \$25,000 plus a 10% commission on sales. Job B pays a base salary of \$30,000 plus 8% commission on sales.

a. How much would Donovan have to sell for the total earnings from Job A to exceed the total earnings from Job B?

b. If Donovan routinely sells more than \$500,000 in merchandise, which job would result in higher total earnings?

78. Aurora wants to vacation in Viña del Mar, Chile. Hotel A charges \$179 per night with a 14% nightly room tax and free parking. Hotel B charges \$169 per night with an 18% nightly room tax plus a one-time \$40 parking fee. After how many nights will Hotel B be less expensive? After 8 nights (9 or more), Hotel B will be less expensive.

Absolute value inequalities are often used to represent measurement error. For example, suppose that a machine is calibrated to dispense 8 fl oz of orange juice with a measurement error of no more than 0.05 fl oz. If *x* represents the actual amount of orange juice poured into the bottle, then *x* is a solution to the inequality $|x - 8| \le 0.05$. For Exercises 79–82,

a. Write an absolute value inequality to represent each statement.

b. Solve the inequality. Write the solution set in interval notation.

79. The variation between the measured value *v* and 16 oz is less than 0.01 oz.

80. The variation between the measured value *t* and 60 min is less than 0.2 min.

81. The value of *x* differs from 4 by more than 1 unit.

82. The value of *y* differs from 10 by more than 2 units.

83. A refrigerator manufacturer recommends that the temperature *t* (in °F) inside a refrigerator be 36.5°F. If the thermostat has a margin of error of no more than 1.5°F,

a. Write an absolute value inequality that represents an interval in which to estimate *t*.

b. Solve the inequality and interpret the answer.

84. A box of cereal is labeled to contain 16 oz. A consumer group takes a sample of 50 boxes and measures the contents of each box. The individual content of each box differs slightly from 16 oz, but by no more than 0.5 oz.

a. If *x* represents the exact weight of the contents of a box of cereal, write an absolute value inequality that represents an interval in which to estimate *x*.

b. Solve the inequality and interpret the answer.

85. The results of a political poll indicate that the leading candidate will receive 51% of the votes with a margin of error of no more than 3%. Let *x* represent the true percentage of votes received by this candidate.

a. Write an absolute value inequality that represents an interval in which to estimate *x*.

b. Solve the inequality and interpret the answer.

86. A police officer uses a radar detector to determine that a motorist is traveling 34 mph in a 25-mph school zone. The driver goes to court and argues that the radar detector is not accurate. The manufacturer claims that the radar detector is calibrated to be in error by no more than 3 mph.

a. If *x* represents the motorist's actual speed, write an inequality that represents an interval in which to estimate *x*.

b. Solve the inequality and interpret the answer. Should the motorist receive a speeding ticket?

87. For one of the variants of the COVID-19 virus, scientists estimated, with 95% confidence, the mean incubation period (time between exposure and showing symptoms) to be 5.4 days with a margin of error of less than 0.6 days. Let x represent the true mean incubation period.

- **a.** Write an absolute value inequality that represents an interval in which to estimate x.
- **b.** Solve the inequality from part (a). Write the solution set in interval notation and interpret the answer.

88. The infectious period for a patient with COVID-19 varies significantly from patient to patient and according to the severity of the case. One study approximated that, with 95% confidence, the mean time from symptom onset to two consecutive negative COVID tests was 13.4 days with a margin of error of less than 2.5 days. Let x represent the true mean time from symptom onset to two consecutive negative tests.

- **a.** Write an absolute value inequality that represents an interval in which to estimate x.
- **b.** Solve the inequality from part (a). Write the solution set in interval notation, and interpret the answer.

Mixed Exercises

For Exercises 89–106, solve the inequalities and compound inequalities. Write the solution set in interval notation or indicate that the solution set is the empty set.

89. $-1.2b - 0.4 \geq -0.4b$

90. $-0.4t + 1.2 < -2$

91. $-\frac{3}{4}c - \frac{5}{4} \geq 2c$

92. $-\frac{2}{3}q - \frac{1}{3} > \frac{1}{2}q$

93. $\frac{2}{3}(2p - 1) \geq 10$ and $\frac{4}{5}(3p + 4) \geq 20$

94. $\frac{5}{2}(a + 2) < -6$ and $\frac{3}{4}(a - 2) < 1$

95. $-2 < -x - 12$ and $-14 < 5(x - 3) + 6x$

96. $-8 \geq -3y - 2$ and $3(y - 7) + 16 > 4y$

97. $5(x - 1) \geq -5$ or $5 - x \leq 11$

98. $-p + 7 \geq 10$ or $3(p - 1) \leq 12$

99. $\frac{5}{3}v \leq 5$ or $-v - 6 < 1$

100. $\frac{3}{8}u + 1 > 0$ or $-2u \geq -4$

101. $-4 \leq \frac{2 - 4x}{3} < 8$

102. $-1 < \frac{3 - x}{2} \leq 0$

103. $\frac{-x + 3}{2} > \frac{4 + x}{5}$ or $\frac{1 - x}{4} > \frac{2 - x}{3}$

104. $\frac{y - 7}{-3} < \frac{1}{4}$ or $\frac{y + 1}{-2} > -\frac{1}{3}$

105. **a.** $\left|\frac{x - 1}{2} - 3\left(\frac{x + 5}{4}\right)\right| \leq 0$

b. $\left|\frac{x - 1}{2} - 3\left(\frac{x + 5}{4}\right)\right| \geq 2$

c. $\left|\frac{x - 1}{2} - 3\left(\frac{x + 5}{4}\right)\right| \geq -2$

106. **a.** $\left|\frac{2 - x}{9} + \frac{5 - x}{3}\right| < 1$

b. $\left|\frac{2 - x}{9} + \frac{5 - x}{3}\right| < 0$

c. $\left|\frac{2 - x}{9} + \frac{5 - x}{3}\right| \leq 0$

For Exercises 107–112, determine the set of values for x for which the radical expression would produce a real number. For example, the expression $\sqrt{x - 1}$ is a real number if $x - 1 \geq 0$ or equivalently, $x \geq 1$.

107. **a.** $\sqrt{x - 2}$ **b.** $\sqrt{2 - x}$

108. **a.** $\sqrt{x - 6}$ **b.** $\sqrt{6 - x}$

109. **a.** $\sqrt{x + 4}$ **b.** $\sqrt[3]{x + 4}$

110. **a.** $\sqrt{x + 7}$ **b.** $\sqrt[3]{x + 7}$

111. **a.** $\sqrt{2x - 9}$ **b.** $\sqrt[4]{2x - 9}$

112. **a.** $\sqrt{3x - 7}$ **b.** $\sqrt[4]{3x - 7}$

For Exercises 113–116, answer true or false given that $a > 0$, $b < 0$, $c > 0$, and $d < 0$.

113. $cd > a$

114. $ab < c$

115. If $a > c$, then $ad < cd$.

116. If $a < c$, then $ab < bc$.

Write About It

117. How is the process to solve a linear inequality different from the process to solve a linear equation?

118. Explain why $8 < x < 2$ has no solution.

119. Explain the difference between the solution sets for the following inequalities:

$|x - 3| \le 0$ and $|x - 3| > 0$

120. Explain why $x^2 = 4$ is equivalent to the equation $|x| = 2$.

Expanding Your Skills

For Exercises 121–128, solve the inequality and write the solution set in interval notation.

121. $|x| + x < 11$ (*Hint*: Use the definition of $|x|$ to consider two cases.)

Case 1: $x + x < 11$ if $x \ge 0$.
Case 2: $-x + x < 11$ if $x < 0$.

122. $|x| - x > 10$

123. $|x - 1| + 2x < 8$

124. $5x - |x + 2| > 6$

125. $1 < |x| < 9$

126. $2 < |y| < 11$

127. $5 \le |2x + 1| \le 7$

128. $7 \le |3x - 5| \le 13$

129. Solve the inequality for p: $|p - p| < z\sqrt{\frac{pq}{n}}$. (Do not rationalize the denominator.

130. Solve the inequality for μ: $|\mu - x| < \frac{z\sigma}{\sqrt{n}}$. (Do not rationalize the denominator.)

Problem Recognition Exercises

Recognizing and Solving Equations and Inequalities

For Exercises 1–20,

a. Identify the type of equation or inequality (some may fit more than one category).

b. Solve the equation or inequality. Write the solution sets to the inequalities in interval notation if possible.

- Linear equation or inequality
- Radical equation
- Quadratic equation
- Equation in quadratic form
- Rational equation
- Polynomial equation (degree > 2)
- Absolute value equation or inequality
- Compound inequality

1. $(x^2 - 5)^2 - 5(x^2 - 5) + 4 = 0$

2. $2 \le |3t - 1| - 6$

3. $\sqrt[3]{2y - 5} - 4 = -1$

4. $-9|3z - 7| + 1 = 4$

5. $\frac{2}{w - 3} + \frac{5}{w + 1} = 1$

6. $48x^3 + 80x^2 - 3x - 5 = 0$

7. $-2(m + 2) < -m + 5$ and $6 \ge m + 3$

8. $6 \le -2c + 8$ or $\frac{1}{3}c - 2 < 2$

9. $(2p + 1)(p + 5) = 2p + 40$

10. $2x(x - 4) + 7 = 2x^2 - 3[x + 5 - (2 + x)]$

11. $\frac{a - 4}{2} - \frac{3a + 1}{4} \le -\frac{a}{8}$

12. $3x^2 + 11 = 4$

13. $-1 \le \frac{6 - x}{-5} \le 7$

14. $5 = \sqrt{5 + 2n} + \sqrt{2 + n}$

15. $|4x - 5| = |3x - 2|$

16. $\frac{1}{d} - \frac{1}{2d - 1} + \frac{2d}{2d - 1} = 0$

17. $-|x + 4| + 8 > 3$

18. $y - 4\sqrt{y} - 12 = 0$

19. $c^{2/3} = 16$

20. $2|z - 14| + 8 > 4$

Algebra for Calculus

For Exercises 1–2, write an inequality using an absolute value that represents the given condition.

1. The distance between y and L is less than ε (epsilon).

2. The distance between x and c is less than δ (delta).

3. Simplify $\frac{x+8}{|x+8|}$

a. for $x > -8$. **b.** for $x < -8$.

4. Simplify $\frac{14-x}{|x-14|}$

a. for $x > 14$. **b.** for $x < 14$.

For Exercises 5–10,

a. Simplify the expression.

b. Substitute 0 for h in the simplified expression.

5. $\frac{2(x+h)^2 + 3(x+h) - (2x^2 + 3x)}{h}$

6. $\frac{3(x+h)^2 - 4(x+h) - (3x^2 - 4x)}{h}$

7. $\frac{\frac{1}{(x+h)-2} - \frac{1}{x-2}}{h}$

8. $\frac{\frac{1}{2(x+h)+5} - \frac{1}{2x+5}}{h}$

9. $\frac{(x+h)^3 - x^3}{h}$

10. $\frac{(x+h)^4 - x^4}{h}$

For Exercises 11–12,

a. Rationalize the numerator of the expression and simplify.

b. Substitute 0 for h in the simplified expression.

11. $\frac{\sqrt{x+h} + 1 - (\sqrt{x} + 1)}{h}$

12. $\frac{\sqrt{2(x+h)} - \sqrt{2x}}{h}$

For Exercises 13–22, factor completely and write the answer with no negative exponents. Do not rationalize the denominator.

13. $\frac{3}{2}x^{1/2} + \frac{5}{2}x^{3/2}$

14. $\frac{7}{6}x^{1/6} - \frac{1}{6}x^{-5/6}$

15. $4(3x+1)^3(3)(x^2+2)^3 + (3x+1)^4(3)(x^2+2)^2(2x)$

16. $3(-2x+3)^2(-2)(4x^2-5)^2 +$
$(-2x+3)^3(2)(4x^2-5)(8x)$

17. $\frac{6(t-1)^5(2t+5)^6 - 6(2t+5)^5(2)(t-1)^6}{[(2t+5)^6]^2}$

18. $\frac{6x^5(x^2+4)^3 - 3(x^2+4)^2(2x)x^6}{[(x^2+4)^3]^2}$

19. $(x^2+4)^{1/2} + x \cdot \frac{1}{2}(x^2+4)^{-1/2}(2x)$

20. $(2-x^2)^{1/2} + x \cdot \frac{1}{2}(2-x^2)^{-1/2}(-2x)$

21. $(x^2+1)^{-1/2} + x\left(-\frac{1}{2}\right)(x^2+1)^{-3/2}(2x)$

22. $6(3x-1)^{1/3} + 6x\left(-\frac{1}{3}\right)(3x-1)^{-2/3}(3)$

For Exercises 23–28, write the answer as a single term and simplify. It is not necessary to rationalize the denominator.

23. $\frac{2\sqrt{x+4} - \frac{x}{\sqrt{x+4}}}{(\sqrt{x+4})^2}$

24. $\frac{2x\sqrt{16-x^2} + \frac{x^3}{\sqrt{16-x^2}}}{\left(\sqrt{16-x^2}\right)^2}$

25. $(x-4)^{1/3} + \frac{x}{3(x-4)^{2/3}}$

26. $(x+5)^{1/4} + \frac{x}{4(x+5)^{3/4}}$

27. $\frac{1}{2}\left(\frac{2x}{x+1}\right)^{-1/2}\left[\frac{(x+1)(2) - 2x(1)}{(x+1)^2}\right]$

28. $\frac{1}{3}\left(\frac{3x}{x^2+1}\right)^{-2/3}\left[\frac{(x^2+1)\cdot 3 - 3x(2x)}{(x^2+1)^2}\right]$

Equations and Inequalities for Calculus

In Calculus you will see the symbol y'. For Exercises 1–4, treat y' as a variable and solve the equation for y'.

1. $\frac{2x}{25} + \frac{2y}{9}y' = 0$
2. $2xy^3 + 3x^2y^2y' - y' = 1$
3. $3y^2y' + 6xy + 3x^2y' = 2y^2 + 4xyy'$
4. $3(x + y)^2 + 3(x + y)^2\,y' - 3y^2y' = 3x^2$

For Exercises 5–7, simplify the expression. Do not rationalize the denominator.

5. $2x\sqrt{2x-3} + x^2\left(\frac{1}{2}\right)\frac{1}{\sqrt{2x-3}}(2)$
6. $\frac{2x(2x-7)^{1/2} - x^2\left(\frac{1}{2}\right)(2x-7)^{-1/2}(2)}{[(2x-7)^{1/2}]^2}$
7. $\frac{(1)(x^2-9)^{1/2} - x\left(\frac{1}{2}\right)(x^2-9)^{-1/2}(2x)}{[(x^2-9)^{1/2}]^2}$

For Exercises 8–10,

a. Simplify the expression. Do not rationalize the denominator.

b. Find the values of x for which the expression equals zero.

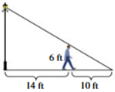
c. Find the values of x for which the denominator is zero.

8. $\frac{4x(4x-5) - 2x^2(4)}{(4x-5)^2}$
9. $\frac{-6x(6x+1) - (-3x^2)(6)}{(6x+1)^2}$
10. $\sqrt{4-x^2} - x\left(\frac{1}{2}\right)\frac{1}{\sqrt{4-x^2}}(2x)$

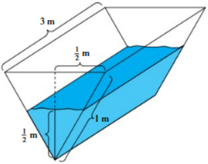
Some applications of calculus use a mathematical structure called a power series. To find the interval of convergence of a power series, it is often necessary to solve an absolute value inequality. For Exercises 11–12, solve the absolute value inequality to find the interval of convergence.

11. $\left|\frac{x+1}{2}\right| < 1$
12. $\left|-\frac{x}{2}\right| < 1$

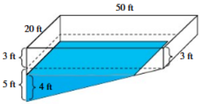
13. A person 6 ft tall walks away from a lamppost. At the instant the person is 14 ft away from the lamppost, the shadow is 10 ft. Find the height of the lamppost.



14. A water trough has a cross section in the shape of an equilateral triangle with sides of length 1 m. The length is 3 m. Determine the volume of water when the water level is $\frac{1}{2}$ m.



15. A contractor builds a swimming pool with cross section in the shape of a trapezoid. The deep end is 8 ft deep and the shallow end is 3 ft deep. The length of the pool is 50 ft and the width is 20 ft. As the pool is being filled, find the volume of water when the depth is 4 ft.



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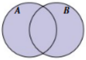
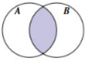
Chapter R Review

Chapter R Key Concepts

Lesson R-1 Sets and the Real Number Line

Key Concepts		Reference
Natural Numbers: N = {1, 2, 3, ...} **Whole Numbers:** W = {0, 1, 2, 3, ...} **Integers:** Z = {..., −3, −2, −1, 0, 1, 2, 3, ...} **Rational Numbers:** $Q = \left\{ \frac{p}{q} \middle\| p, q \in Z \text{ and } q \neq 0 \right\}$ **Irrational Numbers:** H is the set of real numbers that cannot be expressed as a ratio of integers.		p. R5
$A \cup B$ is the **union** of A and B. This is the set of elements that belong to set A or set B or to both sets A and B.	 $A \cup B$	p. R8
$A \cap B$ is the **intersection** of A and B. This is the set of elements common to both A and B.	 $A \cap B$	
For a real number x, the **absolute value of** x is $\|x\| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$		p. R11
The distance between two points a and b on a number line is given by $\|a - b\|$ or $\|b - a\|$.		p. R12
• Commutative property of addition	$a + b = b + a$	p. R15
• Commutative property of multiplication	$a \cdot b = b \cdot a$	
• Associative property of addition	$(a + b) + c = a + (b + c)$	
• Associative property of multiplication	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$	
• Identity property of addition The number 0 is the identity element of addition.	$a + 0 = a$ and $0 + a = a$	
• Identity property of multiplication The number 1 is the identity element of multiplication.	$a \cdot 1 = a$ and $1 \cdot a = a$	
• Inverse property of addition	$a + (-a) = 0$ and $(-a) + a = 0$	
• Inverse property of multiplication	$a \cdot \frac{1}{a} = 1$ and $\frac{1}{a} \cdot a = 1$ where $a \neq 0$	
• Distributive property of multiplication over addition	$a(b + c) = ab + ac$	

Lesson R-2 Exponents and Radicals

Key Concepts	Reference
Properties of exponents and key definitions: $b^m \cdot b^n = b^{m+n}$ $(b^m)^n = b^{m \cdot n}$ $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $b^{-n} = \frac{1}{b^n}$ $\frac{b^m}{b^n} = b^{m-n}$ $(ab)^m = a^m b^m$ $b^0 = 1$	pp. R25–R26
A number expressed in the form $a \times 10^n$, where $1 \leq \lvert a \rvert < 10$ and n is an integer is said to be in **scientific notation**.	p. R28
b is an nth-root of a if $b^n = a$. $\sqrt[n]{a}$ represents the principal nth-root of a.	p. R29
If $n > 1$ is an integer and $\sqrt[n]{a}$ is a real number, then • $a^{1/n} = \sqrt[n]{a}$ • $a^{m/n} = (\sqrt[n]{a})^m$ and $a^{m/n} = \sqrt[n]{a^m}$	p. R30
Let $n > 1$ be an integer and a be a real number. • If n is *even*, then $\sqrt[n]{a^n} = \lvert a \rvert$. • If n is *odd*, then $\sqrt[n]{a^n} = \lvert a \rvert$. Product property of radicals: $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ Quotient property of radicals: $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$ Property of nested radicals: $\sqrt[m]{\sqrt[n]{a}} = \sqrt[m \cdot n]{a}$	p. R32

Lesson R-3 Polynomials and Factoring

Key Concepts	Reference
Special Case Products: $(a + b)(a - b) = a^2 - b^2$ $(a + b)^2 = a^2 + 2ab + b^2$ $(a - b)^2 = a^2 - 2ab + b^2$	p. R47
General factoring strategy: **1.** Factor out the GCF. **2.** Identify the number of terms in the polynomial. 4 terms: Factor by grouping either 2 terms with 2 terms or 3 terms with 1 term. 3 terms: If the trinomial is a perfect square trinomial, factor as the square of a binomial. $a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$ Otherwise, factor by the trial-and-error method. 2 terms: Determine whether the binomial fits one of the following patterns. $a^2 - b^2 = (a + b)(a - b)$ $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	p. R54

Lesson R-4 Rational Expressions and More Operations on Radicals

Key Concepts	Reference
A **rational expression** is a ratio of two polynomials. Values of the variable that make the denominator equal to zero are called restricted values of the variable.	p. R63
To simplify a rational expression, use the property of equivalent algebraic fractions. $\frac{ac}{bc} = \frac{a}{b}$ for $b \neq 0$, $c \neq 0$	p. R64
To multiply or divide rational expressions, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ for $b \neq 0$, $d \neq 0$ and $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ for $b \neq 0$, $c \neq 0$, $d \neq 0$	p. R65
To add or subtract rational expressions, write each fraction as an equivalent fraction with a common denominator. Then apply the following properties. $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$ and $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$ for $b \neq 0$	p. R66
Removing a radical from the denominator of a fraction is called **rationalizing the denominator.**	p. R71

Lesson R-5 Equations with Real Solutions

Key Concepts	Reference
A **linear equation in one variable** is an equation that can be written in the form $ax + b = 0$, where a and b are real numbers and $a \neq 0$.	p. R76
A **conditional equation** is an equation that is true for some values of the variable but false for others. A **contradiction** is false for all values of the variable. An **identity** is true for all values of the variable for which the expressions in the equation are defined.	p. R78
Let a, b, and c represent real numbers. A **quadratic equation** in the variable x is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$.	p. R80
Zero product property: If $mn = 0$, then $m = 0$ or $n = 0$.	p. R80
Square root property: If $x^2 = k$, then $x = \pm\sqrt{k}$.	p. R81
A quadratic equation can be solved by completing the square and applying the square root property.	p. R82
The solutions to $ax^2 + bx + c = 0$ $(a \neq 0)$ are given by the quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	p. R84

Rational equations: Solve a rational equation by multiplying both sides of the equation by the LCD of all fractions in the equation.	p. R85
Absolute value equations: To solve an absolute value equation, isolate the absolute value. Then use one of the following properties for a positive real number k. **1.** $\lvert u \rvert = k$ is equivalent to $u = k$ or $u = -k$. **2.** $\lvert u \rvert = 0$ is equivalent to $u = 0$. **3.** $\lvert u \rvert = -k$ has no solution. **4.** $\lvert u \rvert = \lvert w \rvert$ is equivalent to $u = w$ or $u = -w$.	p. R87
Solving radical equations: **1.** Isolate the radical. If an equation has more than one radical, choose one of the radicals to isolate. **2.** Raise each side of the equation to a power equal to the index of the radical. **3.** Solve the resulting equation. If the equation still has a radical, repeat steps 1 and 2. **4.** Check the potential solutions in the original equation and write the solution set.	p. R89

Lesson R-6 Complex Numbers and More Quadratic Equations

Key Concepts	Reference
$i = \sqrt{-1}$ and $i^2 = -1$. For a real number $b > 0$, $\sqrt{-b} = i\sqrt{b}$.	p. R98
To add or subtract complex numbers, combine the real parts, and combine the imaginary parts.	p. R101
Multiply complex numbers by using the distributive property.	p. R102
The product of complex conjugates: $(a + bi)(a - bi) = a^2 + b^2$	p. R103
Divide complex numbers by multiplying the numerator and denominator by the conjugate of the denominator.	p. R103
The discriminant to the equation $ax^2 + bx + c = 0$ $(a \neq 0)$ is given by $b^2 - 4ac$. The discriminant indicates the number of and type of solutions to the equation. • If $b^2 - 4ac < 0$, the equation has 2 nonreal solutions. • If $b^2 - 4ac = 0$, the equation has 1 real solution. • If $b^2 - 4ac > 0$, the equation has 2 real solutions.	p. R105
Substitution can be used to solve equations that are in quadratic form.	p. R106

Lesson R-7 Applications of Equations

Key Concepts	**Reference**
Equations in algebra can be used to organize information from a physical situation.	p. R114
Quadratic equations are used to model applications with the Pythagorean theorem, volume, area, and objects moving vertically under the influence of gravity.	pp. R120–R121
The vertical position s of an object moving vertically under the influence of gravity is approximated by $s = -\frac{1}{2}gt^2 + v_0t + s_0$, where • g is the acceleration due to gravity (at sea level on Earth: $g = 32$ ft/sec^2 or 9.8 m/sec^2). • t is the time after the start of the experiment. • v_0 is the initial velocity. • s_0 is the initial position (height). • s is the position of the object at time t.	p. R121

Lesson R-8 Linear, Compound, and Absolute Value Inequalities

Key Concepts	**Reference**
An inequality that can be written in one of the following forms is a **linear inequality in one variable.** $ax + b < 0$, $ax + b \leq 0$, $ax + b > 0$, or $ax + b \geq 0$	p. R132
Solving compound inequalities: • If two inequalities are joined by the word "and," take the *intersection* of the individual solution sets. • The inequality $a < x < b$ is equivalent to $a < x$ and $x < b$. • If two inequalities are joined by the word "or," take the *union* of the individual solution sets.	p. R135
Solving absolute value inequalities: For a positive real number k, **1.** $\lvert u\rvert < k$ is equivalent to $-k < u < k$. **2.** $\lvert u\rvert > k$ is equivalent to $u < -k$ or $u > k$.	p. R137

Go online for a detailed chapter summary.

Chapter R Test

1. Write the interval notation representing $\{x|2.7 < x\}$.

2. Write the set-builder notation representing the interval $[-3, 5)$.

3. Given $A = \{x|x < 2\}$, $B = \{x|x \geq 0\}$, and $C = (x|x < -1\}$, write the union or intersection in set notation.

- **a.** $A \cup B$
- **b.** $A \cap B$
- **c.** $A \cup C$
- **d.** $A \cap C$
- **e.** $B \cup C$
- **f.** $B \cap C$

4. Simplify without absolute value bars. $|5 - t|$ for $t > 5$.

5. **a.** Find the distance between $\sqrt{2}$ and 2 on the number line.

b. Simplify without absolute value bars.

6. **a.** Determine the restrictions on y.

$$\frac{2y-22}{y^2-9y-22}$$

b. Simplify the expression.

For Exercises 7–23, perform the indicated operations and simplify the expression. Assume that all radical expressions represent real numbers.

7. $-\{6 + 4[9x - 2(3x - 5)] + 3\}$

8. $\dfrac{7^2 - 4[8 - (-2)] - 4^2}{-2|-4 + 6|}$

9. $(2a^3b^{-4})^2(4a^{-2}b^7)^{-1}$

10. $\left(\frac{1}{2}\right)^0 + \left(\frac{1}{3}\right)^{-3} + \left(\frac{1}{4}\right)^{-1}$

11. $\sqrt[3]{80k^{15}m^2n^7}$

12. $\frac{3}{4}\sqrt[4]{8p^2q^3} \cdot \sqrt[4]{2p^3q}$

13. $\sqrt{125ab^3} - 3b\sqrt{20ab}$

14. $(12n^2 - 4)\left(\frac{1}{2}n^2 - 3n + 5\right)$

15. $[(3a + b) - c][(3a + b) + c]$

16. $\left(\frac{1}{4}\sqrt{z} - p^2\right)\left(\frac{1}{4}\sqrt{z} + p^2\right)$

17. $(5\sqrt{x} + \sqrt{2})(3\sqrt{x} - 2\sqrt{2})$

18. $(\sqrt{x} - 6z)^2$

19. $\dfrac{3x^2}{x^3 + 14x^2 + 49x} \div \dfrac{8x - 4}{4x^2 + 26x - 14}$

20. $\dfrac{x^2}{x - 5} + \dfrac{10x - 25}{5 - x}$

21. $\dfrac{-12}{y^3 + 4y^2} + \dfrac{1}{y} - \dfrac{3}{y^2 + 4y}$

22. $\dfrac{\frac{x}{4} - \frac{9}{4x}}{\frac{1}{4} - \frac{3}{4x}}$

23. $\dfrac{6}{\sqrt{13} + \sqrt{10}}$

For Exercises 24–31, factor completely. Write the answer with positive exponents only.

24. $30x^3 + 2x^2 - 4x$

25. $xy + 5ay + 10ac + 2xc$

26. $x^5 + 2x^4 - 81x - 162$

27. $c^2 - 4a^2 - 44a - 121$

28. $27u^3 - v^6$

29. $4w^{-6} + 2w^{-5} + 7w^{-4}$

30. $y(2y - 1)^{-3/4} + (2y - 1)^{1/4}$

31. $2x(9 - x^2)^{1/2} + x^2 \cdot \frac{1}{2}(9 - x^2)^{-1/2}(-2x)$

32. Perform the indicated operations.

$$\frac{(8.4 \times 10^{11})(6.0 \times 10^{-3})}{(4.2 \times 10^{-5})}$$

33. Write the expression in terms of i and simplify. $\sqrt{-25} \cdot \sqrt{-4}$

34. Simplify the powers of i.

- **a.** i^{89}
- **b.** i^{46}
- **c.** i^{35}
- **d.** i^{120}
- **e.** i^{-11}

For Exercises 35–56, perform the indicated operations. Write the answers in the form $a + bi$.

35. $(3 - 5i)^2$

36. $\dfrac{4 + 3i}{2 - 5i}$

For Exercises 37–39, (a) evaluate the discriminant and (b) determine the number and type of solutions to the equation.

37. $2x^2 - 4x + 7 - 0$

38. $x^2 + 25 = 10x$

39. $3x(x + 4) = 2x - 2$

For Exercises 40–50, solve the equation.

40. $3y + 2[5(y - 4) - 2] = 5y + 6(7 + y) - 3$

41. $(3x - 4)^2 - 2 = 11$

42. $y^2 + 10y = 4$

43. $6t(2t + 1) = 5 - 5t$

44. $\frac{3x^2}{4} - x = -\frac{1}{2}$

45. $(2y - 3)^{1/3} - (4y + 5)^{1/3} = 0$

46. $\sqrt{2d} = 1 - \sqrt{d + 7}$

47. $\frac{c}{c + 6} - 4 = \frac{72}{c^2 - 36}$

48. $\left(5 - \frac{2}{k}\right)^2 - 6\left(5 - \frac{2}{k}\right) - 27 = 0$

49. $-2 = |x - 3| - 6$

50. $|2v + 5| = |2v - 1|$

For Exercises 51–53, solve for the indicated variable.

51. $aP - 4 = Pt + 2$ for P

52. $-16t^2 + v_0t + 2 = 0$ for t

53. $a^2 + b^2 + c^2 = 49$ for $c > 0$

For Exercises 54–60, solve the equations and inequalities. Write the answers to the inequalities in interval notation if possible.

54. $-2 \le \frac{4 - x}{3} \le 6$

55. $-\frac{4}{3}y < -24$ or $y + 7 \le 2y - 3$

56. $3(x - 5) + 1 \le 4(x + 2) + 6$ and

$0.3x - 1.6 > 0.2$

57. $2 < -1 + |4w - 3|$

58. $-|8 - v| \ge -6$

59. a. $|7x + 4| + 11 = 2$

b. $|7x + 4| + 11 < 2$

c. $|7x + 4| + 11 > 2$

60. a. $|x - 13| + 4 = 4$

b. $|x - 13| + 4 < 4$

c. $|x - 13| + 4 \le 4$

d. $|x - 13| + 4 > 4$

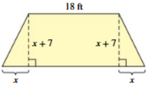
e. $|x - 13| + 4 \ge 4$

61. How much of an 80% antifreeze solution should be mixed with 2 gal of a 50% antifreeze solution to make a 60% antifreeze solution?

62. Two passengers leave the airport at Denver, Colorado. One takes a 2.3-hr flight to Seattle, Washington, and the other takes a 3.3-hr flight to New York City. The plane flying to New York flies 60 mph faster than the plane flying to Seattle. If the total distance traveled by both planes is 2662 mi, determine the average speed of each plane.

63. A homeowner has an aboveground swimming pool. Using water from one hose requires 3 hr to fill the pool. If a second hose is used, the pool can be filled in 1.2 hr. How long would it take the second hose to fill the pool if it worked alone?

64. A garden area is configured in the shape of a rectangle with two right triangles of equal size attached at the ends. The length of the rectangle is 18 ft. The height of the right triangles is 7 ft longer than the base. If the total area of the garden is 276 ft², determine the base and height of the triangular portions.



65. A varsity soccer player kicks a soccer ball approximately straight upward with an initial velocity of 60 ft/sec. The ball leaves the player's foot at a height of 2 ft.

a. Use the formula $s = -\frac{1}{2}gt^2 + v_0t + s_0$ to write a model representing the height of the ball s (in ft), t seconds after being kicked. Assume that the acceleration due to gravity is $g = 32$ ft/sec^2.

b. Determine the times at which the ball is 52 ft in the air.

66. A golfer plays 5 rounds of golf with the following scores: 92, 88, 85, 90, and 89. What score would be needed in the sixth round to have an average below 88.

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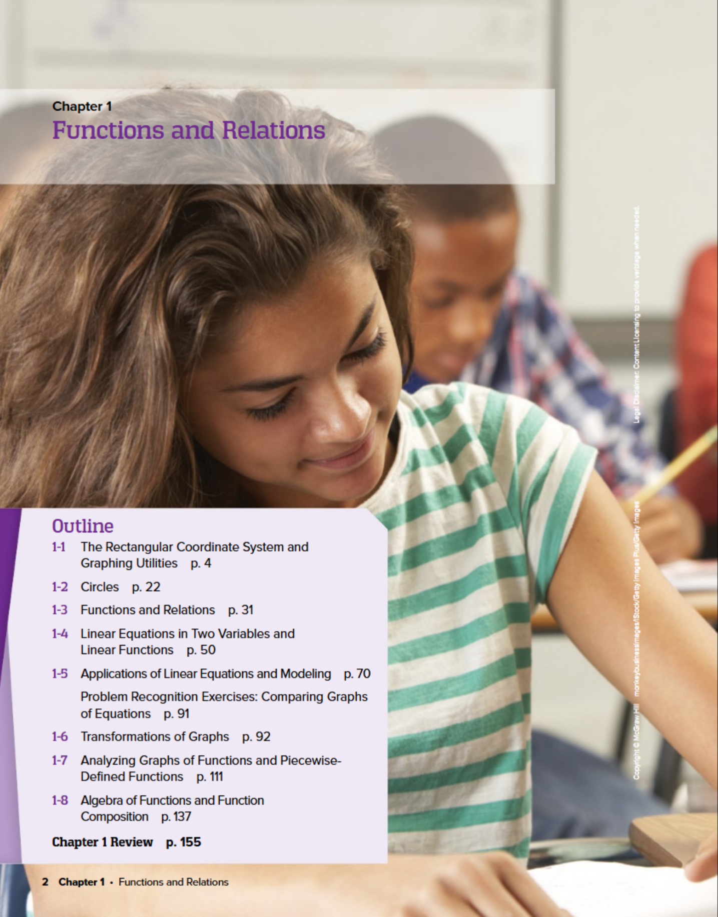
Chapter 1

Functions and Relations



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Launch Activity: Mathematical Modeling

Making the Grade

In Tariq and Allison's algebra classes, there will be six 100-point tests and 20 homework assignments per quarter. Both students expect that they will average 80 points per test. Tariq and Allison's teachers use different methods for calculating grades. Tariq's teacher takes 2 points off the total of Tariq's test scores for each homework assignment that is not fully completed. Allison's teacher counts homework as a seventh test, where Allison can earn 5 points per fully completed homework assignment.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer this question: In a given quarter, when will Tariq's grade be greater than Allison's grade?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

GO ONLINE to answer these questions.

Lesson 1-1

The Rectangular Coordinate System and Graphing Utilities

What Will You Learn?

After completing this lesson, you should be able to:

- Plot Points on a Rectangular Coordinate System
- Use the Distance and Midpoint Formulas
- Graph Equations by Plotting Points
- Identify *x*- and *y*-Intercepts
- Graph Equations Using a Graphing Utility

Websites, newspapers, sporting events, and the workplace all utilize graphs and tables to present data. Therefore, it is important to learn how to create and interpret meaningful graphs. Understanding how points are located relative to a fixed origin is important for many graphing applications. For example, computer game developers use a rectangular coordinate system to define the locations of objects moving around the screen.

Learn: Plot Points on a Rectangular Coordinate System

Mathematician René Descartes (pronounced "day cart") (1597–1650) was the first to identify points in a plane by a pair of coordinates. He did this by intersecting two perpendicular number lines with the point of intersection called the **origin**. These lines form a **rectangular coordinate system** (also known in his honor as the **Cartesian coordinate system**) or simply a **coordinate plane**. The horizontal line is called the ***x*-axis** and the vertical line is called the ***y*-axis**. The *x*- and *y*-axes divide the plane into four **quadrants**. The quadrants are labeled counterclockwise as I, II, III, and IV (Figure 1-1).

Every point in the plane can be uniquely identified by using an ordered pair (x, y) to specify its coordinates with respect to the origin. In an ordered pair, the first coordinate is called the ***x*-coordinate**, and the second is called the ***y*-coordinate**. The origin is identified as (0, 0). In Figure 1-2, six points have been graphed.

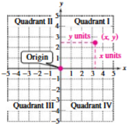

Figure 1-1

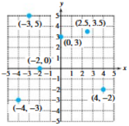

Figure 1-2

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The point (−3, 5), for example, is placed 3 units in the negative x direction (to the left) and 5 units in the positive y direction (upward).

Learn: Use the Distance and Midpoint Formulas

Recall that the distance between two points A and B on a number line can be represented by $|A - B|$ or $|B - A|$. Now we want to find the distance between two points in a coordinate plane. For example, consider the points (1, 5) and (4, 9). The distance d between the points is labeled in Figure 1-3. The dashed horizontal and vertical line segments form a right triangle with hypotenuse d.

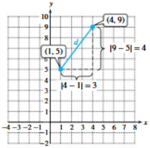

Figure 1-3

The horizontal distance between the points is $|4 - 1| = 3$.

The vertical distance between the points is $|9 - 5| = 4$.

Applying the Pythagorean theorem, we have

$d^2 = (3)^2 + (4)^2$ Since d is a distance, reject the negative square root.

$d = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$ The distance between the points is 5 units.

We can make this process generic by labeling the points $P(x_1, y_1)$ and $Q(x_2, y_2)$. See Figure 1-4.

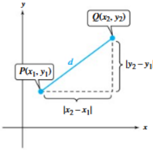

Figure 1-4

- The horizontal leg of the right triangle is $|x_2 - x_1|$ or equivalently $|x_1 - x_2|$.
- The vertical leg of the right triangle is $|y_2 - y_1|$ or equivalently $|y_1 - y_2|$.

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Applying the Pythagorean theorem, we have

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	We can drop the absolute value bars because $\lvert a\rvert^2 = (a)^2$ for all real numbers a. Likewise $\lvert x_2 - x_1\rvert^2 = (x_2 - x_1)^2$ and $\lvert y_2 - y_1\rvert^2 = (y_2 - y_1)^2$.

Distance Formula

The distance between points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Insights

Since $(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$, the distance formula can also be expressed as

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Example 1: Finding the Distance Between Two Points

Find the distance between the points $(-5, 1)$ and $(7, -3)$. Give the exact distance and an approximation to 2 decimal places.

Solution:

$(-5, 1)$ and $(7, -3)$ (x_1, y_1) and (x_2, y_2)	Label the points. Note that the choice for (x_1, y_1) and (x_2, y_2) will not affect the outcome.
$d = \sqrt{[7 - (-5)]^2 + (-3 - 1)^2}$	Apply the distance formula. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(12)^2 + (-4)^2}$	Simplify the radical.
$= \sqrt{160}$	
$= 4\sqrt{10} \approx 12.65$	The exact distance is $4\sqrt{10}$ units.
	This is approximately 12.65 units.

Apply the Skills

1. Find the distance between the points $(-1, 4)$ and $(3, -6)$. Give the exact distance and an approximation to 2 decimal places.

The Pythagorean theorem tells us that if a right triangle has legs of lengths a and b and hypotenuse of length c, then $a^2 + b^2 = c^2$. A statement of this form, “if p, then q,” is called a **conditional statement**. The **converse** is the statement “if q, then p.” The converse of a true statement is not necessarily true.

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However, in the case of the Pythagorean theorem, the converse is a true statement: If $a^2 + b^2 = c^2$, then a triangle with sides of lengths a, b, and c is a right triangle. We use this important concept in Example 2.

Example 2: Determining if Three Points Form the Vertices of a Right Triangle

Determine if the points $M(-2, -3)$, $P(4, 1)$, and $Q(-1, 7)$ form the vertices of a right triangle.

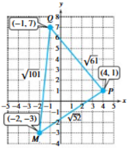
Solution:

Determine the distance between each pair of points.

Apply the distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\begin{aligned} d(M, P) &= \sqrt{[4 - (-2)]^2 + [1 - (-3)]^2} \\ &= \sqrt{52} \\ d(P, Q) &= \sqrt{(-1 - 4)^2 + (7 - 1)^2} \\ &= \sqrt{61} \\ d(M, Q) &= \sqrt{[-1 - (-2)]^2 + [7 - (-3)]^2} \\ &= \sqrt{101} \end{aligned}$$



The line segment $\overline{MQ}$ is the longest and would potentially be the hypotenuse, c. Label the shorter sides as a and b.

Check the condition that $a^2 + b^2 = c^2$.

$$(\sqrt{52})^2 + (\sqrt{61})^2 \overset{?}{=} (\sqrt{101})^2$$

$$52 + 61 \neq 101$$

The points M, P, and Q do not form the vertices of a right triangle.

Apply the Skills

2. Determine if the points $X(-6, -4)$, $Y(2, -2)$, and $Z(0, 5)$ form the vertices of a right triangle.

Good Practices

We denote the distance between points P and Q as

$$d(P, Q) \quad \text{or} \quad PQ.$$

The second notation is the length of the line segment with endpoints P and Q.

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Now suppose that we want to find the midpoint of the line segment between the distinct points (x_1, y_1) and (x_2, y_2). The **midpoint** of a line segment is the point equidistant (the same distance) from the endpoints (Figure 1-5).

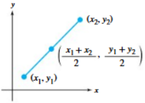


Figure 1-5

The x-coordinate of the midpoint is the average of the x-coordinates from the endpoints. Likewise, the y-coordinate of the midpoint is the average of the y-coordinates from the endpoints.

Midpoint Formula

The midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is

$$M = \left(\underset{\substack{\uparrow \\ \text{average of} \\ x\text{-coordinates}}}{\frac{x_1+x_2}{2}}, \underset{\substack{\uparrow \\ \text{average of} \\ y\text{-coordinates}}}{\frac{y_1+y_2}{2}}\right)$$

Check Your Work

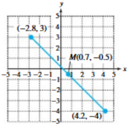
The midpoint of a line segment is an ordered pair (with two coordinates), not a single number.

Example 3: Finding the Midpoint of a Line Segment

Find the midpoint of the line segment with endpoints (4.2, –4) and (–2.8, 3).

Solution:

$(4.2, -4)$ and $(-2.8, 3)$ (x_1, y_1) and (x_2, y_2)	Label the points.
$M = \left(\frac{4.2+(-2.8)}{2}, \frac{-4+3}{2}\right)$	Apply the midpoint formula.
$= \left(0.7, -\frac{1}{2}\right)$ or $(0.7, -0.5)$	Simplify.



Apply the Skills

3. Find the midpoint of the line segment with endpoints (–1.5, –9) and (–8.7, 4).

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Learn: Graph Equations by Plotting Points

The relationship between two variables can often be expressed as a graph or expressed algebraically as an equation. For example, suppose that two variables, x and y, are related such that y is 2 more than x. An equation to represent this relationship is $y = x + 2$. A **solution to an equation** in the variables x and y is an ordered pair (x, y) that when substituted into the equation makes the equation a true statement.

For example, the following ordered pairs are solutions to the equation $y = x + 2$.

Solution	$y = x + 2$
(0, 2)	$2 = 0 + 2$ ✓
(−4, −2)	$-2 = -4 + 2$ ✓
(2, 4)	$4 = 2 + 2$ ✓

The set of all solutions to an equation is called the **solution set of the equation**. The graph of all solutions to an equation is called the **graph of the equation**. The graph of $y = x + 2$ is shown in Figure 1-6.

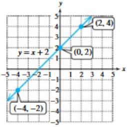


Figure 1-6

One of the goals of this text is to identify families of equations and the characteristics of their graphs. As we proceed through the text, we will develop tools to graph equations efficiently. For now, we present the point-plotting method to graph the solution set of an equation. In Example 4, we start by selecting several values of x and using the equation to calculate the corresponding values of y. Then we plot the points to form a general outline of the curve.

Example 4: Graphing an Equation by Plotting Points

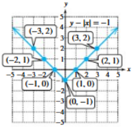
Graph the equation by plotting points. $y - |x| = -1$

Solution:

$y - |x| = -1$ Solve for y in terms of x.

$y = |x| - 1$ Arbitrarily select negative and positive values for x such as −3, −2, −1, 0, 1, 2, and 3. Then use the equation to calculate the corresponding y values.

x	y	$y = \|x\| - 1$	Ordered pair
−3	2	$y = \|-3\| - 1 = 2$	(−3, 2)
−2	1	$y = \|-2\| - 1 = 1$	(−2, 1)
−1	0	$y = \|-1\| - 1 = 0$	(−1, 0)
0	−1	$y = \|0\| - 1 = -1$	(0, −1)
1	0	$y = \|1\| - 1 = 0$	(1, 0)
2	1	$y = \|2\| - 1 = 1$	(2, 1)
3	2	$y = \|3\| - 1 = 2$	(3, 2)



Apply the Skills

4. Graph the equation by plotting points. $x^2 + y = 4$

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The graph of an equation in the variables x and y represents a relationship between a real number x and a corresponding real number y. Therefore, the values of x must be chosen so that when substituted into the equation, they produce a real number for y. Sometimes the values of x must be restricted to produce real numbers for y. This is demonstrated in Example 5.

Example 5: Graphing an Equation by Plotting Points

Graph the equation by plotting points. $y^2 - 1 = x$

Solution:

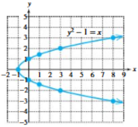
$y^2 - 1 = x$ Solve for y in terms of x.

$y^2 = x + 1$

$y = \pm\sqrt{x+1}$ Apply the square root property.

Choose $x \geq -1$ so that the radicand is nonnegative.

x	y	$y = \pm\sqrt{x+1}$	Ordered pairs
-1	0	$y = \pm\sqrt{(-1)+1} = 0$	$(-1, 0)$
0	± 1	$y = \pm\sqrt{(0)+1} = \pm 1$	$(0, 1), (0, -1)$
1	$\pm\sqrt{2}$	$y = \pm\sqrt{(1)+1} = \pm\sqrt{2}$ ≈ 1.4	$(1, \sqrt{2}), (1, \sqrt{2})$
3	± 2	$y = \pm\sqrt{(3)+1} = \pm 2$	$(3, 2), (3, -2)$
8	± 3	$y = \pm\sqrt{(8)+1} = \pm 3$	$(8, 3), (8, -3)$



Good Practices

In Example 5, we choose several convenient values of x such as -1, 0, 3, and 8 so that the radicand will be a perfect square.

Apply the Skills

5. Graph the equation by plotting points. $x + y^2 = 2$

Learn: Identify x- and y-Intercepts

When analyzing graphs, we want to examine their most important features. Two key features are the x- and y-intercepts of a graph. These are the points where a graph intersects the x- and y-axes.

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Any point on the x-axis has a y-coordinate of zero. Therefore, an **x-intercept** is a point $(a, 0)$ where a graph intersects the x-axis (Figure 1-7). Any point on the y-axis has an x-coordinate of zero. Therefore, a **y-intercept** is a point $(0, b)$ where a graph intersects the y-axis (Figure 1-7).

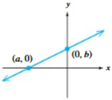

Figure 1-7

In some applications, we may refer to an x-intercept as the *x-coordinate* of a point of intersection that a graph makes with the x-axis. For example, if an x-intercept is $(-4, 0)$, then the x-intercept may be stated simply as -4 (the y-coordinate is understood to be zero). Similarly, we may refer to a y-intercept as the *y-coordinate* of a point of intersection that a graph makes with the y-axis. For example, if a y-intercept is $(0, 2)$, then it may be stated simply as 2.

To find the x- and y-intercepts from an equation in x and y, follow these steps.

Determining x- and y-Intercepts from an Equation

Given an equation in x and y,

- Find the x-intercept(s) by substituting 0 for y in the equation and solving for x.
- Find the y-intercept(s) by substituting 0 for x in the equation and solving for y.

Example 6: Finding x- and y-Intercepts

Given the equation $y = |x| - 1$,

a. Find the x-intercept(s). **b.** Find the y-intercept(s).

Solution:

a. $y = |x| - 1$

$0 = |x| - 1$ To find the x-intercept(s), substitute 0 for y and solve for x.

$|x| = 1$ Isolate the absolute value.

$x = 1$ or $x = -1$ Recall that for $k > 0$, $|x| = k$ is equivalent to $x = k$ or $x = -k$.

The x-intercepts are $(1, 0)$ and $(-1, 0)$.

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b. $y = |x| - 1$

$= |0| - 1$ To find the y-intercept(s), substitute 0 for x and solve for y.

$= -1$

The y-intercept is $(0, -1)$.

The intercepts $(1, 0)$, $(-1, 0)$, and $(0, -1)$ are consistent with the graph of the equation $y = |x| - 1$ found in Example 4 (Figure 1-8).

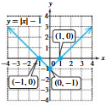


Figure 1-8

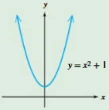
Apply the Skills

6. Given the equation $y = x^2 - 4$,

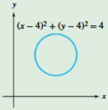
a. Find the x-intercept(s). **b.** Find the y-intercept(s).

Insights

Sometimes when solving for an x- or y-intercept, we encounter an equation with an imaginary solution. In such a case, the graph has no x- or y-intercept.



No x-intercept



No x- or y-intercept

Learn: Graph Equations Using a Graphing Utility

Graphing by the point-plotting method should only be considered a beginning strategy for creating the graphs of equations in two variables. We will quickly enhance this method with other techniques that are less cumbersome and use more analysis and strategy.

One weakness of the point-plotting method is that it may be slow to execute by pencil and paper. Also, the selected points must fairly represent the shape of the graph. Otherwise the sketch will be inaccurate. Graphing utilities can help with both of these weaknesses. They can graph many points quickly, and the more points that are plotted, the greater the likelihood that we see the key features of the graph. Graphing utilities include graphing calculators, spreadsheets, specialty graphing programs, and apps on phones.

Figures 1-9 and 1-10 show a table and a graph for $y = x^2 - 3$.

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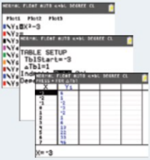
TECHNOLOGY CONNECTIONS

Using the Table Feature and Graphing an Equation

In Figure 1-9, we first enter the equation into the graphing editor. Notice that the calculator expects the equation represented with the *y* variable isolated.



```
Plot1  Plot2  Plot3
\Y1=X²-3
```

```
TABLE SETUP
 TblStart=-3
 ΔTbl=1
```

```
X=-3
```

Figure 1-9

To set up a table, enter the starting value for *x*, in this case, –3. Then set the increment by which to increase *x*, in this case 1. The *x*-increment is entered as ΔTbl (read "delta table").

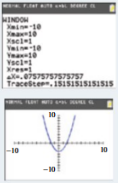
Insights

The Greek letter Δ ("delta") written before a variable represents an increment of change in that variable. In this context, it represents the change from one value of *x* to the next.

Using the "Auto" setting means that the table of values for X and Y_1 will be automatically generated.

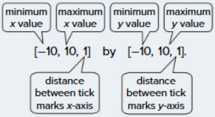
The table shows eleven *x-y* pairs, but more can be accessed by using the up and down arrow keys on the keypad.

The graph in Figure 1-10 is shown between *x* and *y* values from –10 to 10. The tick marks on the axes are 1 unit apart. The viewing window with these parameters is denoted [–10, 10, 1] by [–10, 10, 1].



```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
 ΔX=.07575757575757
 TraceStep=.15151515151515
```

Figure 1-10



minimum x value
maximum x value
minimum y value
maximum y value
[–10, 10, 1] by [–10, 10, 1].
distance between tick marks x-axis
distance between tick marks y-axis

Insights

The calculator plots a large number of points and then connects the points. So instead of graphing a single smooth curve, it graphs a series of short line segments. This may give the graph a jagged look (see Figure 1-10 on page 13).

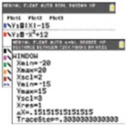
Example 7: Graphing Equations Using a Graphing Utility

Use a graphing utility to graph $y = |x| - 15$ and $y = -x^2 + 12$ on the viewing window defined by $[-20, 20, 2]$ by $[-15, 15, 3]$.

Solution:

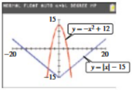
Enter the equations using the Y = editor.

Use the WINDOW editor to change the viewing window parameters. The variables Xmin, Xmax, and Xscl relate to $[-20, 20, 2]$. The variables Ymin, Ymax, and Yscl relate to $[-15, 15, 3]$.



```
Plot1  Plot2  Plot3
\Y1■|X|-15
\Y2■-X²+12
WINDOW
 Xmin=-20
 Xmax=20
 Xscl=2
 Ymin=-15
 Ymax=15
 Yscl=3
 Xres=1
 ΔX=.15151515151515
 TraceStep=.30303030303030303
```

Select the GRAPH feature. Notice that the graphs of both equations appear. This provides us with a tool for visually examining two different models at the same time.



Apply the Skills

7. Use a graphing utility to graph $y = -x + 2$ and $y = 0.5x^2 - 2$ on the viewing window $[-6, 6, 1]$ by $[-4, 8, 1]$.

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Practice 1-1

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Practice Exercises

Prerequisite Review

R.1. Simplify the radical. $\sqrt{48}$

R.2. Given a right triangle with a leg of length 7 km and hypotenuse of length 25 km, find the length of the unknown leg.

R.3. Solve for y. $ax + by = c$

R.4. Evaluate $x^2 + 4x + 5$ for $x = -5$

Concept Connections

1. In a rectangular coordinate system, the point where the x- and y-axes meet is called the __________.

2. The x- and y-axes divide the coordinate plane into four regions called __________.

3. The distance between two distinct points (x_1, y_1) and (x_2, y_2) is given by the formula __________.

4. The midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is given by the formula __________.

5. A __________ to an equation in the variables x and y is an ordered pair (x, y) that makes the equation a true statement.

6. An x-intercept of a graph has a y-coordinate of __________.

7. A y-intercept of a graph has an x-coordinate of __________.

8. Given an equation in the variables x and y, find the y-intercept by substituting __________ for x and solving for __________.

Learn: Plot Points on a Rectangular Coordinate System

For Exercises 9–10, plot the points on a rectangular coordinate system.

9. $A(-3, -4)$ $B\left(\frac{5}{3}, \frac{7}{4}\right)$ $C(-1.2, 3.8)$

$D(\pi, -5)$ $E(0, 4.5)$ $F(\sqrt{5}, 0)$

10. $A(-2, -5)$ $B\left(\frac{9}{2}, \frac{7}{3}\right)$ $C(-3.6, 2.1)$

$D(5, -\pi)$ $E(3.4, 0)$ $F(0, \sqrt{3})$

Learn: Use the Distance and Midpoint Formulas

For Exercises 11–18,

a. Find the exact distance between the points. **(See Example 1)**

b. Find the midpoint of the line segment whose endpoints are the given points. **(See Example 3)**

11. $(-2, 7)$ and $(-4, 11)$

12. $(-1, -3)$ and $(3, -7)$

13. $(-7, -4)$ and $(2, 5)$

14. $(3, 6)$ and $(-4, -1)$

15. $(2.2, -2.4)$ and $(5.2, -6.4)$

16. $(37.1, -24.7)$ and $(31.1, -32.7)$

17. $(\sqrt{5}, -\sqrt{2})$ and $(4\sqrt{5}, -7\sqrt{2})$

18. $(\sqrt{7}, -3\sqrt{5})$ and $(2\sqrt{7}, \sqrt{5})$

For Exercises 19–22, determine if the given points form the vertices of a right triangle. (See Example 2)

19. $(1, 3)$, $(3, 1)$, and $(0, -2)$

20. $(1, 2)$, $(3, 0)$, and $(-3, -2)$

21. $(-2, 4)$, $(5, 0)$, and $(-5, 1)$

22. $(-6, 2)$, $(3, 1)$, and $(1, -2)$

Learn: Graph Equations by Plotting Points

For Exercises 23–24, determine if the given points are solutions to the equation.

23. $x^2 + y = 1$

a. $(-2, -3)$ **b.** $(4, -17)$ **c.** $\left(\frac{1}{2}, \frac{3}{4}\right)$

24. $|x - 3| - y = 4$

a. $(1, -2)$ **b.** $(-2, -3)$ **c.** $\left(\frac{1}{10}, -\frac{11}{10}\right)$

For Exercises 25–30, identify the set of values *x* for which *y* will be a real number.

25. $y = \frac{2}{x-3}$

26. $y = \frac{2}{x+7}$

27. $y = \sqrt{x-10}$

28. $y = \sqrt{x+11}$

29. $y = \sqrt{1.5-x}$

30. $y = \sqrt{2.2-x}$

For Exercises 31–44, graph the equations by plotting points. (See Examples 4–5)

31. $y = x$

32. $y = x^2$

33. $y = \sqrt{x}$

34. $y = |x|$

35. $y = x^3$

36. $y = \frac{1}{x}$

37. $y - |x| = 2$

38. $|x| + y = 3$

39. $y^2 - x - 2 = 0$

40. $y^2 - x + 1 = 0$

41. $x = |y| + 1$

42. $x = |y| - 3$

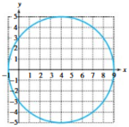
43. $y = |x+1|$

44. $y = |x-2|$

Learn: Identify *x*- and *y*-Intercepts

For Exercises 45–50, estimate the *x*- and *y*-intercepts from the graph.

45.



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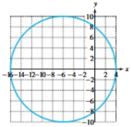

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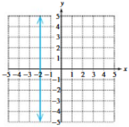

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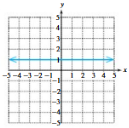
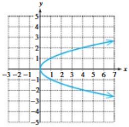
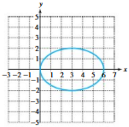


49.



50.



For Exercises 51–62, find the *x*- and *y*-intercepts. (See Example 6)

51. $-2x + 4y = 12$

52. $-3x - 5y = 60$

53. $x^2 + y = 9$

54. $x^2 = -y + 16$

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55. $y = |x - 5| - 2$

56. $y = |x + 4| - 3$

57. $x = y^2 - 1$

58. $x = y^2 - 4$

59. $|x| = |y|$

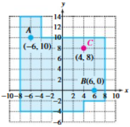
60. $x = |5y|$

61. $\frac{(x-3)^2}{4} + \frac{(y-4)^2}{9} = 1$

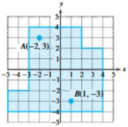
62. $\frac{(x+6)^2}{16} + \frac{(y+3)^2}{4} = 1$

Mixed Exercises

63. A map of a wilderness area is drawn with the origin placed at the parking area. Two fire observation platforms are located at points *A* and *B*. If a fire is located at point *C*, determine the distance to the fire from each observation platform. Assume that the units of measurement are in miles.



64. A map of a state park is drawn so that the origin is placed at the visitor center. The distance between grid lines is 1 mi. Suppose that two hikers are located at points *A* and *B*.



a. Determine the distance between the hikers.

b. If the hikers want to meet for lunch, determine the location of the midpoint between the hikers.

The position of an object in a video game is represented by an ordered pair. The coordinates of the ordered pair give the number of pixels horizontally and vertically from the origin. Use this scenario for Exercises 65–66.

65. a. Suppose that player A is located at (36, 315) and player B is located at (410, 53). How far apart are the players? Round to the nearest pixel.

b. If the two players move directly toward each other at the same speed, where will they meet?

c. If player A moves three times faster than player B, where will they meet? Round to the nearest pixel.

66. Suppose that a player is located at point *A*(460, 420) and must move in a direct line to point *B*(80, 210) and then in a direct line to point *C*(120, 60) to pick up prizes before a 5-sec timer runs out. If the player moves at 120 pixels per second, will the player have enough time to pick up both prizes? Explain.

67. Verify that the points *A*(0, 0), *B*(*x*, 0), and $C\left(\frac{1}{2}x, \frac{\sqrt{3}}{2}x\right)$ make up the vertices of an equilateral triangle.

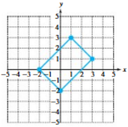
68. Verify that the points *A*(0, 0), *B*(*x*, 0), and *C*(0, *x*) make up the vertices of an isosceles right triangle (an isosceles triangle has two sides of equal length).

For Exercises 69–70, assume that the units shown in the grid are in feet.

a. Determine the exact length and width of the rectangle shown.

b. Determine the perimeter and area.

69.



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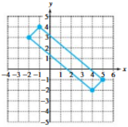
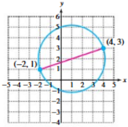
70.

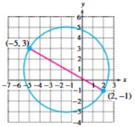

y
x
-4 -3 -2 -1 1 2 3 4 5 6
5 4 3 2 1 -1 -2 -3 -4 -5

For Exercises 71–72, the endpoints of a diameter of a circle are shown. Find the center and radius of the circle.

71.

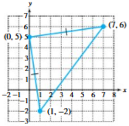

y
x
(4, 3)
(−2, 1)
-5 -4 -3 -2 -1 1 2 3 4 5
6 5 4 3 2 1 -1 -2 -3 -4

72.

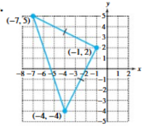

y
x
(−5, 3)
(2, −1)
-7 -6 -5 -4 -3 -2 -1 1 2 3
6 5 4 3 2 1 -1 -2 -3 -4

For Exercises 73–74, an isosceles triangle is shown. Find the area of the triangle. Assume that the units shown in the grid are in meters.

73.


y
x
(0, 5)
(7, 6)
(1, −2)
-2 -1 1 2 3 4 5 6 7 8
7 6 5 4 3 2 1 -1 -2 -3

74.


y
x
(−7, 5)
(−1, 2)
(−4, −4)
-8 -7 -6 -5 -4 -3 -2 -1 1 2
5 4 3 2 1 -1 -2 -3 -4 -5

For Exercises 75–78, determine if points *A*, *B*, and C are collinear. Three points are collinear if they all fall on the same line. There are several ways that we can determine if three points, *A*, *B*, and *C* are collinear. One method is to determine if the sum of the lengths of the line segments $\overline{AB}$ and $\overline{BC}$ equals the length of $\overline{AC}$.

75. (2, 2), (4, 3), and (8, 5)

76. (2, 1.5), (4, 2), and (8, 3)

77. (−2, 8), (1, 2), and (4, −3)

78. (−1, 5), (0, 3), and (5, −13)

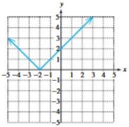
For Exercises 79–80, match each equation with a graph and give the *x*- and *y*-intercepts.

79. a. $y = |x| + 2$

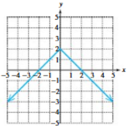
b. $y = |x + 2|$

c. $y = 2 - |x|$

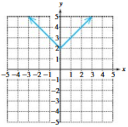
i.



ii.



iii.



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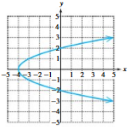
80. **a.** $y = x^2 - 4$

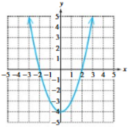
b. $y = (x - 2)^2$

c. $x = y^2 - 4$

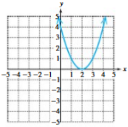
i.



ii.



iii.



Write About It

81. Suppose that d represents the distance between two points (x_1, y_1) and (x_2, y_2). Explain how the distance formula is developed from the Pythagorean theorem.

82. Explain how you might remember the midpoint formula to find the midpoint of the line segment between (x_1, y_1) and (x_2, y_2).

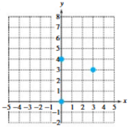
83. Explain how to find the x- and y-intercepts from an equation in the variables x and y.

84. Given an equation in the variables x and y, what does the graph of the equation represent?

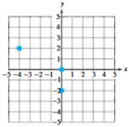
Expanding Your Skills

For Exercises 85–86, three vertices of a parallelogram are shown. Find the possible coordinates of the fourth vertex. Note that there are three solutions.

85.



86.



87. The midpoints of the sides of a triangle are given by (0, 4), (1, −1), and (−4, 1). If the rightmost vertex of the triangle is (5, 2), what are the coordinates of the other two vertices?

88. The midpoints of the sides of a triangle are given by (−6, 1), (−1, 1), and (−1, 5). If the lowest vertex of the triangle is (−6, −3), what are the coordinates of the other two vertices?

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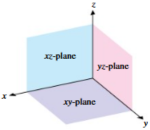
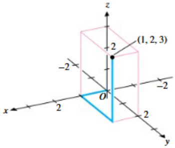
A point in space can be represented in a three-dimensional coordinate system. In such a case, a *z*-axis is taken perpendicular to both the *x*- and *y*-axes. A point is represented by an ordered triple *P*(*x*, *y*, *z*) relative to a fixed origin, (0, 0, 0), where the three axes meet.



For example, to plot the point (1, 2, 3), first move 1 unit from the origin along the positive *x*-axis. Then move parallel to the *y*-axis a distance of 2 units in the positive direction. Finally, move upward in the positive *z* direction 3 units and mark the point.



For Exercises 89–90, plot the indicated points. Note that special graph paper can be used to plot points in three-dimensions. Search for "3D graph paper" online to find free downloads.

89. (3, −2, 4)

90. (3, 1, −2)

The distance between two given points in space can be found in a fashion similar to finding distances in a plane. The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. For Exercises 91–94, find the distance between the points.

91. (5, −3, 2) and (4, 6, −1)

92. (6, −4, −1) and (2, 3, 1)

93. (3, 7, −2) and (0, −5, 1)

94. (9, −5, −3) and (2, 0, 1)

Learn: Graph Equations Using a Graphing Utility (Technology Connections)

95. What is meant by a viewing window on a graphing device?

96. Which of the viewing windows would show both the *x*- and *y*-intercepts of the graph of $780x - 42y = 5460$?

- **a.** [−20, 20, 2] by [−40, 40, 10]
- **b.** [−10, 10, 1] by [−10, 10, 1]
- **c.** [−10, 10, 1] by [−10, 150, 10]
- **d.** [−10, 10, 1] by [−150, 10, 10]

For Exercises 97–100, graph the equation with a graphing utility on the given viewing window. (See Example 7)

97. $y = 2x - 5$ on [−10, 10, 1] by [−10, 10, 1]

98. $y = -4x + 1$ on [−10, 10, 1] by [−10, 10, 1]

99. $y = 1400x^2 - 1200x$ on [−5, 5, 1] by [−1000, 2000, 500]

100. $y = -800x^2 + 600x$ on [−5, 5, 1] by [−1000, 500, 200]

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For Exercises 101–102, graph the equations on the standard viewing window. (See Example 7)

101. a. $y = x^3$

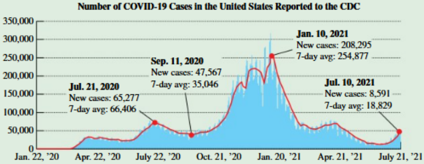
b. $y = |x| - 9$

102. a. $y = \sqrt{x + 4}$

b. $y = |x - 2|$

Point of Interest

Plotting points in a two-dimensional coordinate system provides a graphical representation of numerical data. For example, the graph shows the number of new COVID-19 cases in the United States per day between March 2, 2020, and July 21, 2021. In total, over 500 data points have been plotted in the form of a bar graph. A smooth curve representing consecutive 7-day averages has been superimposed on top of the bar graph. Several data points have been labeled for your reference. (*Source*: Centers for Disease Control, www.cdc.gov)



103. a. Research online how a 7-day moving average is computed, and explain the process.

b. Explain what the peaks and valleys of this curve represent in context.

GO ONLINE **Go online for more practice problems.**

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Lesson 1-2

Circles

What Will You Learn?

After completing this lesson, you should be able to:

- Write an Equation of a Circle in Standard Form
- Write the General Form of an Equation of a Circle

Learn: Write an Equation of a Circle in Standard Form

In addition to graphing equations by plotting points, we will learn to recognize specific categories of equations and the characteristics of their graphs. We begin by presenting the definition of a circle.

Definition of a Circle

A **circle** is the set of all points in a plane that are equidistant from a fixed point called the **center**. The fixed distance from any point on the circle to the center is called the **radius**.

The radius of a circle is often denoted by r, where $r > 0$. It is also important to note that the center is not actually part of the graph of a circle. It will be drawn in the text as an open dot for reference only.

Suppose that a circle is centered at the point (h, k) and has radius r (Figure 1-11). The distance formula can be used to derive an equation of the circle. Let (x, y) be an arbitrary point on the circle. Then by definition the distance between (h, k) and (x, y) must be r.

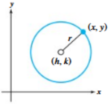

Figure 1-11

Apply the distance formula. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d$

$\sqrt{(x - h)^2 + (y - k)^2} = r$ Distance between (h, k) and (x, y)

$(x - h)^2 + (y - k)^2 = r^2$ Squaring both sides of the equation results in the standard form of an equation of a circle.

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Standard Form of an Equation of a Circle

Given a circle centered at (h, k) with radius r, the **standard form** of an equation of the circle (also called **center-radius form**) is given by

$$(x-h)^2+(y-k)^2=r^2 \text{ where } r>0.$$

Examples	Standard form	Center	Radius
$(x-4)^2+(y+3)^2=25$	$(x-4)^2+[y-(-3)]^2=(5)^2$	$(4, -3)$	5
$x^2+\left(y-\frac{1}{2}\right)^2=12$	$(x-0)^2+\left(y-\frac{1}{2}\right)^2=(\sqrt{12})^2$	$\left(0, \frac{1}{2}\right)$	$2\sqrt{3}$
$x^2+y^2=7$	$(x-0)^2+(y-0)^2=(\sqrt{7})^2$	$(0, 0)$	$\sqrt{7}$

In Example 1, we write an equation of a circle in standard form.

Example 1: Writing an Equation of a Circle in Standard Form

a. Write the standard form of an equation of the circle with center $(-4, 6)$ and radius 2.

b. Graph the circle.

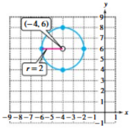
Solution:

a. $(h, k)=(-4, 6)$ and $r=2$ Label the center (h, k) and the radius r.

$[x-(-4)]^2+(y-6)^2=(2)^2$ Standard form: $(x-h)^2+(y-k)^2=r^2$

$(x+4)^2+(y-6)^2=4$ Simplify.

b.



To graph the circle, first locate the center $(-4, 6)$ and draw a small open dot. Then plot points $r=2$ units to the left, right, above, and below the center.

Draw the circle through the points.

Apply the Skills

1. **a.** Write an equation of the circle with center $(3, -1)$ and radius 4.

   b. Graph the circle.

Point of Interest

Among his many contributions to mathematics, René Descartes discovered analytic geometry, which uses algebraic equations to describe geometric shapes. For example, a circle can be described by the algebraic equation $(x-h)^2+(y-k)^2=r^2$.

Example 2: Writing an Equation of a Circle in Standard Form

Write the standard form of an equation of the circle with endpoints of a diameter $(-1, 0)$ and $(3, 4)$.

Solution:

A sketch of this scenario is given in Figure 1-12.

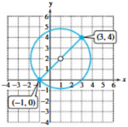


Figure 1-12

Notice that the midpoint of the diameter is the center of the circle.

$(-1, 0)$ and $(3, 4)$

(x_1, y_1) and (x_2, y_2) Label the points.

The center is $\left(\frac{-1+3}{2}, \frac{0+4}{2}\right)=(1, 2)$.

The radius of the circle is the distance between either endpoint of the diameter and the center. Using the endpoint $(-1, 0)$ as (x_1, y_1) and the center $(1, 2)$ as (x_2, y_2), apply the distance formula.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$r=\sqrt{[1-(-1)]^2+(2-0)^2}=\sqrt{(2)^2+(2)^2}=\sqrt{8}$$

An equation of the circle is: $(x-h)^2+(y-k)^2=r^2$.

$$(x-1)^2+(y-2)^2=\left(\sqrt{8}\right)^2$$
$$(x-1)^2+(y-2)^2=8 \quad \text{(Standard form)}$$

Apply the Skills

2. Write the standard form of an equation of the circle with endpoints of a diameter $(-3, 3)$ and $(-1, -1)$.

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Learn: Write the General Form of an Equation of a Circle

In Example 2 we have the equation $(x - 1)^2 + (y - 2)^2 = 8$. If we expand the binomials and combine like terms, we can write the equation in *general form*.

General Form of an Equation of a Circle

An equation of a circle written in the form $x^2 + y^2 + Ax + By + C = 0$ is called the **general form** of an equation of a circle.

$(x - 1)^2 + (y - 2)^2 = 8$	Standard form (center-radius form)
$x^2 - 2x + 1 + y^2 - 4y + 4 = 8$	Expand the binomials
$x^2 + y^2 - 2x - 4y - 3 = 0$	General form

By completing the square we can write an equation of a circle given in general form as an equation in standard form. The purpose of writing an equation of a circle in standard form is to identify the radius and center. This is demonstrated in Example 3.

Example 3: Writing an Equation of a Circle in Standard Form

Write the equation of the circle in standard form. Then identify the center and radius.

$$x^2 + y^2 + 10x - 6y + 25 = 0$$

Solution:

$x^2 + y^2 + 10x - 6y + 25 = 0$ $(x^2 + 10x \quad) + (y^2 - 6y \quad) = -25$	Group the x terms. Group the y terms. Move the constant term to the right.
$(x^2 + 10x + 25) + (y^2 - 6y + 9) = -25 + 25 + 9$	Complete the squares. *Note*: $\left[\frac{1}{2}(10)\right]^2 = 25$, $\left[\frac{1}{2}(-6)\right]^2 = 9$
$(x + 5)^2 + (y - 3)^2 = 9$	Factor.

The center is $(-5, 3)$.
The radius is $\sqrt{9} = 3$.
See Figure 1-13.

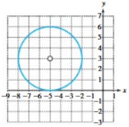

Figure 1-13

Apply the Skills

3. Write the equation of the circle in standard form. Then identify the center and radius. $x^2 + y^2 - 8x + 2y - 8 = 0$

Not all equations of the form $x^2 + y^2 + Ax + By + C = 0$ represent the graph of a circle. Completing the square results in an equation of the form $(x - h)^2 + (y - k)^2 = c$, where c is a constant. In the case where $c > 0$, the graph of the equation is a circle with radius $r = \sqrt{c}$. However, if $c = 0$, or if $c < 0$, the graph will be a single point or nonexistent. These are called **degenerate cases**.

- If $c > 0$, then the graph will be a circle with radius $r = \sqrt{c}$.
- If $c = 0$, then the graph will be a single point, (h, k). The solution set is $\{(h, k)\}$.
- If $c < 0$, then the solution set is the empty set $\{\ \}$.

Example 4: Determining if an Equation Represents the Graph of a Circle

Write the equation in the form $(x - h)^2 + (y - k)^2 = r^2$, and identify the solution set.

$$x^2 + y^2 - 14y + 49 = 0$$

Solution:

$x^2 + y^2 - 14y + 49 = 0$	Group the y terms and complete the square. Note that the x^2 term is already a perfect square: $(x - 0)^2$.
$x^2 + (y^2 - 14y) = -49$	
$x^2 + (y^2 - 14y + 49) = -49 + 49$	Complete the square: $\left[\frac{1}{2}(-14)\right]^2 = 49$.
$x^2 + (y - 7)^2 = 0$	Factor.

Since $r^2 = 0$, the solution set is $\{(0, 7)\}$. The sum of two squares will equal zero only if each individual term is zero. Therefore, $x = 0$ and $y = 7$.

Apply the Skills

4. Write the equation in the form $(x - h)^2 + (y - k)^2 = r^2$, and identify the solution set. $x^2 + y^2 + 2x + 5 = 0$

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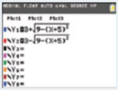
TECHNOLOGY CONNECTIONS

Setting a Square Viewing Window and Graphing a Circle

A graphing calculator expects an equation with the *y* variable isolated. Therefore, to graph an equation of a circle such as $(x + 5)^2 + (y - 3)^2 = 9$, from Example 3, we first solve for *y*.

$$
\begin{aligned}
(x + 5)^2 + (y - 3)^2 &= 9 \\
(y - 3)^2 &= 9 - (x + 5)^2 \\
y - 3 &= \pm\sqrt{9 - (x + 5)^2} \\
y &= 3 \pm \sqrt{9 - (x + 5)^2}
\end{aligned}
$$



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Plot1 Plot2 Plot3

$\backslash Y_1 = 3+\sqrt{9-(X+5)^2}$

$\backslash Y_2 = 3-\sqrt{9-(X+5)^2}$

$\backslash Y_3 =$

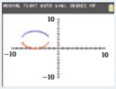
$\backslash Y_4 =$

$\backslash Y_5 =$

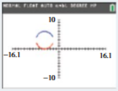
$\backslash Y_6 =$

$\backslash Y_7 =$

$\backslash Y_8 =$



Notice that the graph looks more oval-shaped than circular. This is because the calculator has a rectangular screen. If the scaling is the same on the *x*- and *y*-axes, the graph will appear elongated horizontally.



To eliminate this distortion, use a ZSquare option, located in the Zoom menu.

Also notice that the calculator display may not show the upper and lower semicircles connecting. The viewing window between $x = -16.1$ and $x = 16.1$ is divided by the number of pixels displayed horizontally to get the values of *x* used to graph the equation. These may not include *x* values at the leftmost and rightmost points on the circle. That is, the calculator may graph points *close* to $(-8, 3)$ and $(-2, 3)$ but not exactly at $(-8, 3)$ and $(-2, 3)$. Therefore, the upper and lower semicircles may not connect in the calculator display.

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Practice 1-2

Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, find the value of n so that the expression is a perfect square trinomial. Then factor the trinomial.

R.1. $c^2 - 8c + n$

R.2. $x^2 + \frac{2}{7}x + n$

R.3. Find the distance between (2, 3) and (–3, –2). Express your answer in simplified radical form.

R.4. Multiply by using the special case products. Simplify. $(x - 2)^2$

Concept Connections

1. A __________ is the set of all points in a plane equidistant from a fixed point called the __________.

2. The distance from the center of a circle to any point on the circle is called the __________ and is often denoted by r.

3. The standard form of an equation of a circle with center (h, k) and radius r is given by __________.

4. An equation of a circle written in the form $x^2 + y^2 + Ax + By + C = 0$ is called the __________ form of an equation of a circle.

Learn: Write an Equation of a Circle in Standard Form

5. Is the point (2, 7) on the circle defined by $(x - 2)^2 + (y - 7)^2 = 4$?

6. Is the point (3, 5) on the circle defined by $(x - 3)^2 + (y - 5)^2 = 36$?

7. Is the point (–4, 7) on the circle defined by $(x + 1)^2 + (y - 3)^2 = 25$?

8. Is the point (2, –7) on the circle defined by $(x + 6)^2 + (y + 1)^2 = 100$?

For Exercises 9–16, determine the center and radius of the circle.

9. $(x - 4)^2 + (y + 2)^2 = 81$

10. $(x + 3)^2 + (y - 1)^2 = 16$

11. $x^2 + (y - 2.5)^2 = 6.25$

12. $(x - 1.5)^2 + y^2 = 2.25$

13. $x^2 + y^2 = 20$

14. $x^2 + y^2 = 28$

15. $\left(x - \frac{3}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = \frac{81}{49}$

16. $\left(x + \frac{1}{7}\right)^2 + \left(y - \frac{3}{5}\right)^2 = \frac{25}{9}$

For Exercises 17–32, information about a circle is given.

a. Write an equation of the circle in standard form.

b. Graph the circle. (**See Examples 1–2**)

17. Center: (–2, 5); Radius: 1

18. Center: (–3, 2); Radius: 4

19. Center: (–4, 1); Radius: 3

20. Center: (6, –2); Radius: 6

21. Center: (–4, –3); Radius: $\sqrt{11}$

22. Center: (–5, –2); Radius: $\sqrt{21}$

23. Center: (0, 0); Radius: 2.6

24. Center: (0, 0); Radius: 4.2

25. The endpoints of a diameter are (–2, 4) and (6, –2).

26. The endpoints of a diameter are (7, 3) and (5, –1).

27. The center is (–2, –1) and a point on the circle is (6, 5).

28. The center is (3, 1) and a point on the circle is (6, 5).

29. The center is (4, 6) and the circle is tangent to the y-axis. (Informally, a line is tangent to a circle if it touches the circle in exactly one point.)

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30. The center is $(-2, -4)$ and the circle is tangent to the x-axis.

31. The center is in Quadrant IV, the radius is 5, and the circle is tangent to both the x- and y-axes.

32. The center is in Quadrant II, the radius is 3, and the circle is tangent to both the x- and y-axes.

33. Write an equation that represents the set of points that are 5 units from $(8, -11)$.

34. Write an equation that represents the set of points that are 9 units from $(-4, 16)$.

35. Write an equation of the circle that is tangent to both axes with radius $\sqrt{7}$ and center in Quadrant I.

36. Write an equation of the circle that is tangent to both axes with radius $\sqrt{11}$ and center in Quadrant III.

Learn: Write the General Form of an Equation of a Circle

37. Determine the solution set for the equation $(x + 1)^2 + (y - 5)^2 = 0$.

38. Determine the solution set for the equation $(x - 3)^2 + (y + 12)^2 = 0$.

39. Determine the solution set for the equation $(x - 17)^2 + (y + 1)^2 = -9$.

40. Determine the solution set for the equation $(x + 15)^2 + (y - 3)^2 = -25$.

For Exercises 41–54, write the equation in the form $(x - h)^2 + (y - k)^2 = c$. Then if the equation represents a circle, identify the center and radius. If the equation represents a degenerate case, give the solution set. (See Examples 3–4)

41. $x^2 + y^2 + 6x - 2y + 6 = 0$

42. $x^2 + y^2 + 12x - 14y + 84 = 0$

43. $x^2 + y^2 - 22x + 6y + 129 = 0$

44. $x^2 + y^2 - 10x + 4y - 20 = 0$

45. $x^2 + y^2 - 20y - 4 = 0$

46. $x^2 + y^2 + 22x - 4 = 0$

47. $10x^2 + 10y^2 - 80x + 200y + 920 = 0$

(*Hint*: Divide by 10 to make the x^2 and y^2 term coefficients equal to 1.)

48. $2x^2 + 2y^2 - 32x + 12y + 90 = 0$

49. $x^2 + y^2 - 4x - 18y + 89 = 0$

50. $x^2 + y^2 - 10x - 22y + 155 = 0$

51. $4x^2 + 4y^2 - 20y + 25 = 0$

52. $4x^2 + 4y^2 - 12x + 9 = 0$

53. $x^2 + y^2 - x - \frac{3}{2}y - \frac{3}{4} = 0$

54. $x^2 + y^2 - \frac{2}{3}x - \frac{5}{3}y - \frac{5}{9} = 0$

Mixed Exercises

55. A cell tower is a site where antennas, transmitters, and receivers are placed to create a cellular network. Suppose that a cell tower is located at a point $A(4, 6)$ on a map and its range is 1.5 mi. Write an equation that represents the boundary of the area that can receive a signal from the tower. Assume that all distances are in miles.

56. A radar transmitter on a ship has a range of 20 nautical miles. If the ship is located at a point $(-32, 40)$ on a map, write an equation for the boundary of the area within the range of the ship's radar. Assume that all distances on the map are represented in nautical miles.

57. Suppose that three geological study areas are set up on a map at points $A(-4, 12)$, $B(11, 3)$, and $C(0, 1)$, where all units are in miles. Based on the speed of compression waves, scientists estimate the distances from the study areas to the epicenter of an earthquake to be 13 mi, 5 mi, and 10 mi, respectively. Graph three circles whose centers are located at the study areas and whose radii are the given distances to the earthquake. Then estimate the location of the earthquake.

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58. Three fire observation towers are located at points $A(-6, -14)$, $B(14, 10)$, and $C(-3, 13)$ on a map where all units are in kilometers. A fire is located at distances of 17 km, 15 km, and 13 km, respectively, from the observation towers. Graph three circles whose centers are located at the observation towers and whose radii are the given distances to the fire. Then estimate the location of the fire.

Write About It

59. State the definition of a circle.

60. What are the advantages of writing an equation of a circle in standard form?

Expanding Your Skills

61. Find all values of y such that the distance between $(4, y)$ and $(-2, 6)$ is 10 units.

62. Find all values of x such that the distance between $(x, -1)$ and $(4, 2)$ is 5 units.

63. Find all points on the line $y = x$ that are 6 units from $(2, 4)$.

64. Find all points on the line $y = -x$ that are 4 units from $(-4, 6)$.

The standard form of an equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$. If we solve the equation for x we get equations of the form $x = h \pm \sqrt{r^2 - (y - k)^2}$. The equation $x = h + \sqrt{r^2 - (y - k)^2}$ represents the graph of the corresponding right-side semicircle, and the equation $x = h - \sqrt{r^2 - (y - k)^2}$ represents the graph of the left-side semicircle. Likewise, if we solve for y, we have $y = k \pm \sqrt{r^2 - (x - h)^2}$. These equations represent the top and bottom semicircles. For Exercises 65–68, graph the equations.

65. a. $y = \sqrt{16 - x^2}$ **b.** $y = -\sqrt{16 - x^2}$

c. $x = \sqrt{16 - y^2}$ **d.** $x = -\sqrt{16 - y^2}$

66. a. $y = \sqrt{9 - x^2}$ **b.** $y = -\sqrt{9 - x^2}$

c. $x = \sqrt{9 - y^2}$ **d.** $x = -\sqrt{9 - y^2}$

67. a. $x = -1 - \sqrt{9 - (y - 2)^2}$

b. $x = -1 + \sqrt{9 - (y - 2)^2}$

c. $y = 2 - \sqrt{9 - (x + 1)^2}$

d. $y = 2 + \sqrt{9 - (x + 1)^2}$

68. a. $x = 3 - \sqrt{4 - (y + 2)^2}$

b. $x = 3 + \sqrt{4 - (y + 2)^2}$

c. $y = -2 - \sqrt{4 - (x - 3)^2}$

d. $y = -2 + \sqrt{4 - (x - 3)^2}$

69. Find the shortest distance from the origin to a point on the circle defined by $x^2 + y^2 - 6x - 12y + 41 = 0$.

70. Find the shortest distance from the origin to a point on the circle defined by $x^2 + y^2 + 4x - 12y + 31 = 0$.

For Exercises 71–72, suppose that a point $P(x, y)$ is moving in a circular path in the xy-plane.

a. Find the equation of the circle in general form that traces the path of P.

b. Write the equation in standard form and identify the center and radius of the circular path.

71. The point P is always one-half as far from $(2, -1)$ as from $(0, 4)$.

72. The distance from $(-1, 0)$ to P is 3 times the distance from P to $(0, 4)$.

Technology Connections

For Exercises 73–76, use a graphing calculator to graph the circles on an appropriate square viewing window.

73. $x^2 + y^2 = 36$

74. $x^2 + y^2 = 49$

75. $(x - 18)^2 + (y + 20)^2 = 80$

76. $(x + 0.04)^2 + (y - 0.02)^2 = 0.01$

Go online for more practice problems.

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Lesson 1-3

Functions and Relations

What Will You Learn?

After completing this lesson, you should be able to:

- Determine Whether a Relation Is a Function
- Apply Function Notation
- Determine *x*- and *y*-Intercepts of a Function Defined by $y = f(x)$
- Determine Domain and Range of a Function
- Interpret a Function Graphically

Learn: Determine Whether a Relation Is a Function

In the physical world, many quantities that are subject to change are related to other variables. For example:

- The cost of mailing a package is related to the weight of a package.
- The minimum braking distance of a car depends on the speed of the car.
- The perimeter of a rectangle is a function of its length and width.
- The test score that a student earns is related to the number of hours of study.

In mathematics we can express the relationship between two values as a set of ordered pairs.

Definition of a Relation

A set of ordered pairs (x, y) is called a **relation** in x and y.

- The set of x values in the ordered pairs is called the **domain** of the relation.
- The set of y values in the ordered pairs is called the **range** of the relation.

Example 1: Writing a Relation from Observed Data Points

Table 1-1 shows the score y that a student earned on an algebra test based on the number of hours x spent studying one week prior to the test.

a. Write the set of ordered pairs that defines the relation.

b. Write the domain.

c. Write the range.

TABLE 1-1 Hours of Study and Corresponding Test Scores

Hours of Study, x	Test Score, y
8	92
3	58
11	98
5	72
8	86

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Solution:

a. Relation: $\{(8, 92), (3, 58), (11, 98), (5, 72), (8, 86)\}$

b. Domain: $\{8, 3, 11, 5\}$

c. Range: $\{92, 58, 98, 72, 86\}$

Check Your Work

Do not list the elements in a set more than once. In Example 1(b), the value 8 is listed in the domain one time only.

Apply the Skills

1. Complete the following exercises for the table shown below.
   - **a.** Write the set of ordered pairs that defines the relation.
   - **b.** Write the domain.
   - **c.** Write the range.

x	3	−2	5	1
y	−4	0	3	0

The data in Table 1-1 (see page 31) show two different test scores for 8 hr of study. That is, for $x = 8$, there are two different y values. In many applications, we prefer to work with relations that assign one and only one y value for a given value of x. Such a relation is called a function.

Definition of a Function

Given a relation in x and y, we say that ***y* is a function of *x*** if for each value of x in the domain, there is exactly one value of y in the range.

Example 2: Determining if a Relation Is a Function

Determine if the relation defines y as a function of x.

a. $\{(3, 1), (2, 5), (-4, 2), (-1, 0), (3, -4)\}$

b. $\{(-1, 4), (2, 3), (3, 4), (-4, 5)\}$

Solution:

a.

same x value

$\{(3, 1), (2, 5), (-4, 2), (-1, 0), (3, -4)\}$

different y values

When $x = 3$, there are two different y values: $y = 1$ and $y = -4$.

This relation is *not* a function.

b. $\{(-1, 4), (2, 3), (3, 4), (-4, 5)\}$

This relation *is* a function.

No two ordered pairs have the same x value but different y values.

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Insights

A function may not have the same *x* value paired with different *y* values. However, it is acceptable for a function to have two or more *x* values paired with the same *y* value, as shown in Example 2(b).

Apply the Skills

2. Determine if the relation defines *y* as a function of *x*.

a. $\{(8, 4), (3, -1), (5, 4)\}$ **b.** $\{(-3, 2), (9, 5), (1, 0), (-3, 1)\}$

A relation that is not a function has at least one domain element *x* paired with more than one range element *y*. For example, the ordered pairs (3, 1) and (3, −4) do not make up a function. On a graph, these two points are aligned vertically. A vertical line drawn through one point also intersects the other point (Figure 1-14). This observation leads to the vertical line test.

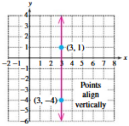


Figure 1-14

Using the Vertical Line Test

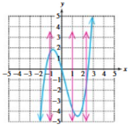
Consider a relation defined by a set of points (*x*, *y*) graphed on a rectangular coordinate system. The graph defines *y* as a function of *x* if no vertical line intersects the graph in more than one point.

Example 3: Applying the Vertical Line Test

The graphs of three relations are shown in blue. In each case, determine if the relation defines *y* as a function of *x*.

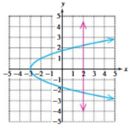
Solution:

a.



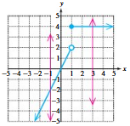
This is a function. No vertical line intersects the graph in more than one point.

b.



This is not a function. There is at least one vertical line that intersects the graph in more than one point.

c.



This is a function. No vertical line intersects the graph in more than one point.

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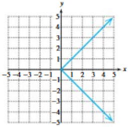
Insights

In Example 3(c) there is only one y value assigned to $x = 1$. This is because the point (1, 2) is *not* included in the graph of the function as denoted by the open dot.

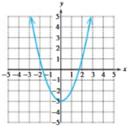
Apply the Skills

3. Determine if the given relation defines y as a function of x.

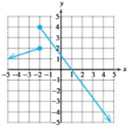
a.



b.



c.

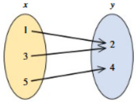


A relation can also be defined by a figure showing a "mapping" between x and y, or by an equation in x and y.

Example 4: Determining if a Relation Is a Function

Determine if the relation defines y as a function of x.

a.



b. $y^2 = x$

c. $(x - 2)^2 + (y + 1)^2 = 9$

Solution:

a. This mapping defines the set of ordered pairs: {(1, 2), (3, 2), (5, 4)}. This relation *is* a function.

No two ordered pairs have the same x value but different y values.

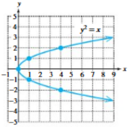
b. $y^2 = x$

Solve the equation for y.

$y = \pm\sqrt{x}$

For any $x > 0$, there are two corresponding y values.

x	y	Ordered Pairs
0	0	(0, 0)
1	1, –1	(1, 1), (1, –1)
4	2, –2	(4, 2), (4, –2)
9	3, –3	(9, 3), (9, –3)



This relation is *not* a function.

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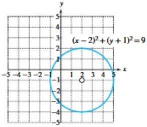
c. $(x - 2)^2 + (y + 1)^2 = 9$
This equation represents the graph of a circle with center (2, −1) and radius 3.

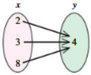
This relation is *not* a function because it fails the vertical line test.



Apply the Skills

4. Determine if the relation defines *y* as a function of *x*.

a.



b. $|y + 1| = x$

c. $x^2 + y^2 = 25$

Learn: Apply Function Notation

A function may be defined by an equation with two variables. For example, the equation $y = x - 2$ defines *y* as a function of *x*. This is because for any real number *x*, the value of *y* is the unique number that is 2 less than *x*.

When a function is defined by an equation, we often use function notation. For example, the equation $y = x - 2$ may be written in function notation as

$f(x) = x - 2$ read as "*f* of *x* equals *x* − 2."

With function notation,

- *f* is the name of the function,
- *x* is an input variable from the domain,
- $f(x)$ is the function value (or *y* value) corresponding to *x*.

A function may be evaluated at different values of *x* by using substitution.

$f(x) = x - 2$

$f(4) = (4) - 2 = 2$ $\quad$ $f(4) = 2$ can be interpreted as (4, 2).

$f(1) = (1) - 2 = -1$ $\quad$ $f(1) = -1$ can be interpreted as (1, −1).

Check Your Work

The notation $f(x)$ does *not* imply multiplication of *f* and *x*.

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Example 5: Evaluating a Function

Evaluate the function defined by $g(x) = 2x + 1$ for the given values of x.

a. $g(-2)$ **b.** $g(-1)$ **c.** $g(0)$ **d.** $g(1)$ **e.** $g(2)$

Solution:

a. $g(-2) = 2(-2) + 1$ Substitute -2 for x.

$= -3$ $g(-2) = -3$

b. $g(-1) = 2(-1) + 1$ Substitute -1 for x.

$= -1$ $g(-1) = -1$

c. $g(0) = 2(0) + 1$ Substitute 0 for x.

$= 1$ $g(0) = 1$

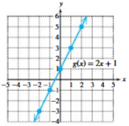
d. $g(1) = 2(1) + 1$ Substitute 1 for x.

$= 3$ $g(1) = 3$

e. $g(2) = 2(2) + 1$ Substitute 2 for x.

$= 5$ $g(2) = 5$

The function values represent the ordered pairs $(-2, -3)$, $(-1, -1)$, $(0, 1)$, $(1, 3)$, and $(2, 5)$. The line through the points represents all ordered pairs defined by this function. This is the graph of the function.



Apply the Skills

5. Evaluate the function defined by $h(x) = 4x - 3$ for the given values of x.

a. $h(-3)$ **b.** $h(-1)$ **c.** $h(0)$ **d.** $h(1)$ **e.** $h(3)$

Good Practices

The name of a function can be represented by any letter or symbol. However, lowercase letters such as *f*, *g*, *h*, and so on are often used.

Example 6: Evaluating a Function

Evaluate the function defined by $f(x) = 3x^2 + 2x$ for the given values of x.

a. $f(a)$ **b.** $f(x + h)$

Solution:

a. $f(a) = 3a^2 + 2a$ Substitute a for x.

b. $f(x + h) = 3(x + h)^2 + 2(x + h)$ Substitute $x + h$ for x.

$= 3(x^2 + 2xh + h^2) + 2x + 2h$ Simplify.

$= 3x^2 + 6xh + 3h^2 + 2x + 2h$ Recall: $(a + b)^2 = a^2 + 2ab + b^2$

Apply the Skills

6. Evaluate the function defined by $f(x) = -x^2 + 4x$ for the given values of x.

a. $f(t)$ **b.** $f(a + h)$

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Learn: Determine x- and y-Intercepts of a Function Defined by $y = f(x)$

Recall that to find an x-intercept(s) of the graph of an equation, we substitute 0 for y in the equation and solve for x. Using function notation, $y = f(x)$, this is equivalent to finding the real solutions of the equation $f(x) = 0$. To find the y-intercept, substitute 0 for x and solve the equation for y. Using function notation, this is equivalent to finding $f(0)$.

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Finding Intercepts Using Function Notation

Given a function defined by $y = f(x)$,

- The x-intercepts are the real solutions to the equation $f(x) = 0$.
- The y-intercept is given by $f(0)$.

Example 7: Finding the x- and y-Intercepts of a Function

Find the x- and y-intercepts of the function defined by $f(x) = x^2 - 4$.

Solution:

To find the x-intercept(s), solve the equation $f(x) = 0$.

$$\begin{aligned} f(x) &= x^2 - 4 \\ 0 &= x^2 - 4 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

The x-intercepts are $(2, 0)$ and $(-2, 0)$.

To find the y-intercept, evaluate $f(0)$.

$$\begin{aligned} f(0) &= (0)^2 - 4 \\ &= -4 \end{aligned}$$

The y-intercept is $(0, -4)$.

Apply the Skills

7. Find the x- and y-intercepts of the function defined by $f(x) = |x| - 5$.

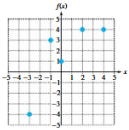
Learn: Determine Domain and Range of a Function

Given a relation defining y as a function of x, the **domain** is the set of x values in the function, and the **range** is the set of y values in the function. In Example 8, we find the domain and range from the graph of a function.

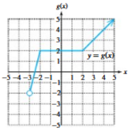
Example 8: Determining Domain and Range

Determine the domain and range for the functions shown.

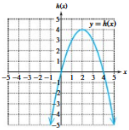
a.



b.



c.



Solution:

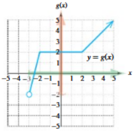
a. The graph defines the set of ordered pairs:

{(−3, −4), (−1, 3), (0, 1), (2, 4), (4, 4)}

Domain: {−3, −1, 0, 2, 4} The domain is the set of x values.

Range: {−4, 1, 3, 4} The range is the set of y values.

b.



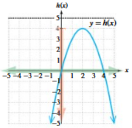
The domain is shown on the x-axis in green tint.

Domain: $\{x|x > -3\}$ or in interval notation: $(-3, \infty)$.

The range is shown on the y-axis in red tint.

Range: $\{y|y > -2\}$ or in interval notation: $(-2, \infty)$.

c.



The graph extends infinitely far downward and infinitely far to the left and right. Therefore, the domain is the set of all real numbers, x.

The domain is shown on the x-axis in green tint.

Domain: $\mathbb{R}$ or in interval notation: $(-\infty, \infty)$.

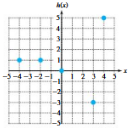
The range is shown on the y-axis in red tint.

Range: $\{y|y \leq 4\}$ or in interval notation: $(-\infty, 4]$.

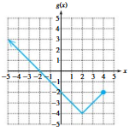
Apply the Skills

8. Determine the domain and range for the functions shown.

a.



b.



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In some cases, a function may have restrictions on the domain. For example, consider the function defined by

$$f(x) = x^2 + 2 \qquad \text{for} \qquad x \geq 0$$

The restriction on x (that is, $x \geq 0$) is explicitly stated along with the definition of the function. If no such restriction is stated, then by default, the domain is all real numbers that when substituted into the function produce real numbers in the range.

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Guidelines to Find Domain of a Function

To determine the implied domain of a function defined by $y = f(x)$,

- Exclude values of x that make the denominator of a fraction zero.
- Exclude values of x that make the radicand negative within an even-indexed root.

Example 9: Determining the Domain of a Function

Write the domain of each function in interval notation.

a. $f(x) = \dfrac{x+3}{2x-5}$

b. $g(x) = \dfrac{x}{x^2+4}$

c. $h(t) = \sqrt{2-t}$

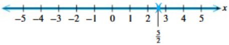
d. $m(a) = |4 + a|$

Solution:

a. $f(x) = \dfrac{x+3}{2x-5}$

The domain is all real numbers except those that make the denominator zero.

The variable x has the restriction that $2x - 5 \neq 0$. Therefore, $x \neq \frac{5}{2}$.



Domain: $\left(-\infty, \frac{5}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$

b. $g(x) = \dfrac{x}{x^2+4}$ Denominator always positive (never zero)

Domain: $(-\infty, \infty)$

The expression $x^2 \geq 0$ for all real numbers x.

Therefore, $x^2 + 4 > 0$ for all real numbers x.

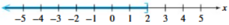
c. $h(t) = \sqrt{2-t}$

The domain is restricted to the real numbers that make the radicand greater than or equal to zero.

$$2 - t \geq 0$$
$$-t \geq -2$$
$$t \leq 2$$

Divide by –1 and reverse the inequality sign.

Domain: $(-\infty, 2)$



d. $m(a) = |4 + a|$

There are no fractions or radicals that would restrict the domain.

Domain: $(-\infty, \infty)$

The expression $|4 + a|$ is a real number for all real numbers a.

Apply the Skills

9. Write the domain of each function in interval notation.

a. $f(x) = \frac{x-2}{3x+1}$

b. $g(x) = \frac{x^2}{5}$

c. $k(x) = \sqrt{x+3}$

d. $p(x) = 2x^2 + 3x$

Learn: Interpret a Function Graphically

In Example 10, we will review the key concepts studied in this lesson by identifying characteristics of a function based on its graph.

Example 10: Identifying Characteristics of a Function

Use the function f pictured to answer the questions.

a. Determine $f(2)$.

b. Determine $f(-5)$.

c. Find all x for which $f(x) = 0$.

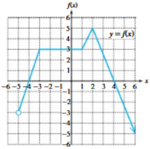
d. Find all x for which $f(x) = 3$.

e. Determine the x-intercept(s).

f. Determine the y-intercept.

g. Determine the domain of f.

h. Determine the range of f.



Solution:

a. $f(2) = 5$

$f(2) = 5$ because the function contains the point $(2, 5)$.

b. $f(-5)$ is not defined.

The point $(-5, -3)$ is not included in the function as indicated by the open dot.

c. $f(x) = 0$ for $x = -4$ and $x = 4$.

The points $(-4, 0)$ and $(4, 0)$ represent the points where $f(x) = 0$.

d. $f(x) = 3$ for all x on the interval $[-3, 1]$ and for $x = \frac{14}{5}$.

e. The x-intercepts are $(-4, 0)$ and $(4, 0)$.

f. The y-intercept is $(0, 3)$.

g. The domain is $(-5, \infty)$.

h. The range is $(-\infty, 5]$.

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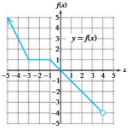
Apply the Skills

10. Use the function f pictured to find:



- **a.** $f(-2)$.
- **b.** $f(4)$.
- **c.** All x for which $f(x) = 3$.
- **d.** All x for which $f(x) = 1$.
- **e.** The x-intercept(s).
- **f.** The y-intercept.
- **g.** The domain of f.
- **h.** The range of f.

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Practice 1-3

Practice Exercises

Prerequisite Review

R.1. Solve the equation using the square root property. $8x^2 - 40 = 0$

R.2. Solve. $4x^2 - 7x - 15 = 0$

R.3. Solve. Write the solution set in interval notation. $-3y - 9 \leq 15$

R.4. Solve. $|2n + 5| = 2$

R.5. Given $2x - 5y = 20$,

a. Find the x-intercept.

b. Find the y-intercept.

Concept Connections

1. A set of ordered pairs (x, y) is called a __________ in x and y. The set of x values in the relation is called the __________ of the relation. The set of __________ values is called the range of the relation.

2. Given a function defined by $y = f(x)$, the statement $f(2) = 4$ is equivalent to what ordered pair?

3. Given a function defined by $y = f(x)$, to find the __________ -intercept, evaluate $f(0)$.

4. Given a function defined by $y = f(x)$, to find the x-intercept(s), substitute 0 for __________ and solve for x.

5. Given $f(x) = \frac{x+1}{x+5}$, the domain is restricted so that $x \neq$ __________.

6. Given $g(x) = \sqrt{x-5}$, the domain is restricted so that $x \geq$ __________.

7. Consider a relation that defines the height y of a tree for a given time t after it is planted. Does this relation define y as a function of t? Explain.

8. Consider a relation that defines a time y during the course of a year when the temperature T in Fort Collins, Colorado, is 70°. Does this relation define y as a function of T? Explain.

Learn: Determine Whether a Relation Is a Function

For Exercises 9–12,

a. Write a set of ordered pairs (x, y) that defines the relation.

b. Write the domain of the relation.

c. Write the range of the relation.

d. Determine if the relation defines y as a function of x. (**See Examples 1–2**)

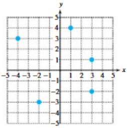
9.

Actor *x*	Number of Oscar Nominations *y*
Frances McDormand	6
Denzel Washington	8
Rami Malek	1
Viola Davis	4

10.

City *x*	Elevation at Airport (ft) *y*
Albany	285
Denver	5883
Miami	11
San Francisco	11

11.



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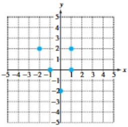
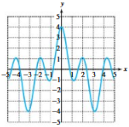


13. Answer true or false. All relations are functions.

14. Answer true or false. All functions are relations.

For Exercises 15–32, determine if the relation defines *y* as a function of *x*. (See Examples 3–4)

15.



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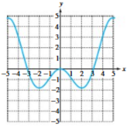

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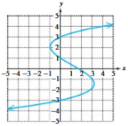

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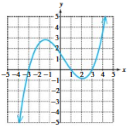

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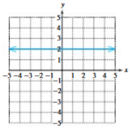

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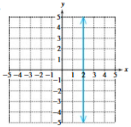

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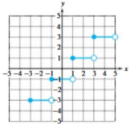

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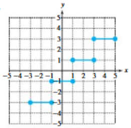

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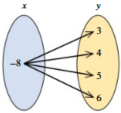

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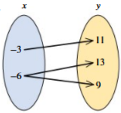

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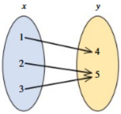

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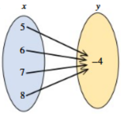

27. $(x+1)^2+(y+5)^2=25$

28. $(x+3)^2+(y+4)^2=1$

29. $y=x+3$

30. $y=x-4$

31. **a.** $y=x^2$

b. $x=y^2$

32. **a.** $y=|x|$

b. $x=|y|$

Learn: Apply Function Notation

33. The statement $f(4)=1$ corresponds to what ordered pair?

34. The statement $g(7)=-5$ corresponds to what ordered pair?

For Exercises 35–56, evaluate the function for the given value of *x*. (See Examples 5–6)

$f(x)=x^2+3x \quad g(x)=\frac{1}{x} \quad h(x)=5 \quad k(x)=\sqrt{x+1}$

35. **a.** $f(-2)$ **b.** $f(-1)$

c. $f(0)$ **d.** $f(1)$

e. $f(2)$

36. **a.** $g(-2)$ **b.** $g(-1)$

c. $g\left(-\frac{1}{2}\right)$ **d.** $g\left(\frac{1}{2}\right)$

e. $g(2)$

37. **a.** $h(-2)$ **b.** $h(-1)$

c. $h(0)$ **d.** $h(1)$

e. $h(2)$

38. **a.** $k(-2)$ **b.** $k(-1)$

c. $k(0)$ **d.** $k(1)$

e. $k(3)$

39. $g(3)$ **40.** $h(-7)$

41. $g\left(\frac{1}{3}\right)$ **42.** $h(7)$

43. $k(-5)$ **44.** $f(5)$

45. $k(8)$ **46.** $f(-5)$

47. $g(t)$ **48.** $f(a)$

49. $k(x+h)$ **50.** $h(x+h)$

51. $f(a+4)$ **52.** $f(t-3)$

53. $g(0)$ **54.** $k(-10)$

55. $f(x+h)$ **56.** $g(x+h)$

For Exercises 57–62, find and simplify $f(x+h)$. (See Example 6)

57. $f(x)=-4x^2-5x+2$

58. $f(x)=-2x^2+6x-3$

59. $f(x)=7-3x^2$

60. $f(x)=11-5x^2$

61. $f(x)=x^3+2x-5$

62. $f(x)=x^3-4x+2$

For Exercises 63–70, refer to the function $f=\{(2,3),(9,7),(3,4),(-1,6)\}$.

63. Determine $f(9)$.

64. Determine $f(-1)$.

65. Determine $f(3)$.

66. Determine $f(2)$.

67. For what value of x is $f(x)=6$?

68. For what value of x is $f(x)=7$?

69. For what value of x is $f(x)=3$?

70. For what value of x is $f(x)=4$?

71. José rides a bicycle an average of 18 mph. The distance José rides $d(t)$ (in miles) is given by $d(t)=18t$, where t is the time in hours of the ride.

a. Evaluate $d(2)$ and interpret the meaning.

b. Determine the distance José travels in 40 min.

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72. Kordell needs to drive 250 mi from Daytona Beach to Miami. After having driven *x* miles, the distance remaining $r(x)$ (in miles) is given by $r(x) = 250 - x$.

a. Evaluate $r(50)$ and interpret the meaning.

b. Determine the distance remaining after 122 mi.

73. At a restaurant, if a party has eight or more people, the gratuity is automatically added to the bill. If *x* is the cost of the meal, then the total bill $C(x)$ with an 18% gratuity and a 6% sales tax is given by: $C(x) = x + 0.06x + 0.18x$. Evaluate $C(225)$ and interpret the meaning in the context of this problem.

74. A bookstore marks up the price of a book by 40% of the cost from the publisher. Therefore, the bookstore's price to the student, $P(x)$ (in \$) after a 7.5% sales tax, is given by $P(x) = 1.075(x + 0.40x)$, where *x* is the cost of the book from the publisher. Evaluate $P(60)$ and interpret the meaning in the context of this problem.

Learn: Determine *x*- and *y*-Intercepts of a Function Defined by $y = f(x)$

For Exercises 75–84, determine the *x*- and *y*-intercepts for the given function. (See Example 7)

75. $f(x) = 2x - 4$

76. $g(x) = 3x - 12$

77. $h(x) = |x| - 8$

78. $k(x) = -|x| + 2$

79. $p(x) = -x^2 + 12$

80. $q(x) = x^2 - 8$

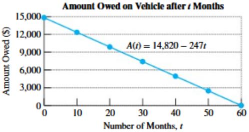
81. $r(x) = |x - 8|$

82. $s(x) = |x + 3|$

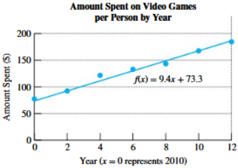
83. $f(x) = \sqrt{x} - 2$

84. $g(x) = -\sqrt{x} + 3$

85. A student decides to finance a used car over a 5-yr (60-month) period. After making a down payment of \$2000, the remaining cost of the car including tax and interest is \$14,820. The amount owed $y = A(t)$ (in \$) is given by $A(t) = 14{,}820 - 247t$, where *t* is the number of months after purchase and $0 \le t \le 60$. Determine the *t*-intercept and *y*-intercept and interpret their meanings in context.



86. The amount spent on video games per person in the United States has been increasing since 2010. (*Source*: www.census.gov) The function defined by $f(x) = 9.4x + 73.3$ represents the amount spent $f(x)$ (in \$) *x* years since 2010. Determine the *y*-intercept and interpret its meaning in context.



Learn: Determine Domain and Range of a Function

For Exercises 87–96, determine the domain and range of the function. (See Example 8)

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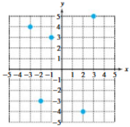

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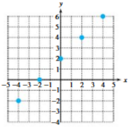

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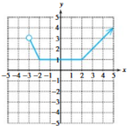

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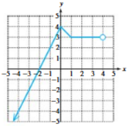

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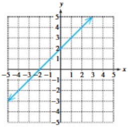

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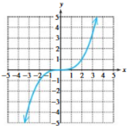

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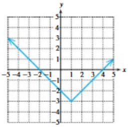

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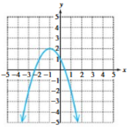

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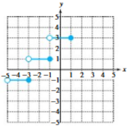

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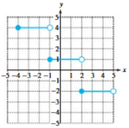

For Exercises 97–110, write the domain in interval notation. (See Example 9)

97. a. $f(x) = \frac{x-3}{x-4}$

b. $g(x) = \frac{x-3}{x^2-4}$

c. $h(x) = \frac{x-3}{x^2+4}$

98. a. $k(x) = \frac{x+6}{x-2}$

b. $j(x) = \frac{x+6}{x^2+2}$

c. $p(x) = \frac{x+6}{x^2-2}$

99. a. $a(x) = \sqrt{x+9}$

b. $b(x) = \sqrt{9-x}$

c. $c(x) = \frac{1}{\sqrt{x+9}}$

100. a. $y(t) = \sqrt{16-t}$

b. $w(t) = \sqrt{t-16}$

c. $z(t) = \frac{1}{\sqrt{16-t}}$

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101. **a.** $f(t) = \sqrt[3]{t-5}$

b. $g(t) = \sqrt[3]{5-t}$

c. $h(t) = \dfrac{1}{\sqrt[3]{t-5}}$

102. **a.** $k(x) = \sqrt[5]{3+x}$

b. $m(x) = \sqrt[5]{x-3}$

c. $n(x) = \dfrac{1}{\sqrt[5]{x-3}}$

103. **a.** $f(x) = x^2 - 3x - 28$

b. $g(x) = \dfrac{x+2}{x^2 - 3x - 28}$

c. $h(x) = \dfrac{x^2 - 3x - 28}{x+2}$

104. **a.** $r(x) = x^2 - 4x - 12$

b. $s(x) = \dfrac{x^2 - 4x - 12}{x+1}$

c. $t(x) = \dfrac{x+1}{x^2 - 4x - 12}$

105. **a.** $w(x) = |x+1| + 4$

b. $y(x) = \dfrac{x}{|x+1| + 4}$

c. $z(x) = \dfrac{x}{|x+1| - 4}$

106. **a.** $f(a) = 8 - |a-2|$

b. $g(a) = \dfrac{5}{8 - |a-2|}$

c. $h(a) = \dfrac{5}{8 + |a-2|}$

107. **a.** $f(x) = \sqrt{x+15}$

b. $g(x) = \sqrt{x+15} - 2$

c. $k(x) = \dfrac{5}{\sqrt{x+15} - 2}$

108. **a.** $f(c) = \sqrt{c+20}$

b. $g(c) = \sqrt{c+20} - 1$

c. $h(c) = \dfrac{-4}{\sqrt{c+20} - 1}$

109. **a.** $p(x) = 2x + 1$

b. $q(x) = 2x + 1;\ x \ge 0$

c. $r(x) = 2x + 1;\ 0 \le x < 7$

110. **a.** $m(x) = 3x - 7$

b. $n(x) = 3x - 7;\ x < 0$

c. $p(x) = 3x - 7;\ -2 < x < 2$

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Learn: Interpret a Function Graphically

For Exercises 111–114, use the graph of $y = f(x)$ to answer the following. (See Example 10)

a. Determine $f(-2)$.

b. Determine $f(3)$.

c. Find all x for which $f(x) = -1$.

d. Find all x for which $f(x) = -4$.

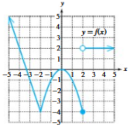
e. Determine the x-intercept(s).

f. Determine the y-intercept.

g. Determine the domain of f.

h. Determine the range of f.

111.



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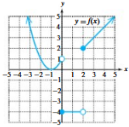

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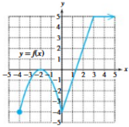

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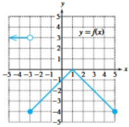


Mixed Exercises

For Exercises 115–122, write a function that represents the given statement.

115. Suppose that a phone card has 400 min. Write a relationship that represents the number of minutes remaining $r(x)$ as a function of the number of minutes already used x.

116. Suppose that a roll of wire has 200 ft. Write a relationship that represents the amount of wire remaining $w(x)$ as a function of the number of feet of wire x already used.

117. Given an equilateral triangle with sides of length x, write a relationship that represents the perimeter $P(x)$ as a function of x.

118. In an isosceles triangle, two angles are equal in measure. If the third angle is x degrees, write a relationship that represents the measure of one of the equal angles $A(x)$ as a function of x.

119. Two adjacent angles form a right angle. If the measure of one angle is x degrees, write a relationship representing the measure of the other angle $C(x)$ as a function of x.

120. Two adjacent angles form a straight angle (180°). If the measure of one angle is x degrees, write a relationship representing the measure of the other angle $S(x)$ as a function of x.

121. Write a relationship for a function whose $f(x)$ values are 2 less than three times the square of x.

122. Write a relationship for a function whose $f(x)$ values are 3 more than the principal square root of x.

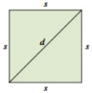
Write About It

123. If two points align vertically, then the points do not define y as a function of x. Explain why.

124. Given a function defined by $y = f(x)$, explain how to determine the x- and y-intercepts.

Expanding Your Skills

125. Given a square with sides of length s, diagonal of length d, perimeter P, and area A,



a. Write P as a function of s.

b. Write A as a function of s.

c. Write A as a function of P.

d. Write P as a function of A.

e. Write d as a function of s.

f. Write s as a function of d.

g. Write P as a function of d.

h. Write A as a function of d.

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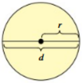
126. Given a circle with radius r, diameter d, circumference C, and area A,



a. Write C as a function of r.

b. Write A as a function of r.

c. Write r as a function of d.

d. Write d as a function of r.

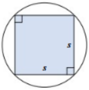
e. Write C as a function of d.

f. Write A as a function of d.

g. Write A as a function of C.

h. Write C as a function of A.

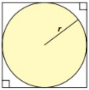
127. A square with sides of length s is circumscribed by a circle.



a. Write the area A of the circle as a function of s.

b. Write the circumference C of the circle as a function of s.

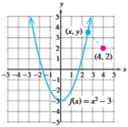
128. A circle of radius r is inscribed within a square.



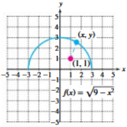
a. Write the area A of the square as a function of r.

b. Write the perimeter P of the square as a function of r.

129. Write a function d that expresses the distance from a point on the graph of $f(x) = x^2 - 3$ to the point $(4, 2)$.



130. Write a function d that expresses the distance from a point on the graph of $f(x) = \sqrt{9 - x^2}$ to the point $(1, 1)$.



Go online for more practice problems.

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Lesson 1-4

Linear Equations in Two Variables and Linear Functions

What Will You Learn?

After completing this lesson, you should be able to:

- Graph Linear Equations in Two Variables
- Determine the Slope of a Line
- Apply the Slope-Intercept Form of a Line
- Compute Average Rate of Change
- Solve Equations and Inequalities Graphically

Learn: Graph Linear Equations in Two Variables

The median incomes for individuals for all levels of education have shown an increasing trend since 1990. However, the median income for individuals with a bachelor's degree is consistently greater than for individuals whose highest level of education is a high school degree or equivalent (Figure 1-15). (*Source*: U.S. Census Bureau, www.census.gov)

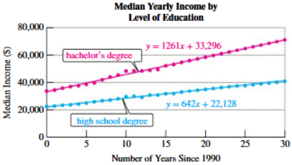

Figure 1-15

The graph in Figure 1-15 is called a scatter plot. A **scatter plot** is a visual representation of a set of points. In this case, the x values represent the number of years since 1990, and the y values represent the median income in dollars. The line that models each set of data is called a **regression line** and is found by using techniques taught in a first course in statistics. The equations that represent the two lines are called linear equations in two variables.

Linear Equation in Two Variables

Let A, B, and C represent real numbers such that A and B are not both zero. A **linear equation** in the variables x and y is an equation that can be written in the form:

$Ax + By = C$ This is called the **standard form** of an equation of a line.

Note: A linear equation $Ax + By = C$ has variables x and y each of first degree.

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Good Practices

For an equation in standard form, the value of *A*, *B*, and *C* are usually taken to be integers where *A*, *B*, and *C* share no common factors.

In Example 1, we demonstrate that the graph of a linear equation $Ax + By = C$ is a line. The line may be slanted, horizontal, or vertical depending on the coefficients *A*, *B*, and *C*.

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Example 1: Graphing Linear Equations

Graph the line represented by each equation.

a. $2x + 3y = 6$ **b.** $x = -3$ **c.** $2y = 4$

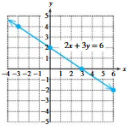
Solution:

a. Solve the equation for *y*. Then substitute arbitrary values of *x* into the equation and solve for the corresponding values of *y*.

$$2x + 3y = 6$$
$$3y = -2x + 6$$
$$y = -\frac{2}{3}x + 2$$

In the table we have selected convenient values of *x* that are multiples of 3.

x	*y*
–3	4
0	2
3	0
6	–2



b. $x = -3$

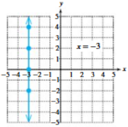
The solutions to this equation must have an *x*-coordinate of –3.

The *y* variable can be *any* real number.

x	*y*
–3	–2
–3	0
–3	2
–3	4

x must be –3.

y can be any real number.



Insights

The graph of a vertical line will have no *y*-intercept unless the line is the *y*-axis itself.

c. $2y = 4$

$y = 2$

Solve for y.

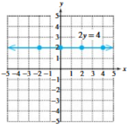
The solutions to this equation must have a y-coordinate of 2.

The x variable can be *any* real number.

x	y
−2	2
0	2
2	2
4	2

x can be any real number.

y must be 2.



Insights

The graph of a horizontal line will have no x-intercept unless the line is the x-axis itself.

Apply the Skills

1. Graph the line represented by each equation.

a. $4x + 2y = 2$ **b.** $y = 1$ **c.** $-3x = 12$

Check Your Work

The graph of a linear equation is a line. Therefore, a minimum of two points is needed to graph the line. A third point can be used to verify that the line is graphed correctly. The points must all line up.

Learn: Determine the Slope of a Line

One of the important characteristics of a nonvertical line is that for every 1 unit of change in the horizontal variable, the vertical change is a constant m called the **slope** of the line. For example, consider the line representing the median income for individuals with a bachelor's degree, x years since the year 1990. The line in Figure 1-16 has a slope of \$1261. This means that median income for individuals with a bachelor's degree increased on average by \$1261 per year during this time period.

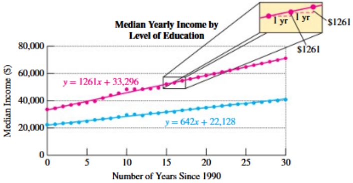


Figure 1-16

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Consider any two distinct points (x_1, y_1) and (x_2, y_2) on a line (Figure 1-17). The slope m of the line through the points is the ratio between the change in the y values $(y_2 - y_1)$ and the change in the x values $(x_2 - x_1)$. In many applications in the sciences, the change in a variable is denoted by the Greek letter Δ(delta). Therefore, $(y_2 - y_1)$ can be represented by Δy and $(x_2 - x_1)$ can be represented by Δx.

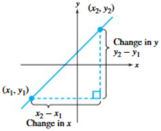


Figure 1-17

Slope Formula

The **slope** of a line passing through the distinct points (x_1, y_1) and (x_2, y_2) is

change in y (rise)

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \text{ provided that } x_2 - x_1 \neq 0$$

change in x (run)

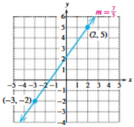
Example 2: Finding the Slope of a Line Through Two Points

Find the slope of the line passing through the given points.

a. $(-3, -2)$ and $(2, 5)$ **b.** $\left(-\frac{5}{2}, 0\right)$ and $(1, -7)$

Solution:

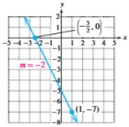
a. $(-3, -2)$ and $(2, 5)$



(x_1, y_1) and (x_2, y_2)	Label the points.
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-2)}{2 - (-3)} = \frac{7}{5}$	Apply the slope formula.

A line with a positive slope "*rises*" upward from left to right.

b. $\left(-\frac{5}{2}, 0\right)$ and $(1, -7)$



(x_1, y_1) and (x_2, y_2)	Label the points.
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7 - 0}{1 - \left(-\frac{5}{2}\right)}$	Apply the slope formula.

$$= \frac{-7}{\frac{7}{2}} = -7 \cdot \frac{2}{7} = -2$$

A line with a negative slope "*falls*" downward from left to right.

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Apply the Skills

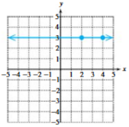
2. Find the slope of the line passing through the given points.

a. $(-4, 1)$ and $(2, -2)$ **b.** $\left(\frac{3}{4}, 2\right)$ and $(-3, 17)$

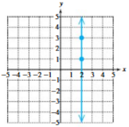
Example 3: Finding the Slope of Horizontal and Vertical Lines

Find the slope of each line.

a.



b.



Solution:

a. By inspection, we see that between any two points on the graph, the vertical change is zero, so the slope is zero.

To compute this numerically, select any two points on the line such as $(2, 3)$ and $(4, 3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 3}{4 - 2} = \frac{0}{2} = 0$$

b. To find the slope, select any two points on the line such as $(2, 1)$ and $(2, 3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{2 - 2} = \frac{2}{0} \text{ (undefined)}$$

By inspection, we see that between any two points on the line, the change in *x* is zero. This makes the slope undefined because the ratio representing the slope has a divisor of zero.

Apply the Skills

3. Fill in the blank.

a. The slope of a vertical line is __________.

b. The slope of a horizontal line is __________.

From Example 1, we see that a linear equation may represent the graph of a slanted line, a horizontal line, or a vertical line. From Examples 2 and 3, we see that a line may have a positive slope, a negative slope, a zero slope, or an undefined slope.

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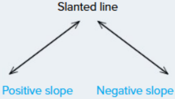
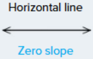
Linear Equations and Slopes of Lines

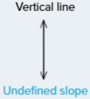
- $Ax + By = C$ where $A \neq 0$ and $B \neq 0$)


Slanted line
Positive slope
Negative slope

- $y = k$ where k is a constant


Horizontal line
Zero slope

- $x = k$ where k is a constant


Vertical line
Undefined slope

Learn: Apply the Slope-Intercept Form of a Line

The slope formula can be used to develop the slope-intercept form of a line. Suppose that a line has a slope *m* and *y*-intercept (0, *b*). Let (*x*, *y*) be any other point on the line. From the slope formula, we have:

$\frac{y-b}{x-0} = m$	Slope formula
$y - b = mx$	Multiply by *x*.
$y = mx + b$	This is slope-intercept form. Slope-intercept form has the *y* variable isolated.

Slope-Intercept Form of a Line

Given a line with slope *m* and *y*-intercept (0, *b*), the **slope-intercept form** of the line is given by $y = mx + b$.

Insight

An equation of a vertical line takes the form $x = k$, where k is a constant. Because there is no *y* variable and because the slope is undefined, an equation of a vertical line cannot be written in slope-intercept form.

The slope-intercept form of a line is particularly useful because we can identify the slope and *y*-intercept by inspection. For example:

$y = \frac{2}{3}x - 5$		$m = \frac{2}{3}$	*y*-intercept: $(0, -5)$
$y = x + 4$		$m = 1$	*y*-intercept: $(0, 4)$
$y = 2x$	(or $y = 2x + 0$)	$m = 2$	*y*-intercept: $(0, 0)$
$y = 6$	(or $y = 0x + 6$)	$m = 0$	*y*-intercept: $(0, 6)$

If the slope and *y*-intercept of a line are known, we can graph the line. This is demonstrated in Example 4.

Example 4: Using the Slope and *y*-Intercept to Graph a Line

Given $3x + 4y = 4$,

a. Write the equation in slope-intercept form.

b. Determine the slope and *y*-intercept.

c. Graph the line by using the slope and *y*-intercept.

Solution:

a. $3x + 4y = 4$

$4y = -3x + 4$

$y = -\frac{3}{4}x + 1$

To write an equation in slope-intercept form, isolate the *y* variable.

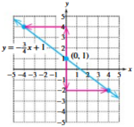
b. The slope is $m = -\frac{3}{4}$.

Slope-intercept form
The slope is the coefficient on *x*.

The *y*-intercept is $(0, 1)$.

The constant term gives the *y*-intercept.

c.



To graph the line, first plot the *y*-intercept $(0, 1)$.

Then begin at the *y*-intercept, and use the slope to find a second point on the line. In this case, the slope can be interpreted as the following two ratios:

$$m = \frac{-3}{4}\begin{matrix}\leftarrow \text{Move down 3 units.}\\ \leftarrow \text{Move to the right 4 units.}\end{matrix}$$

$$m = \frac{3}{-4}\begin{matrix}\leftarrow \text{Move up 3 units.}\\ \leftarrow \text{Move to the left 4 units.}\end{matrix}$$

Apply the Skills

4. Given $2x + 4y = 8$,

a. Write the equation in slope-intercept form.

b. Determine the slope and *y*-intercept.

c. Graph the line by using the slope and *y*-intercept.

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Notice that the slope-intercept form of a line $y = mx + b$ has the y variable isolated and defines y in terms of x. Therefore, an equation written in slope-intercept form defines y as a function of x. In Example 4, $y = -\frac{3}{4}x + 1$ can be written using function notation as $f(x) = -\frac{3}{4}x + 1$.

Definition of Linear and Constant Functions

Let m and b represent real numbers where $m \neq 0$. Then,

- A function defined by $f(x) = mx + b$ is a **linear function**. The graph of a linear function is a slanted line.
- A function defined by $f(x) = b$ is a **constant function**. The graph of a constant function is a horizontal line.

The slope-intercept form of a line can be used as a tool to define a linear function given a point on the line and the slope.

Example 5: Writing an Equation of a Line Given a Point and the Slope

Write an equation of the line that passes through the point (2, –3) and has slope –4. Then write the linear equation using function notation, where $y = f(x)$.

Solution:

Given $m = -4$ and $(2, -3)$. We need to find an equation of the form $y = mx + b$.

$y = mx + b$

$y = -4x + b$ The value of m is given as -4.

$-3 = -4(2) + b$ Substitute $x = 2$ and $y = -3$ from the given point (2, –3).

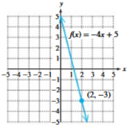
$-3 = -8 + b$ Solve for b.

$b = 5$

$y = mx + b$

$y = -4x + 5$ Substitute $m = -4$ and $b = 5$ into the equation $y = mx + b$.

$f(x) = -4x + 5$ Write the relation using function notation.



From the graph, we see that the graph of $f(x) = -4x + 5$ does indeed pass through the point (2, –3) and has slope –4.

Apply the Skills

5. Write an equation of the line that passes through the point (–1, –4) and has slope 3. Then write the equation using function notation.

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Learn: Compute Average Rate of Change

The graphs of many functions are not linear. However, we often use linear approximations to analyze nonlinear functions on small intervals. For example, the graph in Figure 1-18 shows the concentration of a medication in the blood stream for an individual over a period of 9 hr.

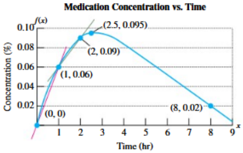


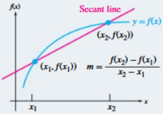
Figure 1-18

A line drawn through at least two points on a curve is called a **secant line.** In Figure 1-18, the average rate of change in concentration between two points on the graph is the slope of the secant line through the points. Notice that the slope of the secant line between $x = 0$ and $x = 1$ (shown in red) is greater than the slope of the secant line between $x = 1$ and $x = 2$ (shown in green). This means that the average increase in concentration is greater over the first hour than over the second hour.

Average Rate of Change of a Function

Suppose that the points (x_1, y_1) and (x_2, y_2) are points on the graph of a function f. Using function notation, these are the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

If f is defined on the interval $[x_1, x_2]$, then the **average rate of change** of f on the interval $[x_1, x_2]$ is the slope of the secant line containing $(x_1, f(x_1))$ and $(x_2, f(x_2))$.



Average rate of change: $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$ or $m = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

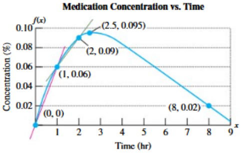
Example 6: Computing Average Rate of Change

Determine the average rate of change of the concentration of a medication

a. from $x_1 = 0$ to $x_2 = 1$.

b. from $x_1 = 1$ to $x_2 = 2$.

c. Interpret the results from parts (a) and (b).



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Solution:

a. Average rate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{0.06 - 0}{1} = 0.06$

b. Average rate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{0.09 - 0.06}{1} = 0.03$

c. The concentration rose by an average of 0.06% per hour during the first hour.

The concentration level rose by an average of 0.03% per hour during the second hour.

Apply the Skills

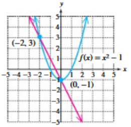
6. Refer to the graph in Example 6.

a. Determine the average rate of change of concentration level from $x_1 = 2.5$ to $x_2 = 8$. Round to 3 decimal places.

b. Interpret the results from part (a).

Example 7: Computing Average Rate of Change

Given the function defined by $f(x) = x^2 - 1$, determine the average rate of change from $x_1 = -2$ to $x_2 = 0$.



Solution:

Average rate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$= \frac{f(0) - f(-2)}{0 - (-2)} = \frac{-1 - 3}{2} = -2$$

The average rate of change is −2.

Apply the Skills

7. Given the function defined by $f(x) = x^3 + 2$, determine the average rate of change from $x_1 = -3$ to $x_2 = 0$.

Learn: Solve Equations and Inequalities Graphically

In many settings, the use of technology can provide a numerical and visual interpretation of an algebraic problem. For example, consider the equation $-x - 1 = x + 5$.

$$-x - 1 = x + 5$$
$$-6 = 2x$$
$$-3 = x \qquad \text{The solution set is } \{-3\}.$$

Now suppose that we create two functions from the left and right sides of the equation. We have $Y_1 = -x - 1$ and $Y_2 = x + 5$.

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Figure 1-19 shows that the graphs of the two lines intersect at $(-3, 2)$. The x-coordinate of the point of intersection is the solution to the equation $-x - 1 = x + 5$. That is, $Y_1 = Y_2$ when $x = -3$.

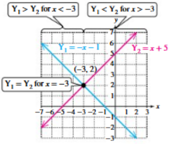


Figure 1-19

The graphs $Y_1 = -x - 1$ and $Y_2 = x + 5$ can also be used to find the solution sets to the related inequalities.

$-x - 1 < x + 5$	The solution set is the set of x values for which $Y_1 < Y_2$. This is the interval where the blue line is below the red line. The solution set is $(-3, \infty)$.
$-x - 1 > x + 5$	The solution set is the set of x values for which $Y_1 > Y_2$. This is the interval where the blue line is *above* the red line. The solution set is $(-\infty, -3)$.

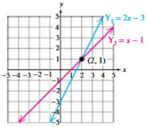
Example 8: Solving Equations and Inequalities Graphically

Solve the equations and inequalities graphically.

a. $2x - 3 = x - 1$ **b.** $2x - 3 < x - 1$ **c.** $2x - 3 > x - 1$

Solution:

a. The left side of the equation is graphed as $Y_1 = 2x - 3$. The right side of the equation is graphed as $Y_2 = x - 1$. The point of intersection is $(2, 1)$. Therefore, $Y_1 = Y_2$ for $x = 2$. The solution set is $\{2\}$.



b. $Y_1 < Y_2$ to the *left* of $x = 2$. (That is, the blue line is below the red line for $x < 2$.) In interval notation the solution set is $(-\infty, 2)$.

c. $Y_1 > Y_2$ to the *right* of $x = 2$. (That is, the blue line is above the red line for $x > 2$.) In interval notation the solution set is $(2, \infty)$.

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Insights

The solution set to the inequality $2x - 3 \leq x - 1$ includes equality, so the right endpoint would be included: $(-\infty, 2]$.

The solution set to the inequality $2x - 3 \geq x - 1$ includes equality, so the left endpoint would be included: $[2, \infty)$.

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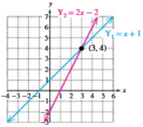
Apply the Skills

8. Use the graph to solve the equations and inequalities.



a. $x + 1 = 2x - 2$

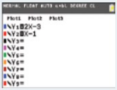
b. $x + 1 \leq 2x - 2$

c. $x + 1 \geq 2x - 2$

TECHNOLOGY CONNECTIONS

Verifying Solutions to an Equation

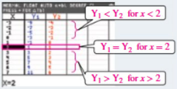
We can verify the solutions to the equations and inequalities from Example 8 on a graphing calculator.



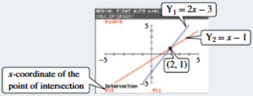
```
Plot1  Plot2  Plot3
\Y1=2X−3
\Y2=X−1
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
```

The solutions can be verified numerically by using the Table feature on the calculator. First enter $Y_1 = 2x - 3$ and $Y_2 = x - 1$.

Then display the table values for Y_1 and Y_2 for $x = 2$ and for x values less than and greater than 2.



Display the graphs of Y_1 and Y_2 and use the Intersect feature to determine the point of intersection.



In Example 9 we solve the equation $6x - 2(x + 2) - 5 = 0$. Notice that one side is zero. We can check the solution graphically by determining where the related function $Y_1 = 6x - 2(x + 2) - 5$ intersects the x-axis.

Example 9: Solving Equations and Inequalities Graphically

a. Solve the equation $6x - 2(x + 2) - 5 = 0$ and verify the solution graphically on a graphing utility.

b. Use the graph to find the solution set to the inequality $6x - 2(x + 2) - 5 \leq 0$.

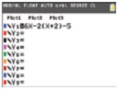
c. Use the graph to find the solution set to the inequality $6x - 2(x + 2) - 5 \geq 0$.

Solution:

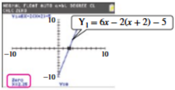
a.
$$\begin{aligned} 6x - 2(x + 2) - 5 &= 0 \\ 6x - 2x - 4 - 5 &= 0 \\ 4x - 9 &= 0 \\ x &= \frac{9}{4} \end{aligned}$$

The solution set is $\left\{\frac{9}{4}\right\}$.

To verify the solution graphically enter the left side of the equation as $Y_1 = 6x - 2(x + 2) - 5$.



```
NORMAL FLOAT AUTO a+bi DEGREE CL
Plot1  Plot2  Plot3
\Y1=6X-2(X+2)-5
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
```



$Y_1 = 6x - 2(x + 2) - 5$

Using the Zero feature, we have $Y_1 = 0$ for $x = 2.25$. This is consistent with the solution $x = \frac{9}{4}$.

b. To solve $6x - 2(x + 2) - 5 \leq 0$ determine the values of x for which $Y_1 \leq 0$ (where the function is on or below the x-axis).

The solution set is $\left(-\infty, \frac{9}{4}\right]$.

c. To solve $6x - 2(x + 2) - 5 \geq 0$ determine the values of x for which $Y_1 \geq 0$ (where the function is on or above the x-axis).

The solution set is $\left[\frac{9}{4}, \infty\right)$.

Apply the Skills

9. **a.** Solve the equation $3x - (x + 4) - 1 = 0$ and verify the solution graphically on a graphing utility.

b. Use the graph to find the solution set to the inequality $3x - (x + 4) - 1 \leq 0$.

c. Use the graph to find the solution set to the inequality $3x - (x + 4) - 1 \geq 0$.

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Practice 1-4

Practice Exercises

Prerequisite Review

R.1. Determine the *x*- and *y*-intercepts for $h(x) = 6x - 42$.

R.2. Solve $-7x - 8y = 1$ for *y*.

For Exercises R.3–R.4, solve the inequality. Write the solution set in interval notation.

R.3. $-4t + 5 < 13$

R.4. $6p - 2 \geq 5p + 8$

R.5. Given the function defined by $g(x) = -x^2 + 3x + 2$, find $g(-1)$.

Concept Connections

1. A __________ equation in the variables *x* and *y* can be written in the form $Ax + By = C$, where *A* and *B* are not both zero.

2. An equation of the form $x = k$ where *k* is a constant represents the graph of a __________ line.

3. An equation of the form $y = k$ where *k* is a constant represents the graph of a __________ line.

4. Write the formula for the slope of a line between the two distinct points (x_1, y_1) and (x_2, y_2).

5. The slope of a horizontal line is __________ and the slope of a vertical line is __________.

6. A function *f* is a linear function if $f(x) =$ __________, where *m* represents the slope and (0, b) represents the *y*-intercept.

7. If *f* is defined on the interval $[x_1, x_2]$, then the average rate of change of *f* on the interval $[x_1, x_2]$ is given by the formula __________.

8. The graph of a constant function defined by $f(x) = b$ is a (horizontal/vertical) line.

Learn: Graph Linear Equations in Two Variables

For Exercises 9–20, graph the equation and identify the *x*- and *y*-intercepts. (See Example 1)

9. $-3x + 4y = 12$

10. $-2x + y = 4$

11. $2y = -5x + 2$

12. $3y = -4x + 6$

13. $x = -6$

14. $y = 4$

15. $5y + 1 = 11$

16. $3x - 2 = 4$

17. $0.02x + 0.05y = 0.1$

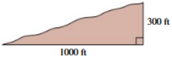
18. $0.03x + 0.07y = 0.21$

19. $2x = 3y$

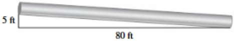
20. $2x = -5y$

Learn: Determine the Slope of a Line

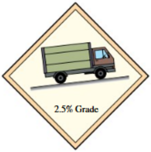
21. Find the average slope of the hill.



22. Find the absolute value of the slope of the storm drainage pipe.



23. The road sign shown in the figure indicates the percent grade of a hill. This gives the slope of the road as the change in elevation per 100 horizontal feet. Given a 2.5% grade, write this as a slope in fractional form.



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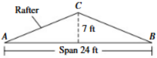
24. The pitch of a roof is defined as $\frac{\text{rafter rise}}{\text{rafter run}}$ and the fraction is typically written with a denominator of 12. Determine the pitch of the roof from point *A* to point *C*.



For Exercises 25–36, determine the slope of the line passing through the given points. (See Example 2)

25. (4, −7) and (2, −1)

26. (−3, −8) and (4, 6)

27. (17, 9) and (42, −6)

28. (−9, 4) and (−1, −6)

29. (30, −52) and (−22, −39)

30. (−100, −16) and (84, 30)

31. (2.6, 4.1) and (9.5, −3.7)

32. (8.5, 6.2) and (−5.1, 7.9)

33. $\left(\frac{3}{4}, 6\right)$ and $\left(\frac{5}{2}, 1\right)$

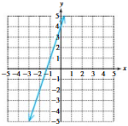
34. $\left(-3, \frac{2}{5}\right)$ and $\left(4, \frac{3}{10}\right)$

35. $(3\sqrt{6}, 2\sqrt{5})$ and $(\sqrt{6}, \sqrt{5})$

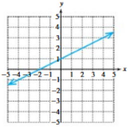
36. $(2\sqrt{11}, -3\sqrt{3})$ and $(\sqrt{11}, -5\sqrt{3})$

For Exercises 37–42, determine the slope of the line. (See Examples 2–3)

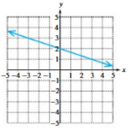
37.



38.



39.



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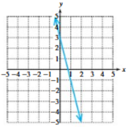

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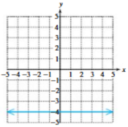
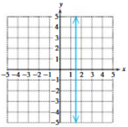


42.



43. What is the slope of a line perpendicular to the *x*-axis?

44. What is the slope of a line parallel to the *x*-axis?

45. What is the slope of a line defined by $y = -7$?

46. What is the slope of a line defined by $x = 2$?

47. If the slope of a line is $\frac{4}{5}$, how much vertical change will be present for a horizontal change of 52 ft?

48. If the slope of a line is $\frac{5}{8}$, how much horizontal change will be present for a vertical change of 216 m?

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49. Suppose that $y = P(t)$ represents the population of a city at time t. What does $\frac{\Delta P}{\Delta t}$ represent?

50. Suppose that $y = d(t)$ represents the distance that an object travels in time t. What does $\frac{\Delta d}{\Delta t}$ represent?

Learn: Apply the Slope-Intercept Form of a Line

For Exercises 51–62,

a. Write the equation in slope-intercept form if possible, and determine the slope and y-intercept.

b. Graph the equation using the slope and y-intercept. (See Example 4)

51. $2x - 4y = 8$ **52.** $3x - y = 6$

53. $3x = 2y - 4$ **54.** $5x = 3y - 6$

55. $3x = 4y$ **56.** $-2x = 3y$

57. $2y - 6 = 8$ **58.** $3y + 9 = 6$

59. $0.02x + 0.06y = 0.06$

60. $0.03x + 0.04y = 0.12$

61. $\frac{x}{4} + \frac{y}{7} = 1.$ **62.** $\frac{x}{3} + \frac{y}{4} = 1$

For Exercises 63–64, determine if the function is linear, constant, or neither.

63. a. $f(x) = -\frac{3}{4}x$ **b.** $g(x) = -\frac{3}{4}x - 3$

c. $h(x) = -\frac{3}{4x}$ **d.** $k(x) = -\frac{3}{4}$

64. a. $m(x) = 5x + 1$ **b.** $n(x) = \frac{5}{x} + 1$

c. $p(x) = 5$ **d.** $q(x) = 5x$

For Exercises 65–74,

a. Use slope-intercept form to write an equation of the line that passes through the given point and has the given slope.

b. Write the equation using function notation where $y = f(x)$. **(See Example 5)**

65. $(0, 9)$; $m = \frac{1}{2}$ **66.** $(0, -4)$; $m = \frac{1}{3}$

67. $(1, -6)$; $m = -3$ **68.** $(2, -8)$; $m = -5$

69. $(-5, -3)$; $m = \frac{2}{3}$ **70.** $(-4, -2)$; $m = \frac{3}{2}$

71. $(2, 5)$; $m = 0$ **72.** $(-1, -3)$; $m = 0$

73. $(3.6, 5.1)$; $m = 1.2$ **74.** $(1.2, 2.8)$; $m = 2.4$

For Exercises 75–78,

a. Use slope-intercept form to write an equation of the line that passes through the two given points.

b. Then write the equation using function notation where $y = f(x)$.

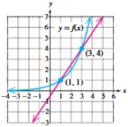
75. $(4, 2)$ and $(0, -6)$ **76.** $(-8, 1)$ and $(0, -3)$

77. $(7, -3)$ and $(4, 1)$ **78.** $(2, -4)$ and $(-1, 3)$

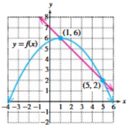
Learn: Compute Average Rate of Change

For Exercises 79–80, find the slope of the secant line pictured in red. (See Example 6)

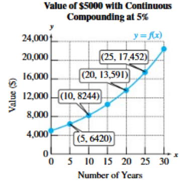
79.



80.



81. The function given by $y = f(x)$ shows the value of $5000 invested at 5% interest compounded continuously, x years after the money was originally invested.

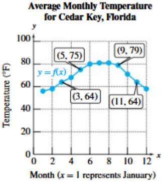


a. Find the average amount earned per year between the 5th year and 10th year.

b. Find the average amount earned per year between the 20th year and the 25th year.

c. Based on the answers from parts (a) and (b), does it appear that the rate at which annual income increases is increasing or decreasing with time?

82. The function given by $y = f(x)$ shows the average monthly temperature (in °F) for Cedar Key. The value of x is the month number and $x = 1$ represents January.



- **a.** Find the average rate of change in temperature between months 3 and 5 (March and May).
- **b.** Find the average rate of change in temperature between months 9 and 11 (September and November).
- **c.** Comparing the results in parts (a) and (b), what does a positive rate of change mean in the context of this problem? What does a negative rate of change mean?

83. The number $N(t)$ of new cases of a flu outbreak for a given city is given by $N(t) = 5000 \cdot 2^{-0.04t^2}$, where t is the number of months since the outbreak began.

- **a.** Find the average rate of change in the number of new flu cases between months 0 and 2, and interpret the result. Round to the nearest whole unit.
- **b.** Find the average rate of change in the number of new flu cases between months 4 and 6, and between months 10 and 12.
- **c.** Use a graphing utility to graph the function. Use the graph and the average rates of change found in parts (a) and (b) to discuss the pattern of the number of new flu cases.

84. The speed $v(L)$ (in m/sec) of an ocean wave in deep water is approximated by $v(L) = 1.2\sqrt{L}$, where L (in meters) is the wavelength of the wave. (The wavelength is the distance between two consecutive wave crests.)

- **a.** Find the average rate of change in speed between waves that are between 1 m and 4 m in length.
- **b.** Find the average rate of change in speed between waves that are between 4 m and 9 m in length.
- **c.** Use a graphing utility to graph the function. Using the graph and the results from parts (a) and (b), what does the difference in the rates of change mean?

For Exercises 85–90, determine the average rate of change of the function on the given interval. (See Example 7)

85. $f(x) = x^2 - 3$

a. [0, 1] **b.** [1, 3]

c. [−2, 0]

86. $g(x) = 2x^2 + 2$

a. [0, 1] **b.** [1, 3]

c. [−2, 0]

87. $h(x) = x^3$

a. [−1, 0] **b.** [0, 1]

c. [1, 2]

88. $k(x) = x^3 - 2$

a. [−1, 0] **b.** [0, 1]

c. [1, 2]

89. $m(x) = \sqrt{x}$

a. [0, 1] **b.** [1, 4] **c.** [4, 9]

90. $n(x) = \sqrt{x - 1}$

a. [1, 2] **b.** [2, 5] **c.** [5, 10]

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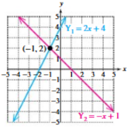
Learn: Solve Equations and Inequalities Graphically

For Exercises 91–98, use the graph to solve the equation and inequalities. Write the solutions to the inequalities in interval notation. (See Examples 8–9)

91.

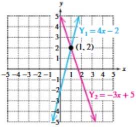


a. $2x + 4 = -x + 1$

b. $2x + 4 < -x + 1$

c. $2x + 4 \geq -x + 1$

92.

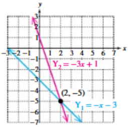


a. $4x - 2 = -3x + 5$

b. $4x - 2 < -3x + 5$

c. $4x - 2 \geq -3x + 5$

93.

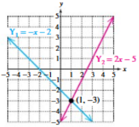


a. $-3x + 1 = -x - 3$

b. $-3x + 1 > -x - 3$

c. $-3x + 1 \leq -x - 3$

94.

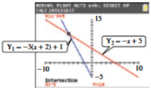


a. $-x - 2 = 2x - 5$

b. $-x - 2 \leq 2x - 5$

c. $-x - 2 > 2x - 5$

95.



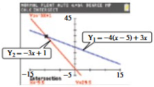
NORMAL FLOAT AUTO a+bi DEGREE MP
CALC INTERSECT
15
$Y_1 = -3(x + 2) + 1$
$Y_2 = -x + 5$
−10
10
−5
Intersection

a. $-3(x + 2) + 1 = -x + 5$

b. $-3(x + 2) + 1 \leq -x + 5$

c. $-3(x + 2) + 1 \geq -x + 5$

96.



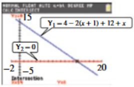
NORMAL FLOAT AUTO a+bi DEGREE MP
CALC INTERSECT
45
$Y_1 = -4(x - 5) + 3x$
$Y_2 = -3x + 1$
−15
−5
15
Intersection

a. $-4(x - 5) + 3x = -3x + 1$

b. $-4(x - 5) + 3x \leq -3x + 1$

c. $-4(x - 5) + 3x \geq -3x + 1$

97.



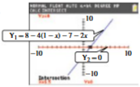
NORMAL FLOAT AUTO a+bi DEGREE MP
CALC INTERSECT
15
$Y_1 = 4 - 2(x + 1) + 12 + x$
$Y_2 = 0$
−2
−5
20
Intersection

a. $4 - 2(x + 1) + 12 + x = 0$

b. $4 - 2(x + 1) + 12 + x < 0$

c. $4 - 2(x + 1) + 12 + x > 0$

98.



NORMAL FLOAT AUTO a+bi DEGREE MP
CALC INTERSECT
10
$Y_1 = 8 - 4(1 - x) - 7 - 2x$
−10
10
$Y_2 = 0$
−10
Intersection

a. $8 - 4(1 - x) - 7 - 2x = 0$

b. $8 - 4(1 - x) - 7 - 2x < 0$

c. $8 - 4(1 - x) - 7 - 2x > 0$

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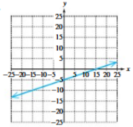
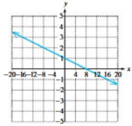
Mixed Exercises

For Exercises 99–102, write an equation of the line from the graph. Write the answer in slope-intercept form.

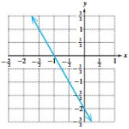
99.



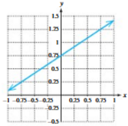
100.



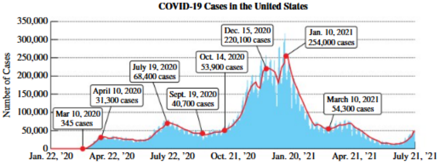
101.



102.



For Exercises 103–106, refer to the graph showing the 7-day average number of COVID-19 cases reported to the Centers for Disease Control (CDC) between March 2, 2020 and July 21, 2021.



Source: Centers for Disease Control, www.cdc.gov

103. If there are 31 days in March, determine the average daily rate of change in the number of cases reported to the CDC from March 10, 2020, to April 10, 2020. Round to the nearest hundred. Interpret the result.

104. If there are 31 days in October and 30 days in November, determine the average daily rate of change in the number of cases reported to the CDC from October 14, 2020, to December 15, 2020. Round to the nearest hundred.

105. If there are 31 days in January and 28 days in February, determine the average daily rate of change in the number of cases reported to the CDC from January 10, 2021, to March 10, 2021. Round to the nearest hundred.

106. If there are 31 days in July and 31 days in August, determine the average daily rate of change in the number of cases reported to the CDC from July 19, 2020, to September 19, 2020. Round to the nearest hundred.

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Write About It

107. Explain how you can determine from a linear equation $Ax + By = C$ (A and B not both zero) whether the line is slanted, horizontal, or vertical.

108. Explain how you can determine from a linear equation $Ax + By = C$ (A and B not both zero) whether the line passes through the origin.

109. What is the benefit of writing an equation of a line in slope-intercept form?

110. Explain how the average rate of change of a function f on the interval $[x_1, x_2]$ is related to slope.

Expanding Your Skills

111. Determine the area in the second quadrant enclosed by the equation $y = 2x + 4$ and the x- and y-axes.

112. Determine the area enclosed by the equations.

$y = x + 6$
$y = -2x + 6$
$y = 0$

113. Determine the area enclosed by the equations.

$y = -\frac{1}{2}x - 2$
$y = \frac{1}{3}x - 2$
$y = 0$

114. Determine the area enclosed by the equations.

$y = \sqrt{4 - (x - 2)^2}$
$y = 0$

115. Consider the standard form of a linear equation $Ax + By = C$ in the case where $B \neq 0$.

- **a.** Write the equation in slope-intercept form.
- **b.** Identify the slope in terms of the coefficients A and B.
- **c.** Identify the y-intercept in terms of the coefficients B and C.

116. Use the results from Exercise 115 to determine the slope and y-intercept for the graphs of the lines.

- **a.** $5x - 9y = 6$
- **b.** $0.052x - 0.013y = 0.39$

Technology Connections

For Exercises 117–120, solve the equation in part (a) and verify the solution on a graphing calculator. Then use the graph to find the solution set to the inequalities in parts (b) and (c). Write the solution sets to the inequalities in interval notation. (See Example 9)

117. **a.** $3.1 - 2.2(t + 1) = 6.3 + 1.4t$
b. $3.1 - 2.2(t + 1) > 6.3 + 1.4t$
c. $3.1 - 2.2(t + 1) < 6.3 + 1.4t$

118. **a.** $-11.2 - 4.6(c - 3) + 1.8c = 0.4(c + 2)$
b. $-11.2 - 4.6(c - 3) + 1.8c > 0.4(c + 2)$
c. $-11.2 - 4.6(c - 3) + 1.8c < 0.4(c + 2)$

119. **a.** $|2x - 3.8| - 4.6 = 7.2$
b. $|2x - 3.8| - 4.6 \geq 7.2$
c. $|2x - 3.8| - 4.6 \leq 7.2$

120. **a.** $|x - 1.7| + 4.95 = 11.15$
b. $|x - 1.7| + 4.95 \geq 11.15$
c. $|x - 1.7| + 4.95 \leq 11.15$

For Exercises 121–122, graph the lines in (a)–(c) on the standard viewing window. Compare the graphs. Are they exactly the same? If not, how are they different?

121. **a.** $y = 3x + 1$
b. $y = 2.99x + 1$
c. $y = 3.01x + 1$

122. **a.** $y = x + 3$
b. $y = x + 2.99$
c. $y = x + 3.01$

Go online for more practice problems.

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Lesson 1-5

Applications of Linear Equations and Modeling

What Will You Learn?

After completing this lesson, you should be able to:

- Apply the Point-Slope Formula
- Determine the Slopes of Parallel and Perpendicular Lines
- Create Linear Functions to Model Data
- Create Models Using Linear Regression

Learn: Apply the Point-Slope Formula

The slope formula can be used to develop the point-slope form of an equation of a line. Suppose that a line has a slope *m* and passes through a known point (x_1, y_1). Let (x, y) be any other point on the line. From the slope formula, we have

$\frac{y-y_1}{x-x_1}=m$ Slope formula

$\left(\frac{y-y_1}{x-x_1}\right)(x-x_1)=m(x-x_1)$ Clear fractions.

$y-y_1=m(x-x_1)$ This is called the point-slope formula for a line.

The point-slope formula is useful for building an equation of a line given a point on the line and the slope of the line.

Point-Slope Formula

The **point-slope formula** for a line is given by $y-y_1=m(x-x_1)$, where *m* is the slope of the line and (x_1, y_1) is a point on the line.

Example 1: Writing an Equation of a Line Given a Point on the Line and the Slope

Use the point-slope formula to find an equation of the line passing through the point (2, −3) and having slope −4. Write the answer in slope-intercept form.

Solution:

Label (2, −3) as (x_1, y_1) and $m=-4$.

$y-y_1=m(x-x_1)$ Apply the point-slope formula.

$y-(-3)=-4(x-2)$ Substitute $x_1=2$, $y_1=-3$, and $m=-4$.

$y+3=-4x+8$ Simplify.

$y=-4x+5$ (slope-intercept form)

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Apply the Skills

1. Use the point-slope formula to find an equation of the line passing through the point (–5, 2) and having slope –3. Write the answer in slope-intercept form.

Insights

The slope-intercept form of a line can also be used to write an equation of a line if a point on the line and the slope are known. See Example 5 in Lesson 1.4.

Example 2: Writing an Equation of a Line Given Two Points

Use the point-slope formula to write an equation of the line passing through the points (4, –6) and (–1, 2). Write the answer in slope-intercept form.

Solution:

To apply the point-slope formula, we first need to know the slope of the line.

(4, –6) and (–1, 2)
(x_1, y_1) and (x_2, y_2)

Label the points. Either point can be labeled (x_1, y_1).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-6)}{-1-4}$$

Apply the slope formula.

$$= \frac{8}{-5} = -\frac{8}{5}$$

$$y - y_1 = m(x - x_1)$$

Apply the point-slope formula.

$$y - (-6) = -\frac{8}{5}(x - 4)$$

Substitute $y_1 = -6$, $x_1 = 4$, and $m = -\frac{8}{5}$.

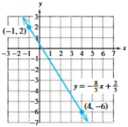
$$y + 6 = -\frac{8}{5}x + \frac{32}{5}$$

$$y = -\frac{8}{5}x + \frac{32}{5} - 6$$

$$y = -\frac{8}{5}x + \frac{32}{5} - \frac{30}{5}$$

$y = -\frac{8}{5}x + \frac{2}{5}$ Slope-intercept form

To check, we see that the graph of the line passes through (4, –6) and (–1, 2) as expected.



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Apply the Skills

2. Write an equation of the line passing through the points (2, –5) and (7, –3).

Insights

In Example 2, the slope-intercept form of a line can also be used to find an equation of the line. Substitute $-\frac{8}{5}$ for m and (4, –6) for (x, y).

$$\begin{aligned} y &= mx + b \\ -6 &= -\tfrac{8}{5}(4) + b \\ -6 &= -\tfrac{32}{5} + b \\ -6 + \tfrac{32}{5} &= b \\ \tfrac{2}{5} &= b \end{aligned}$$

Therefore, $y = mx + b$ is $y = -\frac{8}{5}x + \frac{2}{5}$.

Learn: Determine the Slopes of Parallel and Perpendicular Lines

Lines in the same plane that do not intersect are **parallel lines**. Nonvertical parallel lines have the same slope and different *y*-intercepts (Figure 1-20).

Lines that intersect at a right angle are **perpendicular lines**. If two nonvertical lines are perpendicular, then the slope of one line is the opposite of the reciprocal of the slope of the other line (Figure 1-21).

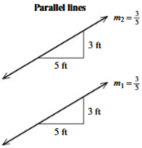

Figure 1-20

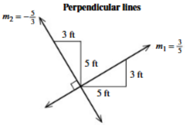

Figure 1-21

Slopes of Parallel and Perpendicular Lines

- If m_1 and m_2 represent the slopes of two nonvertical parallel lines, then $m_1 = m_2$.
- If m_1 and m_2 represent the slopes of two nonvertical perpendicular lines, then $m_1 = -\frac{1}{m_2}$ or equivalently $m_1 m_2 = -1$.

In Examples 3 and 4, we use the point-slope formula to find an equation of a line through a specified point and parallel or perpendicular to another line.

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Example 3: Writing an Equation of a Line Parallel to Another Line

Write an equation of the line passing through the point $(-4, 1)$ and parallel to the line defined by $x + 4y = 3$. Write the answer in slope-intercept form and in standard form.

Solution:

$$x + 4y = 3$$

$$y = -x + 3$$

The slope of the given line can be found from its slope-intercept form. Solve for y.

$$y = -\frac{1}{4}x + \frac{3}{4}$$

The slope of both lines is $-\frac{1}{4}$.

$$y - y_1 = m(x - x_1)$$

Apply the point-slope formula with $x_1 = -4$, $y_1 = 1$, and $m = -\frac{1}{4}$.

$$y - 1 = -\frac{1}{4}[x - (-4)]$$

$$y - 1 = -\frac{1}{4}(x + 4)$$

$$y - 1 = -\frac{1}{4}x - 1$$

$$y = -\frac{1}{4}x \text{ (slope-intercept form)}$$

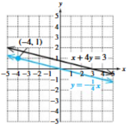
$$4(y) = 4\left(-\frac{1}{4}x\right)$$

$$4y = -x$$

$$x + 4y = 0 \text{ (standard form)}$$

Clearing fractions, and collecting the x and y terms on one side of the equation gives us standard form.

From the graph, we see that the line $y = -\frac{1}{4}x$ passes through the point $(-4, 1)$ and is parallel to the graph of $x + 4y = 3$.



Apply the Skills

3. Write an equation of the line passing through the point $(-3, 2)$ and parallel to the line defined by $x + 3y = 6$. Write the answer in slope-intercept form and in standard form.

Example 4: Writing an Equation of a Line Perpendicular to Another Line

Write an equation of the line passing through the point $(2, -3)$ and perpendicular to the line defined by $y = \frac{1}{2}x - 4$. Write the answer in slope-intercept form and in standard form.

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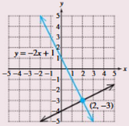
Solution:

From the slope-intercept form, $y = \frac{1}{2}x - 4$, the slope of given line is $\frac{1}{2}$.

$y - y_1 = m(x - x_1)$	The slope of a line perpendicular to the given line is –2.
$y - (-3) = -2(x - 2)$	Apply the point-slope formula with $x_1 = 2$, $y_1 = -3$, and $m = -2$.
$y + 3 = -2x + 4$	Simplify.
$y = -2x + 1$ (slope-intercept form)	Write the equation in slope-intercept form by solving for y.
$2x + y = 1$ (standard form)	Write the equation in standard form by collecting the x and y terms on one side of the equation.

Check Your Work

The solution to Example 4 can be checked by graphing both lines and verifying that they are perpendicular and that the line $y = -2x + 1$ passes through the point (2, –3).



Apply the Skills

4. Write an equation of the line passing through the point (–8, –4) and perpendicular to the line defined by $y = \frac{1}{6}x + 3$.

Learn: Create Linear Functions to Model Data

In many day-to-day applications, two variables are related linearly. By finding an equation of the line, we produce a model that relates the two variables. This is demonstrated in Example 5.

Example 5: Using a Linear Function in an Application

A family cell phone plan costs \$130 per month for four lines. Each additional line costs \$19.99 per month.

a. Write a linear function to model the monthly cost $C(x)$ (in \$) of a family plan for x additional family members added beyond the first four.

b. Evaluate $C(3)$ and interpret the meaning in the context of this problem.

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Solution:

a. $C(x) = mx + b$ The base price $130 is the fixed cost with zero family members added beyond the first four. So the constant *b* is 130.

$C(x) = 19.99x + 130$ The rate of increase, $19.99 per additional family member beyond the first four, is the slope.

b. $C(3) = 19.99(3) + 130$ Substitute 3 for *x*.

$= 189.97$

The total monthly cost of the plan with 3 additional family members (7 family members in total) is $189.97.

Apply the Skills

5. A speeding ticket is $100 plus $5 for every 1 mph over the speed limit.

a. Write a linear function to model the cost $S(x)$ (in $) of a speeding ticket for a person caught driving *x* mph over the speed limit.

b. Evaluate $S(15)$ and interpret the meaning in the context of this problem.

Linear functions can sometimes be used to model the cost, revenue, and profit of producing and selling *x* items.

Linear Cost, Revenue, and Profit Functions

- A **linear cost function** models the cost $C(x)$ to produce *x* items.

  $$C(x) = mx + b$$

  m is the variable cost per item.

  b is the fixed cost.

  The fixed cost does not change relative to the number of items produced. For example, the cost to rent an office is a fixed cost. The variable cost per item is the rate at which cost increases for each additional unit produced. Variable costs include labor, material, and shipping.

- A **linear revenue function** models revenue $R(x)$ for selling *x* items.

  $$R(x) = px$$

  The product *px* represents the price per item *p* times the number of items sold *x*.

- A **linear profit function** models the profit for producing and selling *x* items.

  $$P(x) = R(x) - C(x)$$

  Subtract the cost to produce *x* items from the revenue brought in from selling *x* items.

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Example 6: Writing Linear Cost, Revenue, and Profit Functions

At a summer art show a vendor sells lemonade for \$2.00 per cup. The cost to rent the booth is \$120. Furthermore, the vendor knows that the lemons, sugar, and cups collectively cost \$0.50 for each cup of lemonade produced.

a. Write a linear cost function to produce *x* cups of lemonade.

b. Write a linear revenue function for selling *x* cups of lemonade.

c. Write a linear profit function for producing and selling *x* cups of lemonade.

d. How much profit will the vendor make if 50 cups of lemonade are produced and sold?

e. How much profit will be made for producing and selling 128 cups?

f. Determine the break-even point.

Solution:

a. $C(x) = 0.50x + 120$	The fixed cost is \$120 because it does not change relative to the number of cups of lemonade produced. The variable cost is \$0.50 per lemonade.
b. $R(x) = 2.00x$	The price per cup of lemonade is \$2.00. Therefore, the product 2.00*x* gives the amount of revenue for *x* cups of lemonade sold.
c. $P(x) = R(x) - C(x)$ $P(x) = 2.00x - (0.50x + 120)$ $P(x) = 1.50x - 120$	Profit is defined as the difference of revenue and cost.
d. $P(50) = 1.50(50) - 120$	Substitute 50 for *x*.
$= -45$	The vendor will lose \$45.
e. $P(128) = 1.50(128) - 120$	Substitute 128 for *x*.
$= 72$	The vendor will make \$72.
f. For what value of *x* will $R(x) = C(x)$ or $P(x) = 0$? $P(x) = 0$	The break-even point is defined as the point where revenue equals cost. Alternatively, this can be stated as the point where profit equals zero: $P(x) = 0$.
$1.50x - 120 = 0$	Solve for *x*.
$1.50x = 120$	
$x = 80$	

If the vendor produces and sells 80 cups of lemonade, the cost and revenue will be equal, resulting in a profit of \$0. This is the break-even point.

Apply the Skills

6. Repeat Example 6 in the case where the vendor can cut the cost to \$0.40 per cup of lemonade and sell lemonades for \$1.50 per cup.

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Figure 1-22 shows the graphs of the revenue and cost functions from Example 6. Notice that R and C intersect at (80, 160). This means that if 80 cups of lemonade are produced and sold, the revenue and cost are both \$160. That is, $R(x) = C(x)$ and the company breaks even. The graph of the profit function P is consistent with this result. The value of $P(x)$ is 0 for 80 lemonades produced and sold (Figure 1-23).

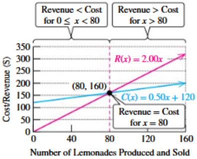


Figure 1-22

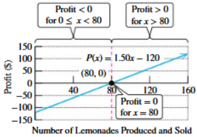


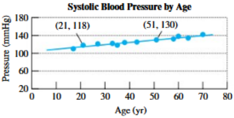
Figure 1-23

From Figures 1-22 and 1-23, we can draw the following conclusions.

- The company experiences a loss if fewer than 80 cups of lemonade are produced and sold. That is, $R(x) < C(x)$, or equivalently $P(x) < 0$.
- The company experiences a profit if more than 80 cups of lemonade are produced and sold. That is, $R(x) > C(x)$, or equivalently $P(x) > 0$.
- The company breaks even if exactly 80 cups of lemonade are produced and sold. That is, $R(x) = C(x)$, or equivalently $P(x) = 0$.

Example 7: Writing a Linear Model to Relate Two Variables

The data shown in the graph represent the age and systolic blood pressure for a sample of 12 randomly selected healthy adults.



a. Suppose that x represents the age of an adult (in years), and y represents the systolic blood pressure (in mmHg). Use the points (21, 118) and (51, 130) to write a linear model relating y as a function of x.

b. Interpret the meaning of the slope in the context of this problem.

c. Use the model to estimate the systolic blood pressure for a 55-year-old. Round to the nearest whole unit.

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Solution:

a. (21, 118) and (51, 130)

(x_1, y_1) and (x_2, y_2)	Label the points.
$m = \frac{130 - 118}{51 - 21} = 0.4$	Determine the slope of the line.
$y - y_1 = m(x - x_1)$	
$y - 118 = 0.4\,(x - 21)$	Apply the point-slope formula.
$y = 0.4x + 109.6$	The equation $y = 0.4x + 109.6$ relates an individual's age to an estimated systolic blood pressure for that age.

Insights

The equation $y = 0.4x + 109.6$ can also be expressed in function notation. For example, we can rename y as $S(x)$.

$$S(x) = 0.4x + 109.6$$

The value $S(x)$ represents the estimated systolic blood pressure for an adult of age x years.

b. The slope is 0.4. This means that the average increase in systolic blood pressure for adults is 0.4 mmHg per year of age.

c. $y = 0.4x + 109.6$

$y = 0.4(55) + 109.6$	Substitute 55 for x.
$y = 131.6$	

Based on the sample of data, the estimated systolic blood pressure for a 55-year-old is 132 mmHg.

Apply the Skills

7. Suppose that y represents the average consumer spending on television services per year (in dollars), and that x represents the number of years since 2010.

   a. Use the data points (2, 744) and (6, 864) to write a linear equation relating y to x.

   b. Interpret the meaning of the slope in the context of this problem.

   c. Interpret the meaning of the y-intercept in the context of this problem.

   d. Use the model from part (a) to predict the average consumer spending on television services for the year 2024 if this trend continues.

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Learn: Create Models Using Linear Regression

In Example 7, we used two given data points to determine a linear model for systolic blood pressure versus age. There are two drawbacks to this method. First, the equation is not necessarily unique. If we use two different data points, we may get a different equation. Second, it is generally preferable to write a model that is based on *all* the data points, rather than just two points. One such model is called the least-squares regression line.

The procedure to find the least-squares regression line is discussed in detail in a statistics course. Here we will give the basic premise and use a graphing utility to perform the calculations. Consider a set of data points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)$. The **least-squares regression line**, $y = mx + b$, is the unique line that minimizes the sum of the squared vertical deviations from the observed data points to the line (Figure 1-24).

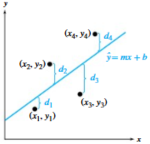

Figure 1-24

On a calculator or spreadsheet, the equation $y = mx + b$ may be denoted as $y = ax + b$ or as $y = b_0 + b_1x$. In any event, the coefficient of x is the slope of the line, and the constant gives us the y-intercept. Although the exact keystrokes on different calculators and graphing utilities may vary, we will use the following guidelines to find the least-squares regression line.

Creating a Linear Regression Model

1. Graph the data in a scatter plot.
2. Inspect the data visually to determine if the data suggest a linear trend.
3. Invoke the linear regression feature on a calculator, graphing utility, or spreadsheet.
4. Check the result by graphing the line with the data points to verify that the line passes through or near the data points.

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Example 8: Finding a Least-Squares Regression Line

The data given in the table represent the age and systolic blood pressure for a sample of 12 randomly selected healthy adults.

Age (yr)	17	21	27	33	35	38	43	51	58	60	64	70
Systolic blood pressure (mmHg)	110	118	121	122	118	124	125	130	132	138	134	142

a. Make a scatter plot of the data using age as the independent variable *x* and systolic pressure as the dependent variable *y*.

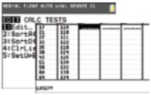
b. Based on the graph, does a linear model seem appropriate?

c. Determine the equation of the least-squares regression line.

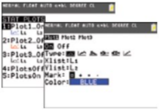
d. Use the least-squares regression line to approximate the systolic blood pressure for a healthy 55-year-old. Round to the nearest whole unit.

Solution:

a. On a graphing calculator hit the STAT button and select EDIT to enter the *x* and *y* data into two lists (shown here as L1 and L2).

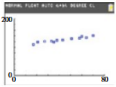

NORMAL FLOAT AUTO a+bi DEGREE CL
EDIT CALC TESTS
1:Edit…
2:SortA(
3:SortD(
4:ClrList
5:SetUpE
17 21 27 33 35 38 43 51 58 60 64
110 118 121 122 118 124 125 130 132 138 134
L2(1)=

Select the STAT PLOT option and turn Plot1 to On. For the type of graph, select the scatter plot image.


NORMAL FLOAT AUTO a+bi DEGREE CL
STAT PLOTS
1:Plot1…On
L1 L2
2:Plot2…Off
L1 L2
3:Plot3…Off
L1 L2
4:PlotsOff
5:PlotsOn
NORMAL FLOAT AUTO a+bi DEGREE CL
Plot1 Plot2 Plot3
On Off

Type:
Xlist:L1
Ylist:L2
Mark:
Color: BLUE

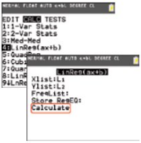
Be sure that the window is set to accommodate *x* values between 17 and 70, and *y* values between 110 and 142, inclusive. Then hit the GRAPH key. The window settings shown here are [0, 80, 10] by [0, 200, 20].



b. From the graph, the data appear to follow a linear trend.

c. Under the STAT menu, select CALC and then the LinReg(ax + b) option.

The command LinReg(ax + b) prompts the user to enter the list names (L1 and L2) containing the *x* and *y* data values. Then highlight Calculate and hit ENTER.


NORMAL FLOAT AUTO a+bi DEGREE CL
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:Cub
7:Quar
8:LinR
9↓LnRe
NORMAL FLOAT AUTO a+bi DEGREE CL
LinReg(ax+b)

Xlist:L1
Ylist:L2
FreqList:
Store RegEQ:
Calculate

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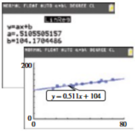
In the regression model $y = ax + b$, the values for the coefficients a and b are placed on the home screen.



```
LinReg
y=ax+b
a=.5105505157
b=104.1704486
```

NORMAL FLOAT AUTO a+bi DEGREE CL
NORMAL FLOAT AUTO a+bi DEGREE CL
200
y = 0.511x + 104
0
0
80

Rounding the values of a and b gives us $y = 0.511x + 104$.

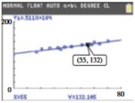
Enter the equation $Y_1 = 0.511x + 104$ into the equation editor and hit the GRAPH key. The graph of the regression line passes near or through the observed data points.

d. $y = 0.511x + 104$

$y = 0.511(55) + 104$ To approximate the systolic blood pressure for a 55-year-old, substitute 55 for x.

$= 132.105$

The value of the function at $x = 55$ can also be found by selecting the CALC menu and selecting the VALUE function. Enter 55 for x and press the ENTER key.



The systolic blood pressure for a healthy 55-year-old would be approximately 132 mmHg.

Apply the Skills

8. The data given represent the class averages for individual students based on the number of absences from class.

Number of Absences (x)	3	7	1	11	2	14	2	5
Average in Class (y)	88	67	96	62	90	56	97	82

a. Find the equation of the least-squares regression line.

b. Use the model from part (a) to approximate the average for a student who misses 6 classes.

Insights

The linear equation found in Example 7 was based on two data points. The least-squares regression line is based on all available data points. The estimate from each model for systolic blood pressure for a 55-year-old rounds to 132 mmHg.

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Practice 1-5

Practice Exercises

Prerequisite Review

R.1. Use slope-intercept form to write an equation of the line that passes through $(3, -7)$ with slope -5.

R.2. Write the equation in slope-intercept form and determine the slope and y-intercept $3x - 5y = -15$.

R.3. Determine the slope of the line containing the points $(-4, -2)$ and $(-4, -7)$.

R.4. Determine the slope of the line containing the points $(3, -2)$ and $(5, -2)$.

R.5. The cost C (in dollars) to rent an apartment is \$950 per month, plus a \$450 nonrefundable security deposit, plus a \$250 deposit for each dog or cat. Write a formula for the total cost to rent an apartment for m months with n cats/dogs.

Concept Connections

1. Given a point (x_1, y_1) on a line with slope m, the point-slope formula is given by ____________.

2. If two nonvertical lines have the same slope but different y-intercepts, then the lines are (parallel/perpendicular).

3. If m_1 and m_2 represent the slopes of two nonvertical perpendicular lines, then $m_1m_2 =$ ____________.

4. Suppose that $y = C(x)$ represents the cost to produce x items, and that $y = R(x)$ represents the revenue for selling x items. The profit $P(x)$ of producing and selling x items is defined by $P(x) =$ ____________.

Learn: Apply the Point-Slope Formula

For Exercises 5–20, write an equation of the line having the given conditions. Write the answer in slope-intercept form (if possible). (See Examples 1–2)

5. Passes through $(-3, 5)$ and $m = -2$.

6. Passes through $(4, -6)$ and $m = 3$.

7. Passes through $(-1, 0)$ and $m = \frac{2}{3}$.

8. Passes through $(-4, 0)$ and $m = \frac{3}{5}$.

9. Passes through $(3.4, 2.6)$ and $m = 1.2$.

10. Passes through $(2.2, 4.1)$ and $m = 2.4$.

11. Passes through $(6, 2)$ and $(-3, 1)$.

12. Passes through $(-4, 8)$ and $(-7, -3)$.

13. Passes through $(0, 8)$ and $(5, 0)$.

14. Passes through $(0, -6)$ and $(11, 0)$.

15. Passes through $(2.3, 5.1)$ and $(1.9, 3.7)$.

16. Passes through $(1.6, 4.8)$ and $(0.8, 6)$.

17. Passes through $(3, -4)$ and $m = 0$.

18. Passes through $(-5, 1)$ and $m = 0$.

19. Passes through $\left(\frac{2}{3}, \frac{1}{5}\right)$ and the slope is undefined.

20. Passes through $\left(-\frac{4}{7}, \frac{3}{10}\right)$ and the slope is undefined.

21. Given a line defined by $x = 4$, what is the slope of the line?

22. Given a line defined by $y = -2$, what is the slope of the line?

Learn: Determine the Slopes of Parallel and Perpendicular Lines

For Exercises 23–28, the slope of a line is given.

a. Determine the slope of a line parallel to the given line, if possible.

b. Determine the slope of a line perpendicular to the given line, if possible.

23. $m = \frac{3}{11}$

24. $m = \frac{6}{7}$

25. $m = -6$

26. $m = -10$

27. $m = 1$

28. m is undefined

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For Exercises 29–36, determine if the lines defined by the given equations are parallel, perpendicular, or neither.

29. $y = 2x - 3$
$y = -\frac{1}{2}x + 7$

30. $y = \frac{4}{3}x - 1$
$y = -\frac{3}{4}x + 5$

31. $8x - 5y = 3$
$2x = \frac{5}{4}y + 1$

32. $2x + 3y = 7$
$4x = -6y + 2$

33. $2x = 6$
$5 = y$

34. $3y = 5$
$x = 1$

35. $6x = 7y$
$\frac{7}{2}x - 3y = 0$

36. $5y = 2x$
$\frac{5}{2}x - y = 0$

For Exercises 37–44, write an equation of the line satisfying the given conditions. Write the answer in slope-intercept form (if possible) and in standard form ($Ax + By = C$) with no fractional coefficients. (See Examples 3–4)

37. Passes through (2, 5) and is parallel to the line defined by $2x + y = 6$.

38. Passes through (3, −1) and is parallel to the line defined by $-3x + y = 4$.

39. Passes through (6, −4) and is perpendicular to the line defined by $x - 5y = 1$.

40. Passes through (5, 4) and is perpendicular to the line defined by $x - 2y = 7$.

41. Passes through (6, 8) and is parallel to the line defined by $3x = 7y + 5$.

42. Passes through (7, −6) and is parallel to the line defined by $2x = 5y - 4$.

43. Passes through (2.2, 6.4) and is perpendicular to the line defined by $2x = 4 - y$.

44. Passes through (3.6, 1.2) and is perpendicular to the line defined by $4x = 9 - y$.

For Exercises 45–50, write an equation of the line that satisfies the given conditions.

45. Passes through (8, 6) and is parallel to the *x*-axis.

46. Passes through (−11, 13) and is parallel to the *y*-axis.

47. Passes through $\left(\frac{5}{11}, -\frac{3}{4}\right)$ and is perpendicular to the *y*-axis.

48. Passes through $\left(-\frac{7}{9}, \frac{7}{3}\right)$ and is perpendicular to the *x*-axis.

49. Passes through (−61.5, 47.6) and is parallel to the line defined by $x = -12$.

50. Passes through (−0.004, 0.009) and is parallel to the line defined by $y = 6$.

Learn: Create Linear Functions to Model Data

51. A salesperson makes a base salary of $400 per week plus 12% commission on sales. **(See Example 5)**

a. Write a linear function to model the salesperson's weekly salary $S(x)$ for x dollars in sales.

b. Evaluate S(8000) and interpret the meaning in the context of this problem.

52. At a parking garage in a large city, the charge for parking consists of a flat fee of $2.00 plus $1.50/hr.

a. Write a linear function to model the cost for parking $P(t)$ for t hours.

b. Evaluate $P(1.6)$ and interpret the meaning in the context of this problem.

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53. Millage rate is the amount per \$1000 that is often used to calculate property tax. For example, a home with a \$160,000 taxable value in a municipality with a 19 mil tax rate would require (0.019)(\$160,000) = \$3040 in property taxes. In one county, homeowners pay a flat tax of \$172 plus a rate of 19 mil on the taxable value of a home.

- **a.** Write a linear function that represents the total property tax $T(x)$ for a home with a taxable value of x dollars.
- **b.** Evaluate $T(180{,}000)$ and interpret the meaning in the context of this problem.

54. The average water level in a retention pond is 6.8 ft. During a time of drought, the water level decreases at a rate of 3 in./day.

- **a.** Write a linear function W that represents the water level $W(t)$ (in ft) t days after a drought begins.
- **b.** Evaluate $W(20)$ and interpret the meaning in the context of this problem.

For Exercises 55–56, the fixed and variable costs to produce an item are given along with the price at which an item is sold. (See Example 6)

- **a.** Write a linear cost function that represents the cost $C(x)$ to produce x items.
- **b.** Write a linear revenue function that represents the revenue $R(x)$ for selling x items.
- **c.** Write a linear profit function that represents the profit $P(x)$ for producing and selling x items.
- **d.** Determine the break-even point.

55. Fixed cost: \$2275

Variable cost per item: \$34.50

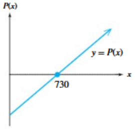
Price at which the item is sold: \$80.00

56. Fixed cost: \$5625

Variable cost per item: \$0.40

Price at which the item is sold: \$1.30

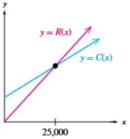
57. The profit function P is shown for producing and selling x items.



Determine the values of x for which

- **a.** $P(x) = 0$ (the company breaks even)
- **b.** $P(x) < 0$ (the company experiences a loss)
- **c.** $P(x) > 0$ (the company makes a profit)

58. The cost and revenue functions C and R are shown for producing and selling x items.



Determine the values of x for which

- **a.** $R(x) = C(x)$ (the company breaks even)
- **b.** $R(x) < C(x)$ (the company experiences a loss)
- **c.** $R(x) > C(x)$ (the company makes a profit)

59. The owner of a bakery is required to close the bakery to walk-in customers for several months. The baker's landlord agrees to reduce the rent during this time, bringing the fixed monthly cost for rent and electricity down to \$790. The cost of take-out bags, disinfectant wipes, and ingredients for cookies amounts to \$0.24 per cookie. The cookies will be delivered curbside for \$6.00 per dozen. **(See Example 6)**

- **a.** Write a linear cost function representing the cost $C(x)$ to produce x dozen cookies per month.

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b. Write a linear revenue function representing the revenue $R(x)$ for selling x dozen cookies.

c. Write a linear profit function representing the profit for producing and selling x dozen cookies in a month.

d. Determine the number of cookies (in dozens) that must be produced and sold for a monthly profit.

e. If 150 dozen cookies are sold in a given month, how much money will the business make or lose?

60. A lawn service company charges $60 for each lawn maintenance call. The fixed monthly cost of $680 includes telephone service and depreciation of equipment. The variable costs include labor, gasoline, and taxes and amount to $36 per lawn.

a. Write a linear cost function representing the monthly cost $C(x)$ for x maintenance calls.

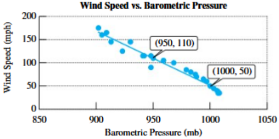
b. Write a linear revenue function representing the monthly revenue $R(x)$ for x maintenance calls.

c. Write a linear profit function representing the monthly profit $P(x)$ for x maintenance calls.

d. Determine the number of lawn maintenance calls needed per month for the company to make money.

e. If 42 maintenance calls are made for a given month, how much money will the lawn service make or lose?

61. The data in the graph show the wind speed y (in mph) for a recent hurricane versus the barometric pressure x (in millibars, mb). (*Source*: NOAA: www.noaa.gov) **(See Example 7)**



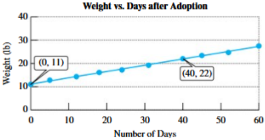
a. Use the points (950, 110) and (1000, 50) to write a linear model for these data.

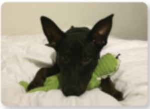
b. Interpret the meaning of the slope in the context of this problem.

c. Use the model from part (a) to estimate the wind speed for a hurricane with a pressure of 900 mb.

d. The lowest barometric pressure ever recorded for an Atlantic hurricane was 882 mb for Hurricane Wilma in 2005. Would it be reasonable to use the model from part (a) to estimate the wind speed for a hurricane with a pressure of 800 mb?

62. Caroline adopted a puppy named Dodger from an animal shelter in Chicago and recorded Dodger's weight during the first two months. The data in the graph show Dodger's weight y (in lb), x days after adoption.





a. Use the points (0, 11) and (40, 22) to write a linear model for these data.

b. Interpret the meaning of the slope in context.

c. Interpret the meaning of the y-intercept in context.

d. If this linear trend continues during Dodger's growth period, how long will it take Dodger to reach 90% of his expected full-grown weight of 70 lb? Round to the nearest day.

e. Is the model from part (a) reasonable long term?

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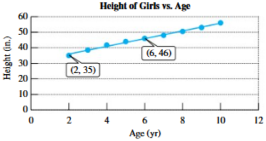
Courtesy of Caroline Celano

63. A pediatrician records the age *x* (in yr) and average height *y* (in inches) for girls between the ages of 2 and 10.



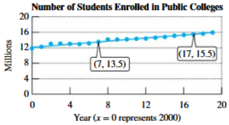
a. Use the points (2, 35) and (6, 46) to write a linear model for these data.

b. Interpret the meaning of the slope in context.

c. Use the model to forecast the average height of 11-yr-old girls.

d. If the height of a girl at age 11 is 90% of her full-grown adult height, use the result of part (c) to estimate the average height of adult women. Round to the nearest tenth of an inch.

64. The graph shows the number of students enrolled in public colleges for selected years (*Source*: U.S. National Center for Education Statistics, www.nces.ed.gov). The *x* variable represents the number of years since 2000 and the *y* variable represents the number of students (in millions).



a. Use the points (7, 13.5) and (17, 15.5) to write a linear model for these data.

b. Interpret the meaning of the slope in the context of this problem.

c. Interpret the meaning of the *y*-intercept in the context of this problem.

d. If the linear trend continues beyond the last observed data point, use the model in part (a) to predict the number of students enrolled in public colleges for the year 2024.

65. The table gives the number of calories and the amount of cholesterol for selected fast food hamburgers.

Hamburger Calories	Cholesterol (mg)
220	35
420	50
460	50
480	60
560	70
590	105
610	65
680	80
720	90

a. Graph the data in a scatter plot using the number of calories as the independent variable *x* and the amount of cholesterol as the dependent variable *y*.

b. Use the data points (480, 60) and (720, 90) to write a linear function that defines the amount of cholesterol $c(x)$ as a linear function of the number of calories *x*.

c. Interpret the meaning of the slope in the context of this problem.

d. Use the model from part (b) to predict the amount of cholesterol for a hamburger with 650 calories.

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66. The table gives the average gestation period for selected animals and their corresponding average longevity.

Animal	Gestation Period (days)	Longevity (yr)
Rabbit	33	7.0
Squirrel	44	8.5
Fox	57	9.0
Cat	60	11.0
Dog	62	11.0
Lion	109	10.0
Pig	115	10.0
Goat	148	12.0
Horse	337	23.0
Elephant	620	35.0

a. Graph the data in a scatter plot using the number of days for gestation as the independent variable *x* and the longevity as the dependent variable *y*.

b. Use the data points (44, 8.5) and (620, 35) to write a linear function that defines longevity $L(x)$ as a linear function of the length of the gestation period *x*. Round the slope to 3 decimal places and the *y*-intercept to 2 decimal places.

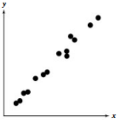
c. Interpret the meaning of the slope in the context of this problem.

d. Use the model from part (b) to predict the longevity for an animal with an 80-day gestation period. Round to the nearest year.

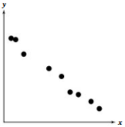
Learn: Create Models Using Linear Regression

For Exercises 67–70, use the scatter plot to determine if a linear regression modal appears to be appropriate.

67.



68.



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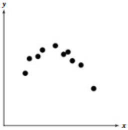
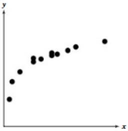


70.



71. The graph in Exercise 61 shows the wind speed *y* (in mph) of a hurricane versus the barometric pressure *x* (in mb). The table gives a partial list of data from the graph. **(See Example 8)**

Barometric Pressure (mb) (*x*)	Wind Speed (mph) (*y*)
1007	35
1003	45
1000	50
994	65
983	80
968	100
950	110
930	145
905	160

a. Use the data in the table to find the least-squares regression line. Round the slope to 2 decimal places and the *y*-intercept to the nearest whole unit.

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b. Use a graphing utility to graph the regression line and the observed data.

c. Use the model in part (a) to approximate the wind speed of a hurricane with a barometric pressure of 900 mb.

d. By how much do the results of part (c) differ from the result of Exercise 61(c)?

72. The graph in Exercise 62 shows the weight of Dodger, a puppy recently adopted from an animal shelter. The data in the table give Dodger's weight y (in lb), x days after adoption.

Number of Days (x)	Weight (lb) (y)
0	11.0
5	12.8
12	14.3
18	16.1
24	17.2
31	19.2
40	22.0
45	23.4
52	24.7
60	27.5

a. Use the data in the table to find the least-squares regression line. Round the slope to 2 decimal places and the y-intercept to 1 decimal place.

b. Use a graphing utility to graph the regression line and the observed data.

c. Use the model in part (a) to approximate the time required for Dodger to reach 90% of his full-grown weight of 70 lb. Round to the nearest day.

d. By how much do the results of part (c) differ from the result of Exercise 62(d)?

73. The graph in Exercise 63 shows the average height of girls based on their age. The data in the table give the average height y (in inches) for girls of age x (in yr).

Age (yr) (x)	Height (in.) (y)
2	35.00
3	38.50
4	41.75
5	44.00
6	46.00
7	48.00
8	50.50
9	53.00
10	56.00

a. Use the data in the table to find the least-squares regression line. Round the slope to 2 decimal places and the y-intercept to 1 decimal place.

b. Use a graphing utility to graph the regression line and the observed data.

c. Use the model in part (a) to approximate the average height of 11-yr-old girls.

d. If the height of a girl at age 11 is 90% of her full-grown adult height, use the result of part (c) to estimate the average height of adult women. Round to the nearest tenth of an inch.

e. By how much do the results of part (d) differ from the result of Exercise 63(d)?

74. The graph in Exercise 64 shows the number of students y enrolled in public colleges for selected years x, where x is the number of years since 2000. The table gives a partial list of data from the graph.

Number of Years Since 2000 (x)	Enrollment (millions) (y)
0	11.8
3	12.9
6	13.2
9	14.1
12	14.4
15	15.1
18	15.7

a. Use the data in the table to find the least-squares regression line. Round the slope and y-intercept to 1 decimal place.

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b. Use a graphing utility to graph the regression line and the observed data.

c. Assuming that the linear trend continues, use the model from part (a) to predict the number of students enrolled in public colleges for the year 2024.

d. By how much do the results of part (c) differ from the result of Exercise 64(d)?

75. The data in Exercise 65 give the amount of cholesterol y for a hamburger with x calories.

a. Use these data to find the least-squares regression line. Round the slope to 3 decimal places and the y-intercept to 2 decimal places.

b. Use a graphing utility to graph the regression line and the observed data.

c. Use the regression line to predict the amount of cholesterol in a hamburger with 650 calories. Round to the nearest milligram.

76. The data in Exercise 66 give the average gestation period x (in days) for selected animals and their corresponding average longevity y (in years).

a. Use these data to find the least-squares regression line. Round the slope to 3 decimal places and the y-intercept to 2 decimal places.

b. Use a graphing utility to graph the regression line and the observed data.

c. Use the regression line to predict the longevity for an animal with an 80-day gestation period. Round to the nearest year.

Mixed Exercises

For Exercises 77–78, write an equation of a line that satisfies the given conditions. Write the answer in slope-intercept form.

77. a. Passes through the point (6, 3) and is perpendicular to the line defined by $y - 1 = -3(x + 5)$.

b. Passes through the point (6, 3) and the y-intercept of the line defined by $y + 2 = 4(x - 1)$.

c. Passes through the point (6, 3) and is parallel to the line passing through (4, 2) and (8, 0).

78. a. Passes through the point (−2, 5) and the y-intercept of the line defined by $y = 5x - 2$.

b. Passes through the point (−2, 5) and is parallel to the line defined by $2x + 5y = 10$.

c. Passes through the point (−2, 5) and is perpendicular to the line passing through (−1, 2) and (5, 10).

79. Suppose that a line passes through the points (4, −6) and (2, −1). Where will it pass through the x-axis?

80. Suppose that a line passes through the point (2, −5) and (−4, 7). Where will it pass through the x-axis?

81. Write a rule for a linear function $y = f(x)$, given that $f(0) = 4$ and $f(3) = 11$.

82. Write a rule for a linear function $y = g(x)$, given that $g(0) = 7$ and $g(-2) = 4$.

83. Write a rule for a linear function $y = h(x)$, given that $h(1) = 6$ and $h(-3) = 2$.

84. Write a rule for a linear function $y = k(x)$, given that $k(-2) = 10$ and $k(5) = -18$.

Write About It

85. Explain how you can use slope to determine if two nonvertical lines are parallel or perpendicular.

86. State one application of using the point-slope formula.

87. Explain how cost and revenue are related to profit.

88. Explain how to determine the break-even point.

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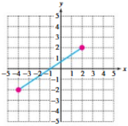
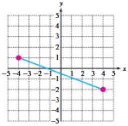
Expanding Your Skills

For Exercises 89–92, determine the equation of the line segment from the graph. Write the equation in slope-intercept form and give the restrictions on *x*.

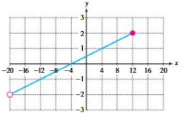
89.



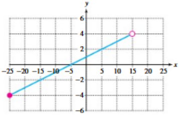
90.



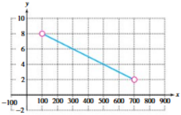
91. a.



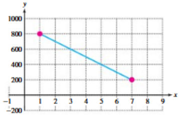
b.



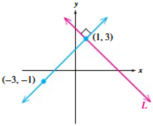
92. a.



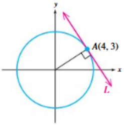
b.



93. Find an equation of line *L*.



94. In geometry, it is known that the tangent line to a circle at a given point *A* on the circle is perpendicular to the radius drawn to point *A*. Suppose that line *L* is tangent to the given circle at the point (4, 3). Write an equation representing line *L*.



95. In calculus, we can show that the slope of the line drawn tangent to the curve $y = x^3 + 1$ at the point $(c, c^3 + 1)$ is given by $3c^2$. Find an equation of the line tangent to $y = x^3 + 1$ at the point $(-2, -7)$.

96. In calculus, we can show that the slope of the line drawn tangent to the curve $y = \frac{1}{x}$ at the point $\left(c, \frac{1}{c}\right)$ is given by $-\frac{1}{c^2}$. Find an equation of the line tangent to $y = \frac{1}{x}$ at the point $\left(2, \frac{1}{2}\right)$.

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For Exercises 97–98, use the fact that a median of a triangle is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side of the triangle.

97. Find an equation of the median of a triangle drawn from vertex $A(5, -2)$ to the side formed by $B(-2, 9)$ and $C(4, 7)$.

98. Find an equation of the median of a triangle drawn from vertex $A(6, -5)$ to the side formed by $B(-4, 1)$ and $C(12, 3)$.

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Problem Recognition Exercises

Comparing Graphs of Equations

In Lesson 1.6, we will learn additional techniques to graph functions by recognizing characteristics of the functions. In many cases, we can also graph families of functions by relating them to one of several basic graphs. To prepare for the discussion in Lesson 1.6, use a graphing utility or plot points to graph the basic functions in Exercises 1–8.

1. $y = 1$

2. $y = x$

3. $y = x^2$

4. $y = x^3$

5. $y = \sqrt{x}$

6. $y = \sqrt[3]{x}$

7. $y = |x|$

8. $y = \frac{1}{x}$

For Exercises 9–18, graph the functions by plotting points or by using a graphing utility. Explain how the graphs are related.

9. a. $f(x) = x^2$
b. $g(x) = x^2 + 2$
c. $h(x) = x^2 - 4$

10. a. $f(x) = |x|$
b. $g(x) = |x| + 2$
c. $h(x) = |x| - 4$

11. a. $f(x) = \sqrt{x}$
b. $g(x) = \sqrt{x - 2}$
c. $h(x) = \sqrt{x + 4}$

12. a. $f(x) = x^2$
b. $g(x) = (x - 2)^2$
c. $h(x) = (x + 3)^2$

13. a. $f(x) = |x|$
b. $g(x) = -|x|$

14. a. $f(x) = \sqrt{x}$
b. $g(x) = -\sqrt{x}$

15. a. $f(x) = x^2$
b. $g(x) = \frac{1}{2}x^2$
c. $h(x) = 2x^2$

16. a. $f(x) = |x|$
b. $g(x) = \frac{1}{3}|x|$
c. $h(x) = 3|x|$

17. a. $f(x) = \sqrt{x}$
b. $g(x) = \sqrt{-x}$

18. a. $f(x) = \sqrt[3]{x}$
b. $g(x) = \sqrt[3]{-x}$

Go online for more practice problems.

Lesson 1-6

Transformations of Graphs

What Will You Learn?

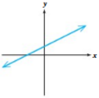
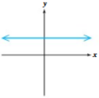
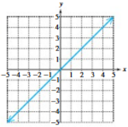
After completing this lesson, you should be able to:

- Recognize Basic Functions
- Apply Vertical and Horizontal Translations (Shifts)
- Apply Vertical and Horizontal Shrinking and Stretching
- Apply Reflections Across the *x*- and *y*-Axes
- Summarize Transformations of Graphs

Learn: Recognize Basic Functions

A function defined by $f(x) = mx + b$ is a linear function and its graph is a line in a rectangular coordinate system. In addition to linear functions, we will learn to identify other categories of functions and the shapes of their graphs (Table 1-2).

TABLE 1-2 Basic Functions and Their Graphs

<table>
<tr><th>Function</th><th>Graph</th></tr>
<tr><td>Linear functions: $f(x) = mx + b$</td><td>
</td></tr>
<tr><td>Constant functions: $f(x) = b$</td><td>
</td></tr>
<tr><td>Identity function: $f(x) = x$</td><td>
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−2</td></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>2</td></tr>
</table>

</td></tr>
</table>

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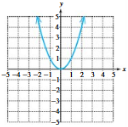
Quadratic function: $f(x) = x^2$

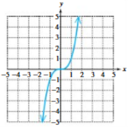
The graph is a parabola.

x	$f(x)$
−2	4
−1	1
0	0
1	1
2	4



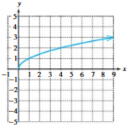
Cube function: $f(x) = x^3$

x	$f(x)$
−2	−8
−1	−1
0	0
1	1
2	8



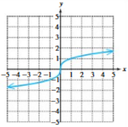
Square root function: $f(x) = \sqrt{x}$

x	$f(x)$
0	0
1	1
4	2
9	3
16	4



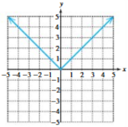
Cube root function: $f(x) = \sqrt[3]{x}$

x	$f(x)$
−8	−2
−1	−1
0	0
1	1
8	2



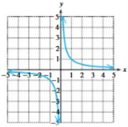
Absolute value function: $f(x) = |x|$

x	$f(x)$
−2	2
−1	1
0	0
1	1
2	2



Reciprocal function: $f(x) = \frac{1}{x}$

x	$f(x)$
−2	$-\frac{1}{2}$
−1	−1
$-\frac{1}{2}$	−2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$



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The functions given in Table 1-2 were introduced in Lesson 1.1, Exercises 31–36, and in the Problem Recognition Exercises on page 91.

Notice that the graph of $f(x) = \frac{1}{x}$ gets close to (but never touches) the y-axis as x gets close to zero. Likewise, as x approaches ∞ and $-\infty$, the graph approaches the x-axis without touching the x-axis. The x- and y-axes are called **asymptotes** of f and will be studied in detail in Lesson 2.5.

Learn: Apply Vertical and Horizontal Translations (Shifts)

We will call the eight basic functions pictured in Table 1-2 "parent" functions. Other functions that share the characteristics of a parent function are grouped as a "family" of functions. For example, consider the functions defined by $g(x) = x^2 + 2$ and $h(x) = x^2 - 4$, pictured in Figure 1-25.

x	$f(x) = x^2$	$g(x) = x^2 + 2$	$h(x) = x^2 - 4$
−3	9	11	5
−2	4	6	0
−1	1	3	−3
0	0	2	−4
1	1	3	−3
2	4	6	0
3	9	11	5

+2

−4

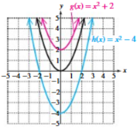

Figure 1-25

The graphs of g and h both resemble the graph of $f(x) = x^2$, but are shifted vertically upward or downward. The table of points reveals that for corresponding x values, the values of $g(x)$ are 2 more than the values of $f(x)$. So, the graph is shifted *upward* 2 units. Likewise, the values of $h(x)$ are 4 less than the values of $f(x)$ and the graph is shifted *downward* 4 units. Such shifts are called **translations**. These observations are consistent with the following rules.

Insights

For each ordered pair (x, y) on the graph of $y = f(x)$, the corresponding point

- $(x, y + k)$ is on the graph of $y = f(x) + k$.
- $(x, y - k)$ is on the graph of $y = f(x) - k$.

Vertical Translations of Graphs

Consider a function defined by $y = f(x)$. Let k represent a positive real number.

- The graph of $y = f(x) + k$ is the graph of $y = f(x)$ shifted k units *upward*.
- The graph of $y = f(x) - k$ is the graph of $y = f(x)$ shifted k units *downward*.

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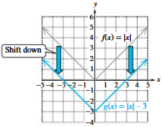
Example 1: Translating a Graph Vertically

Use translations to graph the given functions.

a. $g(x) = |x| - 3$ **b.** $h(x) = x^3 + 2$

Solution:

a.

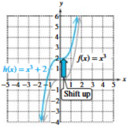


The parent function for $g(x) = |x| - 3$ is $f(x) = |x|$.

The graph of g (shown in blue) is the graph of f shifted *downward* 3 units. For example:

The point (0, 0) on the graph of $f(x) = |x|$ corresponds to (0, −3) on the graph of $g(x) = |x| - 3$.

b.



The parent function for $h(x) = x^3 + 2$ is $f(x) = x^3$.

The graph of h (shown in blue) is the graph of f shifted *upward* 2 units. For example:

The point (0, 0) on the graph of $f(x) = x^3$ corresponds to (0, 2) on the graph of $h(x) = x^3 + 2$.

The point (1, 1) on the graph of $f(x) = x^3$ corresponds to (1, 3) on the graph of $h(x) = x^3 + 2$.

Apply the Skills

1. Use translations to graph the given functions.

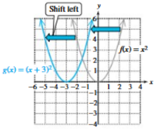
a. $g(x) = \sqrt{x} - 2$ **b.** $h(x) = \sqrt{x} + 3$

The graph of a function will be shifted to the right or left if a constant is added to or subtracted from the input variable x. In Example 2, we consider $g(x) = (x + 3)^2$.

Example 2: Translating a Graph Horizontally

Graph the function defined by $g(x) = (x + 3)^2$.

Solution:



Because 3 is added to the x variable, we might expect the graph of $g(x) = (x + 3)^2$ to be the same as the graph of $f(x) = x^2$, but shifted in the x direction (horizontally). To determine whether the shift is to the left or right, we can locate the x-intercept of the graph of $g(x) = (x + 3)^2$. Substituting 0 for $g(x)$, we have:

$0 = (x + 3)^2$

$x = -3$ The x-intercept is (−3, 0).

Therefore, the new x-intercept (and also the vertex of the parabola) is (−3, 0). This means that the graph is shifted to the left.

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Apply the Skills

2. Graph the function defined by $g(x) = |x + 2|$.

Using similar logic as in Example 2, we can show that the graph of $h(x) = (x - 3)^2$ is the graph of $f(x) = x^2$ translated to the *right* 3 units. These observations are consistent with the following rules.

Horizontal Translations of Graphs

Consider a function defined by $y = f(x)$. Let h represent a positive real number.

- The graph of $y = f(x - h)$ is the graph of $y = f(x)$ shifted h units to the *right*.
- The graph of $y = f(x + h)$ is the graph of $y = f(x)$ shifted h units to the *left*.

Insights

Consider a positive real number h.

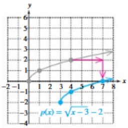
- To graph $y = f(x - h)$ or $y = f(x + h)$, shift the graph of $y = f(x)$ horizontally in the opposite direction of the sign within parentheses.
- The graph of $y = f(x - h)$ is a shift in the positive x direction. The graph of $y = f(x + h)$ is a shift in the negative x direction.

Example 3: Translating a Function Horizontally and Vertically

Use translations to graph the function defined by $p(x) = \sqrt{x - 3} - 2$.

Solution:

The parent function for $p(x) = \sqrt{x - 3} - 2$ is $f(x) = \sqrt{x}$.



The graph of p (shown in blue) is the graph of f shifted to the right 3 units and downward 2 units. We can plot several strategic points as an outline for the new curve.

- The point (0, 0) on the graph of f corresponds to $(0 + 3, 0 - 2) = (3, -2)$ on the graph of p.
- The point (1, 1) on the graph of f corresponds to $(1 + 3, 1 - 2) = (4, -1)$ on the graph of p.
- The point (4, 2) on the graph of f corresponds to $(4 + 3, 2 - 2) = (7, 0)$ on the graph of p.

Apply the Skills

3. Use translations to graph the function defined by $q(x) = \sqrt{x + 2} - 5$.

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Learn: Apply Vertical and Horizontal Shrinking and Stretching

Horizontal and vertical translations of functions are called **rigid transformations** because the shape of the graph is not affected. We now look at **nonrigid transformations**. These operations cause a distortion of the graph (either an elongation or contraction in the horizontal or vertical direction). We begin by investigating the functions defined by $y = f(x)$ and $y = a \cdot f(x)$, where a is a positive real number.

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Example 4: Graphing a Function with a Vertical Stretch or Shrink

Graph the functions.

a. $f(x) = |x|$

b. $g(x) = 2|x|$

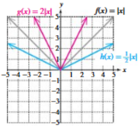
c. $h(x) = \frac{1}{2}|x|$

Solution:

x	$f(x) = \|x\|$	$g(x) = 2\|x\|$	$h(x) = \frac{1}{2}\|x\|$
−3	3	6	$\frac{3}{2}$
−2	2	4	1
−1	1	2	$\frac{1}{2}$
0	0	0	0
1	1	2	$\frac{1}{2}$
2	2	4	1
3	3	6	$\frac{3}{2}$

double

multiply by $\frac{1}{2}$



For a given value of x, the value of $g(x)$ is twice the value of $f(x)$. Therefore, the graph of g is elongated or stretched vertically by a factor of 2.

For a given value of x, the value of $h(x)$ is one-half that of $f(x)$. Therefore, the graph of h is shrunk vertically.

Apply the Skills

4. Graph the functions.

a. $f(x) = x^2$

b. $g(x) = 3x^2$

c. $h(x) = \frac{1}{3}x^2$

Vertical Shrinking and Stretching of Graphs

Consider a function defined by $y = f(x)$. Let a repre]sent a positive real number.

- If $a > 1$, then the graph of $y = af(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of a.
- If $0 < a < 1$, then the graph of $y = af(x)$ is the graph of $y = f(x)$ shrunk vertically by a factor of a.

Note: For any point (x, y) on the graph of $y = f(x)$, the point $y = f(x)$, (x, ay) is on the graph of $y = af(x)$.

A function may also be stretched or shrunk horizontally.

Horizontal Shrinking and Stretching of Graphs

Consider a function defined by $y = f(x)$. Let a represent a positive real number.

- If $a > 1$, then the graph of $y = f(ax)$ is the graph of $y = f(x)$ shrunk horizontally by a factor of $\frac{1}{a}$.
- If $0 < a < 1$, then the graph of $y = f(ax)$ is the graph of $y = f(x)$ stretched horizontally by a factor of $\frac{1}{a}$.

Note: For any point (x, y) on the graph of $y = f(x)$, the point $\left(\frac{x}{a}, y\right)$ is on the graph of $y = f(ax)$.

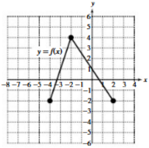
A point (x, y) on the graph of $y = f(x)$ corresponds to the point $\left(\frac{x}{a}, y\right)$ on the graph of $y = f(ax)$. Since the x-coordinate is multiplied by the *reciprocal* of a, values of a greater than 1 actually compress (shrink) the graph horizontally toward the y-axis. Values of a between 0 and 1 *stretch* the graph horizontally away from the y-axis. This is demonstrated in Example 5.

Example 5: Graphing a Function with a Horizontal Shrink or Stretch

The graph of $y = f(x)$ is shown. Graph

a. $y = f(2x)$

b. $y = f\left(\frac{1}{2}x\right)$



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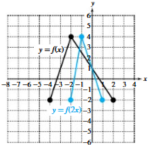
Solution:

a. $f(2x)$ is in the form $f(ax)$ with $a = 2 > 1$. The graph of $y = f(2x)$ is the graph of $y = f(x)$ horizontally compressed.



The graph of f has the following "strategic" points that define the shape of the function: $(-4, -2)$, $(-2, 4)$, and $(2, -2)$.

To graph $y = f(2x)$, divide each x value by 2.

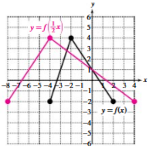
$(-4, -2)$ becomes $\left(\frac{-4}{2}, -2\right) = (-2, -2)$.

$(-2, 4)$ becomes $\left(\frac{-2}{2}, 4\right) = (-1, 4)$.

$(2, -2)$ becomes $\left(\frac{2}{2}, -2\right) = (1, -2)$.

The graph of $y = f(2x)$ is shown in blue.

b. $f\left(\frac{1}{2}x\right)$ is in the form $f(ax)$ with $a = \frac{1}{2}$. The graph of $y = f\left(\frac{1}{2}x\right)$ is the graph of $y = f(x)$ stretched horizontally.



To graph $y = f\left(\frac{1}{2}x\right)$, divide each x value on the graph of $y = f(x)$ by $\frac{1}{2}$. For example:

$(-4, -2)$ becomes $\left(\frac{-4}{\frac{1}{2}}, -2\right) = (-8, -2)$.

$(-2, 4)$ becomes $\left(\frac{-2}{\frac{1}{2}}, 4\right) = (-4, 4)$.

$(2, -2)$ becomes $\left(\frac{2}{\frac{1}{2}}, -2\right) = (4, -2)$.

The graph of $y = f\left(\frac{1}{2}x\right)$ is shown in red.

Insights

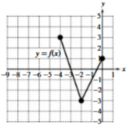
Dividing the x values by $\frac{1}{2}$ is the same as multiplying the x values by 2.

Apply the Skills

5. The graph of $y = f(x)$ is shown. Graph

a. $y = f(2x)$

b. $y = f\left(\frac{1}{2}x\right)$



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Learn: Apply Reflections Across the *x*- and *y*-Axes

The graphs of $f(x) = x^2$ (in black) and $g(x) = -x^2$ (in blue) are shown in Figure 1-26. Notice that a point (x, y) on the graph of f corresponds to the point $(x, -y)$ on the graph of g. Therefore, the graph of g is the graph of f reflected across the x-axis.

The graphs of $f(x) = \sqrt{x}$ (in black) and $g(x) = \sqrt{-x}$ (in blue) are shown in Figure 1-27. Notice that a point (x, y) on the graph of f corresponds to the point $(-x, y)$ on g. Therefore, the graph of g is the graph of f reflected across the y-axis.

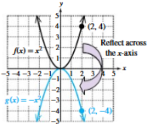


Figure 1-26

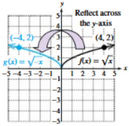


Figure 1-27

Reflections Across the *x*- and *y*-Axes

Consider a function defined by $y = f(x)$.

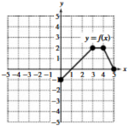
- The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected across the x-axis.
- The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis.

Example 6: Reflecting the Graph of a Function Across the *x*- and *y*-Axes

The graph of $y = f(x)$ is given.

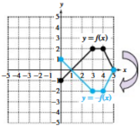
a. Graph $y = -f(x)$.

b. Graph $y = f(-x)$.

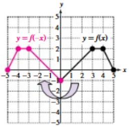


Solution:

a. Reflect $y = f(x)$ across the x-axis.



b. Reflect $y = f(x)$ across the y-axis.



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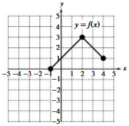
Apply the Skills

6. The graph of $y = f(x)$ is given.

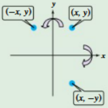
a. Graph $y = -f(x)$.

b. Graph $y = f(-x)$.



Insights

For a given point (x, y), notice that $(-x, y)$ is on the opposite side of and equidistant to the y-axis. Likewise, $(x, -y)$ is on the opposite side of and equidistant from the x-axis as (x, y).



Learn: Summarize Transformations of Graphs

The operations of reflecting a graph of a function about an axis and shifting, stretching, and shrinking a graph are called **transformations**. Transformations give us tools to graph families of functions that are built from basic “parent” functions.

Transformations of Functions

Consider a function defined by $y = f(x)$. If h, k, and a represent positive real numbers, then the graphs of the following functions are related to $y = f(x)$ as follows in Table 1-3 on page 102.

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TABLE 1-3 Transformations of Functions

Transformation	Effect on the Graph of f	Changes to Points on f
Vertical translation (shift) $y = f(x) + k$ $y = f(x) - k$	For real $k > 0$: Shift upward k units Shift downward k units	 Replace (x, y) by $(x, y + k)$. Replace (x, y) by $(x, y - k)$.
Horizontal translation (shift) $y = f(x - h)$ $y = f(x + h)$	For real $h > 0$: Shift to the right h units Shift to the left h units	 Replace (x, y) by $(x + h, y)$. Replace (x, y) by $(x - h, y)$.
Vertical stretch/shrink $y = a[f(x)]$	Vertical stretch (if $a > 1$) Vertical shrink (if $0 < a < 1$) Graph is stretched/shrunk vertically by a factor of a.	Replace (x, y) by (x, ay).
Horizontal stretch/shrink $y = f(a \cdot x)$	Horizontal shrink (if $a > 1$) Horizontal stretch (if $0 < a < 1$) Graph is shrunk/stretched horizontally by a factor of $\frac{1}{a}$.	Replace (x, y) by $\left(\frac{x}{a}, y\right)$.
Reflection $y = -f(x)$ $y = f(-x)$	 Reflection across the x-axis Reflection across the y-axis	 Replace (x, y) by $(x, -y)$. Replace (x, y) by $(-x, y)$.

When graphing a function requiring multiple transformations on the parent function, it is important to follow the correct sequence of steps.

Steps for Graphing Multiple Transformations of Functions

To graph a function requiring multiple transformations, use the following order.

1. Horizontal translation (shift)
2. Horizontal and vertical stretch and shrink
3. Reflections across the x- and y-axes
4. Vertical translation (shift)

Example 7: Using Transformations to Graph a Function

Use transformations to graph the function defined by $n(x) = -\frac{1}{2}(x - 2)^2 + 3$.

Solution:

The graph of $n(x) = -\frac{1}{2}(x - 2)^2 + 3$ is the same as the graph of $f(x) = x^2$, with four transformations in the following order.

1. Shift the graph to the right 2 units.
2. Apply a vertical shrink (multiply the y values by $\frac{1}{2}$).

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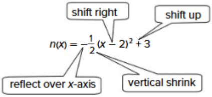
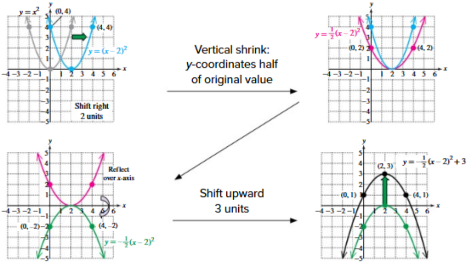
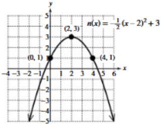
3. Reflect the graph over the *x*-axis.
4. Shift the graph upward 3 units.


shift right
shift up
reflect over x-axis
vertical shrink

$$n(x) = -\frac{1}{2}(x - 2)^2 + 3$$





Check Your Work

As a means to check the graph of $y = n(x)$, substitute the *x*-coordinates of the strategic points (0, 1), (2, 3), and (4, 1) into the function.

$n(0) = -\frac{1}{2}(0 - 2)^2 + 3 = 1$ ✓

$n(2) = -\frac{1}{2}(2 - 2)^2 + 3 = 3$ ✓

$n(4) = -\frac{1}{2}(4 - 2)^2 + 3 = 1$ ✓

Apply the Skills

7. Use transformations to graph the function defined by $m(x) = -3\,|x - 2| - 4$.

Example 8: Using Transformations to Graph a Function

Use transformations to graph the function defined by $v(x) = -\sqrt{-x+2}$.

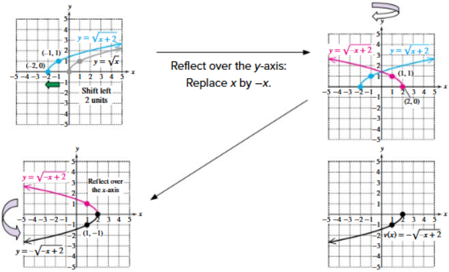
Solution:

The graph of $v(x) = -\sqrt{-x+2}$ is the same as the graph of $f(x) = \sqrt{x}$, with three transformations in the following order.

1. Shift the graph to the left 2 units.
2. Reflect the graph across the y-axis.
3. Reflect the graph across the x-axis.

   (Note that the reflections in steps 2 and 3 can be applied in either order.)

$$v(x) = \underbrace{-}_{\text{reflect across } x\text{-axis}}\sqrt{\overset{\text{reflect across } y\text{-axis}}{-}x + \overset{\text{shift left}}{2}}$$



Check Your Work

Transformations involving a horizontal shrink, stretch, or reflection often introduce confusion when coupled with a horizontal shift. To further illustrate the rationale for the order of steps taken in Example 8, begin with the parent function $y = \sqrt{x}$. Performing a horizontal shift first means that we replace x by $x + 2$. This gives us $y = \sqrt{x+2}$. Then to perform the reflection across the y-axis, we replace x by $-x$ to get $y = \sqrt{-x+2}$. Performing these two transformations in the reverse order would *not* result in the function we want. We would first have $y = \sqrt{-x}$, and then replacing x by $x + 2$ would give $y = \sqrt{-(x+2)} = \sqrt{-x-2}$ rather than $y = \sqrt{-x+2}$.

Apply the Skills

8. Use transformations to graph the function defined by $r(x) = \sqrt[3]{-x+1}$.

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Practice 1-6

Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, graph each equation.

R.1. $y = -3x - 1$

R.2. $y = \frac{3}{5}x + 2$

R.3. $y = 1$

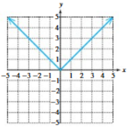
Concept Connections

1. Let c represent a positive real number. The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted (up/down/left/right) c units.
2. Let c represent a positive real number. The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted (up/down/left/right) c units.
3. Let c represent a positive real number. The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted (up/down/left/right) c units.
4. The graph of $y = 3f(x)$ is the graph of $y = f(x)$ with a (choose one: vertical stretch, vertical shrink, horizontal stretch, horizontal shrink).
5. The graph of $y = f(3x)$ is the graph of $y = f(x)$ with a (choose one: vertical stretch, vertical shrink, horizontal stretch, horizontal shrink).
6. The graph of $y = f\left(\frac{1}{3}x\right)$ is the graph of $y = f(x)$ with a (choose one: vertical stretch, vertical shrink, horizontal stretch, horizontal shrink).
7. The graph of $y = \frac{1}{3}f(x)$ is the graph of $y = f(x)$ with a (choose one: vertical stretch, vertical shrink, horizontal stretch, horizontal shrink).
8. The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected across the ____________ -axis.

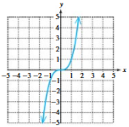
Learn: Recognize Basic Functions

For Exercises 9–14, from memory match the equation with its graph.

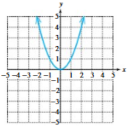
a.



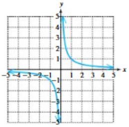
b.



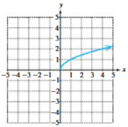
c.



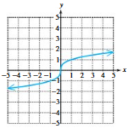
d.



e.



f.



9. $f(x) = \sqrt{x}$
10. $f(x) = \sqrt[3]{x}$
11. $f(x) = x^3$
12. $f(x) = x^2$
13. $f(x) = |x|$
14. $f(x) = \frac{1}{x}$

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Learn: Apply Vertical and Horizontal Translations (Shifts)

For Exercises 15–26, use translations to graph the given functions. (See Examples 1–3)

15. $f(x) = |x| + 1$

16. $g(x) = \sqrt{x} + 2$

17. $k(x) = x^3 - 2$

18. $h(x) = \frac{1}{x} - 2$

19. $g(x) = \sqrt{x + 5}$

20. $m(x) = |x + 1|$

21. $r(x) = (x - 4)^2$

22. $t(x) = \sqrt[3]{x - 2}$

23. $a(x) = \sqrt{x + 1} - 3$

24. $b(x) = |x - 2| + 4$

25. $c(x) = \frac{1}{x - 3} + 1$

26. $d(x) = \frac{1}{x + 4} - 1$

Learn: Apply Vertical and Horizontal Shrinking and Stretching

For Exercises 27–32, use transformations to graph the functions. (See Example 4)

27. $m(x) = 4\sqrt[3]{x}$

28. $n(x) = 3|x|$

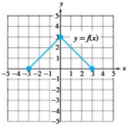
29. $r(x) = \frac{1}{2}x^2$

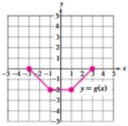
30. $t(x) = \frac{1}{3}|x|$

31. $p(x) = |2x|$

32. $q(x) = \sqrt{2x}$

For Exercises 33–40, use the graphs of $y = f(x)$ and $y = g(x)$ to graph the given function. (See Example 5)





33. $y = \frac{1}{3}f(x)$

34. $y = \frac{1}{2}g(x)$

35. $y = 3f(x)$

36. $y = 2g(x)$

37. $y = f(3x)$

38. $y = g(2x)$

39. $y = f\left(\frac{1}{3}x\right)$

40. $y = g\left(\frac{1}{2}x\right)$

Learn: Apply Reflections Across the *x*- and *y*-Axes

For Exercises 41–46, graph the function by applying an appropriate reflection.

41. $f(x) = -\frac{1}{x}$

42. $g(x) = -\sqrt{x}$

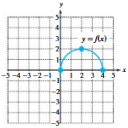
43. $h(x) = -x^3$

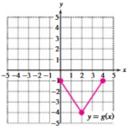
44. $k(x) = -|x|$

45. $p(x) = (-x)^3$

46. $q(x) = \sqrt[3]{-x}$

For Exercises 47–50, use the graphs of $y = f(x)$ and $y = g(x)$ to graph the given function. (See Example 6)





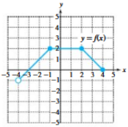
47. $y = f(-x)$

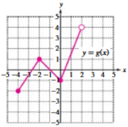
48. $y = g(-x)$

49. $y = -f(x)$

50. $y = -g(x)$

For Exercises 51–54, use the graphs of $y = f(x)$ and $y = g(x)$ to graph the given function. (See Example 6)





51. $y = f(-x)$

52. $y = g(-x)$

53. $y = -f(x)$

54. $y = -g(x)$

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Learn: Summarize Transformations of Graphs

For Exercises 55–62, a function g is given. Identify the parent function from Table 1-2 on pages 92–93. Then use the steps for graphing multiple transformations of functions on page 196 to list, in order, the transformations applied to the parent function to obtain the graph of g.

55. $g(x) = \frac{3}{1+x} - 2$

56. $g(x) = \frac{5}{x-4} + 1$

57. $g(x) = \frac{1}{3}(x - 2.1)^2 + 7.9$

58. $g(x) = \frac{1}{2}\sqrt{x + 4.3} - 8.4$

59. $g(x) = 2\sqrt{-2x + 5}$

60. $g(x) = 3\left|-\frac{1}{2}x - 4\right|$

61. $g(x) = -\sqrt{\frac{1}{3}x - 6}$

62. $g(x) = -|2x| + 8$

For Exercises 63–78, use transformations to graph the functions. (See Examples 7–8)

63. $v(x) = -(x + 2)^2 + 1$

64. $u(x) = -(x - 1)^2 - 2$

65. $f(x) = 2\sqrt{x + 3} - 1$

66. $g(x) = 2\sqrt{x - 1} + 3$

67. $p(x) = \frac{1}{2}|x - 1| - 2$

68. $q(x) = \frac{1}{3}|x + 2| - 1$

69. $r(x) = -\sqrt{-x} + 1$

70. $s(x) = -\sqrt{-x} - 2$

71. $f(x) = \sqrt{-x + 3}$

72. $g(x) = \sqrt{-x - 4}$

73. $n(x) = -\left|\frac{1}{2}x - 3\right|$

74. $m(x) = -\left|\frac{1}{3}x + 1\right|$

75. $f(x) = -\frac{1}{2}(x - 3)^2 + 8$

76. $g(x) = -\frac{1}{3}(x + 2)^2 + 3$

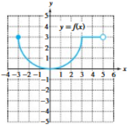
77. $p(x) = -4|x + 2| - 1$

78. $q(x) = -2|x - 1| + 4$

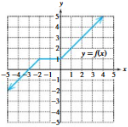
Mixed Exercises

For Exercises 79–86, the graph of $y = f(x)$ is given. Graph the indicated function.

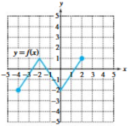
79. Graph $y = -f(x - 1) + 2$.



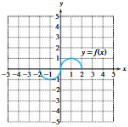
80. Graph $y = -f(x + 1) - 2$.



81. Graph $y = 2f(x - 2) - 3$.



82. Graph $y = 2f(x + 2) - 4$.



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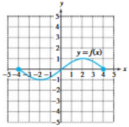
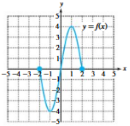
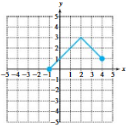
83. Graph $y = -3f(2x)$.



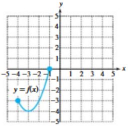
84. Graph $y = -\frac{1}{2}f\left(\frac{1}{2}x\right)$.



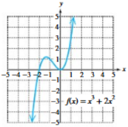
85. Graph $y = f(-x) - 2$.

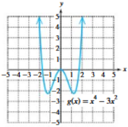


86. Graph $y = f(-x) + 3$.



For Exercises 87–88, match the equations with the graph. Use the parent functions *f* and *g* for reference, and identify the transformation used on the parent function to obtain each graph (choose from: horizontal shift, vertical shift, horizontal shrink, horizontal stretch, vertical shrink, vertical stretch).



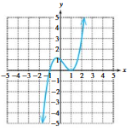


87. a. $m(x) = 2x^3 + 4x^2$

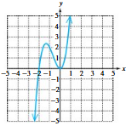
b. $n(x) = x^3 + 2x^2 - 1$

c. $p(x) = (x - 1)^3 + 2(x - 1)^2$

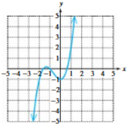
i.



ii.



iii.



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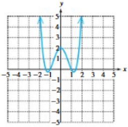
88. a. $q(x) = (x - 2)^4 - 3(x - 2)^2$

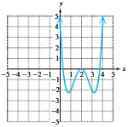
b. $r(x) = x^4 - 3x^2 + 2$

c. $s(x) = \frac{x^4}{16} - \frac{3x^2}{4}$

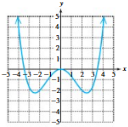
i.



ii.



iii.



For Exercises 89–92, write a function based on the given parent function and transformations in the given order.

89. Parent function: $y = x^3$

1. Shift 4.5 units to the left.
2. Reflect across the y-axis.
3. Shift upward 2.1 units.

90. Parent function $y = \sqrt[3]{x}$

1. Shift 1 unit to the left.
2. Stretch horizontally by a factor of 4.
3. Reflect across the x-axis.

91. Parent function: $y = \frac{1}{x}$

1. Stretch vertically by a factor of 2.
2. Reflect across the x-axis.
3. Shift downward 3 units.

92. Parent function: $y = |x|$

1. Shift 3.7 units to the right.
2. Shrink horizontally by a factor of $\frac{1}{3}$.
3. Reflect across the y-axis.

Write About It

93. Explain why the graph of $g(x) = |2x|$ can be interpreted as a horizontal shrink of the graph of $f(x) = |x|$ or as a vertical stretch of the graph of $f(x) = |x|$.

94. Explain why the graph of $h(x) = \sqrt{\frac{1}{2}x}$ can be interpreted as a horizontal stretch of the graph of $f(x) = \sqrt{x}$ or as a vertical shrink of the graph of $f(x) = \sqrt{x}$.

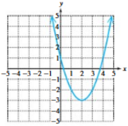
95. Explain the difference between the graphs of $f(x) = |x - 2| - 3$ and $g(x) = |x - 3| - 2$.

96. Explain why $g(x) = \frac{1}{-x + 1}$ can be graphed by shifting the graph of $f(x) = \frac{1}{x}$ one unit to the left and reflecting across the y-axis, or by shifting the graph of f one unit to the right and reflecting across the x-axis.

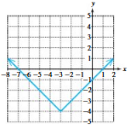
Expanding Your Skills

For Exercises 97–102, use transformations on the basic functions presented in Table 1-2 on pages 92–93 to write a rule $y = f(x)$ that would produce the given graph.

97.



98.



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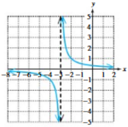
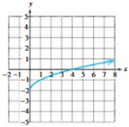
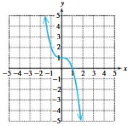
99.



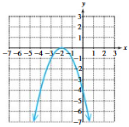
100.



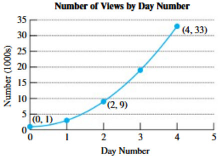
101.



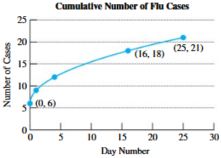
102.



103. The graph shows the number of views y (in thousands) for a new online video, t days after it was posted. Use transformations on one of the parent functions from Table 1-2 on pages 92–93 to model these data.



104. The graph shows the cumulative number y of flu cases among passengers on a 25-day cruise, t days after the cruise began. Use transformations on one of the parent functions from Table 1-2 on pages 92–93 to model these data.



Technology Connections

105. **a.** Graph the functions on the viewing window [−5, 5, 1] by [−2, 8, 1].

$y = x^2$

$y = x^4$

$y = x^6$

b. Graph the functions on the viewing window [−4, 4, 1] by [−10, 10, 1].

$y = x^3$

$y = x^5$

$y = x^7$

c. Describe the general shape of the graph of $y = x^n$ where n is an even integer greater than 1.

d. Describe the general shape of the graph of $y = x^n$ where n is an odd integer greater than 1.

GO ONLINE **Go online for more practice problems.**

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Lesson 1-7

Analyzing Graphs of Functions and Piecewise-Defined Functions

What Will You Learn?

After completing this lesson, you should be able to:

- Test for Symmetry
- Identify Even and Odd Functions
- Graph Piecewise-Defined Functions
- Investigate Increasing, Decreasing, and Constant Behavior of a Function
- Determine Relative Minima and Maxima of a Function

Learn: Test for Symmetry

The photos in Figures 1-28 through 1-30 each show a type of symmetry.

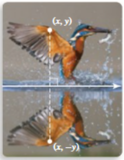

Figure 1-28

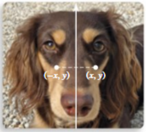

Figure 1-29

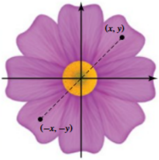

Figure 1-30

The photo of the kingfisher (Figure 1-28) shows an image of the bird reflected in the water. Suppose that we superimpose the *x*-axis at the waterline. Every point (x, y) on the bird has a mirror image $(x, -y)$ below the *x*-axis. Therefore, this image is symmetric with respect to the *x*-axis.

The face shown in Figure 1-29 is symmetric with respect to a vertical line through the center. If we place the *y*-axis along this line, a point (x, y) on one side has a mirror image at $(-x, y)$. This image is symmetric with respect to the *y*-axis.

The flower shown in Figure 1-30 is symmetric with respect to the point at its center. Suppose that we place the origin at the center of the flower. Notice that a point (x, y) on the image has a corresponding point $(-x, -y)$ on the image. This image is symmetric with respect to the origin.

Given an equation in *x* and *y*, the graph of the equation may possess symmetry with respect to the *x*-axis, *y*-axis, or origin. In each graph, we show two points related through symmetry.

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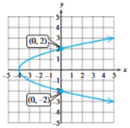
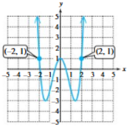
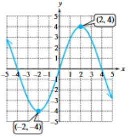
Symmetry with respect to the x-axis



Symmetry with respect to the y-axis



Symmetry with respect to the origin



Given an equation in the variables x and y, use the following rules to determine if the graph is symmetric with respect to the x-axis, the y-axis, or the origin.

Tests for Symmetry

Consider an equation in the variables x and y.

- The graph of the equation is symmetric with respect to the y-axis if substituting $-x$ for x in the equation results in an equivalent equation.
- The graph of the equation is symmetric with respect to the x-axis if substituting $-y$ for y in the equation results in an equivalent equation.
- The graph of the equation is symmetric with respect to the origin if substituting $-x$ for x and $-y$ for y in the equation results in an equivalent equation.

Example 1: Testing for Symmetry

Determine whether the graph is symmetric with respect to the y-axis, x-axis, or origin.

a. $y = |x|$ **b.** $x = y^2 - 4$

Solution:

a. $y = |x|$ Test for symmetry with respect to the y-axis.

$y = |-x|$ Replace x by $-x$. Note that $|-x| = |x|$.

$y = |x|$ The resulting equation *is* equivalent to the original equation.

Same equation: Graph is symmetric with respect to the y-axis.

$y = |x|$ Test for symmetry with respect to the x-axis.

$-y = |x|$ Replace y by $-y$.

$y = -|x|$ The resulting equation is *not* equivalent to the original equation.

not the same

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$y = |x|$

$-y = |-x|$

$-y = |x|$

$y = |-x|$

not the same

Test for symmetry with respect to the origin.

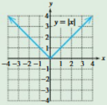
Replace x by $-x$ and y by $-y$.

The resulting equation is *not* equivalent to the original equation.

The graph is symmetric with respect to the y-axis only.

Insights

The graph of $y = |x|$ is one of the basic graphs presented in Lesson 1.6. From our familiarity with the graph we can visualize the symmetry with respect to the y-axis.



b. $x = y^2 - 4$

$-x = y^2 - 4$

$x = -y^2 + 4$

not the same

Test for symmetry with respect to the y-axis.

Replace x by $-x$. The resulting equation is *not* equivalent to the original equation.

$x = y^2 - 4$

$x = (-y)^2 - 4$

$x = y^2 - 4$

Same equation: Graph is symmetric with respect to the x-axis.

Test for symmetry with respect to the x-axis.

Replace y by $-y$. The resulting equation *is* equivalent to the original equation.

$x = y^2 - 4$

$-x = (-y)^2 - 4$

$-x = y^2 - 4$

$x = -y^2 - 4$

not the same

Test for symmetry with respect to the origin.

Replace x by $-x$ and y by $-y$.

The resulting equation is *not* equivalent to the original equation.

The graph is symmetric with respect to the x-axis only (Figure 1-31).

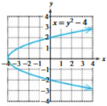


Figure 1-31

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Apply the Skills

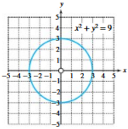
1. Determine whether the graph is symmetric with respect to the y-axis, x-axis, or origin.

   a. $y = x^2$ **b.** $|y| = x + 1$

Example 2: Testing for Symmetry

Determine whether the graph is symmetric with respect to the y-axis, x-axis, or origin.

$$x^2 + y^2 = 9$$



Solution:

The graph of $x^2 + y^2 = 9$ is a circle with center at the origin and radius 3. By inspection, we can see that the graph is symmetric with respect to both axes and the origin.

Test for y-axis symmetry.
Replace x by $=x$.

$$\left.\begin{aligned} x^2 + y^2 &= 9 \\ (-x)^2 + y^2 &= 9 \\ x^2 + y^2 &= 9 \end{aligned}\right] \text{same}$$

Test for x-axis symmetry.
Replace y by $-y$.

$$\left.\begin{aligned} x^2 + y^2 &= 9 \\ x^2 + (-y)^2 &= 9 \\ x^2 + y^2 &= 9 \end{aligned}\right] \text{same}$$

Test for origin symmetry.
Replace x by $-x$ and y by $-y$.

$$\left.\begin{aligned} x^2 + y^2 &= 9 \\ (-x)^2 + (-y)^2 &= 9 \\ x^2 + y^2 &= 9 \end{aligned}\right] \text{same}$$

The graph is symmetric with respect to the y-axis, the x-axis, and the origin.

Apply the Skills

2. Determine whether the graph is symmetric with respect to the y-axis, x-axis, or origin.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Learn: Identify Even and Odd Functions

A function may be symmetric with respect to the y-axis or to the origin. A function that is symmetric with respect to the y-axis is called an *even* function. A function that is symmetric with respect to the origin is called an *odd* function.

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Even and Odd Functions

- A function f is an **even function** if $f(-x) = f(x)$ for all x in the domain of f. The graph of an even function is symmetric with respect to the y-axis.
- A function f is an **odd function** if $f(-x) = -f(x)$ for all x in the domain of f. The graph of an odd function is symmetric with respect to the origin.

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Insights

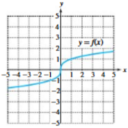
The only functions that are symmetric with respect to the x-axis are functions whose points lie solely on the x-axis.

Example 3: Identifying Even and Odd Functions

By inspection, determine if the function is even, odd, or neither.

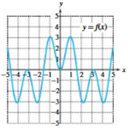
Solution:

a.



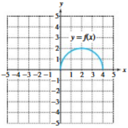
The function is symmetric with respect to the origin. Therefore, the function is an *odd* function.

b.



The function is symmetric with respect to the y-axis. Therefore, the function is an *even* function.

c.

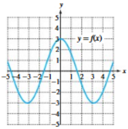


The function is not symmetric with respect to either the y-axis or the origin. Therefore, the function is *neither* even nor odd.

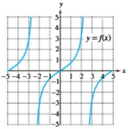
Apply the Skills

3. Determine if the function is even, odd, or neither.

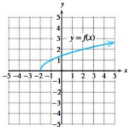
a.



b.



c.



Example 4: Identifying Even and Odd Functions

Determine if the function is even, odd, or neither.

a. $f(x) = -2x^4 + 5|x|$ **b.** $g(x) = 4x^3 - x$ **c.** $h(x) = 2x^2 + x$

Solution:

a.

$$f(x) = -2x^4 + 5|x|$$

$$f(-x) = -2(-x)^4 + 5|-x|$$

$$= -2x^4 + 5|x|$$

same

Determine whether the function is even.

Replace x by $-x$ to determine if $f(-x) = f(x)$.

Since $f(-x) = f(x)$, the function f is an even function.

There is no need to test whether f is an odd function because a function cannot be both even and odd unless all points are on the x-axis.

b. $g(x) = 4x^3 - x$

Each term has x raised to an odd power. Therefore, replacing x by $-x$ will result in the *opposite* of the original term. Therefore, test whether g is an odd function. That is, test whether $g(-x) = -g(x)$.

Evaluate: $g(-x)$

$$g(-x) = 4(-x)^3 - (-x)$$

$$= -4x^3 + x$$

Evaluate: $-g(x)$

$$-g(x) = -(4x^3 - x)$$

$$= -4x^3 + x$$

same

Since $g(-x) = -g(x)$, the function g is an odd function.

c.

$$h(x) = 2x^2 + x$$

$$h(-x) = 2(-x)^2 + (-x)$$

$$= 2x^2 - x$$

not the same

Determine whether the function is even.

Replace x by $-x$ to determine if $h(-x) = h(x)$.

Since $h(-x) \neq h(x)$, the function is not even.

Next, test whether h is an odd function. Test whether $h(-x) = -h(x)$.

Evaluate: $h(-x)$

$$h(-x) = 2(-x)^2 + (-x)$$

$$= 2x^2 - x$$

Evaluate: $-h(x)$

$$-h(x) = -(2x^2 + x)$$

$$= -2x^2 - x$$

not the same

Since $h(-x) \neq -h(x)$, the function is not an odd function. Therefore, h is neither even nor odd.

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Insights

In Example 4(c), $h(x)$ has a mixture of terms of the form x^{odd} and x^{even}. Therefore, we might suspect that the function is neither even nor odd.

Apply the Skills

4. Determine if the function is even, odd, or neither.

a. $m(x) = -x^5 + x^3$ **b.** $n(x) = x^2 - |x| + 1$ **c.** $p(x) = 2\,|x| + x$

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Learn: Graph Piecewise-Defined Functions

Suppose that a car is stopped for a red light. When the light turns green, the car under-goes a constant acceleration for 20 sec until it reaches a speed of 45 mph. It travels 45 mph for 1 min (60 sec) and then decelerates for 30 sec to stop at another red light. The graph of the car's speed y (in mph) versus the time x (in sec) after leaving the first red light is shown in Figure 1-32.

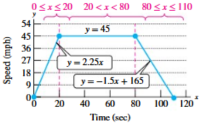


Figure 1-32

Notice that the graph can be segmented into three pieces. The first 20 sec is represented by a linear function with a positive slope, $y = 2.25x$. The next 60 sec is represented by the constant function $y = 45$. And the last 30 sec is represented by a linear function with a negative slope, $y = -1.5x + 165$.

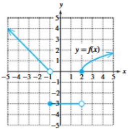
To write a rule defining this function we use a **piecewise-defined function** in which we define each "piece" on a restricted domain.

$$f(x) = \begin{cases} 2.25x & \text{for } 0 \le x \le 20 \\ 45 & \text{for } 20 < x < 80 \\ -1.5x + 165 & \text{for } 80 \le x \le 110 \end{cases}$$

Example 5: Interpreting a Piecewise-Defined Function

Evaluate the function for the given values of x.

$$f(x) = \begin{cases} -x - 1 & \text{for } x < -1 \\ -3 & \text{for } -1 \le x < 2 \\ \sqrt{x - 2} & \text{for } x \ge 2 \end{cases}$$



a. $f(-3)$ **b.** $f(-1)$

c. $f(2)$ **d.** $f(6)$

Solution:

a. $$\begin{aligned} f(x) &= -x - 1 \\ f(-3) &= -(-3) - 1 \\ &= 2 \end{aligned}$$

$x = -3$ is on the interval $x < -1$. Use the first rule in the function: $f(x) = -x - 1$.

b. $$\begin{aligned} f(x) &= -3 \\ f(-1) &= -3 \end{aligned}$$

$x = -1$ is on the interval $-1 \leq x < 2$. Use the second rule in the function: $f(x) = -3$.

c. $$\begin{aligned} f(x) &= \sqrt{x-2} \\ f(2) &= \sqrt{2-2} \\ &= 0 \end{aligned}$$

$x = 2$ is on the interval $x \geq 2$. Use the third rule in the function: $f(x) = \sqrt{x-2}$.

d. $$\begin{aligned} f(x) &= \sqrt{x-2} \\ f(6) &= \sqrt{6-2} \\ &= 2 \end{aligned}$$

$x = 6$ is on the interval $x \geq 2$. Use the third rule in the functi)on: $f(x) = \sqrt{x-2}$.

Apply the Skills

5. Evaluate the function for the given values of x.

$$f(x) = \begin{cases} x + 7 & \text{for } x < -2 \\ x^2 & \text{for } -2 \leq x < 1 \\ 3 & \text{for } x \geq 1 \end{cases}$$

a. $f(-3)$ **b.** $f(-2)$ **c.** $f(1)$ **d.** $f(4)$

TECHNOLOGY CONNECTIONS

Graphing a Piecewise-Defined Function

A graphing calculator can be used to graph a piecewise-defined function. The format to enter the function is as follows:

Y_1 = (first piece)/(first condition)

Y_2 = (second piece)/(second condition)

⋮

Each condition in parentheses is an inequality, and the calculator assigns it a value of 1 or 0 depending on whether the inequality is true or false. If an inequality is true, the function is divided by 1 on that interval and is “activated.” If an inequality is false, then the function is divided by 0. Since division by zero is undefined, the calculator does not graph the function on that interval, and the function is effectively “deactivated.”

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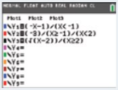
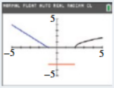
Enter the function from Example 5 as shown. Note that the inequality symbols can be found in the TEST menu.

$$f(x) = \begin{cases} -x-1 & \text{for } x < -1 \\ -3 & \text{for } -1 \le x < 2 \\ \sqrt{x-2} & \text{for } x \ge 2 \end{cases}$$



```
NORMAL FLOAT AUTO REAL RADIAN CL
 Plot1  Plot2  Plot3
\Y1■(⁻X−1)/(X<⁻1)
\Y2■(⁻3)/(X≥⁻1)/(X<2)
\Y3■(√(X−2))/(X≥2)
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
```



Notice that the individual "pieces" of the graph do not meet. For this reason, it is also a good practice to put the calculator in DOT mode in the MODE menu.

In Examples 6 and 7, we graph piecewise-defined functions.

Example 6: Graphing a Piecewise-Defined Function

Graph the function defined by $f(x) = \begin{cases} -3x & \text{for } x < 1 \\ -3 & \text{for } x \ge 1 \end{cases}$.

Solution:

- The first rule $f(x) = -3x$ defines a line with slope -3 and y-intercept $(0, 0)$. This line should be graphed only to the left of $x = 1$. The point $(1, -3)$ is graphed as an open dot, because the point is not part of the rule $f(x) = -3x$. See the blue portion of the graph in Figure 1-33.
- The second rule $f(x) = -3$ is a horizontal line for $x \ge 1$. The point $(1, -3)$ is a closed dot to show that it is part of the rule $f(x) = -3$. The closed dot from the red segment of the graph "overrides" the open dot from the blue segment. Taken together, the closed dot "plugs" the hole in the graph.

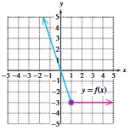


Figure 1-33

Apply the Skills

6. Graph the function.

$$f(x) = \begin{cases} 2 & \text{for } x \le -1 \\ -2x & \text{for } x > -1 \end{cases}$$

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The function in Example 6 has no "gaps," and therefore we say that the function is **continuous.** Informally, this means that we can draw the function without lifting our pencil from the page. The formal definition of a continuous function will be studied in calculus.

Example 7: Graphing a Piecewise-Defined Function

Graph the function. $f(x) = \begin{cases} x+3 & \text{for } x < -1 \\ x^2 & \text{for } -1 \le x < 2 \end{cases}$

Solution:

- The first rule $f(x) = x + 3$ defines a line with slope 1 and y-intercept (0, 3). This line should be graphed only for $x < -1$ (that is, to the left of $x = -1$). The point $(-1, 2)$ is graphed as an open dot, because the point is not part of the function. See the red portion of the graph in Figure 1-34.
- The second rule $f(x) = x^2$ is one of the basic functions learned in Lesson 1.6. It is a parabola with vertex at the origin. We sketch this function only for x values on the interval $-1 \le x < 2$. The point $(-1, 1)$ is a closed dot to show that it is part of the function. The point (2, 4) is displayed as an open dot to indicate that it is not part of the function.

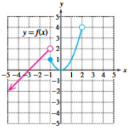

Figure 1-34

Check Your Work

Note that the function cannot have a closed dot at both $(-1, 1)$ and $(-1, 2)$ because it would not pass the vertical line test.

Apply the Skills

7. Graph the function.

$$f(x) = \begin{cases} |x| & \text{for } -4 \le x < 2 \\ -x+2 & \text{for } x \ge 2 \end{cases}$$

We now look at a special category of piecewise-defined functions called **step functions.** The graph of a step function is a series of discontinuous "steps." One important step function is called the **greatest integer function** or **floor function.** It is defined by

$f(x) = [\![x]\!]$ where $[\![x]\!]$ is the greatest integer less than or equal to x.

The operation $[\![x]\!]$ may also be denoted as **int(*x*)** or by **floor(*x*).** These alternative notations are often used in computer programming.

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In Example 8, we graph the greatest integer function.

Example 8: Graphing the Greatest Integer Function

Graph the function defined by $f(x) = \llbracket x \rrbracket$.

Solution:

x	$f(x) = \llbracket x \rrbracket$	Evaluate f for several values of x.
-1.7	-2	Greatest integer less than or equal to -1.7 is -2.
-1	-1	Greatest integer less than or equal to -1 is -1.
-0.6	-1	Greatest integer less than or equal to -0.6 is -1.
0	0	Greatest integer less than or equal to 0 is 0.
0.4	0	Greatest integer less than or equal to 0.4 is 0.
1	1	Greatest integer less than or equal to 1 is 1.
1.8	1	Greatest integer less than or equal to 1.8 is 1.
2	2	Greatest integer less than or equal to 2 is 2.
2.5	2	Greatest integer less than or equal to 2.5 is 2.

From the table, we see a pattern and from the pattern, we form the graph.

If $-3 \le x < -2$, then $\llbracket x \rrbracket = -3$

If $-2 \le x < -1$, then $\llbracket x \rrbracket = -2$

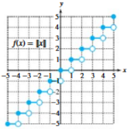
If $-1 \le x < 0$, then $\llbracket x \rrbracket = -1$

If $0 \le x < 1$, then $\llbracket x \rrbracket = 0$

If $1 \le x < 2$, then $\llbracket x \rrbracket = 1$

If $2 \le x < 3$, then $\llbracket x \rrbracket = 2$

...



Apply the Skills

8. Evaluate $f(x) = \llbracket x \rrbracket$ for the given values of x.

a. $f(1.7)$ **b.** $f(5.5)$ **c.** $f(-4)$ **d.** $f(-4.2)$

TECHNOLOGY CONNECTIONS

On many graphing calculators, the greatest integer function is denoted by int( ) and is found under the MATH menu followed by NUM.

In Example 9, we use a piecewise-defined function to model an application.

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Example 9: Using a Piecewise-Defined Function in an Application

A salesperson makes a monthly salary of \$3000 along with a 5% commission on sales over \$20,000 for the month. Write a piecewise-defined function to represent the salesperson's monthly income $I(x)$ (in \$) for x dollars in sales.

Solution:

Let x represent the amount in sales.

Then $x - 20{,}000$ represents the amount in sales over \$20,000.

There are two scenarios for the salesperson's income.

Scenario 1: The salesperson sells \$20,000 or less. In this case, the monthly income is a constant \$3000. This is represented by

$$y = 3000 \text{ for } 0 \le x \le 20{,}000$$

Scenario 2: The salesperson sells over \$20,000. In this case, the monthly income is \$3000 plus 5% of sales over \$20,000. This is represented by

$$y = 3000 + 0.05(x - 20{,}000) \text{ for } x > 20{,}000$$

Therefore, a piecewise-defined function for monthly income is

$$I(x) = \begin{cases} 3000 & \text{for } 0 \le x \le 20{,}000 \\ 3000 + 0.05(x - 20{,}000) & \text{for } x > 20{,}000 \end{cases}$$

Alternatively, we can simplify to get

$$I(x) = \begin{cases} 3000 & \text{for } 0 \le x \le 20{,}000 \\ 0.05x + 2000 & \text{for } x > 20{,}000 \end{cases}$$

A graph of $y = I(x)$ is shown in Figure 1-35. Notice that for $x = \$20{,}000$, both equations within the piecewise-defined function yield a monthly salary of \$3000. Therefore, the two line segments in the graph meet at (20,000, 3000).

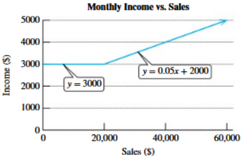

Figure 1-35

Apply the Skills

9. A retail store buys T-shirts from the manufacturer. The cost is \$7.99 per shirt for 1 to 100 shirts, inclusive. Then the price is decreased to \$6.99 per shirt thereafter. Write a piecewise-defined function that expresses the cost $C(x)$ (in \$) to buy x shirts.

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Learn: Investigate Increasing, Decreasing, and Constant Behavior of a Function

The graph in Figure 1-36 approximates the altitude of an airplane, $f(t)$, at a time t minutes after takeoff.

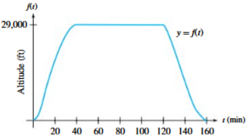


Figure 1-36

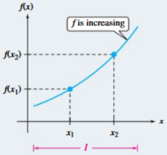
Notice that the plane's altitude increases up to the first 40 min of the flight. So we say that the function f is increasing on the interval (0, 40). The plane flies at a constant altitude for the next 1 hr 20 min, so we say that f is constant on the interval (40, 120). Finally, the plane's altitude decreases for the last 40 min, so we say that f is decreasing on the interval (120, 160).

Informally, a function is increasing on an interval in its domain if its graph rises from left to right. A function is decreasing on an interval in its domain if the graph "falls" from left to right. A function is constant on an interval in its domain if its graph is horizontal over the interval. These ideas are stated formally using mathematical notation.

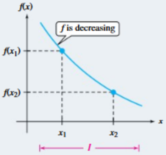
Intervals of Increasing, Decreasing, and Constant Behavior

Suppose that I is an interval contained within the domain of a function f.

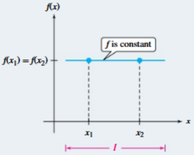
- f is increasing on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ on I.
- f is decreasing on I if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ on I.
- f is constant on I if $f(x_1) = f(x_2)$ for all x_1 and x_2 on I.



For all $x_1 < x_2$ on I, $f(x_1) < f(x_2)$



For all $x_1 < x_2$ on I, $f(x_1) > f(x_2)$



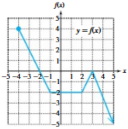
For all x_1 and x_2 on I, $f(x_1) = f(x_2)$

Example 10: Determining the Intervals Over Which a Function Is Increasing, Decreasing, and Constant

Use interval notation to write the interval(s) over which *f* is

a. Increasing **b.** Decreasing

c. Constant

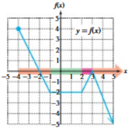


Solution:

a. *f* is increasing on the interval (2, 3). (Highlighted in red tint.)

b. *f* is decreasing on the intervals (–4, –1), (3, ∞). (Highlighted in orange tint.)

c. *f* is constant on the interval (–1, 2). (Highlighted in green tint.)



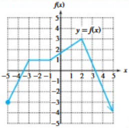
Apply the Skills

10. Use interval notation to write the interval(s) over which *f* is

a. Increasing

b. Decreasing

c. Constant



Learn: Determine Relative Minima and Maxima of a Function

The intervals over which a function changes from increasing to decreasing behavior or vice versa tell us where to look for relative maximum values and relative minimum values of a function. Consider the function pictured in Figure 1-37.

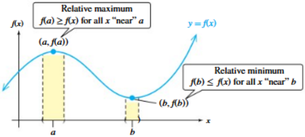

Relative maximum
$f(a) \geq f(x)$ for all x "near" a
f(x)
(a, f(a))
y = f(x)
Relative minimum
$f(b) \leq f(x)$ for all x "near" b
(b, f(b))
x
a
b

Figure 1-37

- The function has a relative maximum of *f*(*a*). Informally, this means that *f*(*a*) is the greatest function value relative to other points on the function nearby.
- The function has a relative minimum of *f*(*b*). Informally, this means that *f*(*b*) is the smallest function value relative to other points on the function nearby.

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This is stated formally in the following definition.

Relative Minimum and Relative Maximum Values

- $f(a)$ is a **relative maximum** of f if there exists an open interval containing a such that $f(a) \geq f(x)$ for all x in the interval.
- $f(b)$ is a **relative minimum** of f if there exists an open interval containing b such that $f(b) \leq f(x)$ for all x in the interval.

Note: An open interval is an interval in which the endpoints are not included.

The plural of maximum and minimum are **maxima** and **minima.** Note that relative maxima and minima are also called *local* maxima and minima.

If an ordered pair $(a, f(a))$ corresponds to a relative minimum or relative maximum, we interpret the coordinates of the ordered pair as follows:

- The x-coordinate is the *location* of the relative maximum or minimum within the domain of the function.
- The y-coordinate is the *value* of the relative maximum or minimum. This tells us how "high" or "low" the graph is at that point.

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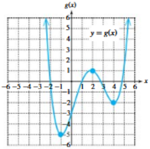
Example 11: Finding Relative Maxima and Minima

For the graph of $y = g(x)$ shown,



a. Determine the location and value of any relative maxima.

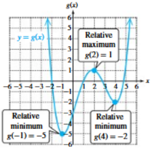
b. Determine the location and value of any relative minima.

Solution:

a. The point (2, 1) is the highest point in a small interval surrounding $x = 2$. Therefore, at $x = 2$, the function has a relative maximum of 1.

b. The point (–1, –5) is the lowest point in a small interval surrounding $x = –1$. Therefore, at $x = –1$, the function has a relative minimum of –5.

The point (4, –2) is the lowest point in a small interval surrounding $x = 4$. Therefore, at $x = 4$, the function has a relative minimum of –2.



Check Your Work

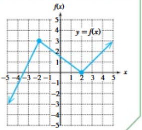
Be sure to note that the value of a relative minimum or relative maximum is the *y* value of a function, not the *x* value.

Apply the Skills

11. For the graph of $y = f(x)$ shown,

a. Determine the location and value of any relative maxima.

b. Determine the location and value of any relative minima.



TECHNOLOGY CONNECTIONS

Determining Relative Maxima and Minima

Relative maxima and relative minima are often difficult to find analytically and require techniques from calculus. However, a graphing utility can be used to approximate the location and value of relative maxima and minima. To do so, we use the Minimum and Maximum features.

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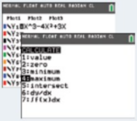
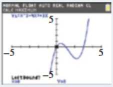
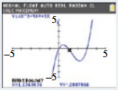
For example, enter the function defined by $Y_1 = x^3 - 4x^2 + 3x$. Then access the Maximum feature from the CALC menu.


Plot1 Plot2 Plot3
Y1=X^3−4X²+3X
Y2
Y3
Y4
Y5
Y6
Y7
Y8
Y9
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx

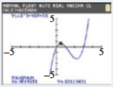
The calculator asks for a left bound. This is a point slightly to the left of the relative maximum. Then hit ENTER.


5
−5
5
Left Bound?
−5

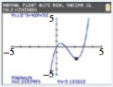
The calculator asks for a right bound. This is a point slightly to the right of the relative maximum. Hit ENTER.


5
−5
5
Right Bound?
−5

The calculator asks for a guess. This is a point close to the relative maximum. Hit ENTER and the approximate coordinates of the relative maximum point are shown (0.45, 0.63).


5
−5
5
Maximum
−5

To find the relative minimum, repeat these steps using the Minimum feature. The coordinates of the relative minimum point are approximately (2.22, −2.11).


5
−5
5
Minimum
−5

Practice 1–7

Practice Exercises

Prerequisite Review

R.1. Given the function defined by $f(x) = 7x - 2$, find $f(-a)$.

For Exercises R.2–R.4, graph the set and express the set in interval notation.

R.2. $\{x \mid x < 8\}$

R.3. $\{x \mid -2.4 \le x < 5.8\}$

R.4. $\left\{x \mid x \ge -\frac{9}{2}\right\}$

Concept Connections

1. A graph of an equation is symmetric with respect to the ________ -axis if replacing x by $-x$ results in an equivalent equation.
2. A graph of an equation is symmetric with respect to the ________ -axis if replacing y by $-y$ results in an equivalent equation.
3. A graph of an equation is symmetric with respect to the ________ if replacing x by $-x$ and y by $-y$ results in an equivalent equation.
4. An even function is symmetric with respect to the ________.
5. An odd function is symmetric with respect to the ________.
6. The expression ________ represents the greatest integer, less than or equal to x.

Learn: Test for Symmetry

For Exercises 7–18, determine whether the graph of the equation is symmetric with respect to the *x*-axis, *y*-axis, origin, or none of these. (See Examples 1–2)

7. $y = x^2 + 3$

8. $y = -|x| - 4$

9. $x = -|y| - 4$

10. $x = y^2 + 3$

11. $x^2 + y^2 = 3$

12. $|x| + |y| = 4$

13. $y = |x| + 2x + 7$

14. $y = x^2 + 6x + 1$

15. $x^2 = 5 + y^2$

16. $y^4 = 2 + x^2$

17. $y = \frac{1}{2}x - 3$

18. $y = \frac{2}{5}x + 1$

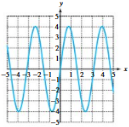
Learn: Identify Even and Odd Functions

19. What type of symmetry does an even function have?

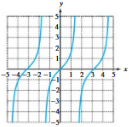
20. What type of symmetry does an odd function have?

For Exercises 21–26, use the graph to determine if the function is even, odd, or neither. (See Example 3)

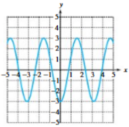
21.



22.



23.



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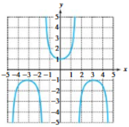
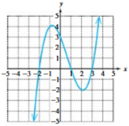
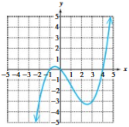
24.



25.



26.



27. a. Given $f(x) = 4x^2 - 3\,|x|$, find $f(-x)$.

b. Is $f(-x) = f(x)$?

c. Is this function even, odd, or neither?

28. a. Given $g(x) = -x^8 + |3x|$, find $g(-x)$.

b. Is $g(-x) = g(x)$?

c. Is this function even, odd, or neither?

29. a. Given $h(x) = 4x^3 - 2x$, find $h(-x)$.

b. Find $-h(x)$.

c. Is $h(-x) = -h(x)$?

d. Is this function even, odd, or neither?

30. a. Given $k(x) = -8x^5 - 6x^3$, find $k(-x)$.

b. Find $-k(x)$.

c. Is $k(-x) = -k(x)$?

d. Is this function even, odd, or neither?

31. a. Given $m(x) = 4x^2 + 2x - 3$, find $m(-x)$.

b. Find $-m(x)$.

c. Is $m(-x) = m(x)$?

d. Is $m(-x) = -m(x)$?

e. Is this function even, odd, or neither?

32. a. Given $n(x) = 7|x| + 3x - 1$, find $n(-x)$.

b. Find $-n(x)$.

c. Is $n(-x) = n(x)$?

d. Is $n(-x) = -n(x)$?

e. Is this function even, odd, or neither?

For Exercises 33–46, determine if the function is even, odd, or neither. (See Example 4)

33. $f(x) = 3x^6 + 2x^2 + |x|$

34. $p(x) = -|x| + 12x^{10} + 5$

35. $k(x) = 13x^3 + 12x$

36. $m(x) = -4x^5 + 2x^3 + x$

37. $n(x) = \sqrt{16 - (x-3)^2}$

38. $r(x) = \sqrt{81 - (x+2)^2}$

39. $q(x) = \sqrt{16 + x^2}$

40. $z(x) = \sqrt{49 + x^2}$

41. $h(x) = 5x$

42. $g(x) = -x$

43. $f(x) = \frac{x^2}{3(x-4)^2}$

44. $g(x) = \frac{x^3}{2(x-1)^3}$

45. $vx = \frac{-x^5}{|x| + 2}$

46. $w(x) = \frac{-\sqrt[3]{x}}{x^2 + 1}$

Learn: Graph Piecewise-Defined Functions

For Exercises 47–50, evaluate the function for the given values of *x*. (See Example 5)

47. $$f(x) = \begin{cases} -3x + 7 & \text{for } x < -1 \\ x^2 + 3 & \text{for } -1 \le x < 4 \\ 5 & \text{for } x \ge 4 \end{cases}$$

a. $f(3)$ **b.** $f(-2)$ **c.** $f(-1)$

d. $f(4)$ **e.** $f(5)$

48. $$g(x) = \begin{cases} -2|x| - 3 & \text{for } x \le -2 \\ 5x + 6 & \text{for } -2 < x < 3 \\ 4 & \text{for } x \ge 3 \end{cases}$$

a. $g(-3)$ **b.** $g(3)$ **c.** $g(-2)$

d. $g(0)$ **e.** $g(4)$

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49. $h(x) = \begin{cases} 2 & \text{for } -3 \le x < -2 \\ 1 & \text{for } -2 \le x < -1 \\ 0 & \text{for } -1 \le x < 0 \\ -1 & \text{for } 0 \le x < 1 \end{cases}$

a. $h(-1.7)$ **b.** $h(-2.5)$ **c.** $h(0.05)$

d. $h(-2)$ **e.** $h(0)$

50. $t(x) = \begin{cases} x & \text{for } 0 < x \le 1 \\ 2x & \text{for } 1 < x \le 2 \\ 3x & \text{for } 2 < x \le 3 \\ 4x & \text{for } 3 < x \le 4 \end{cases}$

a. $t(1.99)$ **b.** $t(0.4)$ **c.** $t(3)$

d. $t(1)$ **e.** $t(3.001)$

51. A sled accelerates down a hill and then slows down after it reaches a flat portion of ground. The speed of the sled $s(t)$ (in ft/sec) at a time t (in sec) after movement begins can be approximated by:

$$s(t) = \begin{cases} 1.5t & \text{for } 0 \le t \le 20 \\ \dfrac{30}{t-19} & \text{for } 20 < t \le 40 \end{cases}$$

Determine the speed of the sled after 10 sec, 20 sec, 30 sec, and 40 sec. Round to 1 decimal place if necessary.

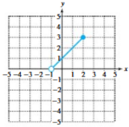
52. A car starts from rest and accelerates to a speed of 60 mph in 12 sec. It travels 60 mph for 1 min and then decelerates for 20 sec until it comes to rest. The speed of the car $s(t)$ (in mph) at a time t (in sec) after the car begins motion can be modeled by:

$$s(t) = \begin{cases} \frac{5}{12}t^2 & \text{for } 0 \le t \le 12 \\ 60 & \text{for } 12 < t \le 72 \\ \frac{3}{20}(92 - t^2) & \text{for } 72 < t \le 92 \end{cases}$$

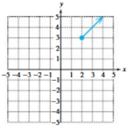
Determine the speed of the car 6 sec, 12 sec, 45 sec, and 80 sec after the car begins motion.

For Exercises 53–56, match the function with its graph from the options shown.

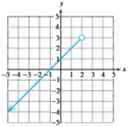
a.



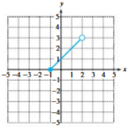
b.



c.



d.



53. $f(x) = x + 1$ for $x < 2$

54. $f(x) = x + 1$ for $-1 < x \le 2$

55. $f(x) = x + 1$ for $-1 \le x < 2$

56. $f(x) = x + 1$ for $x \ge 2$

57. a. Graph $p(x) = x + 2$ for $x \le 0$. (**See Examples 6–7**)

b. Graph $q(x) = -x^2$ for $x > 0$.

c. Graph $r(x) = \begin{cases} x + 2 & \text{for } x \le 0 \\ -x^2 & \text{for } x > 0 \end{cases}$

58. a. Graph $f(x) = |x|$ for $x < 0$.

b. Graph $g(x) = \sqrt{x}$ for $x \ge 0$.

c. Graph $h(x) = \begin{cases} |x| & \text{for } x < 0 \\ \sqrt{x} & \text{for } x \ge 0 \end{cases}$

59. a. Graph $m(x) = \frac{1}{2}x - 2$ for $x \le -2$.

b. Graph $n(x) = -x + 1$ for $x > -2$.

c. Graph $t(x) = \begin{cases} \frac{1}{2}x - 2 & \text{for } x \le -2 \\ -x + 1 & \text{for } x > -2 \end{cases}$

60. a. Graph $a(x) = x$ for $x < 1$.

b. Graph $b(x) = \sqrt{x-1}$ for $x \ge 1$.

c. Graph $c(x) = \begin{cases} x & \text{for } x < 1 \\ \sqrt{x-1} & \text{for } x \ge 1 \end{cases}$

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For Exercises 61–70, graph the function. (See Examples 6–7)

61. $f(x) = \begin{cases} |x| & \text{for } x < 2 \\ -x + 4 & \text{for } x \geq 2 \end{cases}$

62. $h(x) = \begin{cases} -2x & \text{for } x < 0 \\ \sqrt{x} & \text{for } x \geq 0 \end{cases}$

63. $g(x) = \begin{cases} x + 2 & \text{for } x < -1 \\ -x + 2 & \text{for } x \geq -1 \end{cases}$

64. $k(x) = \begin{cases} 3x & \text{for } x < 1 \\ -3x & \text{for } x \geq 1 \end{cases}$

65. $r(x) = \begin{cases} x^2 - 4 & \text{for } x \leq 2 \\ 2x - 4 & \text{for } x > 2 \end{cases}$

66. $s(x) = \begin{cases} -x - 1 & \text{for } x \leq -1 \\ \sqrt{x + 1} & \text{for } x > -1 \end{cases}$

67. $t(x) = \begin{cases} -3 & \text{for } -4 \leq x < -2 \\ -1 & \text{for } -2 \leq x < 0 \\ 1 & \text{for } 0 \leq x < 2 \end{cases}$

68. $z(x) = \begin{cases} -1 & \text{for } -3 < x \leq -1 \\ 1 & \text{for } -1 < x \leq 1 \\ 3 & \text{for } 1 < x \leq 3 \end{cases}$

69. $m(x) = \begin{cases} 3 & \text{for } -4 < x < -1 \\ -x & \text{for } -1 < x < 3 \\ \sqrt{x - 3} & \text{for } x \geq 3 \end{cases}$

70. $n(x) = \begin{cases} -4 & \text{for } -3 < x < -1 \\ x & \text{for } -1 \leq x < 2 \\ -x^2 + 4 & \text{for } x \geq 2 \end{cases}$

71. a. Graph $f(x) = \begin{cases} -x & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}$

b. To what basic function from Lesson 1.6 is the graph of f equivalent?

For Exercises 72–80, evaluate the step function defined by $f(x) = \llbracket x \rrbracket$ for the given values of x. (See Example 8)

72. $f(-3.7)$

73. $f(-4.2)$

74. $f(-0.5)$

75. $f(-0.09)$

76. $f(0.5)$

77. $f(0.09)$

78. $f(6)$

79. $f(-9)$

80. $f(-5)$

For Exercises 81–84, graph the function. (See Example 8)

81. $f(x) = \llbracket x + 3 \rrbracket$

82. $g(x) = \llbracket x - 3 \rrbracket$

83. $k(x) = \text{int}\left(\frac{1}{2}x\right)$

84. $h(x) = \text{int}(2x)$

85. For a recent year, the rate for first class postage was as follows. **(See Example 9)**

Weight not Over	Price
1 oz	\$0.44
2 oz	\$0.61
3 oz	\$0.78
3.5 oz	\$0.95

Write a piecewise-defined function to model the cost $C(x)$ to mail a letter first class if the letter is x ounces.

86. The water level in a retention pond started at 5 ft (60 in.) and decreased at a rate of 2 in./day during a 14-day drought. A tropical depression moved through at the beginning of the 15th day and produced rain at an average rate of 2.5 in./day for 5 days. Write a piecewise-defined function to model the water level $L(x)$ (in inches) as a function of the number of days x since the beginning of the drought.

87. A salesperson makes a base salary of \$2000 per month plus an additional 5% commission on the amount in sales over \$40,000. Write a piecewise-defined function to model the salesperson's total monthly salary $S(x)$ (in \$) as a function of the amount in sales x.

88. A cell phone plan charges \$49.95 per month plus \$14.02 in taxes, plus \$0.40 per minute for calls beyond the 600-min monthly limit. Write a piecewise-defined function to model the monthly cost $C(x)$ (in \$) as a function of the number of minutes used x for the month.

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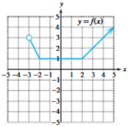
Learn: Investigate Increasing, Decreasing, and Constant Behavior of a Function

For Exercises 89–96, use interval notation to write the intervals over which *f* is (a) increasing, (b) decreasing, and (c) constant. (See Example 10)

89.



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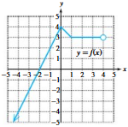

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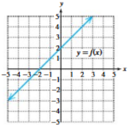

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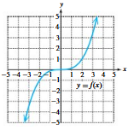

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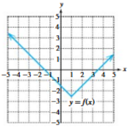

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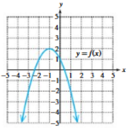

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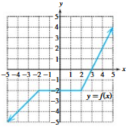

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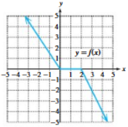
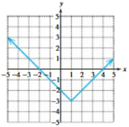


Learn: Determine Relative Minima and Maxima of a Function

For Exercises 97–102, identify the location and value of any relative maxima or minima of the function. (See Example 11)

97.



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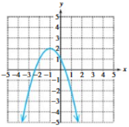

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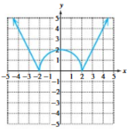
99.



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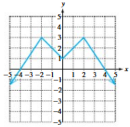

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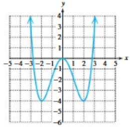
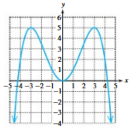
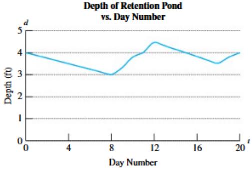


102.



103. The graph shows the depth *d* (in ft) of a retention pond, *t* days after recording began.



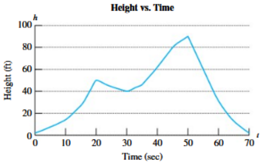
a. Over what interval(s) does the depth increase?

b. Over what interval(s) does the depth decrease?

c. Estimate the times and values of any relative maxima or minima on the interval (0, 20).

d. If rain is the only water that enters the pond, explain what the intervals of increasing and decreasing behavior mean in the context of this problem.

104. The graph shows the height *h* (in meters) of a roller coaster *t* seconds after the ride starts.



a. Over what interval(s) does the height increase?

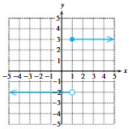
b. Over what interval(s) does the height decrease?

c. Estimate the times and values of any relative maxima or minima on the interval (0, 70).

Mixed Exercises

For Exercises 105–110, produce a rule for the function whose graph is shown. (*Hint*: Consider using the basic functions learned in Lesson 1.6 and transformations of their graphs.)

105.



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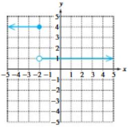
106.



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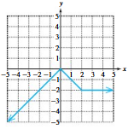

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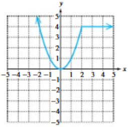

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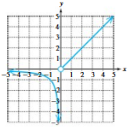

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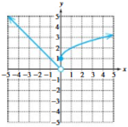

For Exercises 111–112,

a. Graph the function.

b. Write the domain in interval notation.

c. Write the range in interval notation.

d. Evaluate $f(-1)$, $f(1)$, and $f(2)$.

e. Find the value(s) of x for which $f(x) = 6$.

f. Find the value(s) of x for which $f(x) = -3$.

g. Use interval notation to write the intervals over which f is increasing, decreasing, or constant.

111. $f(x) = \begin{cases} -x^2 + 1 & \text{for } x \le 1 \\ 2x & \text{for } x > 1 \end{cases}$

112. $f(x) = \begin{cases} |x| & \text{for } x < 2 \\ -x & \text{for } x \ge 2 \end{cases}$

In computer programming, the greatest integer function is sometimes called the "floor" function. Programmers also make use of the "ceiling" function, which returns the smallest integer not less than *x*. For example: ceil(3.1) = 4. For Exercises 113–114, evaluate the floor and ceiling functions for the given value of *x*.

floor(x) is the greatest integer less than or equal to x.

ceil(x) is the smallest integer not less than x.

113. a. floor(2.8) **b.** floor(−3.1)
c. floor(4) **d.** ceil(2.8)
e. ceil(−3.1) **f.** ceil(4)

114. a. floor(5.5) **b.** floor(−0.1)
c. floor(−2) **d.** ceil(5.5)
e. ceil(−0.1) **f.** ceil(−2)

Write About It

115. From an equation in x and y, explain how to determine whether the graph of the equation is symmetric with respect to the x-axis, y-axis, or origin.

116. From the graph of a function, how can you determine if the function is even or odd?

117. Explain why the relation is not a function.

$$y = \begin{cases} 2x & \text{for } x \le 1 \\ 3 & \text{for } x \ge 1 \end{cases}$$

118. Explain why the function is discontinuous at $x = 1$.

$$f(x) = \begin{cases} 3x & \text{for } x < 1 \\ 3 & \text{for } x > 1 \end{cases}$$

119. Provide an informal explanation of a relative maximum.

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120. Explain what it means for a function to be increasing on an interval.

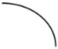
Expanding Your Skills

121. Suppose that the average rate of change of a continuous function between any two points to the left of $x = a$ is negative, and the average rate of change of the function between any two points to the right of $x = a$ is positive. Does the function have a relative minimum or maximum at a?

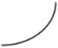
122. Suppose that the average rate of change of a continuous function between any two points to the left of $x = a$ is positive, and the average rate of change of the function between any two points to the right of $x = a$ is negative. Does the function have a relative minimum or maximum at a?

A graph is *concave up* on a given interval if it "bends" upward. A graph is *concave down* on a given interval if it "bends" downward. For Exercises 123–126, determine whether the curve is (a) concave up or concave down and (b) increasing or decreasing.

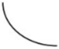
123.



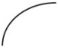
124.



125.

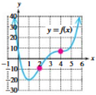


126.

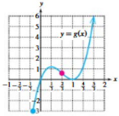


For Exercises 127–128, the graph of a function is shown. Use interval notation to write the open interval(s) over which each function is (a) concave up, (b) concave down, (c) increasing, and (d) decreasing. (Note: An open interval does not include the endpoints.) Then, (e) determine the location and value of the highest and lowest point of the function.

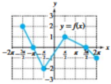
127.

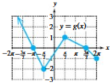


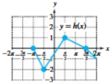
128.



The least value of a function over a given domain is referred to as an *absolute minimum*. Similarly, the greatest value of a function over a given domain is referred to as an *absolute maximum*. For Exercise 129, refer to the graphs of functions *f*, *g*, and *h*.







129. a. Write the domain of each function in interval notation.

b. Which functions have an absolute maximum? What is the absolute maximum?

c. Which functions have an absolute minimum? What is the absolute minimum?

d. What do you observe about the concavity of each function?

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130. For a recent year, the federal income tax owed by a taxpayer (single—no dependents) was based on the individual's taxable income. (*Source*: Internal Revenue Service, www.irs.gov.)

Schedule X–If your filing status is "Single"

If taxable income is over	But not over	The tax is	Of the amount over
\$0	\$9,525	\$0.00 + 10%	\$0
\$9,525	\$38,700	\$952.50 + 12%	\$9,525
\$38,700	\$82,500	\$4,453.50 + 22%	\$38,700

Write a piecewise-defined function that expresses an individual's federal income tax $f(x)$ (in \$) as a function of the individual's taxable income x (in \$).

Technology Connections

For Exercises 131–134, use a graphing utility to graph the piecewise-defined function.

131. $f(x) = \begin{cases} 2.5x + 2 & \text{for } x \le 1 \\ x^2 - x - 1 & \text{for } x > 1 \end{cases}$

132. $g(x) = \begin{cases} -3.1x - 4 & \text{for } x < -2 \\ -x^3 + 4x - 1 & \text{for } x \ge -2 \end{cases}$

133. $k(x) = \begin{cases} -2.7x - 4.1 & \text{for } x \le -1 \\ -x^3 + 2x + 5 & \text{for } -1 < x < 2 \\ 1 & \text{for } x \ge 2 \end{cases}$

134. $z(x) = \begin{cases} 2.5x + 8 & \text{for } x < -2 \\ -2x^2 + x + 4 & \text{for } -2 \le x < 2 \\ -2 & \text{for } x \ge 2 \end{cases}$

For Exercises 135–138, use a graphing utility to

- **a.** Find the locations and values of the relative maxima and relative minima of the function on the standard viewing window. Round to 3 decimal places.
- **b.** Use interval notation to write the intervals over which f is increasing or decreasing.

135. $f(x) = -0.6x^2 + 2x + 3$

136. $f(x) = 0.4x^2 - 3x - 2.2$

137. $f(x) = 0.5x^3 + 2.1x^2 - 3x - 7$

138. $f(x) = -0.4x^3 - 1.1x^2 + 2x + 3$

Go online for more practice problems.

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Lesson 1-8

Algebra of Functions and Function Composition

What Will You Learn?

After completing this lesson, you should be able to:

- Perform Operations on Functions
- Evaluate a Difference Quotient
- Compose and Decompose Functions

Learn: Perform Operations on Functions

In Lesson 1.5, we learned that a profit function can be constructed from the difference of a revenue function and a cost function according to the following rule:

$$P(x) = R(x) - C(x)$$

As this example illustrates, the difference of two functions makes up a new function. New functions can also be formed from the sum, product, and quotient of two functions.

Sum, Difference, Product, and Quotient of Functions

Given functions f and g, the functions $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$ are defined by

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ provided that } g(x) \neq 0$$

The domains of the functions $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$ are all real numbers in the intersection of the domains of the individual functions f and g. For $\frac{f}{g}$, we further restrict the domain to exclude values of x for which $g(x) = 0$.

Example 1: Adding Two Functions

Given $f(x) = \sqrt{25 - x^2}$ and $g(x) = 5$, find $(f + g)(x)$.

Solution:

By definition: $(f + g)(x) = f(x) + g(x)$

$$= \sqrt{25 - x^2} + 5$$

Apply the Skills

1. Given $m(x) = -|x|$ and $n(x) = 4$, find $(m + n)(x)$.

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In Example 1, the graph of function *f* is a semicircle and the graph of function *g* is a horizontal line (Figure 1-38). Therefore, the graph of $y = (f + g)(x)$ is the graph of *f* with a vertical shift (shown in blue). Notice that each *y* value on $f + g$ is the sum of the *y* values from the individual functions *f* and *g*.

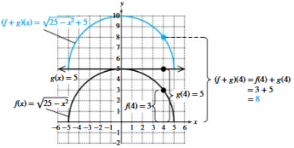

Figure 1-38

In Example 2, we evaluate the difference, product, and quotient of functions for given values of *x*.

Example 2: Evaluating Functions for Given Values of *x*

Given $m(x) = 4x$, $n(x) = |x - 3|$, and $p(x) = \frac{1}{x+1}$, determine the function values if possible.

a. $(m - n)(-2)$ **b.** $(m \cdot p)(1)$ **c.** $\left(\frac{p}{n}\right)(3)$

Solution:

a.
$$\begin{aligned}(m - n)(-2) &= m(-2) - n(-2)\\ &= 4(-2) - |-2 - 3|\\ &= -8 - 5\\ &= -13\end{aligned}$$

b.
$$\begin{aligned}(m \cdot p)(1) &= m(1) \cdot p(1)\\ &= 4(1) \cdot \frac{1}{1+1}\\ &= 2\end{aligned}$$

c.
$$\begin{aligned}\left(\frac{p}{n}\right)(3) &= \frac{p(3)}{n(3)}\\ &= \frac{\frac{1}{3+1}}{|3-3|}\\ &= \frac{\frac{1}{4}}{0} \quad \text{(undefined)}\end{aligned}$$

The domain of $\frac{p}{n}$ excludes any values of *x* that make $n(x) = 0$. In this case, $x = 3$ is excluded from the domain.

Apply the Skills

2. Use the functions defined in Example 2 to find

a. $(n - m)(-6)$ **b.** $(n \cdot p)(0)$ **c.** $\left(\frac{p}{m}\right)(0)$

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When combining two or more functions to create a new function, always be sure to determine the domain of the new function. Notice that in Example 2(c), the function $\frac{p}{n}$ is not defined for $x = -1$ or for $x = 3$.

$$\left(\frac{p}{n}\right)(x) = \frac{p(x)}{n(x)} = \frac{\frac{1}{x+1}}{|x-3|} \begin{array}{l} \longleftarrow \text{Denominator is zero for } x = -1. \\ \longleftarrow \text{Denominator of the complex fraction is zero for } x = 3. \end{array}$$

Example 3: Combining Functions and Determining Domain

Given $g(x) = 2x$, $h(x) = x^2 - 4x$, and $k(x) = \sqrt{x-1}$,

a. Find $(g - h)(x)$ and write the domain of $g - h$ in interval notation.

b. Find $(g \cdot k)(x)$ and write the domain of $g \cdot k$ in interval notation.

c. Find $\left(\frac{k}{h}\right)(x)$ and write the domain of $\frac{k}{h}$ in interval notation.

Solution:

a.
$$\begin{aligned}(g-h)(x) &= g(x) - h(x) \\ &= 2x - (x^2 - 4x) \\ &= -x^2 + 6x\end{aligned}$$

The domain is $(-\infty, \infty)$.

The domain of g is $(-\infty, \infty)$.

The domain of h is $(-\infty, \infty)$.

Therefore, the intersection of their domains is $(-\infty, \infty)$.

b.
$$\begin{aligned}(g \cdot k)(x) &= g(x) \cdot k(x) \\ &= 2x\sqrt{x-1}\end{aligned}$$

The domain is $[1, \infty)$.

The domain of g is $(-\infty, \infty)$.

The domain of k is $[1, \infty)$.

Therefore, the intersection of their domains is $[1, \infty)$.

c.
$$\begin{aligned}\left(\frac{k}{h}\right)(x) &= \frac{k(x)}{h(x)} \\ &= \frac{\sqrt{x-1}}{x^2 - 4x}\end{aligned}$$

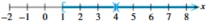
The domain of k is $[1, \infty)$.

The domain of h is $(-\infty, \infty)$.

The intersection of their domains is $[1, \infty)$.

However, we must also exclude values of x that make the denominator zero. In this case, exclude $x = 0$ and $x = 4$. The value $x = 0$ is already excluded because it is not on the interval $[1, \infty)$. Excluding $x = 4$, the domain of $\frac{k}{h}$ is $[1, 4) \cup (4, \infty)$.

The domain is $[1, 4) \cup (4, \infty)$.



Apply the Skills

3. Given $m(x) = x + 3$, $n(x) = x^2 - 9$, and $p(x) = \sqrt{x+1}$

a. Find $(n - m)(x)$ and write the domain of $n - m$ in interval notation.

b. Find $(m \cdot p)(x)$ and write the domain of $m \cdot p$ in interval notation.

c. Find $\left(\frac{p}{n}\right)(x)$ and write the domain of $\frac{p}{n}$ in interval notation.

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Learn: Evaluate a Difference Quotient

In Lesson 1.4, we learned that if f is defined on an interval $[x_1, x_2]$, then the average rate of change of f between $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \quad \text{(Figure 1-39)}$$

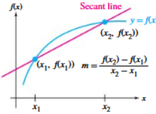

Figure 1-39

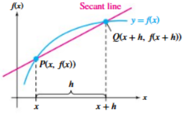

Figure 1-40

Now we look at a related idea. Let P be an arbitrary point $(x, f(x))$ on the function f. Let h be a positive real number and let Q be the point $(x + h, f(x + h))$. See Figure 1-40. The average rate of change between P and Q is the slope of the secant line and is given by:

$$\begin{aligned} m &= \frac{f(x+h) - f(x)}{(x+h) - x} \\ &= \frac{f(x+h) - f(x)}{h} \quad \text{(Difference quotient)} \end{aligned}$$

The expression on the right is called the **difference quotient** and is very important for the foundation of calculus. In Examples 4 and 5, we practice evaluating the difference quotient for two functions.

Insights h is taken to be a positive real number, implying that $h \neq 0$.

Example 4: Finding a Difference Quotient

Given $f(x) = 3x - 5$,

a. Find $f(x + h)$. **b.** Find the difference quotient, $\frac{f(x+h) - f(x)}{h}$.

Solution:

a. $f(x + h) = 3(x + h) - 5$ Substitute $(x + h)$ for x.

$= 3x + 3h - 5$

b.
$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\overbrace{(3x + 3h - 5)}^{\boxed{f(x+h)}} - \overbrace{(3x - 5)}^{\boxed{f(x)}}}{h} \\ &= \frac{3x + 3h - 5 - 3x + 5}{h} && \text{Clear parentheses.} \\ &= \frac{3h}{h} && \text{Combine like terms.} \\ &= 3 && \text{Simplify the fraction.} \end{aligned}$$

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Apply the Skills

4. Given $f(x) = 4x - 2$,

a. Find $f(x + h)$.

b. Find the difference quotient, $\dfrac{f(x+h) - f(x)}{h}$.

Example 5: Finding a Difference Quotient

Given $f(x) = -2x^2 + 4x - 1$,

a. Find $f(x + h)$.

b. Find the difference quotient, $\dfrac{f(x+h) - f(x)}{h}$.

Solution:

a.
$$\begin{aligned} f(x+h) &= -2(x+h)^2 + 4(x+h) - 1 \\ &= -2(x^2 + 2xh + h^2) + 4x + 4h - 1 \\ &= -2x^2 - 4xh - 2h^2 + 4x + 4h - 1 \end{aligned}$$

Substitute $(x + h)$ for x.

b.
$$\frac{f(x+h) - f(x)}{h}$$

$$= \frac{\overbrace{(-2x^2 - 4xh - 2h^2 + 4x + 4h - 1)}^{\boxed{f(x+h)}} - \overbrace{(-2x^2 + 4x - 1)}^{\boxed{f(x)}}}{h}$$

$$= \frac{-2x^2 - 4xh - 2h^2 + 4x + 4h - 1 + 2x^2 - 4x + 1}{h}$$

Clear parentheses.

$$= \frac{-4xh - 2h^2 + 4h}{h}$$

Combine like terms.

$$= \frac{\overset{1}{\cancel{h}}(-4x - 2h + 4)}{\cancel{h}}$$

$$= -4x - 2h + 4$$

Factor numerator and denominator, and simplify the fraction.

Apply the Skills

5. Given $f(x) = -x^2 - 5x + 2$,

a. Find $f(x + h)$.

b. Find the difference quotient, $\dfrac{f(x+h) - f(x)}{h}$.

Learn: Compose and Decompose Functions

The next operation on functions we present is called the composition of functions. Informally, this involves a substitution process in which the output from one function becomes the input to another function.

Composition of Functions

The **composition of f and g,** denoted $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$.

The domain of $f \circ g$ is the set of real numbers x in the domain of g such that $g(x)$ is in the domain of f.

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It is important to note that the notation $(f \circ g)(x)$ represents the composition of functions, *not* multiplication of f, g, and x.

To visualize the composition of functions $(f \circ g)(x) = f(g(x))$, consider Figure 1-41.

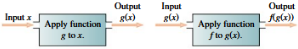

Figure 1-41

Example 6: Composing Functions

Given $f(x) = x^2 + 2x$ and $g(x) = x - 4$, find

a. $f(g(6))$ **b.** $g(f(-3))$ **c.** $(f \circ g)(0)$ **d.** $(g \circ f)(5)$

Solution:

a. $f(g(6)) = f(2)$ Evaluate $g(6)$ first. $g(6) = 6 - 4 = 2$.
$= 8$ $f(2) = (2)^2 + 2(2) = 8$

Good Practices

When composing functions, apply the order of operations. In Example 6(a), the value of $g(6)$ is found first.

b. $g(f(-3)) = g(3)$ Evaluate $f(-3)$ first. $f(-3) = (-3)^2 + 2(-3) = 3$.
$= -1$ $g(3) = 3 - 4 = -1$

c. $(f \circ g)(0) = f(g(0))$ Evaluate $g(0)$ first.
$= f(-4)$ $g(0) = 0 - 4 = -4$.
$= 8$ $f(-4) = (-4)^2 + 2(-4) = 8$

d. $(g \circ f)(5) = g(f(5))$ Evaluate $f(5)$ first.
$= g(35)$ $f(5) = (5)^2 + 2(5) = 35$.
$= 31$ $g(35) = 35 - 4 = 31$

Apply the Skills

6. Refer to functions f and g given in Example 6. Find

a. $f(g(-4))$ **b.** $g(f(-5))$ **c.** $(f \circ g)(9)$ **d.** $(g \circ f)(10)$

In Example 7, we practice composing functions and identifying the domain of the composite function. This Example also illustrates that function composition is not commutative. That is, $(f \circ g)(x) \neq (g \circ f)(x)$ for all functions f and g.

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Example 7: Composing Functions and Determining Domain

Given $f(x) = 2x - 6$ and $g(x) = \frac{1}{x+4}$, write a rule for each function and write the domain in interval notation.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$

Solution:

a. $(f \circ g)(x) = f(g(x)) = 2(g(x)) - 6$

$$= 2\left(\frac{1}{x+4}\right) - 6$$

$$= \frac{2}{x+4} - 6 \text{ provided } x \neq -4$$

Function g has the restriction $x \neq -4$.

The domain of f is all real numbers. Therefore, no further restrictions need to be imposed.

The domain is $(-\infty, -4) \cup (-4, \infty)$.

b. $(g \circ f)(x) = g(f(x)) = \frac{1}{f(x) + 4}$

$$= \frac{1}{(2x-6)+4} \quad f(x) \neq -4$$

$$= \frac{1}{2x-2} \quad \text{provided } x \neq 1$$

The domain of f has no restrictions.

However, function g must not have an input value of -4. Therefore, we have the restriction $f(x) \neq -4$. So,

$$2x - 6 = -4$$

$$2x = 2$$

The domain is $(-\infty, 1) \cup (1, \infty)$.

$x = 1$ must be excluded.

Apply the Skills

7. Given $f(x) = 3x + 4$ and $g(x) = \frac{1}{x-1}$, write a rule for each function and write the domain in interval notation.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$

Example 8: Composing Functions and Determining Domain

Given $m(x) = \frac{1}{x-5}$ and $p(x) = \sqrt{x-2}$, find $(m \circ p)(x)$ and write the domain in interval notation.

Solution:

$$(m \circ p)(x) = m(p(x)) = \frac{1}{p(x) - 5}$$

$$p(x) \neq 5$$

$$= \frac{1}{\sqrt{x-2} - 5}$$

$$(m \circ p)(x) = \frac{1}{\sqrt{x-2} - 5}$$

The domain is $[2, 27) \cup (27, \infty)$.

First note that function p has the restriction that $x \geq 2$.

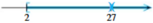
The input value for function m must not be 5. Therefore, $p(x) \neq 5$. We have

$$\sqrt{x-2} \neq 5$$

$$(\sqrt{x-2})^2 \neq (5)^2$$

$$x - 2 \neq 25$$

$$x \neq 27$$



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Apply the Skills

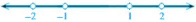
8. Given $f(x) = \sqrt{x-1}$ and $g(x) = \frac{1}{x-3}$, find $(g \circ f)(x)$ and write the domain of $g \circ f$ in interval notation.

Example 9: Composing Functions and Determining Domain

Given $k(x) = \frac{x}{x-2}$ and $m(x) = \frac{6}{x^2-1}$, find $(k \circ m)(x)$ and write the domain in interval notation.

Solution:

$(k \circ m)(x) = k(m(x))$	Evaluate k at $m(x)$.
$= \dfrac{\left(\frac{6}{x^2-1}\right)}{\left(\frac{6}{x^2-1}\right) - 2}$	Substitute $\frac{6}{x^2-1}$ for x in $k(x)$. m has the restriction that $x^2 - 1 \neq 0$. Therefore, $x \neq \pm 1$.
$= \dfrac{(x^2-1)}{(x^2-1)} \cdot \dfrac{\left(\frac{6}{x^2-1}\right)}{\left(\frac{6}{x^2-1} - 2\right)}$	Simplify the complex fraction by multiplying numerator and denominator by the LCD $x^2 - 1$.
$= \dfrac{6}{6 - 2(x^2-1)}$	Apply the distributive property.
$= \dfrac{6}{-2x^2 + 8}$	Simplify the denominator.
$= \dfrac{6}{-2(x^2-4)}$	Factor out -2 from the denominator.
$= -\dfrac{3}{x^2-4}$	Simplify the rational expression. Note the added restriction that $x^2 - 4 \neq 0$ which means that $x \neq \pm 2$.



The domain is $(-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, 2) \cup (2, \infty)$.

Apply the Skills

9. Given $r(x) = \frac{x}{x+1}$ and $t(x) = \frac{5}{x^2-9}$, find $(r \circ t)(x)$ and write the domain in interval notation.

Example 10: Applying Function Composition

At a popular website, the cost to download individual songs is \$1.49 per song. In addition, a first-time visitor to the website has a one-time coupon for \$1.00 off.

- **a.** Write a function to represent the cost $C(x)$ (in \$) for a first-time visitor to purchase x songs.
- **b.** The sales tax for online purchases depends on the location of the business and customer. If the sales tax rate on a purchase is 6%, write a function to represent the total cost $T(a)$ for a first-time visitor who buys a dollars in songs.
- **c.** Find $(T \circ C)(x)$ and interpret the meaning in context.
- **d.** Evaluate $(T \circ C)(10)$ and interpret the meaning in context.

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Solution:

a. $C(x) = 1.49x - 1.00;\ x \geq 1$ The cost function is a linear function with \$1.49 as the variable rate per song.

b. $T(a) = a + 0.06a$ The total cost is the sum of the cost of the songs plus the sales tax.

$$= 1.06a$$

c. $(T \circ C)(x) = T(C(x))$

$$= 1.06(C(x))$$

$$= 1.06(1.49x - 1.00)$$ Substitute $1.49x - 1.00$ for $C(x)$.

$$= 1.5794x - 1.06$$

$(T \circ C)(x) = 1.5794x - 1.06$ represents the total cost to buy x songs for a first-time visitor to the website.

d. $(T \circ C)(x) = 1.5794x - 1.06$

$$(T \circ C)(10) = 1.5794(10) - 1.06$$

$$= 14.734$$

The total cost for a first-time visitor to buy 10 songs is \$14.73.

Apply the Skills

10. An artist shops online for tubes of watercolor paint. The cost is \$16 for each 14-mL tube.

a. Write a function representing the cost $C(x)$ (in \$) for x tubes of paint.

b. There is a 5.5% sales tax on the cost of merchandise and a fixed cost of \$4.99 for shipping. Write a function representing the total cost $T(a)$ for a dollars spent in merchandise.

c. Find $(T \circ C)(x)$ and interpret the meaning in context.

d. Evaluate $(T \circ C)(18)$ and interpret the meaning in context.

The composition of two functions creates a new function in which the output from one function becomes the input to the other. We can also reverse this process. That is, we can decompose a composite function into two or more simpler functions.

For example, consider the function h defined by $h(x) = (x - 3)^2$. To write h as a composition of two functions, we have $h(x) = (f \circ g)(x) = f(g(x))$. The function g is the "inside" function and f is the "outside" function. So one natural choice for g and f would be:

$g(x) = x - 3$ Function g subtracts 3 from the input value.

$f(x) = x^2$ Function f squares the result.

$$h(x) = f(g(x)) = (g(x))^2 = (x - 3)^2$$

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Insights

The decomposition of functions is not unique. For Example, $h(x) = (x - 3)^2$ can also be written as $h(x) = f(g(x))$, where $g(x) = x^2 - 6x$ and $f(x) = x + 9$.

Example 11: Decomposing Two Functions

Given $h(x) = |2x^2 - 5|$, find two functions f and g such that $h(x) = (f \circ g)(x)$.

Solution:

We need to find two functions f and g such that $h(x) = (f \circ g)(x) = f(g(x))$.

The function h first evaluates the expression $2x^2 - 5$, and then takes the absolute value. Therefore, it would be natural to take the absolute value of $g(x) = 2x^2 - 5$.

We have: $g(x) = 2x^2 - 5$ and $f(x) = |x|$

Check:
$$\begin{aligned} h(x) &= (f \circ g)(x) = f(g(x)) = |g(x)| \\ &= |2x^2 - 5| \end{aligned}$$

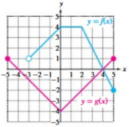
Apply the Skills

11. Given $m(x) = \sqrt[3]{5x + 1}$, find two functions f and g such that $m(x) = (f \circ g)(x)$.

In Example 12, we have the graphs of two functions, and we apply function addition, subtraction, multiplication, and composition for selected values of x.

Example 12: Estimating Function Values from a Graph

The graphs of f and g are shown.



Evaluate the functions at the given values of x if possible.

- **a.** $(f + g)(1)$
- **b.** $(fg)(0)$
- **c.** $(g - f)(-3)$
- **d.** $(f \circ g)(3)$
- **e.** $(g \circ f)(4)$
- **f.** $f(g(1))$

Solution:

a.
$$\begin{aligned} (f + g)(1) &= f(1) + g(1) \\ &= 4 + (-3) \\ &= 1 \end{aligned}$$

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b. $(fg)(0) = f(0) \cdot g(0)$

$= (4)(-4)$

$= -16$

c. $(g - f)(-3) = g(-3) - \underbrace{f(-3)}$

The open dot at $(-3, 1)$ indicates that -3 is not in the domain of f.

$(g - f)(-3)$ is undefined.

$f(-3)$ is undefined.

d. $(f \circ g)(3) = f(g(3))$

$= f(-1)$

$= 3$

e. $(g \circ f)(4) = g(f(4))$

$= g(0)$

$= -4$

f. $f(g(1)) = f(-3)$ is undefined.

The value $g(1) = -3$, but $f(-3)$ is undefined. Therefore, $f(g(1))$ is undefined.

Apply the Skills

12. Refer to the functions f and g pictured in Example 12. Evaluate the functions at the given values of x if possible.

a. $(f - g)(-2)$

b. $\left(\frac{f}{g}\right)(3)$

c. $(gf)(5)$

d. $(g \circ f)(5)$

e. $(f \circ g)(5)$

f. $f(g(0))$

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Practice 1-8

Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, write the domain in interval notation.

R.1. $f(x) = \frac{x-1}{x+2}$

R.2. $r(x) = \sqrt{x+3}$

R.3. $h(x) = \frac{4}{\sqrt{3-x}}$

R.4. $p(x) = 2x^2 - 3x + 1$

R.5. Given $k(x) = x^2 - 2x + 3$, find $k(x + 3)$.

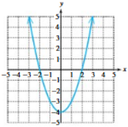
Concept Connections

1. The function $f + g$ is defined by $(f + g)(x) =$ __________ + __________.
2. The function $\frac{f}{g}$ is defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ provided that __________ $\neq 0$.
3. Let h represent a positive real number. Given a function defined by $y = f(x)$, the difference quotient is given by __________.
4. The composition of f and g, denoted by $f \circ g$, is defined by $(f \circ g)(x) =$ __________.

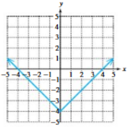
Learn: Perform Operations on Functions

For Exercises 5–8, find $(f + g)(x)$ and identify the graph of $f + g$ from the following options. (See Example 1 on page 137)

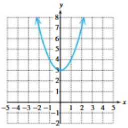
a.



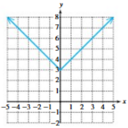
b.



c.



d.



5. $f(x) = |x|$ and $g(x) = 3$
6. $f(x) = |x|$ and $g(x) = -4$
7. $f(x) = x^2$ and $g(x) = -4$
8. $f(x) = x^2$ and $g(x) = 3$

For Exercises 9–18, evaluate the functions for the given values of x. (See Example 2 on page 138)

$f(x) = -2x \qquad g(x) = |x + 4| \qquad h(x) = \frac{1}{x-3}$

9. $(f - g)(3)$

10. $(g - h)(2)$

11. $(f \cdot g)(-1)$

12. $(h \cdot g)(4)$

13. $(g + h)(0)$

14. $(f + h)(5)$

15. $\left(\frac{f}{g}\right)(8)$

16. $\left(\frac{h}{f}\right)(7)$

17. $\left(\frac{g}{f}\right)(0)$

18. $\left(\frac{h}{g}\right)(-4)$

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For Exercises 19–26, refer to the functions *r*, *p*, and *q*. Find the indicated function and write the domain in interval notation. (See Example 3 on page 139)

$r(x) = -3x \qquad p(x) = x^2 + 3x \qquad q(x) = \sqrt{1 - x}$

19. $(r - p)(x)$

20. $(p - r)(x)$

21. $(p \cdot q)(x)$

22. $(r \cdot q)(x)$

23. $\left(\frac{q}{p}\right)(x)$

24. $\left(\frac{q}{r}\right)(x)$

25. $\left(\frac{p}{q}\right)(x)$

26. $\left(\frac{r}{q}\right)(x)$

For Exercises 27–32, refer to functions *s*, *t*, and *v*. Find the indicated function and write the domain in interval notation. (See Example 3 on page 139)

$s(x) = \frac{x - 2}{x^2 - 9} \qquad t(x) = \frac{x - 3}{2 - x} \qquad v(x) = \sqrt{x + 3}$

27. $(s \cdot t)(x)$

28. $\left(\frac{s}{t}\right)(x)$

29. $(s + t)(x)$

30. $(s - t)(x)$

31. $(s \cdot v)(x)$

32. $\left(\frac{v}{s}\right)(x)$

Learn: Evaluate a Difference Quotient

For Exercises 33–36, a function is given. (See Examples 4–5 on pages 140–141)

a. Find $f(x + h)$. **b.** Find $\frac{f(x + h) - f(x)}{h}$.

33. $f(x) = 5x + 9$

34. $f(x) = 8x + 4$

35. $f(x) = x^2 + 4x$

36. $f(x) = x^2 - 3x$

For Exercises 37–44, find the difference quotient and simplify. (See Examples 4–5 on pages 140–141)

37. $f(x) = -2x + 5$

38. $f(x) = -3x + 8$

39. $f(x) = -5x^2 - 4x + 2$

40. $f(x) = -4x^2 - 2x + 6$

41. $f(x) = x^3 + 5$

42. $f(x) = x^3 - 2$

43. $f(x) = \frac{1}{x}$

44. $f(x) = \frac{1}{x + 2}$

45. Given $f(x) = 4\sqrt{x}$,

a. Find the difference quotient (do not simplify).

b. Evaluate the difference quotient for $x = 1$, and the following values of h: $h = 1$, $h = 0.1$, $h = 0.01$, and $h = 0.001$. Round to 4 decimal places.

c. What value does the difference quotient seem to be approaching as *h* gets close to 0?

46. Given $f(x) = \frac{12}{x}$,

a. Find the difference quotient (do not simplify).

b. Evaluate the difference quotient for $x = 2$, and the following values of h: $h = 0.1$, $h = 0.01$, $h = 0.001$, and $h = 0.0001$. Round to 4 decimal places.

c. What value does the difference quotient seem to be approaching as *h* gets close to 0?

Learn: Compose and Decompose Functions

For Exercises 47–62, refer to functions *f*, *g*, and *h*. Evaluate the functions for the given values of *x*. (See Example 6 on page 142)

$f(x) = x^3 - 4x \quad g(x) = \sqrt{2x} \quad h(x) = 2x + 3$

47. $f(g(8))$

48. $h(g(2))$

49. $h(f(1))$

50. $g(f(3))$

51. $(f \circ g)(18)$

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52. $(f \circ h)(-1)$

53. $(g \circ f)(5)$

54. $(h \circ f)(-2)$

55. $(h \circ f)(-3)$

56. $(h \circ g)(72)$

57. $(g \circ f)(1)$

58. $(g \circ f)(-4)$

59. $(f \circ f)(3)$

60. $(h \circ h)(-4)$

61. $(f \circ h \circ g)(2)$

62. $(f \circ h \circ g)(8)$

63. Given $f(x) = 2x + 4$ and $g(x) = x^2$,

- **a.** Find $(f \circ g)(x)$.
- **b.** Find $(g \circ f)(x)$.
- **c.** Is the operation of function composition commutative?

64. Given $k(x) = -3x + 1$ and $m(x) = \frac{1}{x}$,

- **a.** Find $(k \circ m)(x)$.
- **b.** Find $(m \circ k)(x)$.
- **c.** Is $(k \circ m)(x) = (m \circ k)(x)$?

For Exercises 65–76, refer to functions *m*, *n*, *p*, *q*, and *r*. Find the indicated function and write the domain in interval notation. (See Examples 7–9 on pages 143–144)

$m(x) = \sqrt{x+8}$ $\quad n(x) = x - 5$ $\quad p(x) = x^2 - 9x$

$q(x) = \frac{1}{x-10}$ $\quad r(x) = |2x + 3|$

65. $(n \circ p)(x)$

66. $(p \circ n)(x)$

67. $(m \circ n)(x)$

68. $(n \circ m)(x)$

69. $(q \circ n)(x)$

70. $(q \circ p)(x)$

71. $(q \circ r)(x)$

72. $(q \circ m)(x)$

73. $(n \circ r)(x)$

74. $(r \circ n)(x)$

75. $(q \circ q)(x)$

76. $(p \circ p)(x)$

For Exercises 77–80, find $(f \circ g)(x)$ and write the domain in interval notation. (See Example 9 on page 144)

77. $f(x) = \frac{3}{x^2 - 16}$, $g(x) = \sqrt{2 - x}$

78. $f(x) = \frac{4}{x^2 - 9}$, $g(x) = \sqrt{3 - x}$

79. $f(x) = \frac{x}{x - 1}$, $g(x) = \frac{9}{x^2 - 16}$

80. $f(x) = \frac{x}{x + 4}$, $g(x) = \frac{3}{x^2 - 1}$

81. Given $f(x) = \frac{1}{x - 2}$, find $(f \circ f)(x)$ and write the domain in interval notation.

82. Given $g(x) = \sqrt{x - 3}$, find $(g \circ g)(x)$ and write the domain in interval notation.

For Exercises 83–86, find the indicated functions.

$f(x) = 2x + 1$ $\quad g(x) = x^2$ $\quad h(x) = \sqrt[3]{x}$

83. $(f \circ g \circ h)(x)$

84. $(g \circ f \circ h)(x)$

85. $(h \circ g \circ f)(x)$

86. $(g \circ h \circ f)(x)$

87. A law office orders business stationery. The cost is \$21.95 per box. (**See Example 10 on page 144**)

- **a.** Write a function that represents the cost $C(x)$ (in \$) for x boxes of stationery.
- **b.** There is a 6% sales tax on the cost of merchandise and \$10.99 for shipping. Write a function that represents the total cost $T(a)$ for a dollars spent in merchandise and shipping.
- **c.** Find $(T \circ C)(x)$.
- **d.** Find $(T \circ C)(4)$ and interpret its meaning in the context of this problem.

88. The cost to buy tickets online for a dance show is \$60 per ticket.

- **a.** Write a function that represents the cost $C(x)$ (in \$) for x tickets to the show.
- **b.** There is a sales tax of 5.5% and a processing fee of \$8.00 for a group of tickets. Write a function that represents the total cost $T(a)$ for a dollars spent on tickets.

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c. Find $(T \circ C)(x)$.

d. Find $(T \circ C)(6)$ and interpret its meaning in the context of this problem.

89. A bicycle wheel turns at a rate of 80 revolutions per minute (rpm).

a. Write a function that represents the number of revolutions $r(t)$ in t minutes.

b. For each revolution of the wheels, the bicycle travels 7.2 ft. Write a function that represents the distance traveled $d(r)$ (in ft) for r revolutions of the wheel.

c. Find $(d \circ r)(t)$ and interpret the meaning in the context of this problem.

d. Evaluate $(d \circ r)(30)$ and interpret the meaning in the context of this problem.

90. While on vacation in France, Emmanuel bought a box of almond croissants. Each croissant cost €2.40 (euros).

a. Write a function that represents the cost $C(x)$ (in euros) for x croissants.

b. At the time of the purchase, the exchange rate was \$1 = €0.80. Write a function that represents the amount $D(C)$ (in \$) for C euros spent.

c. Find $(D \circ C)(x)$ and interpret the meaning in the context of this problem.

d. Evaluate $(D \circ C)(12)$ and interpret the meaning in the context of this problem.

For Exercises 91–98, find two functions *f* and *g* such that *h*(*x*) = (*f* ∘ *g*)(*x*). (See Example 11 on page 146)

91. $h(x) = (x + 7)^2$

92. $h(x) = (x - 8)^2$

93. $h(x) = \sqrt[3]{2x + 1}$

94. $h(x) = \sqrt[4]{9x - 5}$

95. $h(x) = |2x^2 - 3|$

96. $h(x) = |4 - x^2|$

97. $h(x) = \dfrac{5}{x + 4}$

98. $h(x) = \dfrac{11}{x - 3}$

Mixed Exercises

For Exercises 99–102, the graphs of two functions are shown. Evaluate the function at the given values of *x*, if possible. (See Example 12 on pages 146–147)

99. a. $(f + g)(0)$

b. $(g - f)(2)$

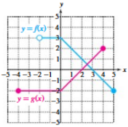
c. $(g \cdot f)(-1)$

d. $\left(\frac{g}{f}\right)(1)$

e. $(f \circ g)(4)$

f. $(g \circ f)(0)$

g. $g(f(4))$



100. a. $(f + g)(0)$

b. $(g - f)(1)$

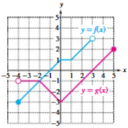
c. $(g \cdot f)(2)$

d. $\left(\frac{f}{g}\right)(-3)$

e. $(f \circ g)(3)$

f. $(g \circ f)(0)$

g. $g(f(-4))$



101. a. $(h + k)(-1)$

b. $(h \cdot k)(4)$

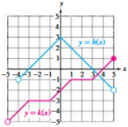
c. $\left(\frac{k}{h}\right)(-3)$

d. $(k - h)(1)$

e. $(k \circ h)(4)$

f. $(h \circ k)(-2)$

g. $h(k(3))$



102. a. $(m + p)(1)$

b. $(p - m)(-4)$

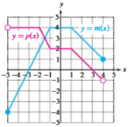
c. $\left(\frac{m}{p}\right)(3)$

d. $(m \cdot p)(3)$

e. $(m \circ p)(0)$

f. $(p \circ m)(0)$

g. $p(m(-4))$



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For Exercises 103–110, refer to the functions f and g and evaluate the functions for the given values of x.

$f = \{(2, 4), (6, -1), (4, -2), (0, 3), (-1, 6)\}$ and $g = \{(4, 3), (0, 6), (5, 7), (6, 0)\}$

103. $(f + g)(4)$

104. $(g \cdot f)(0)$

105. $(g \circ f)(2)$

106. $(f \circ g)(0)$

107. $(g \circ g)(6)$

108. $(f \circ f)(-1)$

109. $(f \circ g)(5)$

110. $(g \circ f)(0)$

111. The diameter d of a sphere is twice the radius r. The volume of the sphere as a function of its radius is given by $V(r) = \frac{4}{3}\pi r^3$.

- **a.** Write the diameter d of the sphere as a function of the radius r.
- **b.** Write the radius r as a function of the diameter d.
- **c.** Find $(V \circ r)(d)$ and interpret its meaning.

112. Consider a right circular cone with given height h. The volume of the cone as a function of its radius r is given by $V(r) = \frac{1}{3}\pi r^2 h$. Consider a right circular cone with fixed height $h = 6$ in.

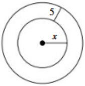
- **a.** Write the diameter d of the cone as a function of the radius r.
- **b.** Write the radius r as a function of the diameter d.
- **c.** Find $(V \circ r)(d)$ and interpret its meaning. Assume that $h = 6$ in.

113. An investment earns 4.5% interest paid at the end of 1 yr. If x is the amount of money initially invested, then $A(x) = 1.045x$ represents the amount of money in the account 1 yr later. Find $(A \circ A)(x)$ and interpret the result.

114. The population in a certain town has been decreasing at a rate of 2% per year. If x is the population at a certain fixed time, then $P(x) = 0.98x$ represents the population 1 yr later. Find $(P \circ P)(x)$ and interpret the result.

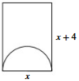
115. Suppose that a function H gives the high temperature $H(x)$ (in°F) for day x. Suppose that a function L gives the low temperature $L(x)$ (in°F) for day x. What does $\left(\frac{H+L}{2}\right)(x)$ represent?

116. For the given figure,



- **a.** What does $A_1(x) = \pi(x + 5)^2$ represent?
- **b.** What does $A_2(x) = \pi x^2$ represent?
- **c.** Find $(A_1 - A_2)(x)$ and interpret its meaning.

117. For the given figure,



- **a.** Write an expression $S_1(x)$ that represents the area of the rectangle.
- **b.** Write an expression $S_2(x)$ that represents the area of the semicircle.
- **c.** Find $(S_1 - S_2)(x)$ and interpret its meaning.

Write About It

118. Given functions f and g, explain how to determine the domain of $\left(\frac{f}{g}\right)(x)$.

119. Given functions f and g, explain how to determine the domain of $(f \circ g)(x)$.

120. Explain what the difference quotient represents.

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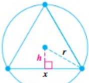
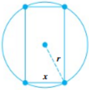
Expanding Your Skills

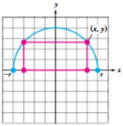
121. A circle of radius r circumscribes an equilateral triangle with sides of length x. Write a function that gives the area of the circle in terms of x. Use the fact that the height of an equilateral triangle with sides of length x is $\frac{\sqrt{3}}{2}x$. (*Hint*: Begin by writing h in terms of x and r.)



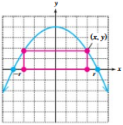
122. A circle of radius r circumscribes a rectangle with the height measuring twice the width x. Write a function that gives the area of the circle in terms of x.



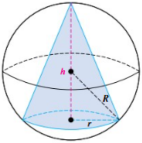
123. The function defined by $f(x) = \sqrt{r^2 - x^2}$ describes the graph of a semicircle of fixed radius r. Determine a function A for the area of the inscribed rectangle (as shown in the figure).



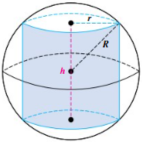
124. Given a fixed $r > 0$, the function defined by $f(x) = \frac{r^2 - x^2}{r}$ describes the graph of a parabola as shown in the figure. Determine a function A for the area of the inscribed rectangle.



125. A right circular cone with a base radius r and height h is inscribed in a sphere of fixed radius R. Determine a function V for the volume of the inscribed cone in terms of r.



126. A right circular cylinder with a base radius of r and height h is inscribed in a sphere of fixed radius R. Determine a function V for the volume of the inscribed cylinder in terms of h.



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127. Given $f(x) = \sqrt{x+3}$,

- **a.** Find the difference quotient.
- **b.** Rationalize the numerator of the expression in part (a) and simplify.
- **c.** Evaluate the expression in part (b) for $h = 0$.

128. Given $f(x) = \sqrt{x-4}$,

- **a.** Find the difference quotient.
- **b.** Rationalize the numerator of the expression in part (a) and simplify.
- **c.** Evaluate the expression in part (b) for $h = 0$.

129. Given $f(x) = \sqrt[3]{x}$,

- **a.** Find the difference quotient.
- **b.** Rationalize the numerator of the expression in part (a) and simplify. Recall that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.
- **c.** Evaluate the expression in part (b) for $h = 0$.

130. Given $f(x) = (3x)^{1/3}$,

- **a.** Find the difference quotient.
- **b.** Rationalize the numerator of the expression in part (a) and simplify.
- **c.** Evaluate the expression in part (b) for $h = 0$.

131. A car traveling 60 mph (88 ft/sec) undergoes a constant deceleration until it comes to rest approximately 9.09 sec later. The distance $d(t)$ (in ft) that the car travels t seconds after the brakes are applied is given by $d(t) = -4.84t^2 + 88t$, where $0 \le t \le 9.09$. **(See Example 5)**

- **a.** Find the difference quotient $\frac{d(t+h) - d(t)}{h}$.

Use the difference quotient to determine the average rate of speed on the following intervals for t.

- **b.** $[0, 2]$ (*Hint*: $t = 0$ and $h = 2$)
- **c.** $[2, 4]$ (*Hint*: $t = 2$ and $h = 2$)
- **d.** $[4, 6]$ (*Hint*: $t = 4$ and $h = 2$)
- **e.** $[6, 8]$ (*Hint*: $t = 6$ and $h = 2$)

132. A car accelerates from 0 to 60 mph (88 ft/sec) in 8.8 sec. The distance $d(t)$ (in ft) that the car travels t seconds after motion begins is given by $d(t) = 5t^2$, where $0 \le t \le 8.8$.

- **a.** Find the difference quotient $\frac{d(t+h) - d(t)}{h}$.

Use the difference quotient to determine the average rate of speed on the following intervals for t.

- **b.** $[0, 2]$
- **c.** $[2, 4]$
- **d.** $[4, 6]$
- **e.** $[6, 8]$

133. If a is b plus eight, and c is the square of a, write c as a function of b.

134. If q is r minus seven, and s is the square root of q, write s as a function of r.

135. If x is twice y, and z is four less than x, write z as a function of y.

136. If m is one-third of n, and p is two less than m, write p as a function of n.

137. Given $f(x) = \sqrt[3]{4x^2 + 1}$, define functions m, n, h, and k such that $f(x) = (m \circ n \circ h \circ k)(x)$.

138. Given $f(x) = |-2x^3 - 4|$, define functions m, n, h, and k such that $f(x) = (m \circ n \circ h \circ k)(x)$.

Go online for more practice problems.

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Chapter 1 Review

Chapter 1 Key Concepts

Lesson 1.1 The Rectangular Coordinate System and Graphing Utilities

Key Concepts	Reference
The **distance** between two points (x_1, y_1) and (x_2, y_2) in a rectangular coordinate system is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.	p. 6
The **midpoint** between the points is given by $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.	p. 8
• To find an x-intercept $(a, 0)$ of the graph of an equation, substitute 0 for y and solve for x. • To find a y-intercept $(0, b)$ of the graph of an equation, substitute 0 for x and solve for y.	p. 11

Lesson 1.2 Circles

Key Concepts	Reference
The **standard form** of an equation of a circle with radius r and center (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.	p. 23
An equation of a circle written in the form $x^2 + y^2 + Ax + By + C = 0$ is called the **general form** of an equation of a circle.	p. 25

Lesson 1.3 Functions and Relations

Key Concepts	Reference
A set of ordered pairs (x, y) is called a **relation** in x and y. The set of x values is the **domain** of the relation, and the set of y values is the **range** of the relation.	p. 31
Given a relation in x and y, we say that ***y* is a function of *x*** if for each value of x in the domain, there is exactly one value of y in the range.	p. 32
The vertical line test tells us that the graph of a relation defines y as a function of x if no vertical line intersects the graph in more than one point.	p. 33
Given a function defined by $y = f(x)$, • The x-intercepts are the real solutions to $f(x) = 0$. • The y-intercept is given by $f(0)$.	p. 37
Given $y = f(x)$, the domain of f is the set of real numbers x that when substituted into the function produce a real number. This excludes • Values of x that make the denominator zero. • Values of x that make a radicand negative within an even-indexed root.	p. 39

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Lesson 1.4 Linear Equations in Two Variables and Linear Functions

Key Concepts	**Reference**
Let A, B, and C represent real numbers where A and B are not both zero. A **linear equation** in the variables x and y is an equation that can be written as $Ax + By = C$.	p. 50
The **slope** of a line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.	p. 53
Given a line with slope m and y-intercept $(0, b)$, the **slope-intercept form** of the line is given by $y = mx + b$.	p. 55
If f is defined on the interval $[x_1, x_2]$, then the **average rate of change** of f on the interval $[x_1, x_2]$ is the slope of the secant line containing $(x_1, f(x_1))$ and $(x_2, f(x_2))$ and is given by $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.	p. 58
The x-coordinates of the points of intersection between the graphs of $y = f(x)$ and $y = g(x)$ are the solutions to the equation $f(x) = g(x)$.	p. 59

Lesson 1.5 Applications of Linear Equations and Modeling

Key Concepts	**Reference**
The **point-slope formula** for a line is given by $y - y_1 = m(x - x_1)$ where m is the slope of the line and (x_1, y_1) is a point on the line.	p. 70
• If m_1 and m_2 represent the slopes of two nonvertical parallel lines, then $m_1 = m_2$. • If m_1 and m_2 represent the slopes of two nonvertical perpendicular lines, then $m_1 = -\frac{1}{m_2}$ or equivalently $m_1m_2 = -1$.	p. 72
In many-day-to-day applications, two variables are related linearly. • A linear model can be made from two data points that represent the general trend of the data. • Alternatively, the **least-squares regression line** is a model that utilizes *all* observed data points.	p. 77 p. 79

Lesson 1.6 Transformations of Graphs

Key Concepts	**Reference**
Consider a function defined by $y = f(x)$. Let h, k, and a represent positive real numbers. The graphs of the following functions are related to $y = f(x)$ as follows: • Vertical translation (shift) $y = f(x) + k$ Shift upward $y = f(x) - k$ Shift downward • Horizontal translation (shift) $y = f(x - h)$ Shift to the right $y = f(x + h)$ Shift to the left	pp. 94, 96

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• Vertical stretch/shrink $y = af(x)$ Vertical stretch (if $a > 1$) Vertical shrink (if $0 < a < 1$) • Horizontal stretch/shrink $y = f(ax)$ Horizontal shrink (if $a > 1$) Horizontal stretch (if $0 < a < 1$)	p. 98
• Reflection $y = -f(x)$ Reflection across the *x*-axis $y = f(-x)$ Reflection across the *y*-axis	p. 100
To graph a function requiring multiple transformations, use the following order. **1.** Horizontal translation (shift) **2.** Horizontal and vertical stretch and shrink **3.** Reflections across the *x*- and *y*-axes **4.** Vertical translation (shift)	p. 102

Lesson 1.7 Analyzing Graphs of Functions and Piecewise-Defined Functions

Key Concepts	Reference
Consider the graph of an equation in x and y. • The graph of the equation is symmetric to the *y*-axis if substituting $-x$ for x results in an equivalent equation. • The graph of the equation is symmetric to the *x*-axis if substituting $-y$ for y results in an equivalent equation. • The graph of the equation is symmetric to the origin if substituting $-x$ for x and $-y$ for y results in an equivalent equation.	p. 112
• f is an even function if $f(-x) = f(x)$ for all x in the domain of f. • f is an odd function if $f(-x) = -f(x)$ for all x in the domain of f.	p. 115
To graph a piecewise-defined function, graph each individual function on its domain.	pp. 118–119
The **greatest integer function,** denoted by $f(x) = \llbracket x \rrbracket$ or $f(x) = \text{int}(x)$ or $f(x) = \text{floor}(x)$ defines $f(x)$ as the greatest integer less than or equal to x.	p. 121
Suppose that I is an interval contained within the domain of a function f. • f is increasing on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ on I. • f is decreasing on I if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ on I. • f is constant on I if $f(x_1) = f(x_2)$ for all x_1 and x_2 on I.	p. 123
• $f(a)$ is a **relative maximum** of f if there exists an open interval containing a such that $f(a) \geq f(x)$ for all x in the interval. • $f(b)$ is a **relative minimum** of f if there exists an open interval containing b such that $f(b) \leq f(x)$ for all x in the interval.	p. 125

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Lesson 1.8 Algebra of Functions and Function Composition

Key Concepts	Reference
Given functions f and g, the functions $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$ are defined by $(f + g)(x) = f(x) + g(x)$ $(f - g)(x) = f(x) - g(x)$ $(f \cdot g)(x) = f(x) \cdot g(x)$ $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ Provided that $g(x) \neq 0$	p. 137
The **difference quotient** represents the average rate of change of a function f between two points $(x, f(x))$ and $(x + h, f(x + h))$. $\frac{f(x + h) - f(x)}{h}$ Difference quotient	p. 140
The **composition of f and g,** denoted $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of real numbers x in the domain of g such that $g(x)$ is in the domain of f.	p. 141

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Chapter 1 Test

1. The endpoints of a diameter of a circle are $(-2, 3)$ and $(8, -5)$.

- **a.** Determine the center of the circle.
- **b.** Determine the radius of the circle.
- **c.** Write an equation of the circle in standard form.

2. Given $x = |y| - 4$,

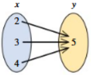
- **a.** Determine the x- and y-intercepts of the graph of the equation.
- **b.** Does the equation define y as a function of x?

3. Given $x^2 + y^2 + 14x - 10y + 70 = 0$,

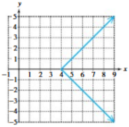
- **a.** Write the equation of the circle in standard form.
- **b.** Identify the center and radius.

For Exercises 4–5, determine if the relation defines y as a function of x.

4.



5.

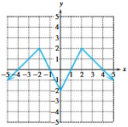


6. Given $f(x) = -2x^2 + 7x - 3$, find

- **a.** $f(-1)$.
- **b.** $f(x + h)$.
- **c.** The difference quotient: $\dfrac{f(x+h) - f(x)}{h}$.
- **d.** The x-intercepts of the graph of f.
- **e.** The y-intercept of the graph of f.
- **f.** The average rate of change of f on the interval $[1, 3]$.

7. Use the graph of $y = f(x)$ to estimate

- **a.** $f(0)$.
- **b.** $f(-4)$.
- **c.** The values of x for which $f(x) = 2$.



- **d.** The interval(s) over which f is increasing.
- **e.** The interval(s) over which f is decreasing.
- **f.** Determine the location and value of any relative minima.
- **g.** Determine the location and value of any relative maxima.
- **h.** The domain.
- **i.** The range.
- **j.** Whether f is even, odd, or neither.

For Exercises 8–9, write the domain in interval notation.

8. $f(w) = \dfrac{2w}{3w + 7}$

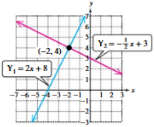
9. $f(c) = \sqrt{4 - c}$

10. Given $3x = -4y + 8$,

- **a.** Identify the slope.
- **b.** Identify the y-intercept.
- **c.** Graph the line.
- **d.** What is the slope of a line perpendicular to this line?
- **e.** What is the slope of a line parallel to this line?

11. Write an equation of the line passing through the point $(-2, 6)$ and perpendicular to the line defined by $x + 3y = 4$.

12. Use the graph to solve the equation and inequalities. Write the solutions to the inequalities in interval notation.



- **a.** $2x + 8 = -\frac{1}{2}x + 3$
- **b.** $2x + 8 < -\frac{1}{2}x + 3$
- **c.** $2x + 8 \geq -\frac{1}{2}x + 3$

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For Exercises 13–16, graph the equation.

13. $x^2 + \left(y + \frac{5}{2}\right)^2 = 9$

14. $f(x) = 2|x + 3|$

15. $g(x) = -\sqrt{x+4} + 3$

16. $h(x) = \begin{cases} -x + 3 & \text{for } x < 1 \\ \sqrt{x-1} & \text{for } x \geq 1 \end{cases}$

17. Determine if the graph of the equation is symmetric to the *y*-axis, *x*-axis, origin, or none of these.

$$x^2 + |y| = 8$$

For Exercises 18–19, determine if the function is even, odd, or neither.

18. $f(x) = x^3 - x$

19. $g(x) = x^4 + x^3 + x$

20. Evaluate the greatest integer function for the following values of *x*.

a. 4.27 **b.** −4.27

For Exercises 21–26, refer to the functions *f*, *g*, and *h* defined here.

$f(x) = x - 4$ $g(x) = \frac{1}{x-3}$ $h(x) = \sqrt{x-5}$

21. Evaluate $(f - h)(6)$.

22. Evaluate $(g \cdot h)(5)$.

23. Evaluate $(h \circ f)(1)$.

24. Find $(f \cdot g)(x)$ and state the domain in interval notation.

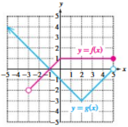
25. Find $\left(\frac{g}{f}\right)(x)$ and state the domain in interval notation.

26. Find $(g \circ h)(x)$ and state the domain in interval notation.

27. Write two functions *f* and *g* such that $h(x) = (f \circ g)(x)$.

$$h(x) = \sqrt[3]{x-7}$$

28. For *f* and *g* pictured, estimate the following:



a. $(f + g)(3)$

b. $(f \cdot g)(0)$

c. $g(f(3))$

d. $(f \circ g)(2)$

e. The interval(s) over which *f* is increasing.

f. The interval(s) over which *g* is decreasing.

29. The number of people *y* that seek vaccinations at a doctor's office is given in the table for selected weeks, *x*.

Week Number, *x*	Number of People, *y*
1	8
3	21
6	30
9	40
12	46
15	56
18	68

a. Graph the data in a scatter plot.

b. Use the points (1, 8) and (9, 40) to write a linear function that defines the number of people as a function of week number.

c. Interpret the meaning of the slope in the context of this problem.

d. Use the model in part (b) to predict the number of people in week 24 assuming that the linear trend continues.

30. Refer to the data given in Exercise 29.

a. Use a graphing utility to find the least-squares regression line. Round the slope and *y*-intercept to 1 decimal place.

b. Use a graphing utility to graph the regression line and the observed data.

c. In the event that the linear trend continues, use the model from part (a) to predict the number of people seeking vaccinations in week 24.

GO ONLINE **Go online for more practice problems.**

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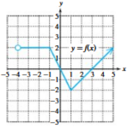
Chapter 1 Cumulative Review Exercises

1. Use the graph of $y = f(x)$ to complete the exercises.



a. Evaluate $f(2)$.

b. Find all x such that $f(x) = 0$.

c. Determine the domain of f.

d. Determine the range of f.

e. Determine the interval(s) over which f is increasing.

f. Determine the interval(s) over which f is decreasing.

g. Determine the intervals(s) over which f is constant.

h. Evaluate $(f \circ f)(-1)$.

2. Given the equation of the circle $x^2 + y^2 + 12x - 4y + 31 = 0$,

a. Write the equation in standard form.

b. Identify the center and radius.

For Exercises 3–7, refer to the functions f, g, and h defined here.

$f(x) = -x^2 + 3x \qquad g(x) = \frac{1}{x} \qquad h(x) = \sqrt{x+2}$

3. Find $(g \circ f)(x)$ and write the domain in interval notation.

4. Find $(g \cdot h)(x)$ and write the domain in interval notation.

5. Find the difference quotient. $\frac{f(x+h) - f(x)}{h}$

6. Find the average rate of change of f over the interval $[0, 3]$.

7. Determine the x- and y-intercepts of f.

For Exercises 8–9, graph the function.

8. $f(x) = -\sqrt{x+3}$

9. $g(x) = \begin{cases} -4 & \text{for } x < -2 \\ 1 & \text{for } -2 \le x < 0 \\ x^2 + 1 & \text{for } x \ge 0 \end{cases}$

10. Write an equation of the line passing through the points $(8, -3)$ and $(-2, 1)$. Write the final answer in slope-intercept form.

11. Write an absolute value expression that represents the distance between the points x and 7 on the number line.

12. Factor. $2x^3 - 128$

For Exercises 13–17, solve the equation or inequality. Write the solutions to the inequalities in interval notation.

13. $-3t(t-1) = 2t + 6$

14. $7 = |4x - 2| + 5$

15. $x^{2/5} - 3x^{1/5} + 2 = 0$

16. $|3a + 1| - 2 \le 9$

17. $3 \le -2x + 1 < 7$

For Exercises 18–20, perform the indicated operations and simplify.

18. $\frac{6}{\sqrt{15} + \sqrt{11}}$

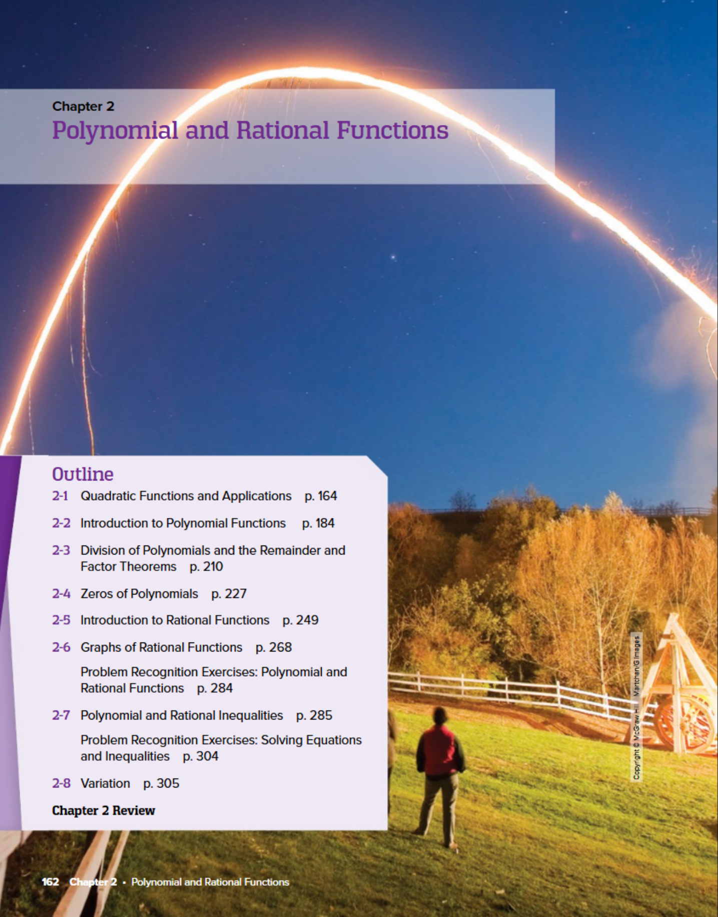
19. $3c\sqrt{8c^2d^3} + c^2\sqrt{50d^3} - 2d\sqrt{2c^4d}$

20. $\frac{2u^{-1} - w^{-1}}{4u^{-2} - w^{-2}}$

Go online for more practice problems.

Chapter 2

Polynomial and Rational Functions



Outline

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2-3 Division of Polynomials and the Remainder and Factor Theorems p. 210

2-4 Zeros of Polynomials p. 227

2-5 Introduction to Rational Functions p. 249

2-6 Graphs of Rational Functions p. 268

Problem Recognition Exercises: Polynomial and Rational Functions p. 284

2-7 Polynomial and Rational Inequalities p. 285

Problem Recognition Exercises: Solving Equations and Inequalities p. 304

2-8 Variation p. 305

Chapter 2 Review

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Launch Activity: Attend to Precision

Height of the Ball

A physics class creates a trebuchet that launches a golf ball. The class agrees upon two launch locations to determine two values: how long the golf ball stays in the air and how high the golf ball will travel. Once the trebuchet is setup and in place, the projectile launches with an initial velocity of 74.3 m/s. First, the class launches from atop the Hawaiian volcano Kaleakala, which has a height of 3 km. Next, the class was able to send the trebuchet to Mercury where the acceleration due to gravity is 3.7 m/s^2. The class launches from atop the Caloris Montes mountain range which also has a height of 3 km.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer this question: Will the maximum height the golf ball be greater on Mercury or Earth?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

GO ONLINE to answer these questions

Lesson 2-1

Quadratic Functions and Applications

What Will You Learn?

After completing this lesson, you should be able to:

- Graph a Quadratic Function Written in Vertex Form
- Write $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in Vertex Form
- Find the Vertex of a Parabola by Using the Vertex Formula
- Solve Applications Involving Quadratic Functions
- Create Quadratic Models Using Regression

Learn: Graph a Quadratic Function Written in Vertex Form

In Chapter 1, we defined a function of the form $f(x) = mx + b$ $(m \neq 0)$ as a linear function. The function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is called a *quadratic function.* Notice that a quadratic function has a leading term of second degree. We are already familiar with the graph of $f(x) = x^2$ (Figure 2-1). The graph is a parabola opening upward with vertex at the origin. Also note that the graph is symmetric with respect to the vertical line through the vertex called the **axis of symmetry**.

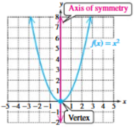


Figure 2-1

We can write $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in the form $f(x) = a(x - h)^2 + k$ by completing the square. Furthermore, from Lesson 1.6 we know that the graph of $f(x) = a(x - h)^2 + k$ is related to the graph of $y = x^2$ by a vertical shrink or stretch determined by a, a horizontal shift determined by h, and a vertical shift determined by k. Therefore, the graph of a quadratic function is a parabola with vertex at (h, k).

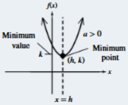
Insights

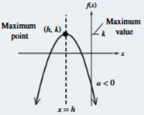
A quadratic function is often used as a model for projectile motion. This is motion followed by an object influenced by an initial force and by the force of gravity.

Quadratic Function

A function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is called a **quadratic function**. By completing the square, $f(x)$ can be expressed in **vertex form** as $f(x) = a(x - h)^2 + k$.

- The graph of f is a parabola with vertex (h, k).
- If $a > 0$, the parabola opens upward, and the vertex is the minimum point. The minimum *value* of f is k.
- If $a < 0$, the parabola opens downward, and the vertex is the maximum point. The maximum *value* of f is k.
- The axis of symmetry is $x = h$. This is the vertical line that passes through the vertex.





In Example 1, we analyze and graph a quadratic function by identifying the vertex, axis of symmetry, and x- and y-intercepts. From the graph, the minimum or maximum value of the function is readily apparent.

Example 1: Analyzing and Graphing a Quadratic Function

Given $f(x) = -2(x - 1)^2 + 8$,

- **a.** Determine whether the graph of the parabola opens upward or downward.
- **b.** Identify the vertex.
- **c.** Determine the x-intercept(s).
- **d.** Determine the y-intercept.
- **e.** Sketch the function.
- **f.** Determine the axis of symmetry.
- **g.** Determine the maximum or minimum value of f.
- **h.** Write the domain and range in interval notation.

Solution:

a. $f(x) = -2(x - 1)^2 + 8$

The parabola opens downward.

The function is written as $f(x) = a(x - h)^2 + k$, where $a = -2$, $h = 1$, and $k = 8$. Since $a < 0$, the parabola opens downward.

b. The vertex is $(1, 8)$.

The vertex is (h, k), which is $(1, 8)$.

c. $f(x) = -2(x-1)^2 + 8$

To find the x-intercept(s), find all real solutions to the equation $f(x) = 0$.

$0 = -2(x-1)^2 + 8$

$-8 = -2(x-1)^2$

$4 = (x-1)^2$

$\pm\sqrt{4} = x - 1$

$1 \pm 2 = x$

$x = 3$ or $x = -1$

The x-intercepts are $(3, 0)$ and $(-1, 0)$.

d. $f(0) = -2(0-1)^2 + 8$

To find the y-intercept, evaluate $f(0)$.

$= 6$

The y-intercept is $(0, 6)$.

e. The graph of f is shown in Figure 2-2.

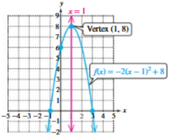


Figure 2-2

f. The axis of symmetry is the vertical line through the vertex: $x = 1$.

g. The maximum value is 8.

h. The domain is $(-\infty, \infty)$. The range is $(-\infty, 8]$.

Apply the Skills

1. Repeat Example 1 with $g(x) = (x+2)^2 - 1$.

Learn: Write $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in Vertex Form

In Lesson 1.2, we learned how to complete the square to write an equation of a circle $x^2 + y^2 + Ax + By + C = 0$ in standard form $(x-h)^2 + (y-k)^2 = r^2$. We use the same process to write a quadratic function $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in vertex form $f(x) = a(x-h)^2 + k$. However, we will work on the right side of the equation only. This is demonstrated in Example 2.

Example 2: Writing a Quadratic Function in Vertex Form

Given $f(x) = 3x^2 + 12x + 5$,

- **a.** Write the function in vertex form: $f(x) = a(x - h)^2 + k$.
- **b.** Identify the vertex.
- **c.** Identify the x-intercept(s).
- **d.** Identify the y-intercept.
- **e.** Sketch the function.
- **f.** Determine the axis of symmetry.
- **g.** Determine the minimum or maximum value of f.
- **h.** Write the domain and range in interval notation.

Solution:

a. $f(x) = 3x^2 + 12x + 5$

$= 3(x^2 + 4x \qquad\quad) + 5$

Factor out the leading coefficient of the x^2 term from the two terms containing x. The leading term within parentheses now has a coefficient of 1.

$= 3(x^2 + 4x + 4 - 4) + 5$

Complete the square within parentheses. Add and subtract $\left[\frac{1}{2}(4)\right]^2 = 4$ within parentheses.

$= 3(x^2 + 4x + 4) + 3(-4) + 5$

Remove –4 from within parentheses, along with a factor of 3.

$= 3(x + 2)^2 - 7$ (vertex form)

b. The vertex is $(-2, -7)$.

c. $f(x) = 3x^2 + 12x + 5$

$0 = 3x^2 + 12x + 5$

To find the x-intercept(s), find the real solutions to the equation $f(x) = 0$.

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4(3)(5)}}{2(3)}$$

$$= \frac{-12 \pm \sqrt{84}}{6}$$

$$= \frac{-12 \pm 2\sqrt{21}}{6}$$

The right side is not factorable. Apply the quadratic formula.

$$= \frac{-6 \pm \sqrt{21}}{3} \begin{array}{l} \nearrow x \approx -0.47 \\ \searrow x \approx -3.53 \end{array}$$

The x-intercepts are $\left(\frac{-6 + \sqrt{21}}{3}, 0\right)$ and $\left(\frac{-6 - \sqrt{21}}{3}, 0\right)$ or approximately $(-0.47, 0)$ and $(-3.53, 0)$.

d. $f(0) = 3(0)^2 + 12(0) + 5$

$= 5$

To find the y-intercept, evaluate $f(0)$.

The y-intercept is $(0, 5)$.

e. The graph of f is shown in Figure 2-3.

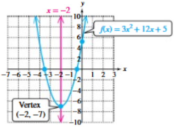


Figure 2-3

f. The axis of symmetry is $x = -2$.

g. The minimum value is -7.

h. The domain is $(-\infty, \infty)$. The range is $[-7, \infty)$.

Apply the Skills

2. Repeat Example 2 with $f(x) = 3x^2 - 6x + 1$.

Learn: Find the Vertex of a Parabola by Using the Vertex Formula

Completing the square and writing a quadratic function in the form $f(x) = a(x - h)^2 + k$ is one method to find the vertex of a parabola. Another method is to use the vertex formula. The vertex formula can be derived by completing the square on $f(x) = ax^2 + bx + c$.

$$f(x) = ax^2 + bx + c \quad (a \neq 0)$$

$$= a\left(x^2 + \frac{b}{a}x\right) + c$$

Factor out a from the x terms

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$

Complete the square within parentheses.

$$\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \frac{b^2}{4a^2}$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + a\left(-\frac{b^2}{4a^2}\right) + c$$

Remove the term $-\frac{b^2}{4a^2}$ from within parentheses along with a factor of a.

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

Factor the trinomial.

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + \frac{4ac}{4a}$$

Obtain a common denominator and add the terms outside parentheses.

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2}{4a} + \frac{4ac}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$= a\left[x - \left(\frac{-b}{2a}\right)\right]^2 + \frac{4ac - b^2}{4a}$$

$f(x)$ is now written in vertex form.

$$\qquad\quad \downarrow \qquad\qquad\quad \downarrow$$

$$f(x) = a(x - h)^2 \quad + \quad k$$

$h = \frac{-b}{2a}$ and $k = \frac{4ac - b^2}{4a}$

The vertex is $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$.

The y-coordinate of the vertex is given by $\frac{4ac - b^2}{4a}$ and is often hard to remember.

Therefore, it is usually easier to evaluate the x-coordinate first from $\frac{-b}{2a}$, and then evaluate $f\left(\frac{-b}{2a}\right)$.

Vertex Formula to Find the Vertex of a Parabola

For $f(x) = ax^2 + bx + c$ $(a \neq 0)$, the vertex is given by $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

Example 3: Using the Vertex Formula

Given $f(x) = -x^2 + 4x - 5$,

a. State whether the graph of the parabola opens upward or downward.

b. Determine the vertex of the parabola by using the vertex formula.

c. Determine the x-intercept(s).

d. Determine the y-intercept.

e. Sketch the graph.

f. Determine the axis of symmetry.

g. Determine the minimum or maximum value of f.

h. Write the domain and range in interval notation.

Solution:

a. $f(x) = -x^2 + 4x - 5$

The parabola opens downward.

The function is written as $f(x) = ax^2 + bx + c$ where $a = -1$. Since $a < 0$, the parabola opens downward.

b. x-coordmate: $\frac{-b}{2a} = \frac{-(4)}{2(-1)} = 2$

y-coordinate: $f(2) = -(2)^2 + 4(2) - 5$

$= -1$

The vertex is $(2, -1)$.

For $f(x) = ax^2 + bx + c$ $(a \neq 0)$, the vertex is given by $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

c. Since the vertex of the parabola is below the x-axis and the parabola opens downward, the parabola cannot cross or touch the x-axis.

Therefore, there are no x-intercepts.

Solving the equation $f(x) = 0$ to find the x-intercepts results in imaginary solutions:

$$0 = -x^2 + 4x - 5$$

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(-1)(-5)}}{2(-1)}$$

$$x = 2 \pm i$$

d. To find the y-intercept, evaluate $f(0)$.

$f(0) = -(0)^2 + 4(0) - 5 = -5$

The y-intercept is $(0, -5)$.

e. The graph of f is shown in Figure 2-4.

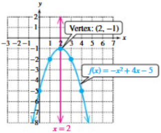


Figure 2-4

f. The axis of symmetry is $x = 2$.

g. The maximum value of f is -1.

h. The domain is $(-\infty, \infty)$. The range is $(-\infty, -1]$.

Good Practices

For more accuracy in the graph, plot one or two points near the vertex. Then use the symmetry of the curve to find additional points on the graph.

For example, the points $(1, -2)$ and $(0, -5)$ are on the left branch of the parabola. The corresponding points to the right of the axis of symmetry are $(3, -2)$ and $(4, -5)$.

Apply the Skills

3. Repeat Example 3 with $f(x) = -x^2 - 4x - 7$.

The x-intercepts of a quadratic function defined by $f(x) = ax^2 + bx + c$ are the real solutions to the equation $f(x) = 0$. The discriminant $b^2 - 4ac$ enables us to determine the number of real solutions to the equation and therefore, the number of x-intercepts of the graph of the function

Using the Discriminant to Determine the Number of x-Intercepts

Given a quadratic function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$,

- If $b^2 - 4ac = 0$, the graph of $y = f(x)$ has one x-intercept.
- If $b^2 - 4ac > 0$, the graph of $y = f(x)$ has two x-intercepts.
- If $b^2 - 4ac < 0$, the graph of $y = f(x)$ has no x-intercept.

From Example 2, the discriminant of $3x^2 + 12x + 5 = 0$ is $(12)^2 - 4(3)(5) = 84 > 0$. Therefore, the graph of $f(x) = 3x^2 + 12x + 5$ has two x-intercepts (Figure 2-3).

From Example 3, the discriminant of $-x^2 + 4x - 5 = 0$ is $(4)^2 - 4(-1)(-5) = -4 < 0$. Therefore, the graph of $f(x) = -x^2 + 4x - 5$ has no x-intercept (Figure 2-4).

Learn: Solve Applications Involving Quadratic Functions

Quadratic functions can be used in a variety of applications in which a variable is optimized. That is, the vertex of a parabola gives the maximum or minimum value of the dependent variable. We show three such applications in Examples 4-6.

Example 4: Using a Quadratic Function for Projectile Motion

A stone is thrown from a 100-m cliff at an initial speed of 20 m/sec at an angle of 30° from the horizontal. The height of the stone can be modeled by $h(t) = -4.9t^2 + 10t + 100$, where $h(t)$ is the height in meters and t is the time in seconds after the stone is released.

a. Determine the time at which the stone will be at its maximum height. Round to 2 decimal places.

b. Determine the maximum height. Round to the nearest meter.

c. Determine the time at which the stone will hit the ground.

Solution:

a. The time at which the stone will be at its maximum height is the t-coordinate of the vertex.

$$t = \frac{-b}{2a} = \frac{-10}{2(-4.9)} \approx 1.02$$

The stone will be at its maximum height approximately 1.02 sec after release.

Given $h(t) = -4.9t^2 + 10t + 100$, the coefficients are $a = -4.9$, $b = 10$, and $c = 100$.

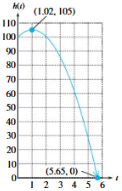
The vertex is given by $\left(\frac{-b}{2a}, h\left(\frac{-b}{2a}\right)\right)$.

b. The maximum height is the value of $h(t)$ at the vertex.

$$h(1.02) = -4.9(1.02)^2 + 10(1.02) + 100$$

$$\approx 105$$

The maximum height is 105 m.



c. The stone will hit the ground when $h(t) = 0$.

$$h(t) = -4.9t^2 + 10t + 100$$

$$0 = -4.9t^2 + 10t + 100$$

$$t = \frac{-10 \pm \sqrt{(10)^2 - 4(-4.9)(100)}}{2(-4.9)}$$

$t \approx 5.65$ or $t \cancel{\approx -}3.61$

Reject the negative solution since t represents time.

The stone will hit the ground in approximately 5.65 sec.

Apply the Skills

4. A quarterback throws a football with an initial velocity of 72 ft/sec at an angle of 25°. The height of the ball can be modeled by $h(t) = -16t^2 + 30.4t + 5$, where $h(t)$ is the height (in ft) and t is the time in seconds after release.

a. Determine the time at which the ball will be at its maximum height.

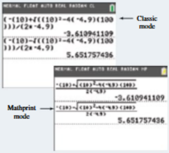
b. Determine the maximum height of the ball.

c. Determine the amount of time required for the ball to reach the receiver's hands if the receiver catches the ball at a point 3 ft off the ground.

TECHNOLOGY CONNECTIONS

Compute Solutions to a Quadratic Equation

The syntax to compute the expressions from Example 4(c) is shown for a calculator in Classic mode and in Mathprint mode. In Classic mode, parentheses are required around the numerator and denominator of the fraction and around the radicand within the square root. In Mathprint mode, select the ALPHA key followed by FI to access the fraction template.



Classic mode

```
NORMAL FLOAT AUTO REAL RADIAN CL
(-(10)+√(((10)²-4(-4.9)(100
)))/(2*-4.9)
                  -3.610941109
(-(10)-√(((10)²-4(-4.9)(100
)))/(2*-4.9)
                   5.651757436
```

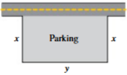
Mathprint mode

```
NORMAL FLOAT AUTO REAL RADIAN MP
                  -3.610941109
                   5.651757436
```

In Example 5, we present a type of application called an optimization problem. The goal is to maximize or minimize the value of the dependent variable by finding an optimal value of the independent variable.

Example 5: Applying a Quadratic Function to Geometry

A parking area is to be constructed adjacent to a road. The developer has purchased 340 ft of fencing. Determine dimensions for the parking lot that would maximize the area. Then find the maximum area.



Solution:

Let x represent the width of the parking area.

Let y represent the length.

Let A represent the area.

Read the problem carefully, draw a representative diagram, and label the unknowns.

We need to find the values of x and y that maximize the area A of the rectangular region. The area is given by $A = (\text{length})(\text{width}) = yx = xy$.

To write the area as a function of one variable only, we need an equation that relates x and y. We know that the parking area is limited by a fixed amount of fencing. That is, the sum of the lengths of the three sides to be fenced can be at most 340 ft. or $2x + y = 340$,

The equation $2x + y = 340$ is called a **constraint equation.** This equation gives an implied restriction on x and y due to the limited amount of fencing.

$2x + y = 340$ — Solve the constraint equation, $2x + y = 340$ for either x or y.

Solve for y.

$y = 340 - 2x$ — In this case, we have solved for y.

$A = xy$

$A(x) = x(340 - 2x)$ — Substitute $340 - 2x$ for y in the equation $A = xy$.

$= -2x^2 + 340x$ — Function A is a quadratic function with a negative leading coefficient. The graph of the parabola opens downward, so the vertex is the maximum point on the function.

x-coordinate of vertex: — The x-coordinate of the vertex $\frac{-b}{2a}$ is the value of x that will maximize the area.

$$x = \frac{-b}{2a} = \frac{-340}{2(-2)}$$

$$= 85$$

$y = 340 - 2(85)$ — The second dimension of the parking lot can be determined from the constraint equation.

$= 170$

The values of x and y that would maximize the area are $x = 85$ ft and $y = 170$ ft.

$A(85) = -2(85)^2 + 340(85) = 14{,}450$ — The value of the function at $x = 85$ gives the maximum area.

The maximum area is 14,450 ft^2.

Check Your Work

To check, verify that the value $A(85)$ is the same as the product of length and width, xy.

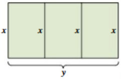
$$A(85) = 14{,}450 \text{ ft}^2$$

$$xy = (85 \text{ ft})(170 \text{ ft})$$

$$= 14{,}450 \text{ ft}^2 \checkmark$$

Apply the Skills

5. A farmer has 200 ft of fencing and wants to build three adjacent rectangular corrals. Determine the dimensions that should be used to maximize the area, and find the area of each individual corral.



Learn: Create Quadratic Models Using Regression

In Lesson 1.5, we introduced linear regression. A regression line is a linear model based on all observed data points. In a similar fashion, we can create a quadratic function using regression. For example, suppose that a scientist growing bacteria measures the population of bacteria as a function of time. A scatter plot reveals that the data follow a curve that is approximately parabolic (Figure 2-5). In Example 6, we use a graphing calculator to find a quadratic function that models the population of the bacteria as a function of time.

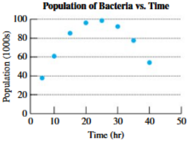


Figure 2-5

Example 6: Creating a Quadratic Function Using Regression

The data in the table represent the population of bacteria $P(t)$ (in 1000s) versus the number of hours t since the culture was started.

Time (hr) t	Population (1000s) $P(t)$
5	37.7
10	60.9
15	85.3
20	96.3
25	98.6
30	92.4
35	77.5
40	54.1

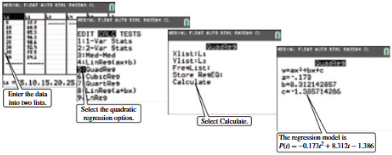
a. Use regression to find a quadratic function to model the data. Round the coefficients to 3 decimal places.

b. Use the model to determine the time at which the population is the greatest. Round to the nearest hour.

c. What is the maximum population? Round to the nearest hundred.

Solution:

a. From the graph in Figure 2-5, it appears that the data follow a parabolic curve. Therefore, a quadratic model would be reasonable.


L1
L2
L3
L4
L1 = {5,10,15,20,25
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
7:QuartReg
8:LinReg(a+bx)
9:LnReg
QuadReg
Xlist:L1
Ylist:L2
FreqList:
Store RegEQ:
Calculate

```
QuadReg
y=ax²+bx+c
a=-.173
b=8.312142857
c=-1.385714286
```

Enter the data into two lists.

Select the quadratic regression option.

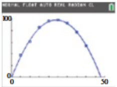
Select Calculate.

The regression model is $P(t) = -0.173t^2 + 8.312t - 1.386$

b. From the graph, the time when the population is greatest is the *t*-coordinate of the vertex.

$$t = \frac{-b}{2a} = \frac{-(8.312)}{2(-0.173)} \approx 24$$

The population is greatest 24 hr after the culture is started.



c. The maximum population of the bacteria is the $P(t)$ value at the vertex.

$$P(24) = -0.173(24)^2 + 8.312(24) - 1.386$$

≈ 98.5 The maximum number of bacteria is approximately 98,500.

Apply the Skills

6. The funding $f(t)$ (in \$ millions) for a drug rehabilitation center is given in the table for selected years *t*.

t	0	3	6	9	12	15
f(*t*)	3.5	2.2	2.1	3	4.9	8

a. Use regression to find a quadratic function to model the data.

b. During what year is the funding the least? Round to the nearest year.

c. What is the minimum yearly amount of funding received? Round to the nearest million.

Practice 2-1

Practice Exercises

Prerequisite Review

R.1. Solve the equation. $x^2 + 3x - 18 = 0$

R.2. Find the following for $f(x) = 3x^2 - 7x - 20$.

  a. Find the values of x for which $f(x) = 0$.

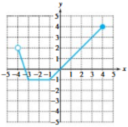
  b. Find $f(0)$.

R.3. Solve the equation by completing the square and applying the square root property.

$$x^2 + 8x + 12 = 0$$

R.4. Find $g\left(-\frac{1}{2}\right)$ for $g(x) = -x^2 + 2x - 4$.

R.5. Write the domain and range in interval notation.



Concept Connections

1. A function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is called a ________ function.

2. The vertical line drawn through the vertex of a quadratic function is called the of ________ symmetry.

3. Given $f(x) = a(x - h)^2 + k$ $(a \neq 0)$, the vertex of the parabola is the point ________.

4. Given $f(x) = a(x - h)^2 + k$, if $a < 0$, the parabola opens (upward/downward) and the (minimum/maximum) value is ________.

5. Given $f(x) = a(x - h)^2 + k$, if $a > 0$, the parabola opens (upward/downward) and the (minimum/maximum) value is ________.

6. The graph of $f(x) = a(x - h)^2 + k$, $a \neq 0$, is a parabola and the axis of symmetry is the line given by ________.

Learn: Graph a Quadratic Function Written in Vertex Form

For Exercises 7–14,

a. Determine whether the graph of the parabola opens upward or downward.

b. Identify the vertex.

c. Determine the x-intercept(s).

d. Determine the y-intercept.

e. Sketch the function.

f. Determine the axis of symmetry.

g. Determine the minimum or maximum value of the function.

h. Write the domain and range in interval notation. **(See Example 1 on p. 165)**

7. $f(x) = -(x - 4)^2 + 1$

8. $g(x) = -(x + 2)^2 + 4$

9. $h(x) = 2(x + 1)^2 - 8$

10. $k(x) = 2(x - 3)^2 - 2$

11. $m(x) = 3(x - 1)^2$

12. $n(x) = \frac{1}{2}(x + 2)^2$

13. $p(x) = -\frac{1}{5}(x + 4)^2 + 1$

14. $q(x) = -\frac{1}{3}(x - 1)^2 + 1$

Learn: Write $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in Vertex Form

For Exercises 15–24,

a. Write the function in vertex form.

b. Identify the vertex.

c. Determine the x-intercept(s).

d. Determine the y-intercept.

e. Sketch the function.

f. Determine the axis of symmetry.

g. Determine the minimum or maximum value of the function.

h. Write the domain and range in interval notation. (**See Example 2 on p. 167**)

15. $f(x) = x^2 + 6x + 5$

16. $g(x) = x^2 + 8x + 7$

17. $p(x) = 3x^2 - 12x - 7$

18. $q(x) = 2x^2 - 4x - 3$

19. $c(x) = -2x^2 - 10x + 4$

20. $d(x) = -3x^2 - 9x + 8$

21. $h(x) = -2x^2 + 7x$

22. $k(x) = 3x^2 - 8x$

23. $p(x) = x^2 + 9x + 17$

24. $q(x) = x^2 + 11x + 26$

Learn: Find the Vertex of a Parabola by Using the Vertex Formula

For Exercises 25–32, find the vertex of the parabola by applying the vertex formula.

25. $f(x) = 3x^2 - 42x - 91$

26. $g(x) = 4x^2 - 64x + 107$

27. $k(a) = -\frac{1}{3}a^2 + 6a + 1$

28. $j(t) = -\frac{1}{4}t^2 + 10t - 5$

29. $f(c) = 4c^2 - 5$

30. $h(a) = 2a^2 + 14$

31. $P(x) = 1.2x^2 + 1.8x - 3.6$ (Write the coordinates of the vertex as decimals.)

32. $Q(x) = 7.5x^2 - 2.25x + 4.75$ (Write the coordinates of the vertex as decimals.)

For Exercises 33–42,

a. State whether the graph of the parabola opens upward or downward.

b. Identify the vertex.

c. Determine the x-intercept(s).

d. Determine the y-intercept.

e. Sketch the graph.

f. Determine the axis of symmetry.

g. Determine the minimum or maximum value of the function.

h. Write the domain and range in interval notation. (**See Example 3 on p. 169**)

33. $g(x) = -x^2 + 2x - 4$

34. $h(x) = -x^2 - 6x - 10$

35. $f(x) = 5x^2 - 15x + 3$

36. $k(x) = 2x^2 - 10x - 5$

37. $f(x) = 2x^2 + 3$

38. $g(x) = -x^2 - 1$

39. $f(x) = -2x^2 - 20x - 50$

40. $m(x) = 2x^2 - 8x + 8$

41. $n(x) = x^2 - x + 3$

42. $r(x) = x^2 - 5x + 7$

Learn: Solve Applications Involving Quadratic Functions

43. The monthly profit for a small company that makes long-sleeve T-shirts depends on the price per shirt. If the price is too high, sales will drop. If the price is too low, the revenue brought in may not cover the cost to produce the shirts. After months of data collection, the sales team determines that the monthly profit is approximated by $f(p) = -50p^2 + 1700p - 12{,}000$, where p is the price per shirt and $f(p)$ is the monthly profit based on that price. (**See Example 4 on p. 171**)

a. Find the price that generates the maximum profit.

b. Find the maximum profit.

c. Find the price(s) that would enable the company to break even.

44. The monthly profit for a company that makes decorative picture frames depends on the price per frame. The company determines that the profit is approximated by $f(p) = -80p^2 + 3440p - 36{,}000$, where p is the price per frame and $f(p)$ is the monthly profit based on that price.

a. Find the price that generates the maximum profit.

b. Find the maximum profit.

c. Find the price(s) that would enable the company to break even.

45. A long jumper leaves the ground at an angle of 20° above the horizontal, at a speed of 11 m/sec. The height of the jumper can be modeled by $h(x) = -0.046x^2 + 0.364x$, where h is the jumper's height in meters and x is the horizontal distance from the point of launch.

a. At what horizontal distance from the point of launch does the maximum height occur? Round to 2 decimal places.

b. What is the maximum height of the long jumper? Round to 2 decimal places.

c. What is the length of the jump? Round to 1 decimal place.

46. A firefighter holds a hose 3 m off the ground and directs a stream of water toward a burning building. The water leaves the hose at an initial speed of 16 m/sec at an angle of 30°. The height of the water can be approximated by $h(x) = -0.026x^2 + 0.577x + 3$, where $h(x)$ is the height of the water in meters at a point x meters horizontally from the firefighter to the building.

a. Determine the horizontal distance from the firefighter at which the maximum height of the water occurs. Round to 1 decimal place.

b. What is the maximum height of the water? Round to 1 decimal place.

c. The flow of water hits the house on the downward branch of the parabola at a height of 6 m. How far is the firefighter from the house? Round to the nearest meter.

47. The population $P(t)$ of a culture of the bacterium *Pseudomonas aeruginosa* is given by $P(t) = -1718t^2 + 82{,}000t + 10{,}000$, where t is the time in hours since the culture was started.

a. Determine the time at which the population is at a maximum. Round to the nearest hour.

b. Determine the maximum population. Round to the nearest thousand.

48. The gas mileage $m(x)$ (in mpg) for a certain vehicle can be approximated by $m(x) = -0.028x^2 + 2.688x - 35.012$, where x is the speed of the vehicle in mph.

a. Determine the speed at which the car gets its maximum gas mileage.

b. Determine the maximum gas mileage.

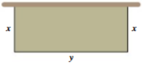
49. The sum of two positive numbers is 24. What t wo numbers will maximize the product? (**See Example 5 on p. 172**)

50. The sum of two positive numbers is 1. What two numbers will maximize the product?

51. The difference of two numbers is 10. What two numbers will minimize the product?

52. The difference of two numbers is 30. What two numbers will minimize the product?

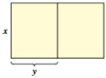
53. Suppose that a family wants to fence in an area of their yard for a vegetable garden to keep out deer. One side is already fenced from the neighbor's property. (**See Example 5 on p. 172**)



a. If the family has enough money to buy 160ft of fencing, what dimensions would produce the maximum area for the garden?

b. What is the maximum area?

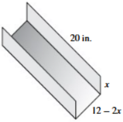
54. Two chicken coops are to be built adjacent to one another using 120 ft of fencing.



a. What dimensions should be used to maximize the area of an individual coop?

b. What is the maximum area of an individual coop?

55. A trough at the end of a gutter spout is meant to direct water away from a house. The homeowner makes the trough from a rectangular piece of aluminum that is 20 in. long and 12 in. wide. He makes a fold along the two long sides a distance of *x* inches from the edge.

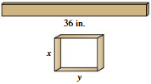


a. Write a function to represent the volume in terms of *x*.

b. What value of *x* will maximize the volume of water that can be carried by the gutter?

c. What is the maximum volume?

56. A rectangular frame of uniform depth for a shadow box is to be made from a 36-in. piece of wood.



a. Write a function to represent the display area in terms of *x*.

b. What dimensions should be used to maximize the display area?

c. What is the maximum area?

Learn: Create Quadratic Models Using Regression

57. *Tetanus bacillus* bacteria are cultured to produce tetanus toxin used in an inactive form for the tetanus vaccine. The amount of toxin produced per batch increases with time and then decreases as the culture becomes unstable. The variable *t* is the time in hours after the culture has started, and *y*(*t*) is the yield of toxin in grams. (**See Example 6 on p. 174**)

t	8	16	24	32	40	48
***y*(*t*)**	0.60	1.12	1.60	1.78	1.90	2.00

t	56	64	72	80	88	96
***y*(*t*)**	1.94	1.80	1.48	1.30	0.66	0.10

a. Use regression to find a quadratic function to model the data.

b. At what time is the yield the greatest? Round to the nearest hour.

c. What is the maximum yield? Round to the nearest gram.

58. Gas mileage is tested for a car under different driving conditions. At lower speeds, the car is driven in stop-and-go traffic. At higher speeds, the car must overcome more wind resistance. The variable *x* given in the table represents the speed (in mph) for a compact car, and *m*(*x*) represents the gas mileage (in mpg).

x	25	30	35	40	45
***m*(*x*)**	22.7	25.1	27.9	30.8	31.9

x	50	55	60	65
***m*(*x*)**	30.9	28.4	24.2	21.9

a. Use regression to find a quadratic function to model the data.

b. At what speed is the gas mileage the greatest? Round to the nearest mile per hour.

c. What is the maximum gas mileage? Round to the nearest mile per gallon.

59. Fluid runs through a drainage pipe with a 10-cm radius and a length of 30 m (3000 cm). The velocity of the fluid gradually decreases from the center of the pipe toward the edges as a result of friction with the walls of the pipe. For the data shown, $v(x)$ is the velocity of the fluid (in cm/sec) and x represents the distance (in cm) from the center of the pipe toward the edge.

x	0	1	2	3	4
$v(x)$	195.6	195.2	194.2	193.0	191.5

x	5	6	7	8	9
$v(x)$	189.8	188.0	185.5	183.0	180.0

a. The pipe is 30 m long (3000 cm). Determine how long it will take fluid to run the length of the pipe through the center of the pipe. Round to 1 decimal place.

b. Determine how long it will take fluid at a point 9 cm from the center of the pipe to run the length of the pipe. Round to 1 decimal place.

c. Use regression to find a quadratic function to model the data.

d. Use the model from part (c) to predict the velocity of the fluid at a distance 5.5 cm from the center of the pipe. Round to 1 decimal place.

60. The braking distance required for a car to stop depends on numerous variables such as the speed of the car, the weight of the car, reaction time of the driver, and the coefficient of friction between the tires and the road. For a certain vehicle on one stretch of highway, the braking distances $d(s)$ (in ft) are given for several different speeds s (in mph).

s	30	35	40	45	50
$d(s)$	109	134	162	191	223

s	55	60	65	70	75
$d(s)$	256	291	328	368	409

a. Use regression to find a quadratic function to model the data.

b. Use the model from part (a) to predict the stopping distance for the car if it is traveling 62 mph before the brakes are applied. Round to the nearest foot.

c. Suppose that the car is traveling 53 mph before the brakes are applied. If a deer is standing in the road at a distance of 245 ft from the point where the brakes are applied, will the car hit the deer?

Mixed Exercises

For Exercises 61–64, given a quadratic function defined by $f(x) = ax^2 + bx + c(a \neq 0)$, answer true or false. If an answer is false, explain why.

61. The graph of fcan have two y-intercepts.

62. The graph of fcan have two x-intercepts.

63. If $a < 0$, then the vertex of the parabola is the maximum point on the graph of f.

64. The axis of symmetry of the graph of f is the line defined by $y = c$.

For Exercises 65–70, determine the number of x-intercepts of the graph of $f(x) = ax^2 + bx + c$ $(a \neq 0)$, based on the discriminant of the related equation $f(x) = 0$. (*Hint:* Recall that the discriminant is $(b^2 - 4ac$.)

65. $f(x) = 4x^2 + 12x + 9$

66. $f(x) = 25x^2 - 20x + 4$

67. $f(x) = -x^2 - 5x + 8$

68. $f(x) = -3x^2 + 4x + 9$

69. $f(x) = -3x^2 + 6x - 11$

70. $f(x) = -2x^2 + 5x - 10$

For Exercises 71–78, given a quadratic function defined by $f(x) = a(x - h)^2 + k$ $(a \neq 0)$, match the graph with the function based on the conditions given.

71. $a > 0, h < 0, k > 0$

72. $a > 0, h < 0, k < 0$

73. $a < 0, h < 0, k < 0$

74. $a < 0, h < 0, k > 0$

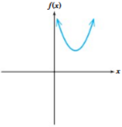
75. $a > 0$, axis of symmetry $x = 2, k < 0$

76. $a < 0$, axis of symmetry $x = 2$, $k > 0$

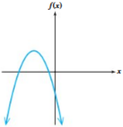
77. $a < 0$, $h = 2$, maximum value equals -2

78. $a > 0$, $h = 2$, minimum value equals 2

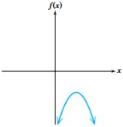
a.



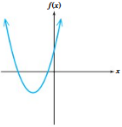
b.



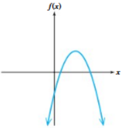
c.



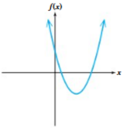
d.



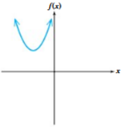
e.



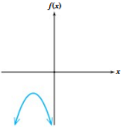
f.



g.



h.



Write About It

79. Explain why a parabola opening upward has a minimum value but no maximum value. Use the graph of $f(x) = x^2$ to explain.

80. Explain why a quadratic function whose graph opens downward with vertex $(4, -3)$ has no x-intercept.

81. Explain why a quadratic function given by $f(x) = ax^2 + bx + c$ cannot have two y-intercepts.

82. Explain how to use the discriminant to determine the number of x-intercepts for the graph of $f(x) = ax^2 + bx + c$.

83. If a quadratic function given by $y = f(x)$ has x-intercepts of $(2, 0)$ and $(6, 0)$, explain why the vertex must be $(4, f(4))$.

84. Given an equation of a parabola in the form $y = a(x - h)^2 + k$, explain how to determine by inspection if the parabola has no x-intercepts.

Expanding Your Skills

For Exercises 85–88, define a quadratic function $y = f(x)$ that satisfies the given conditions.

85. Vertex $(2, -3)$ and passes through $(0, 5)$

86. Vertex $(-3, 1)$ and passes through $(0, -17)$

87. Axis of symmetry $x = 4$, maximum value 6, passes through $(1, 3)$

88. Axis of symmetry $x = -2$, minimum value 5, passes through $(2, 13)$

For Exercises 89–92, find the value of *b* or *c* that gives the function the given minimmn or maximum value.

89. $f(x) = 2x^2 + 12x + c$; minimum value -9

90. $f(x) = 3x^2 + 12x + c$; minimum value -4

91. $f(x) = -x^2 + bx + 4$; maximum value 8

92. $f(x) = -x^2 + bx - 2$; maximum value 7

93. Find the point on the graph of $2x + y = 1$ closest to $(1, 1)$. (*Hint:* find the minimum value of the *square* of the distance.)

94. Find the point on the graph of $y = 2x + 4$ closest to $(-2, -3)$.

95. An entrepreneur sells handmade glass ornaments at \$10 each for any order up to a dozen. As an enticement for online orders of more than a dozen, the price of each ornament will be reduced by 50 cents. (For example, for 13 ornaments, the unit price is \$9.50 each; for 14 ornaments, the unit price is \$9.00 each, and so on.) Due to a scarcity of materials, only orders of up to 20 can be filled.

a. Determine a function R that models the revenue earned in terms of x, where x represents the number of ornaments in excess of a dozen. State the range of values for x.

b. How many ornaments should be sold to produce the maximum revenue, and what is the maximum revenue?

96. A food truck parked in an office plaza sells 40 cups of coffee at \$2 each during the morning rush. The workers on the truck try different pricing options for the coffee and notice that for every 10-cent drop in price they sell 5 more cups of coffee. What price should they charge for maximum revenue? What is the maximum revenue they can expect?

Point of Interest

The movie *Apollo 13*, starring Tom Hanks, was filmed in part on an aircraft that uses a parabolic flight trajectory to produce weightlessness. As the plane climbs toward the top of the parabolic path, occupants experience a force of nearly 2 Gs (twice their body weight). Once the plane goes over the vertex of the parabola, passengers free-fall inside the plane.

During the filming of *Apollo 13,* the director, Ron Howard, had roughly 25 seconds in which to shoot a scene while the plane followed the downward branch of the parabola before it leveled off. Real astronauts who spend days in space face weightlessness around the clock. In 1992, astronaut Dr. Mae Carol Jemison, a mathematician, engineer, and physician, became the first African American woman to visit space aboard NASA's Space Shuttle *Endeavour.* As a practicing physician, Dr. Jemison was a science mission specialist who studied the effects of motion sickness and weightlessness on the human body. The important experiments conducted by Dr. Jemison paved the way for longer human space flight aboard the International Space Station (ISS). In the long term, weightlessness can bring loss of muscle and bone mass. Therefore, the ISS has a stationary bicycle, a treadmill, and an Advanced Resistive Exercise Device to simulate weightlifting in space.

97. a. Dr. Jemison's career in science began with a degree in chemical engineering at Stanford University, but ultimately, her education and work took her around the world. Literally. Read more about Dr. Mae Carol Jemison's fascinating career spanning the globe from Cornell University Medical College in upstate New York to Cuba, Kenya, Thailand, and finally outer space. Then write a report summarizing her educational experiences and professional accomplishments.

b. An airplane flying in a parabolic trajectory is an important tool for astronaut training. Use online resources to read more about the how NASA simulates free fall.

Further Reading

Holt, Nathalia. *Rise ofthe Rocket Girls: The Women Who Propelled Us, from Missiles to the Moon to Mars.* Little, Brown and Company, 2017.

Ackmann, Martha. *The Mercury Thirteen: The True Story* of *Thirteen Women and the Dream* of *Space Flight.* Random House, 2004.

Go online for more practice problems.

Space Frontiers/Archive Photos/Getty Images

Lesson 2-2

Introduction to Polynomial Functions

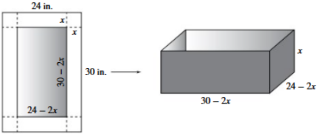
What Will You Learn?

After completing this lesson, you should be able to:

- Determine the End Behavior of a Polynomial Function
- Identify Zeros and Multiplicities of Zeros
- Apply the Intermediate Value Theorem
- Sketch a Polynomial Function

Learn: Determine the End Behavior of a Polynomial Function

A solar oven is to be made from an open box with reflective sides. Each box is made from a 30-in. by 24-in. rectangular sheet of aluminum with squares of length x (in inches) removed from each corner. Then the flaps are folded up to form an open box.



The volume $V(x)$ (in cubic inches) of the box is given by

$$V(x) = 4x^3 - 108x^2 + 720x, \text{ where } 0 < x < 12.$$

From the graph of $y = V(x)$ (Figure 2-6), the maximum volume appears to occur when squares of approximately 4 inches in length are cut from the corners of the sheet of aluminum. See Exercise 107 in the Practice Exercises of this lesson.

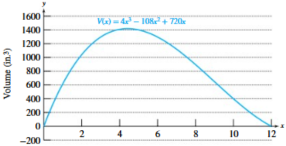


Figure 2-6

The function defined by $V(x) = 4x^3 - 108x^2 + 720x$ is an example of a polynomial function of degree 3.

Definition of a Polynomial Function

Let n be a whole number and $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ be real numbers, where $a_n \neq 0$. Then a function defined by

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

is called a **polynomial function of degree *n***.

The coefficients of each term of a polynomial function are real numbers, and the exponents on x must be whole numbers.

Polynomial Function

$f(x) = 4x^5 - 3x^4 + 2x^2$

Not a Polynomial Function

$f(x) = 4\sqrt{x} - \frac{3}{x} + (3 + 2i)x^2$

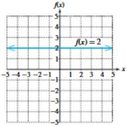
$\sqrt{x} = x^{1/2}$
Exponent not a whole number

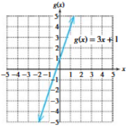
$3/x = 3x^{-1}$
Exponent not a whole number

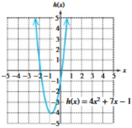
$(3 + 2i)$
Coefficient not a real number

We have already studied several special cases of polynomial functions. For example:

$f(x) = 2$	constant function	(polynomial function, degree 0)
$g(x) = 3x + 1$	linear function	(polynomial function, degree 1)
$h(x) = 4x^2 + 7x - 1$	quadratic function	(polynomial function, degree 2)







Insights

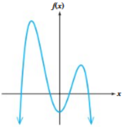
A third-degree polynomial function is referred to as a *cubic* polynomial function.

A fourth-degree polynomial function is referred to as *quartic* polynomial function.

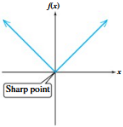
The domain of a polynomial function is all real numbers. Furthermore, the graph of a polynomial function is both continuous and smooth. Informally, a continuous function can be drawn without lifting the pencil from the paper. A smooth function has no sharp corners or points.

For example, the first curve shown here could be a polynomial function, but the last three are not polynomial functions.

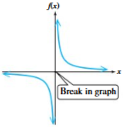
Smooth and Continuous



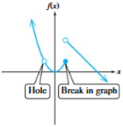
Not Smooth



Not Continuous



Not Continuous



To begin our analysis of polynomial functions, we first consider the graphs of functions of the form $f(x) = ax^n$, where a is a real number and n is a positive integer. These fall into a category of functions called **power functions.** The graphs of three power functions with even degrees and positive coefficients are shown in Figure 2-7. The graphs of three power functions with odd degrees and positive coefficients are shown in Figure 2-8.

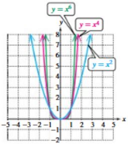


Figure 2-7

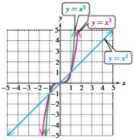


Figure 2-8

Insights

For a positive integer n, the graph of the power function $y = x^n$ becomes "flatter" near the x-intercept for higher powers of n.

From Figure 2-7, notice that for even powers of n, the behavior of $y = x^n$ is similar to the graph of $y = x^2$ with variations on the "steepness" of the curve. Figure 2-8 shows that for odd powers, the behavior of $y = x^n$ with $n \geq 3$ is similar to the graph of $y = x^3$. For any power function $y = ax^n$, the coefficient a will impose a vertical shrink or stretch on the graph of $y = x^n$ by a factor of $|a|$ If $a < 0$, then the graph is reflected across the x-axis.

Power functions are helpful to analyze the "end behaviour" of a polynomial function with multiple terms. The end behavior is the general direction that the function follows as x approaches ∞ or $-\infty$. To describe end behavior, we have the following notation.

Notation for Infinite Behavior of $y = f(x)$

$x \to \infty$	is read as "x approaches infinity." This means that x becomes infinitely large in the positive direction.
$x \to -\infty$	is read as "x approaches negative infinity." This means that x becomes infinitely "large" in the negative direction.
$f(x) \to \infty$	is read as "$f(x)$ approaches infinity." This means that the y value becomes infinitely large in the positive direction.
$f(x) \to -\infty$	is read as "$f(x)$ approaches negative infinity." This means that the y value becomes infinitely "large" in the negative direction.

Consider the function defined by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

The leading term has the greatest exponent on x.

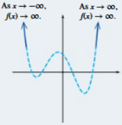
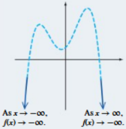
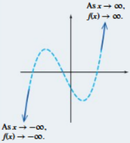
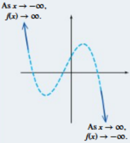
The leading term has the greatest exponent on x. Therefore, as $|x|$ gets large (that is, as $x \to \infty$ or as $x \to -\infty$), the leading term will be relatively larger in absolute value than all other terms. In fact, x^n will eventually be greater in absolute value than the *sum* of all other terms. Therefore, the end behavior of the function is dictated only by the leading term, and the graph of the function far to the left and far to the right will follow the general behavior of the power function $y = ax^n$.

The Leading Term Test

Consider a polynomial function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0.$$

As $x \to \infty$ or as $x \to -\infty$, f eventually becomes forever increasing or forever decreasing and will follow the general behavior of $y = a_n x^n$.

if n is even:		if n is odd:	
a_n positive	a_n negative	a_n positive	a_n negative
			
End behavior: up left/up right	End behavior: down left/down right	End behavior: down left/up right	End behavior: up left/down right

Example 1: Determining End Behavior

Use the leading term to determine the end behavior of the graph of the function.

a. $f(x) = -4x^5 + 6x^4 + 2x$

b. $g(x) = \frac{1}{4}x(2x - 3)^3(x + 4)^2$

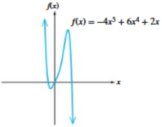
Solution:

a. $f(x) = \underset{\text{negative}}{-4}x^{\underset{\text{odd}}{5}} + 6x^4 + 2x$

The leading coefficient is negative and the degree is odd. By the leading term test, the end behavior is up to the left and down to the right.

As $x \to -\infty$, $f(x) \to \infty$.

As $x \to -\infty$, $f(x) \to -\infty$.



b. $g(x) = \frac{1}{4}x(2x - 3)^3(x + 4)^2$

$g(x) = \frac{1}{4}x(2x - 3)^3(x + 4)^2$

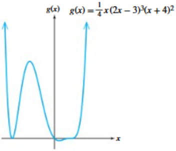
$= \underset{\text{positive}}{2}x^{\underset{\text{even}}{6}} + \cdots$

To determine the leading term, multiply the leading terms from each factor. That is, $\frac{1}{4}x(2x)^3(x)^2 = 2x^6$.

The leading coefficient is positive and the degree is even. By the leading term test, the end behavior is up to the left and up to the right.

As $x \to -\infty$, $f(x) \to \infty$.

As $x \to \infty$, $f(x) \to \infty$.



Apply the Skills

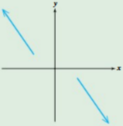
1. Use the leading term to determine the end behavior of the graph of the function.

   a. $f(x) = -0.3x^4 - 5x^2 - 3x + 4$

   b. $g(x) = \frac{6}{7}(x - 9)^4(x + 4)^2(3x - 5)$

Insights

The graph of $y = f(x)$ from Example 1(a) will exhibit the same behavior as the graph of the power function $y = -4x^5$ for values of x far to the right and far to the left. This is similar to the graph of $y = x^5$ reflected across the x-axis.



Learn: Identify Zeros and Multiplicities of Zeros

Consider a polynomial function defined by $y = f(x)$. The values of x in the domain off for which $f(x) = 0$ are called the **zeros** of the function. These are the real solutions (or **roots**) of the equation $f(x) = 0$ and correspond to the x-intercepts of the graph of $y = f(x)$.

Example 2: Determining the Zeros of a Polynomial Function

Find the zeros of the function defined by $f(x) = x^3 + x^2 - 9x - 9$.

Solution:

$f(x) = x^3 + x^2 - 9x - 9$ To find the zeros of f, set $f(x) = 0$ and solve for x.

$0 = x^3 + x^2 - 9x - 9$

$0 = x^2(x + 1) - 9(x + 1)$ Factor by grouping.

$0 = (x + 1)(x^2 - 9)$

$0 = (x + 1)(x - 3)(x + 3)$ Factor the difference of squares.

$x + 1 = 0$ Set each factor equal to zero and solve for x.

$x = -1$

$x - 3 = 0$

$x = 3$

$x + 3 = 0$

$x = -3$

The zeros of f are -1, 3, and -3.

The graph of f is shown in Figure 2-9. The zeros of the function are real numbers and correspond to the x-intercepts of the graph. By inspection, we can evaluate $f(0) = -9$, indicating that the y-intercept is $(0, -9)$.

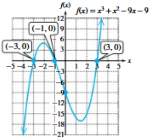
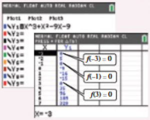


Figure 2-9

Check Your Work

A table of points can be used to check that $f(-1)$, $f(3)$, and $f(-3)$ all equal 0.


NORMAL FLOAT AUTO REAL RADIAN CL
Plot1 Plot2 Plot3
Y1=X^3+X²−9X−9
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=
Y9=

NORMAL FLOAT AUTO REAL RADIAN CL
X= -3

Apply the Skills

2. Find the zeros of the function defined by $f(x) = 4x^3 - 4x^2 - 25x + 25$.

Example 3: Determining the Zeros of a Polynomial Function

Find the zeros of the function defined by $f(x) = -x^3 + 8x^2 - 16x$.

Solution:

$f(x) = -x^3 + 8x^2 - 16x$	To find the zeros of f, set $f(x) = 0$ and solve for x.
$0 = -x(x^2 - 8x + 16)$	Factor out the GCF.
$0 = -x(x - 4)^2$	Factor the perfect square trinomial.
$-x = 0$	Set each factor equal to zero and solve for x.
$x = 0$	

$(x - 4)^2 = 0$

$x - 4 = 0$

$x = 4$

The zeros of f are 0 and 4.

The graph of f is shown in Figure 2-10. The zeros of the function are real numbers and correspond to the x-intercepts (0, 0) and (4, 0).

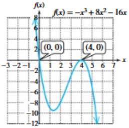


Figure 2-10

The leading term of $f(x)$ is $-x^3$. The coefficient is negative and the exponent is odd. The graph shows the end behavior up to the left and down to the right as expected.

Apply the Skills

3. Find the zeros of the function defined by $f(x) = x^3 + 10x^2 + 25x$.

From Example 3, $f(x) - x^3 + 8x^2 - 16x$ can be written as a product of linear factors:

$$f(x) = -x(x - 4)^2$$

Notice that the factor $(x - 4)$ appears to the second power. Therefore, we say that the corresponding zero, 4, has a multiplicity of 2.

In general, we say that if a polynomial function has a factor $(x - c)$ that appears exactly k times, then c is a **zero of multiplicity k**. For example, consider:

$g(x) = x^2(x - 2)^3(x + 4)^7$

$$\begin{aligned} x^2 &= 0 \\ x &= 0 \qquad \text{0 is a zero of multiplicity 2.} \end{aligned}$$

$$\begin{aligned} (x - 2)^3 &= 0 \\ x - 2 &= 0 \\ x &= 2 \qquad \text{2 is a zero of multiplicity 3.} \end{aligned}$$

$$\begin{aligned} (x + 4)^7 &= 0 \\ x + 4 &= 0 \\ x &= -4 \qquad \text{-4 is a zero of multiplicity 7.} \end{aligned}$$

The graph of a polynomial function behaves in the following manner based on the multiplicity of the zeros.

Touch Points and Cross Points

Let f be a polynomial function and let c be a real zero of f. Then the point $(c, 0)$ is an x-intercept of the graph of f. Furthermore,

- If c is a zero of odd multiplicity, then the graph *crosses* the x-axis at c. The point $(c, 0)$ is called a cross **point**.
- If c is a zero of even multiplicity, then the graph *touches* the x-axis at c and turns back around (does not cross the x-axis). The point $(c, 0)$ is called a **touch point**.

To illustrate the behavior of a polynomial function at its real zeros, consider the graph of $f(x) = -x(x - 4)^2$ from Example 3 (Figure 2-11).

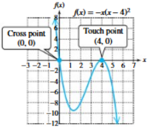


Figure 2-11

- 0 has a multiplicity of 1 (odd multiplicity). The graph *crosses* the x-axis at $(0, 0)$.
- 4 has a multiplicity of 2 (even multiplicity). The graph *touches* the x-axis at $(4, 0)$ and turns back around.

Example 4: Determining Zeros and Multiplicities

Determine the zeros and their multiplicities for the given functions.

a. $m(x) = \frac{1}{10}(x-4)^2(2x+5)^3$

b. $n(x) = x^4 - 2x^2$

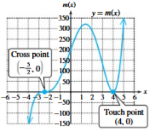
Solution:

a. $m(x) = \frac{1}{10}(x-4)^{\overset{\text{even}}{2}}(2x+5)^{\overset{\text{odd}}{3}}$

The function is factored into linear factors. The zeros are 4 and $-\frac{5}{2}$.

The function has a zero of 4 with multiplicity 2 (even). The graph has a touch point at (4, 0).

The function has a zero of $-\frac{5}{2}$ with multiplicity 3 (odd). The graph has a cross point at $\left(-\frac{5}{2}, 0\right)$.



b. $n(x) = x^4 - 2x^2$

$= x^2(x^2 - 2)$

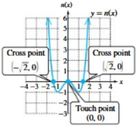
$= x^2(x - \sqrt{2})^1(x + \sqrt{2})^1$

Factor the function into linear factors.

The function has a zero of 0 with multiplicity 2 (even). The graph has a touch point at (0, 0).

The function has a zero of $\sqrt{2}$ with multiplicity 1 (odd). The graph has a cross point at $(-\sqrt{2}, 0) \approx (1.41, 0)$.

The function has a zero of $-\sqrt{2}$ with multiplicity 1 (odd). The graph has a cross point at $(-\sqrt{2}, 0) \approx (-1.41, 0)$.



Apply the Skills

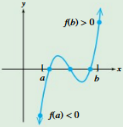
4. Determine the zeros and their multiplicities for the given functions.

a. $p(x) = -\frac{3}{5}(x+3)^4(5x-1)^5$

b. $q(x) = 2x^6 - 14x^4$

Insights

For a polynomial function *f*, if *f*(*a*) and *f*(*b*) have opposite signs, then *f* must have at least one zero on the interval [*a*, *b*]. This includes the possibility that *f* may have more than one zero on [*a*, *b*].



Learn: Apply the Intermediate Value Theorem

In Examples 2-4, the zeros of the functions were easily identified by first factoring the polynomial. However, in most cases, the real zeros of a polynomial are difficult or impossible to determine algebraically. For example, the function given by $f(x) = x^4 + 6x^3 - 26x + 15$ has zeros of $-1 \pm \sqrt{6}$ and $-2 \pm \sqrt{7}$. At this point, we do not have the tools to find the zeros of this function analytically. However, we can use the intermediate value theorem to help us search for zeros of a polynomial function and approximate their values.

Intermediate Value Theorem

Let f be a polynomial function. For $a < b$, if $f(a)$ and $f(b)$ have opposite signs, then f has at least one zero on the interval $[a, b]$.

Insights

It is important to note that if the signs of $f(a)$ and $f(b)$ are the same, then the intermediate value theorem is inconclusive.

Example 5: Applying the Intermediate Value Theorem

Show that $f(x) = x^4 + 6x^3 - 26x + 15$ has a zero on the interval [1, 2].

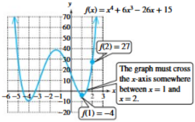
Solution:

$f(x) = x^4 + 6x^3 - 26x + 15$

$f(1) = (1)^4 + 6(1)^3 - 26(1) + 15 = -4$

$f(2) = (2)^4 + 6(2)^3 - 26(2) + 15 = 27$

Since $f(1)$ and $f(2)$ have opposite signs, then by the intermediate value theorem, we know that the function must have at least one zero on the interval [1, 2].



The actual value of the zero on the interval [1, 2] is $-1 + \sqrt{6} \approx 1.45$.

Apply the Skills

5. Show that $f(x) = x^4 + 6x^3 - 26x + 15$ has a zero on the interval [−4, −3].

The intermediate value theorem can be used repeatedly in a technique called the bisection method to approximate the value of a zero. See the online group activity "Investigating the Bisection Method for Finding Zeros" and Exercise 112 in the Practice Exercises for this lesson.

Learn: Sketch a Polynomial Function

The graph of a polynomial function may also have “turning points.” These correspond to relative maxima and minima. For example, consider $f(x) = x(x + 2)(x - 2)^2$. See Figure 2-12.

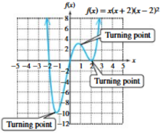


Figure 2-12

Multiplying the leading terms within the factors, we have a leading term of $(x)(x)(x)^2 = x^4$. Therefore, the end behavior of the graph is up to the left and up to the right.

Starting from the far left, the graph of f decreases to the x-intercept of -2. Since -2 is a zero with an odd multiplicity, the graph must cross the x-axis at -2. For the same reason, the graph must cross the x-axis again at the origin. Therefore, somewhere between $x = -2$ and $x = 0$, the graph must “turn around.” This point is called a “turning point.”

The turning points of a polynomial function are the points where the function changes from increasing to decreasing or vice versa.

Insights

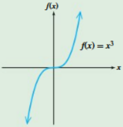
Even with advanced techniques from calculus or the use of a graphing utility, it is often difficult or impossible to find the exact location of the turning points of a polynomial function.

Number of Turning Points of a Polynomial Function

Let f represent a polynomial function of degree n. Then the graph of f has at most $n - 1$ turning points.

Insights

A polynomial of degree n may have fewer than $n - 1$ turning points. For example, $f(x) = x^3$ is a degree 3 polynomial function (indicating that it could have a maximum of two turning points), yet the graph has no turning points.



At this point we are ready to outline a strategy for sketching a polynomial function.

Graphing a Polynomial Function

To graph a polynomial function defined by $y = f(x)$,

1. Use the leading term to determine the end behavior of the graph.
2. Determine the y-intercept by evaluating $f(0)$.
3. Determine the real zeros of f and their multiplicities (these are the x-intercepts of the graph of f).
4. Plot the x- and y-intercepts and sketch the end behavior.
5. Draw a sketch starting from the left-end behavior. Connect the x- and y-intercepts in the order that they appear from left to right using these rules:
    - The curve will cross the x-axis at an x-intercept if the corresponding zero has an odd multiplicity.
    - The curve will touch but not cross the x-axis at an x-intercept if the corresponding zero has an even multiplicity.
6. If a test for symmetry is easy to apply, use symmetry to plot additional points. Recall that
    - f is an even function (symmetric to the y-axis) if $f(-x) = f(x)$.
    - f is an odd function (symmetric to the origin) if $f(-x) = -f(x)$.
7. Plot more points if a greater level of accuracy is desired. In particular, to estimate the location of turning points, find several points between two consecutive x-intercepts.

In Examples 6 and 7, we demonstrate the process of graphing a polynomial function.

Example 6: Graphing a Polynomial Function

Graph $f(x) = x^3 - 9x$.

Solution:

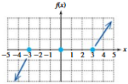
$f(x) = x^3 - 9x$

1. The leading term is x^3. The end behavior is down to the left and up to the right.	The exponent on the leading term is odd and the leading coefficient is positive.
2. $f(0) = (0)^3 - 9(0) = 0$ The y-intercept is $(0, 0)$.	Determine the y-intercept by evaluating $f(0)$.
3. $0 = x^3 - 9x$ $0 = x(x^2 - 9)$ $0 = x(x - 3)(x + 3)$	Find the real zeros of f by solving for the real solutions to the equation $f(x) = 0$.

The zeros of the function are 0, 3, and −3, and each has a multiplicity of 1.

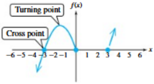
The zeros are real numbers and correspond to *x*-intercepts on the graph. Since the multiplicity of each zero is an odd number, the graph will cross the *x*-axis at the zeros.

4.



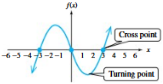
Plot the *x*- and *y*-intercepts and sketch the end behavior.

5.



Draw a sketch starting from the left-end behavior.

Moving from left to right, the curve increases from the far left and then crosses the *x*-axis at −3. The graph must have a turning point between $x = -3$ and $x = 0$ so that the curve can pass through the next *x*-intercept of (0, 0).



The graph crosses the *x*-axis at $x = 0$. The graph must then have another turning point between $x = 0$ and $x = 3$ so that the curve can pass through the next *x*-intercept of (3, 0). Finally, the graph crosses the *x*-axis at $x = 3$ and continues to increase to the far right.

6. $f(x) = x^3 - 9x$

$$f(-x) = (-x)^3 - 9(-x) \qquad -f(x) = -(x^3 - 9x)$$

$$= -x^3 + 9x \underset{\substack{f(-x) = -f(x)\\ \text{(same)}}}{\longleftrightarrow} = -x^3 + 9x$$

Testing for symmetry, we see that $f(-x) = -f(x)$. Therefore, *f* is an odd function and is symmetric with respect to the origin.

7. If more accuracy is desired, plot additional points. In this case, since *f* is symmetric to the origin, if a point (x, y) is on the graph, then so is $(-x, -y)$. The graph of *f* is shown in Figure 2-13.

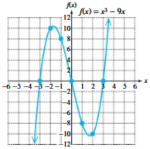


Figure 2-13

x	f(x)
1	−8
2	−10
4	28

Use symmetry. →

x	f(x)
−1	8
−2	10
−4	−28

Apply the Skills

6. Graph $g(x) = -x^3 + 4x$.

Techniques of calculus can be used to find the exact coordinates of the turning points of the polynomial function in Example 6.

Example 7: Graphing a Polynomial Function

Graph $g(x) = -0.1(x - 1)(x + 2)(x - 4)^2$.

Solution:

$g(x) = -0.1(x - 1)(x + 2)(x - 4)^2$

1. Multiplying the leading terms within the factors, we have a leading term of $-0.1(x)(x)(x)^2 = -0.1x^4$.

The end behavior is down to the left and down to the right.

The exponent on the leading term is even and the leading coefficient is negative.

2. $g(0) = -0.1(0 - 1)(0 + 2)(0 - 4)^2 = 3.2$
The y-intercept is $(0, 3.2)$.

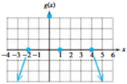
Determine the y-intercept by evaluating $g(0)$.

3. $0 = -0.1(x - 1)(x + 2)(x - 4)^2$
The zeros of the function are 1, −2, and 4.
The multiplicity of 1 is 1.
The multiplicity of −2 is 1.
The multiplicity of 4 is 2.

Find the real zeros of g by solving for the real solutions of the equation $g(x) = 0$.

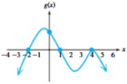
The zeros are real numbers and correspond to x-intercepts on the graph: $(1, 0)$, $(-2, 0)$, and $(4, 0)$.

4.



Plot the x- and y-intercepts and sketch the end behavior.

5.



Draw a sketch starting from the left-end behavior.

Moving from left to right, the curve increases from the far left. It then crosses the x-axis at $x = -2$ and turns back around to pass through the next x-intercept at $x = 1$.

The curve has another turning point between $x = 1$ and $x = 4$ so that it can touch the x-axis at 4. From there it turns back downward and continues to decrease to the far right.

6. From our preliminary sketch in step 5, we see that the function is not symmetric with respect to either the *y*-axis or origin.

7. If more accuracy is desired, plot additional points. The graph is shown in Figure 2-14.

x	$g(x)$
−3	−19.6
−1	5
2	−1.6
3	−1
5	−2.8

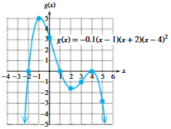


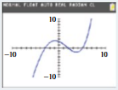
Figure 2-14

Apply the Skills

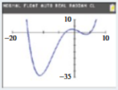
7. Graph $h(x) = 0.5x(x-1)(x+3)^2$.

TECHNOLOGY CONNECTIONS

Using a Graphing Utility to Graph a Polynomial Function



It is important to have a strong knowledge of algebra to use a graphing utility effectively. For example, consider the graph of $f(x) = 0.005(x-2)(x+3)(x-5)(x+15)$ on the standard viewing window.



From the leading term, $0.005x^4$, we know that the end behavior should be up to the left and up to the right. Furthermore, the function has four real zeros (2, −3, 5, and −15), and should have four corresponding *x*-intercepts. Therefore, on the standard viewing window, the calculator does not show the key features of the graph.

By graphing *f* on the window [−20, 10, 2] by [−35, 10, 5], we see the end behavior displayed correctly, all four *x*-intercepts, and the turning points (there should be at most 3).

Practice 2-2

Practice Exercises

Prerequisite Review

For Exercises R.I–R.2, solve the equation.

R.1. $3x^3 + 21x^2 - 54x = 0$

R.2. $5x^3 + 6x^2 - 20x - 24 = 0$

For Exercises R.3–R.5, use transformations to graph the given function.

R.3. $m(x) = x^3 - 5$

R.4. $f(x) = (x + 2)^2 - 4$

R.5. $g(x) = (3x - 6)^2$

Concept Connections

1. A function defined by $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ where $a_n, a_{n-1}, a_{n-2}, \cdots, a_1, a_0$ are real numbers and $a_n \neq 0$ is called a ______________ function.

2. The function given by $f(x) = -3x^5 + \sqrt{2}x + \frac{1}{2}x$ (is/is not) a polynomial function.

3. The function given by $f(x) = -3x^5 + 2\sqrt{x} + \frac{2}{x}$ (is/is not) a polynomial function.

4. A quadratic function is a polynomial function of degree ______________.

5. A linear function is a polynomial function of degree ______________.

6. The values of x in the domain of a polynomial function f for which $f(x) = 0$ are called the ______________ of the function.

7. What is the maximum number of turning points of the graph of $f(x) = -3x^6 - 4x^5 - 5x^4 + 2x^2 + 6$?

8. If the graph of a polynomial function has 3 turning points, what is the minimum degree of the function?

9. If c is a real zero of a polynomial function and the multiplicity is 3, does the graph of the function cross the x-axis or touch the x-axis (without crossing) at $(c, 0)$?

10. If c is a real zero of a polynomial function and the multiplicity is 6, does the graph of the function cross the x-axis or touch the x-axis (without crossing) at $(c, 0)$?

11. Suppose that f is a polynomial function and that $a < b$. If $f(a)$ and $f(b)$ have opposite signs, then what conclusion can be drawn from the intermediate value theorem?

12. What is the leading term of $f(x) = -\frac{1}{3}(x - 3)^4(3x + 5)^2$?

Learn: Determine the End Behavior of a Polynomial Function

For Exercises 13–20, determine the end behavior of the graph of the function. (See Example 1 on p. 188)

13. $f(x) = -3x^4 - 5x^2 + 2x - 6$

14. $g(x) = -\frac{1}{2}x^6 + 8x^4 - x^3 + 9$

15. $h(x) = 12x^5 + 8x^4 - 4x^3 - 8x + 1$

16. $k(x) = 11x^7 - 4x^2 + 9x + 3$

17. $m(x) = -4(x - 2)(2x + 1)^2(x + 6)^4$

18. $n(x) = -2(x + 4)(3x - 1)^3(x + 5)$

19. $p(x) = -2x^2(3 - x)(2x - 3)^3$

20. $q(x) = -5x^4(2 - x)^3(2x + 5)$

Learn: Identify Zeros and Multiplicities of Zeros

21. Given the function defined by $g(x) = -3(x - 1)^3(x + 5)^4$, the value 1 is a zero with multiplicity ______________, and the value -5 is a zero with multiplicity ______________.

22. Given the function defined by $h(x) = \frac{1}{2}x^5(x + 0.6)^3$, the value 0 is a zero with multiplicity ______________, and the value -0.6 is a zero with multiplicity ______________.

For Exercises 23–38, find the zeros of the function and state the multiplicities. (See Examples 2–4 on pp. 189–192)

23. $f(x) = x^3 + 2x^2 - 25x - 50$

24. $g(x) = x^3 + 5x^2 - x - 5$

25. $h(x) = -6x^3 - 9x^2 + 60x$

26. $k(x) = -6x^3 + 26x^2 - 28x$

27. $m(x) = x^5 - 10x^4 + 25x^3$

28. $n(x) = x^6 + 4x^5 + 4x^4$

29. $p(x) = -3x(x + 2)^3(x + 4)$

30. $q(x) = -2x^4(x + 1)^3(x - 2)^2$

31. $t(x) = 5x(3x - 5)(2x + 9)(x - \sqrt{3})(x + \sqrt{3})$.

32. $z(x) = 4x(5x - 1)(3x + 8)(x - \sqrt{5})(x + \sqrt{5})$

33. $c(x) = [x - (3 - \sqrt{5})]\,[x - (3 + \sqrt{5})]$

34. $d(x) = [x - (2 - \sqrt{11})]\,[x - (2 + \sqrt{11})]$

35. $f(x) = 4x^4 - 37x^2 + 9$

36. $k(x) = 4x^4 - 65x^2 + 16$

37. $n(x) = x^6 - 7x^4$

38. $m(x) = x^5 - 5x^3$

Learn: Apply the Intermediate Value Theorem

For Exercises 39–40, determine whether the intermediate value theorem guarantees that the function has a zero on the given interval. (See Example 5 on p. 193)

39. $f(x) = 2x^3 - 7x^2 - 14x + 30$

- **a.** $[1, 2]$
- **b.** $[2, 3]$
- **c.** $[3, 4]$
- **d.** $[4, 5]$

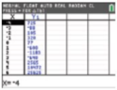
40. $g(x) = 2x^3 - 13x^2 + 18x + 5$

- **a.** $[1, 2]$
- **b.** $[2, 3]$
- **c.** $[3, 4]$
- **d.** $[4, 5]$

For Exercises 41–42, a table of values is given for $Y_1 = f(x)$. Determine whether the intermediate value theorem guarantees that the function has a zero on the given interval.

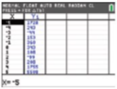
41. $Y_1 = 21x^4 + 46x^3 - 238x^2 - 506x + 77$

- **a.** $[-4, -3]$
- **b.** $[-3, -2]$
- **c.** $[-2, -1]$
- **d.** $[-1, 0]$



42. $Y_1 = 10x^4 + 21x^3 - 119x^2 - 147x + 343$

- **a.** $[-4, -3]$
- **b.** $[-3, -2]$
- **c.** $[-2, -1]$
- **d.** $[-1, 0]$



43. Given $f(x) = 4x^3 - 8x^2 - 25x + 50$,

- **a.** Determine if f has a zero on the interval $[-3, -2]$.
- **b.** Find a zero of fon the interval $[-3, -2]$.

44. Given $f(x) = 9x^3 - 18x^2 - 100x + 200$,

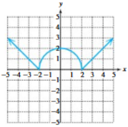
- **a.** Determine if f has a zero on the interval $[-4, -3]$.
- **b.** Find a zero of f on the interval $[-4, -3]$.

Learn: Sketch a Polynomial Function

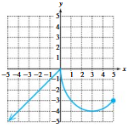
For Exercises 45–52, determine if the graph can represent a polynomial function. If so, assume that the end behavior and all turning points are represented in the graph.

- **a.** Determine the minimum degree of the polynomial.
- **b.** Determine whether the leading coefficient is positive or negative based on the end behavior and whether the degree of the polynomial is odd or even.
- **c.** Approximate the real zeros of the function, and determine if their multiplicities are even or odd

45.



46.



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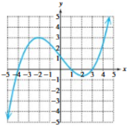

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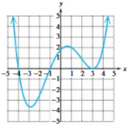

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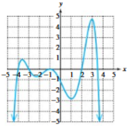

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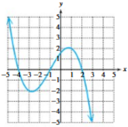

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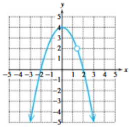

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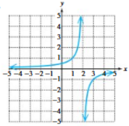


For Exercises 53–58,

a. Identify the power function of the form $y = x^n$ that is the parent function to the given graph.

b. In order, outline the transformations that would be required on the graph of $y = x^n$ to make the graph of the given function. See Lesson 1-6, page 102.

c. Match the function with the graph of i–vi.

53. $g(x) = -\frac{1}{3}x^6 - 2$

54. $f(x) = -\frac{1}{2}(x - 3)^4$

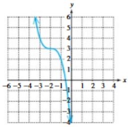
55. $k(x) = -(x + 2)^3 + 3$

56. $p(x) = 2(x + 4)^3 - 3$

57. $m(x) = (-x - 3)^5 + 1$

58. $n(x) = (-x + 3)^4 - 1$

i.



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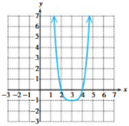

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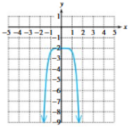

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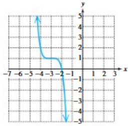

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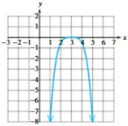
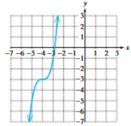


vi.



For Exercises 59–76, sketch the function. (See Examples 6–7 on pp. 195, 197)

59. $f(x) = x^3 - 5x^2$

60. $g(x) = x^5 - 2x^4$

61. $f(x) = \frac{1}{2}(x - 2)(x + 1)(x + 3)$

62. $h(x) = \frac{1}{4}(x - 1)(x - 4)(x + 2)$

63. $k(x) = x^4 + 2x^3 - 8x^2$

64. $h(x) = x^4 - x^3 - 6x^2$

65. $k(x) = 0.2(x + 2)^2(x - 4)^3$

66. $m(x) = 0.1(x - 3)^2(x + 1)^3$

67. $p(x) = 9x^5 + 9x^4 - 25x^3 - 25x^2$

68. $q(x) = 9x^5 + 18x^4 - 4x^3 - 8x^2$

69. $t(x) = -x^4 + 11x^2 - 28$

70. $v(x) = -x^4 + 15x^2 - 44$

71. $g(x) = -x^4 + 5x^2 - 4$

72. $h(x) = -x^4 + 10x^2 - 9$

73. $c(x) = 0.1x(x - 2)^4(x + 2)^3$

74. $d(x) = 0.05x(x - 2)^4(x + 3)^2$

75. $m(x) = -\frac{1}{10}(x + 3)(x - 3)(x + 1)^3$

76. $f(x) = \frac{1}{10}(x - 1)(x + 3)(x - 4)^2$

Mixed Exercises

For Exercises 77–88, determine if the statement is true or false. If a statement is false, explain why.

77. The function defined by $f(x) = (x + 1)^5(x - 5)^2$ crosses the x-axis at 5.

78. The function defined by $g(x) = -3(x + 4)(2x - 3)^4$ touches but does not cross the x-axis at $\left(\frac{3}{2}, 0\right)$.

79. A third-degree polynomial has three turning points.

80. A third-degree polynomial has two turning points.

81. There is more than one polynomial function with zeros of 1, 2, and 6.

82. There is exactly one polynomial with integer coefficients with zeros of 2, 4, and 6.

83. The graph of a polynomial function with leading term of even degree is up to the far left and up to the far right.

84. If c is a real zero of an even polynomial function, then $-c$ is also a zero of the function.

85. The graph of $f(x) = x^3 - 27$ has three x-intercepts.

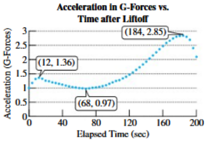
86. The graph of $f(x) = 3x^2(x - 4)^4$ has no points in Quadrants III or IV.

87. The graph of $p(x) = -5x^4(x + 1)^2$ has no points in Quadrants I or II.

88. A fourth-degree polynomial has exactly two relative minima and two relative maxima.

89. A rocket will carry a communications satellite into low Earth orbit. Suppose that the thrust during the first 200 sec of flight is provided by solid rocket boosters at different points during liftoff.

The graph shows the acceleration in G-forces (that is, acceleration in 9.8-m/sec^2 increments) versus time after launch.



a. Approximate the interval(s) over which the acceleration is increasing.

b. Approximate the interval(s) over which the acceleration is decreasing

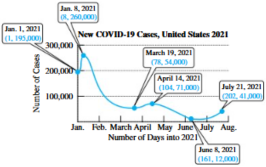
c. How many turning points does the graph show?

d. Based on the number of turning points, what is the minimum degree of a polynomial function that could be used to model acceleration versus time? Would the leading coefficient be positive or negative?

e. Approximate the time when the acceleration was the greatest.

f. Approximate the value of the maximum acceleration.

90. The graph shows the 7-day average number of new COVID-19 cases for the United States from January 1, 2021, to July 21, 2021. The day-to-day fluctuations in the number of cases has been smoothed in the graph so that the prominent features (relative maxima and minima) stand out. The horizontal axis gives the number of days into the year 2021.



a. Use the given points to find the interval(s) over which the 7-day average number of new COVID-19 cases increased.

b. Use the given points to find the interval(s) over which the 7-day average number of new COVID-19 cases decreased.

c. How many turning points does the graph show?

d. Based on the number of turning points, what is the minimum degree of a polynomial function that could be used to model the data? Would the leading coefficient be positive or negative?

e. For the time period represented in this graph, when was the 7-day average number of new COVID-19 cases the greatest?

f. What was the maximum 7-day average number of new cases during this period?

For Exercises 91–92, match each polynomial function with its graph.

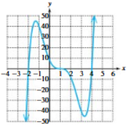
91. a. $f(x) = (x + 2)^2(x - 1)(x - 4)$

b. $g(x) = (x + 2)(x - 1)(x - 4)$

c. $k(x) = (x + 2)(x - 1)^3(x - 4)$

d. $z(x) = (x + 2)(x - 1)(x - 4)^2$

i.



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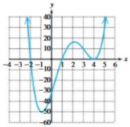

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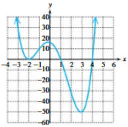

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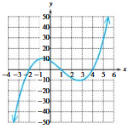


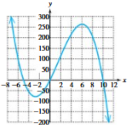
92. a. $h(x) = x^2(x + 10)(x - 5)$

b. $p(x) = -x^2(x + 10)(x - 5)$

c. $q(x) = -x^2(x - 10)(x + 5)$

d. $t(x) = -x(x - 10)(x + 5)$

i.



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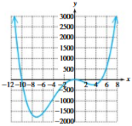

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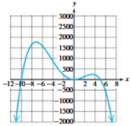

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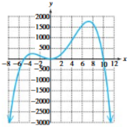


For Exercises 93–96, sketch the graph of a polynomial function satisfying the given conditions.

93. Crosses the x-axis at $(-2, 0)$ and $(5, 0)$

Touches the x-axis at $(1, 0)$

Local minimum points at $(-1.1, -3)$ and $(3.9, -6.8)$

Local maximum point at $(1, 0)$

Increasing on the intervals $(-1.1, 1)$ and $(3.9, \infty)$

Decreasing on the intervals $(-\infty, -1.1)$ and $(1, 3.9)$

94. Crosses the x-axis at $(-3, 0)$, $(1, 0)$, $(3, 0)$, and $(5, 0)$

Local minimum point at $(1.9, -0.8)$

Local maximum points at $(-1.7, 5.5)$ and $(4.2, 1.1)$

y-intercept at $(0, 2.25)$

Increasing on the intervals $(-\infty, -1.7)$ and $(1.9, 4.2)$

Decreasing on the intervals $(-1.7, 1.9)$ and $(4.2, \infty)$

95. Crosses the x-axis at $(3, 0)$

Touches the x-axis at $(-1, 0)$

Local minimum point at $(1.7, -8.5)$

Local maximum point at $(-1, 0)$

y-intercept at $(0, -2.7)$

Increasing on the intervals $(-\infty, -1)$ and $(1.7, \infty)$

Decreasing on the interval $(-1, 1.7)$

96. Crosses the x-axis at $(3, 0)$, $(8, 0)$, and $(-6, 0)$

Local minimum point at $(-2.4, -6.7)$

Local maximum point at $(5.8, 2.4)$

y-intercept at $(0, -4.8)$

Increasing on the interval $(-2.4, 5.8)$

Decreasing on the intervals $(-\infty, -2.4)$ and $(5.8, \infty)$

Write About It

97. Given a polynomial function defined by $y = f(x)$, explain how to find the x-intercepts.

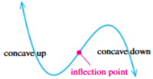
98. Given a polynomial function, explain how to determine whether an x-intercept is a touch point or a cross point.

99. Write an informal explanation of what it means for a function to be continuous.

100. Write an informal explanation of the intermediate value theorem.

Expanding Your Skills

Recall from Chapter 1 that a graph is *concave up* on a given interval if it "bends" upward. A graph is *concave down* on a given interval if it "bends" downward. The point at which a graph changes concavity is called an *inflection point.*



For Exercises 101–102, sketch the graph of a polynomial function satisfying the given conditions.

101. Crosses the x-axis at $x = 0$ and $x = -6$

Lowest point on the graph is $(-4.5, -11.4)$.

No maximum point Inflection points at $(0, 0)$ and $(-3, -6.75)$

Concave up on the intervals $(-\infty, -3)$ and $(0, \infty)$

Concave down on the interval $(-3, 0)$

102. Touches the x-axis at $x = -3$ and $x = 4$

Local minimum point at $(0.5, -7.5)$

Highest points on the graph are $(-3, 0)$ and $(4, 0)$.

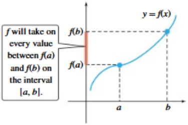
Inflection points at $(-1.5, -3.3)$ and $(2.5, -3.3)$

Concave up on the interval $(-1.5, 2.5)$

Concave down on the intervals $(-\infty, -1.5)$ and $(2.5, \infty)$

The intermediate value theorem given on page 193 is actually a special case of a broader statement of the theorem. Consider the following:

Let *f* be a polynomial function. For $a < b$, if $f(a) \neq f(b)$, then *f* takes on every value between $f(a)$ and $f(b)$ on the interval $[a, b]$.


y = f(x)
f will take on every value between f(a) and f(b) on the interval [a, b].
f(b)
f(a)
a
b

Use this broader statement of the intermediate value theorem for Exercises 103–104.

103. Given $f(x) = x^2 - 3x + 2$,

- **a.** Evaluate $f(3)$ and $f(4)$.
- **b.** Use the intermediate value theorem to show that there exists at least one value of x for which $f(x) = 4$ on the interval $[3, 4]$.
- **c.** Find the value(s) of x for which $f(x) = 4$ on the interval $[3, 4]$.

104. Given $f(x) = -x^2 - 4x + 3$,

- **a.** Evaluate $f(-4)$ and $f(-3)$.
- **b.** Use the intermediate value theorem to show that there exists at least one value of x for which $f(x) = 5$ on the interval $[-4, -3]$.
- **c.** Find the value(s) of x for which $f(x) = 5$ on the interval $[-4, -3]$.

Technology Connections

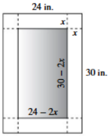
105. For a certain individual, the volume (in liters) of air in the lungs during a 4.5-sec respiratory cycle is shown in the table for 0.5-sec intervals. Graph the points and then find a third-degree polynomial function to model the volume $V(t)$ for t between 0 sec and 4.5 sec. (*Hint*: Use a CubicReg option or polynomial degree 3 option on a graphing utility.)

Time (sec)	Volume (L)
0.0	0.00
0.5	0.11
1.0	0.29
1.5	0.47
2.0	0.63
2.5	0.76
3.0	0.81
3.5	0.75
4.0	0.56
4.5	0.20

106. The torque (in ft-lb) produced by a certain automobile engine turning at x thousand revolutions per minute is shown in the table. Graph the points and then find a third-degree polynomial function to model the torque $T(x)$ for $1 \leq x \leq 5$.

Engine speed (1000 rpm)	Torque (ft-lb)
1.0	165
1.5	180
2.0	188
2.5	190
3.0	186
3.5	176
4.0	161
4.5	142
5.0	120

107. A solar oven is to be made from an open box with reflective sides. Each box is made from a 30-in. by 24-in. rectangular sheet of aluminum with squares of length x (in inches) removed from each corner. Then the flaps are folded up to form an open box.



a. Show that the volume of the box is given by

$V(x) = 4x^3 - 108x^2 + 720x \quad \text{for} \quad 0 < x < 12.$

b. Graph the function from part (a) and use a "Maximum" feature on a graphing utility to approximate the length of the sides of the squares that should be removed to maximize the volume. Round to the nearest tenth of an inch.

c. Approximate the maximum volume. Round to the nearest cubic inch.

For Exercises 108–109, two viewing windows are given for the graph of $y = f(x)$. Choose the window that best shows the key features of the graph.

108. $f(x) = 2(x - 0.5)(x - 0.1)(x + 0.2)$

a. $[-10, 10, 1]$ by $[-10, 10, 1]$

b. $[-1, 1, 0.1]$ by $[-0.05, 0.05, 0.01]$

109. $g(x) = 0.08(x - 16)(x + 2)(x - 3)$

a. $[-10, 10, 1]$ by $[-10, 10, 1]$

b. $[-5, 20, 5]$ by $[-50, 30, 10]$

For Exercises 110–111, graph the function defined by $y = f(x)$ on an appropriate viewing window.

110. $k(x) = \frac{1}{100}(x - 20)(x + 1)(x + 8)(x - 6)$

111. $p(x) = (x - 0.4)(x + 0.5)(x + 0.1)(x - 0.8)$

Point of Interest

The modern definition of a computer is a programmable device designed to carry out a sequence of arithmetic or logical operations. However, the word "computer" originally referred to a person who did such calculations using paper and pencil. "Human computers" were notably used in the eighteenth century to predict the path of Halley's comet and to produce astronomical tables critical to surveying and navigation. Later, during World Wars I and II, human computers developed ballistic firing tables that would describe the trajectory of a shell.

Computing tables of values was very time consuming, and the "computers" would often interpolate to find intermediate values within a table. Interpolation is a method by which intermediate values between two numbers are estimated. Often the interpolated values were based on a polynomial function.

In many applications of mathematics, using analytical methods to find the exact zeros of a polynomial is either not possible, or too tedious to employ. Instead, mathematicians often use approximation techniques. One method, called the "bisection method," is used to approximate the zeros of a polynomial based on the intermediate value theorem. In Exercise 112, we outline the process to find a zero of the polynomial $p(x) = x^3 - 4x^2 - 2x - 3$. Use a calculator or computer to work through this exercise, but take a moment to think of the days before such devices were available. The calculations would have to have been done by hand by human computers.

112. a. Consider the interval [4, 5]. Evaluate $p(4)$ and $p(5)$. Round to four decimal places if necessary.

b. What do you notice about the signs of $p(4)$ and $p(5)$? Does $p(x)$ have a real zero on the interval [4, 5]?

c. From part (b), we know that $p(x)$ has at least one zero somewhere between 4 and 5. As such, we can take the midpoint 4.5 as an *approximation* of a zero on that interval. To get a better approximation, we can repeat this process using a smaller interval. To find a smaller interval, first evaluate $p[4.5]$. Round to four decimal places if necessary.

d. Based on the signs of $p(4)$, $p(5)$, and $p(4.5)$, which subinterval, [4, 4.5] or [4.5, 5], is guaranteed to have a zero of $p(x)$? What is the midpoint of this interval? So, what is the new approximation of a zero on this interval?

e. Create a table as shown below. Repeat steps (a)–(d) to complete your table. In each row, the midpoint of the interval $[a, b]$ is the new approximation of a zero of $p(x)$. Repeat this process until two consecutive values of the midpoint differ by less than 0.1.

Interval $[a, b]$	$p(a)$	$p(b)$	Midpoint (approximation of zero)	p(midpoint)	New Interval
[4, 5]	−11	12	4.5	−1.875	[4.5, 5]
[4.5, 5]					

Further Reading

Kierman, Denise. *The Girls of Atomic City: The Untold Story of the Women Who Helped Win WWII*, Atria Books, 2014.

Go online for more practice problems.

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Lesson 2-3

Division of Polynomials and the Remainder and Factor Theorems

What Will You Learn?

After completing this lesson, you should be able to:

- Divide Polynomials Using Long Division
- Divide Polynomials Using Synthetic Division
- Apply the Remainder and Factor Theorems

Learn: Divide Polynomials Using Long Division

In this lesson, we use the notation $f(x)$, $g(x)$, and so on to represent polynomials in x. We also present two types of polynomial division: long division and synthetic division. Polynomial division can be used to factor a polynomial, solve a polynomial equation, and find the zeros of a polynomial.

When dividing polynomials, if the divisor has two or more terms we can use a long division process similar to the division of real numbers. This process is demonstrated in Examples 1–3.

Insights

Take a minute to review long division of whole numbers: $2273 \div 5$

$$
\begin{array}{rl}
454 & \leftarrow \text{Quotient} \\
5\overline{)2273} & \\
\underline{-20}\downarrow & \\
27 & \\
\underline{-25}\downarrow & \\
23 & \\
\underline{-20} & \\
3 & \leftarrow \text{Remainder}
\end{array}
$$

Answer: $454 + \frac{3}{5}$ or $454\frac{3}{5}$

Example 1: Dividing Polynomials Using Long Division

Use long division to divide. $(6x^3 - 5x^2 - 3) \div (3x + 2)$

Solution:

First note that the dividend can be written as $6x^3 - 5x^2 + 0x - 3$. The term $0x$ is used as a placeholder for the missing power of x. The placeholder is helpful to keep the powers of x lined up. We also set up long division with both the dividend and divisor written in descending order.

$3x + 2\overline{)6x^3 - 5x^2 + 0x - 3}$ Divide the leading term in the dividend by the leading term in the divisor.

$$\begin{array}{r} 2x^2 \\ 3x+2\overline{)6x^3-5x^2+0x-3} \\ \underline{-(6x^3+4x^2)} \end{array}$$

Subtract.

$\boxed{\frac{6x^3}{3x}=2x^2}$ This is the first term in the quotient.

Multiply the divisor by $2x^2$:
$2x^2(3x+2)=6x^3+4x^2$, and subtract the result.

$$\begin{array}{r} 2x^2 \\ 3x+2\overline{)6x^3-5x^2+0x-3} \\ \underline{-(6x^3+4x^2)} \\ -9x^2+0x \end{array}$$

Bring down the next term from the dividend and repeat the process.

$$\begin{array}{r} 2x^2-3x \\ 3x+2\overline{)6x^3-5x^2+0x-3} \\ \underline{-(6x^3+4x^2)} \\ -9x^2+0x \\ \underline{-(-9x^2-6x)} \end{array}$$

Subtract.

Divide $-9x^2$ by the first term in the divisor.

$\boxed{\frac{-9x^2}{3x}=-3x}$

Multiply the divisor by $-3x$:
$-3x(3x+2)=-9x^2-6x$,
and subtract the result.

$$\begin{array}{r} 2x^2-3x+2 \\ 3x+2\overline{)6x^3-5x^2+0x-3} \\ \underline{-(6x^3+4x^2)} \\ -9x^2+0x \\ \underline{-(-9x^2-6x)} \\ 6x-3 \\ \underline{-(6x+4)} \\ -7 \end{array}$$

Subtract.

The remainder is -7.

Bring down the next term from the dividend and repeat the process.

Divide $6x$ by the first term in the divisor.
$\frac{6x}{3x}=2$. This is the next term in the quotient.

Multiply the divisor by 2: $2(3x+2)=6x+4$,
and subtract the result.

Long division is complete when the remainder is either zero or has degree less than the degree of the divisor.

The quotient is $2x^2-3x+2$.

The remainder is -7.

The divisor is $3x+2$.

The dividend is $6x^3-5x^2-3$.

The result of a long division problem is usually written as the quotient plus the remainder divided by the divisor.

$$\underbrace{\frac{\overset{\text{Dividend}}{6x^3-5x^2-3}}{\underset{\text{Divisor}}{3x+2}}}=\overbrace{2x^2+3x+3}^{\text{Quotient}}+\frac{\overset{\text{Remainder}}{-7}}{\underset{\text{Divisor}}{3x+2}}$$

Check Your Work

By clearing fractions, the result of Example 1 can be checked by multiplication.

$$\begin{aligned} \text{Dividend} &= (\text{Divisor})(\text{Quotient})\ +\text{Remainder} \\ 6x^3-5x^2-3 &\overset{?}{=} (3x+2)(2x^2-3x+2)+(-7) \\ &\overset{?}{=} 6x^3-5x^2+4+(-7) \\ &\overset{?}{=} 6x^3-5x^2-3\ ✓ \end{aligned}$$

Apply the Skills

1. Use long division to divide: $(4x^3 - 23x + 3) \div (2x - 5)$.

The work shown for checking the result of Example 1 illustrates the division algorithm.

Division Algorithm

Suppose that $f(x)$ and $d(x)$ are polynomials where $d(x) \neq 0$ and the degree of $d(x)$ is less than or equal to the degree of $f(x)$. Then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x) \cdot q(x) + r(x)$$

where either the degree of $r(x)$ is less than $d(x)$, or $r(x)$ is the zero polynomial.

Note: The polynomial $f(x)$ is the **dividend**, $d(x)$ is the **divisor**, $q(x)$ is the **quotient**, and $r(x)$ is the **remainder**.

Example 2: Dividing Polynomials Using Long Division

Use long division to divide $(-5 + x + 4x^2 + 2x^3 + 3x^4) \div (x^2 + 2)$.

Solution:

Write the dividend and divisor in descending order and insert placeholders for missing powers of x.

$$(3x^4 + 2x^3 + 4x^2 + x - 5) \div (x^2 + 0x + 2)$$

$$\begin{array}{r} 3x^2+2x-2 \\ x^2+0x+2\overline{)3x^4+2x^3+4x^2+x-5} \\ \underline{-(3x^4+0x^3+6x^2)} \\ 2x^3-2x^2+x \\ \underline{-(2x^3+0x^2+4x)} \\ -2x^2-3x-5 \\ \underline{-(-2x^2+0x-4)} \\ -3x-1 \end{array}$$

To begin, divide the leading term in the dividend by the leading term in the divisor.

$$\frac{3x^4}{x^2} = 3x^2$$

Multiply the divisor by $3x^2$ and subtract the result.

Bring down the next term from the dividend and repeat the process.

The process is complete when the remainder is either 0 or has degree less than the degree of the divisor.

The result is $3x^2 + 2x - 2 + \frac{-3x-1}{x^2+2}$.

Check Your Work

Check by using the division algorithm.

$$\begin{aligned} 3x^4+2x^3+4x^2+x-5 &\overset{?}{=} (x^2+2)(3x^2+2x-2)+(-3x-1) \\ &\overset{?}{=} 3x^4+2x^3-2x^2+6x^2+4x-4+(-3x-1) \\ &\overset{?}{=} 3x^4+2x^3+4x^2+x-5\ \checkmark \end{aligned}$$

Apply the Skills

2. Use long division to divide: $(1-7x+5x^2-3x^3+2x^4) \div (x^2+3)$.

In Example 3, we discuss the implications of obtaining a remainder of zero when performing division of polynomials.

Example 3: Dividing Polynomials Using Long Division

Use long division to divide. $\dfrac{2x^2+3x-14}{x-2}$

Solution:

$$\begin{array}{r} 2x+7 \\ x-2\overline{)2x^2+3x-14} \\ \underline{-(2x^2-4x)} \\ 7x-14 \\ \underline{-(7x-14)} \\ 0 \end{array}$$

$$\frac{2x^2+3x-14}{x-2}=2x+7$$

To begin, divide the leading term in the dividend by the leading term in the divisor.

$\dfrac{2x^2}{x}=2x$

Multiply the divisor by $2x$ and subtract the result. Bring down the next term from the dividend and repeat the process.

The process is complete when the remainder is either 0 or has degree less than the degree of the divisor.

The remainder is zero. This implies that the divisor divides evenly into the dividend. Therefore, both the divisor and quotient are factors of the dividend. This is easily verified by the division algorithm.

$$\begin{aligned} \overset{\text{Dividend}}{2x^2+3x-14} &\overset{?}{=} \overset{\text{Divisor}}{(x-2)}\overset{\text{Quotient}}{(2x+7)}+\overset{\text{Remainder}}{0} \\ &\overset{?}{=} \underbrace{(x-2)(2x+7)}_{\text{Factored form of } 2x^2+3x-14} \end{aligned}$$

Apply the Skills

3. Use long division to divide: $(3x^2-14x+15) \div (x-3)$.

Learn: Divide Polynomials Using Synthetic Division

When dividing polynomials where the divisor is a binomial of the form $(x - c)$ and c is a constant, we can use synthetic division. Synthetic division enables us to find the quotient and remainder more quickly than long division. It uses an algorithm that manipulates the coefficients of the dividend, divisor, and quotient without the accompanying variable factors.

The division of polynomials from Example 3 is shown at the top left of page 285. The equivalent synthetic division is shown on the right. Notice that the same coefficients are used in both cases.

$$\begin{array}{r} 2x + 7 \\ x - 2\overline{)2x^2 + 3x - 14} \\ \underline{-(2x^2 - 4x)} \\ 7x - 14 \\ \underline{-(7x - 14)} \\ 0 \end{array}$$

Coefficients of dividend

$$\begin{array}{r|rrr} 2 & 2 & 3 & -14 \\ & & 4 & 14 \\ \hline & 2 & 7 & \boxed{0} \end{array}$$

Remainder

Coefficients of quotient

In Example 4, we demonstrate the process to divide polynomials by synthetic division.

Example 4: Dividing Polynomials Using Synthetic Division

Use synthetic division to divide. $(-10x^2 + 2x^3 - 5) \div (x - 4)$

Solution:

As with long division, the terms of the dividend and divisor must be written in descending order with placeholders for missing powers of x.

$$(2x^3 - 10x^2 + 0x - 5) \div (x - 4)$$

To use synthetic division, the divisor must be in the form $x - c$. In this case, $c = 4$.

Check Your Work

It is important to check that the divisor is in the form $(x - c)$ before applying synthetic division. The variable x in the divisor must be of first degree, and its coefficient must be 1.

Step 1: Write the value of c in a box.

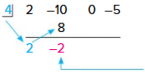
Step 2: Write the coefficients of the dividend to the right of the box.

$$\begin{array}{r|rrrr} 4 & 2 & -10 & 0 & -5 \\ & \downarrow & & & \\ \hline & 2 & & & \end{array}$$

Step 3: Skip a line and draw a horizontal line below the list of coefficients.

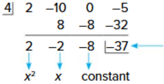
Step 4: Bring down the leading coefficient from the dividend and write it below the line.

Step 5: Multiply the value of c by the number below the line ($4 \times 2 = 8$). Write the result in the next column above the line.



Step 6: Add the numbers in the column above the line ($-10 + 8 = -2$), and write the result below the line.

Repeat steps 5 and 6 until all columns have been completed.


4
2 −10 0 −5
8 −8 −32
2 −2 −8 −37
x²
x
constant

A box is often drawn around the remainder.

The rightmost number below the line is the remainder. The other numbers below the line are the coefficients of the quotient in order by the degree of the term.

Since the divisor is linear (first degree), the degree of the quotient is 1 less than the degree of the dividend. In this case, the dividend is of degree 3. Therefore, the quotient will be of degree 2.

The quotient is $2x^2 - 2x - 8$ and the remainder is -37. Therefore,

$$\frac{2x^3 - 10x^2 - 5}{x - 4} = 2x^2 - 2x - 8 + \frac{-37}{x - 4}$$

Apply the Skills

4. Use synthetic division to divide: $(4x^3 - 28x - 7) \div (x - 3)$.

Example 5: Dividing Polynomials Using Synthetic Division

Use synthetic division to divide. $(-2x + 4x^3 + 18 + x^4) \div (x + 2)$

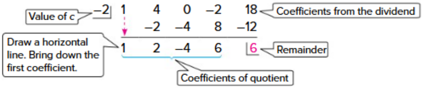
Solution:

Write the dividend and divisor in descending order and insert placeholders for missing powers of x. $(x^4 + 4x^3 + 0x^2 - 2x + 18) \div (x + 2)$

To use synthetic division, the divisor must be of the form $x - c$. In this case, we have $x + 2 = x - (-2)$. Therefore, $c = -2$.

Insights

Given a divisor of the form $(x - c)$, we can determine the value of c by setting the divisor equal to zero and solving for x. In Example 5, we have $x + 2 = 0$, which implies that $x = -2$. The value of c is -2.


Value of c
−2
1 4 0 −2 18
−2 −4 8 −12
1 2 −4 6 6
Coefficients from the dividend
Draw a horizontal line. Bring down the first coefficient.
Remainder
Coefficients of quotient

The dividend is a fourth-degree polynomial and the divisor is a first-degree polynomial. Therefore, the quotient is a third-degree polynomial. The coefficients of the quotient are found below the line: 1, 2, −4, 6. The quotient is $x^3 + 2x^2 - 4x + 6$, and the remainder is 6.

$$\frac{x^4 + 4x^3 - 2x + 18}{x + 2} = x^3 + 2x^2 - 4x + 6 + \frac{6}{x + 2}$$

Apply the Skills

5. Use synthetic division to divide: $(-3x + 7x^3 + 5 + 2x^4) \div (x + 1)$.

Learn: Apply the Remainder and Factor Theorems

Consider the special case of the division algorithm where $f(x)$ is the dividend and $(x - c)$ is the divisor.

$$f(x) = (x - c) \cdot q(x) + r$$

The remainder r is constant because its degree must be one less than the degree of $x - c$.

Now evaluate $f(c)$:

$$f(c) = (c - c) \cdot q(c) + r$$

$$f(c) = 0 \cdot q(c) + r$$

$$f(c) = r$$

This result is stated formally as the remainder theorem.

Remainder Theorem

If a polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Note: The remainder theorem tells us that the value of $f(c)$ is the same as the remainder we get from dividing $f(x)$ by $x - c$.

Example 6: Using the Remainder Theorem to Evaluate a Polynomial

Given $f(x) = x^4 + 6x^3 - 12x^2 - 30x + 35$, use the remainder theorem to evaluate

a. $f(2)$

b. $f(-7)$

Solution:

a. If $f(x)$ is divided by $x - 2$, then the remainder is $f(2)$.

$$\begin{array}{r|rrrrr} 2 & 1 & 6 & -12 & -30 & 35 \\ & & 2 & 16 & 8 & -44 \\ \hline & 1 & 8 & 4 & -22 & \boxed{-9} \end{array}$$

By the remainder theorem, $f(2) = -9$.

b. If $f(x)$ is divided by $x - (-7)$ or equivalently $x + 7$, then the remainder is $f(-7)$.

$$\begin{array}{r|rrrrr} -7 & 1 & 6 & -12 & -30 & 35 \\ & & -7 & 7 & 35 & -35 \\ \hline & 1 & -1 & -5 & 5 & \boxed{0} \end{array}$$

By the remainder theorem, $f(-7) = 0$.

Check Your Work

The results can be checked by direct substitution.

$f(2) = (2)^4 + 6(2)^3 - 12(2)^2 - 30(2) + 35 = -9$ ✓

$f(-7) = (-7)^4 + 6(-7)^3 - 12(-7)^2 - 30(-7) + 35 = 0$ ✓

Apply the Skills

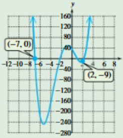
6. Given $f(x) = x^4 + x^3 - 6x^2 - 5x - 15$, use the remainder theorem to evaluate

a. $f(5)$

b. $f(-3)$

Insights

From Example 6, the values $f(2) = -9$ and $f(-7) = 0$, imply that $(2, -9)$ and $(-7, 0)$ are on the graph of $y = f(x)$.



The division algorithm and remainder theorem can be extended over the set of complex numbers. The definition of a polynomial was given in Lesson R.4.

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$ and the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ are real numbers. We now extend our discussion to **complex polynomials**. These are polynomials with complex coefficients.

Insights

Polynomials with complex coefficients include polynomials with real coefficients and with imaginary coefficients. The following are complex polynomials.

$$f(x) = (2 + 3i)x^2 + 4i$$
$$g(x) = \sqrt{2}\,x^2 + 3x + 4i$$
$$h(x) = 2x^2 + 3x + 4$$

We will also evaluate polynomials over the set of complex numbers rather than restricting x to the set of real numbers. A complex number $a + bi$ is a zero of a polynomial $f(x)$ if $f(a + bi) = 0$. For example, given $f(x) = x - (5 + 2i)$, we see that the imaginary number $5 + 2i$ is a zero of $f(x)$.

Example 7: Using the Remainder Theorem to Identify Zeros of a Polynomial

Use the remainder theorem to determine if the given number c is a zero of the polynomial.

a. $f(x) = 2x^3 - 4x^2 - 13x - 9$; $c = 4$

b. $f(x) = x^3 + x^2 - 3x - 3$; $c = \sqrt{3}$

c. $f(x) = x^3 + x + 10$; $c = 1 + 2i$

Solution:

In each case, divide $f(x)$ by $x - c$ to determine the remainder. If the remainder is 0, then the value c is a zero of the polynomial.

a. Divide $f(x) = 2x^3 - 4x^2 - 13x - 9$ by $x - 4$.

$$\begin{array}{r|rrrr} 4 & 2 & -4 & -13 & -9 \\ & & 8 & 16 & 12 \\ \hline & 2 & 4 & 3 & \boxed{3} \end{array}$$

By the remainder theorem, $f(4) = 3$. Since $f(4) \neq 0$, 4 is not a zero of $f(x)$.

b. Divide $f(x) = x^3 + x^2 - 3x - 3$ by $x - \sqrt{3}$.

$$\begin{array}{r|cccc} \sqrt{3} & 1 & 1 & -3 & -3 \\ & & \sqrt{3} & 3 + \sqrt{3} & 3 \\ \hline & 1 & 1 + \sqrt{3} & \sqrt{3} & \boxed{0} \end{array}$$

By the remainder theorem, $f(\sqrt{3}) = 0$. Therefore, $\sqrt{3}$ is a zero of $f(x)$.

c. Divide $f(x) = x^3 + x + 10$ by $x - (1 + 2i)$

$$\begin{array}{r|cccc} 1 + 2i & 1 & 0 & 1 & 10 \\ & & 1 + 2i & -3 + 4i & -10 \\ \hline & 1 & 1 + 2i & -2 + 4i & \boxed{0} \end{array}$$

Note that $(1 + 2i)(1 + 2i) = 1 + 2i + 2i + 4i^2$
$= 1 + 4i + 4(-1)$ since $i^2 = -1$.
$= -3 + 4i$

Note that $(1 + 2i)(-2 + 4i) = -2 + 4i - 4i + 8i^2$
$= -2 - 8 = -10$

By the remainder theorem, $f(1 + 2i) = 0$.

Therefore, $1 + 2i$ is a zero of $f(x)$.

Apply the Skills

7. Use the remainder theorem to determine if the given number, c, is a zero of the function.

   a. $f(x) = 2x^4 - 3x^2 + 5x - 11$; $c = 2$

   b. $f(x) = 2x^3 + 5x^2 - 14x - 35$; $c = \sqrt{7}$

   c. $f(x) = x^3 - 7x^2 + 16x - 10$; $c = 3 + i$

Suppose that we again apply the division algorithm to a dividend of $f(x)$ and a divisor of $x - c$, where c is a complex number.

$$\begin{aligned} f(x) &= (x - c) \cdot q(x) + r \\ &= (x - c) \cdot q(x) + f(c) \end{aligned}$$

By the remainder theorem, $r = f(c)$.

If $f(c) = 0$, then $f(x) = (x - c) \cdot q(x)$

This tells us that if $f(c)$ is a zero of $f(x)$, then $(x - c)$ is a factor of $f(x)$.

Now suppose that $x - c$ is a factor of $f(x)$. Then for some polynomial $q(x)$,

$$\begin{aligned} f(x) &= (x - c) \cdot q(x) \\ f(c) &= (c - c) \cdot q(x) \\ &= 0 \end{aligned}$$

This tells us that if $(x - c)$ is a factor of $f(x)$, then c is a zero of $f(x)$.

These results can be summarized in the factor theorem.

Factor Theorem

Let $f(x)$ be a polynomial.

1. If $f(c) = 0$, then $(x - c)$ is a factor of $f(x)$.

2. If $(x - c)$ is a factor of $f(x)$, then $f(c) = 0$.

Example 8: Identifying Factors of a Polynomial

Use the factor theorem to determine if the given polynomials are factors of $f(x) = x^4 - x^3 - 11x^2 + 11x + 12$.

a. $x - 3$ **b.** $x + 2$

Solution:

a. If $f(3) = 0$, then $x - 3$ is a factor of $f(x)$. Using synthetic division we have:

$$\begin{array}{r|rrrrr} 3 & 1 & -1 & -11 & 11 & 12 \\ & & 3 & 6 & -15 & -12 \\ \hline & 1 & 2 & -5 & -4 & \boxed{0} \end{array}$$

$f(3) = 0$

By the factor theorem, since $f(3) = 0$, $x - 3$ is a factor of $f(x)$.

b. If $f(-2) = 0$, then $x + 2$ is a factor of $f(x)$. Using synthetic division we have:

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -11 & 11 & 12 \\ & & -2 & 6 & 10 & -42 \\ \hline & 1 & -3 & -5 & 21 & \boxed{-30} \end{array}$$

$f(-2) = -30$

By the factor theorem, since $f(-2) \neq 0$, $x + 2$ is not a factor of $f(x)$.

Apply the Skills

8. Use the factor theorem to determine if the given polynomials are factors of $f(x) = 2x^4 - 13x^3 + 10x^2 - 25x + 6$.

a. $x - 6$ **b.** $x + 3$

In Example 9, we illustrate the relationship between the zeros of a polynomial and the solutions (roots) of a polynomial equation.

Example 9: Factoring a Polynomial Given a Known Zero

a. Factor $f(x) = 3x^3 + 25x^2 + 42x - 40$, given that -5 is a zero of $f(x)$.

b. Solve the equation. $3x^3 + 25x^2 + 42x - 40 = 0$

Solution:

a. The value -5 is a zero of $f(x)$, which means that $f(-5) = 0$. By the factor theorem, $x - (-5)$, or equivalently, $x + 5$ is a factor of $f(x)$. Using synthetic division, we have

$$\begin{array}{r|rrrr} -5 & 3 & 25 & 42 & -40 \\ & & -15 & -50 & 40 \\ \hline & 3 & 10 & -8 & \lfloor 0 \end{array}$$

This means that $3x^3 + 25x^2 + 42x - 40 = \overset{\text{divisor}}{(x+5)}\underbrace{\overset{\text{quotient}}{(3x^2 + 10x - 8)}}_{\text{factors as } (3x-2)(x+4)} + \overset{\text{remainder}}{0}$

Therefore, $f(x) = (x + 5)(3x - 2)(x + 4)$.

b. $3x^3 + 25x^2 + 42x - 40 = 0$ To solve the equation, set one side equal to zero.

$(x + 5)(3x - 2)(x + 4) = 0$ Factor the left side.

$x = -5, x = \frac{2}{3}, x = -4$ Set each factor equal to zero and solve for x.

The solution set is $\left\{-5, \frac{2}{3}, -4\right\}$

Apply the Skills

9. a. Factor $f(x) = 2x^3 + 7x^2 - 14x - 40$, given that -4 is a zero of f.

b. Solve the equation. $2x^3 + 7x^2 - 14x - 40 = 0$

Example 10: Using the Factor Theorem to Build a Polynomial

Write a polynomial $f(x)$ of degree 3 that has the zeros $\frac{1}{2}, \sqrt{6}$, and $-\sqrt{6}$.

Solution:

By the factor theorem, if $\frac{1}{2}, \sqrt{6}$, and $-\sqrt{6}$ are zeros of a polynomial $f(x)$, then $\left(x - \frac{1}{2}\right)$, $(x - \sqrt{6})$, and $(x + \sqrt{6})$ are factors of $f(x)$. Therefore,

$f(x) = \left(x - \frac{1}{2}\right)(x - \sqrt{6})(x + \sqrt{6})$ is a third-degree polynomial with the given zeros.

$f(x) = \left(x - \frac{1}{2}\right)(x - \sqrt{6})(x + \sqrt{6})$ Multiply conjugates.

$f(x) = \left(x - \frac{1}{2}\right)(x^2 - 6)$

$f(x) = x^3 - \frac{1}{2}x^2 - 6x + 3$

Apply the Skills

10. Write a polynomial $f(x)$ of degree 3 that has the zeros $\frac{1}{3}, \sqrt{3}$, and $-\sqrt{3}$.

In Example 10, the polynomial $f(x)$ is not unique. If we multiply $f(x)$ by any nonzero constant a, the polynomial will still have the desired factors and zeros.

$g(x) = a\left(x - \frac{1}{2}\right)(x - \sqrt{6})(x + \sqrt{6})$ The zeros are still $\frac{1}{2}, \sqrt{6}$, and $-\sqrt{6}$.

If a is any nonzero multiple of 2, then the polynomial will have integer coefficients.

For example:

$$\begin{aligned} g(x) &= 2\left(x - \tfrac{1}{2}\right)(x - \sqrt{6})(x + \sqrt{6}) \\ &= 2\left(x^3 - \tfrac{1}{2}x^2 - 6x + 3\right) \\ &= 2x^3 - x^2 - 12x + 6 \end{aligned}$$

The zeros of $f(x)$ and $g(x)$ are real numbers and correspond to the x-intercepts of the graphs of the related functions. The graphs of $y = f(x)$ and $y = g(x)$ are shown in Figure 2-15. Notice that the graphs have the same x-intercepts and differ only by a vertical stretch.

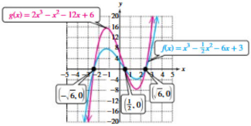


Figure 2-15

Practice 2-3

Practice Exercises

Prerequisite Review

R.1. Evaluate $(3+4i)^2$ and write the answer in standard form, $a+bi$.

R.2. Verify by substitution that the given values of x are solutions to $x^2-12x+39=0$.

a. $6-i\sqrt{3}$ **b.** $6+i\sqrt{3}$

R.3. Solve by using the quadratic formula: $z^2+8z+19=0$.

Concept Connections

1. Given the division algorithm, identify the polynomials representing the dividend, divisor, quotient, and remainder.

$f(x)=d(x)\cdot q(x)+r(x)$

2. Given $\dfrac{2x^3-5x^2-6x+1}{x-3}=2x^2+x-3+\dfrac{-8}{x-3}$, use the division algorithm to check the result

3. The remainder theorem indicates that if a polynomial $f(x)$ is divided by $x-c$, then the remainder is ____________.

4. Given a polynomial $f(x)$, the factor theorem indicates that if $f(c)=0$, then $x-c$ is a ____________ of $f(x)$. Furthermore, if $x-c$ is a factor of $f(x)$, then $f(c)=$ ____________.

5. Answer true or false. If $\sqrt{5}$ is a zero of a polynomial, then $(x-\sqrt{5})$ is a factor of the polynomial.

6. Answer true or false. If $(x+3)$ is a factor of a polynomial, then 3 is a zero of the polynomial.

Learn: Divide Polynomials Using Long Division

For Exercises 7–8, (See Example 1 on p. 210)

a. Use long division to divide.

b. Identify the dividend, divisor, quotient, and remainder.

c. Check the result from part (a) with the division algorithm.

7. $(6x^2+9x+5)\div(2x-5)$

8. $(12x^2+10x+3)\div(3x+4)$

For Exercises 9–22, use long division to divide. (See Examples 1–3 on pp. 210–213)

9. $(3x^3-11x^2-10)\div(x-4)$

10. $(2x^3-7x^2-65)\div(x-5)$

11. $(8+30x-27x^2-12x^3+4x^4)\div(x+2)$

12. $(-48-28x+20x^2+17x^3+3x^4)\div(x+3)$

13. $(-20x^2+6x^4-16)\div(2x+4)$

14. $(-60x^2+8x^4-108)\div(2x-6)$

15. $(x^5+4x^4+18x^2-20x-10)\div(x^2+5)$

16. $(x^5-2x^4+x^3-8x+18)\div(x^2-3)$

17. $\dfrac{6x^4+3x^3-7x^2+6x-5}{2x^2+x-3}$

18. $\dfrac{12x^4-4x^3+13x^2+2x+1}{3x^2-x+4}$

19. $\dfrac{x^3-27}{x-3}$

20. $\dfrac{x^3+64}{x+4}$

21. $(5x^3-2x^2+3)\div(2x-1)$

22. $(2x^3+x^2+1)\div(3x+1)$

Learn: Divide Polynomials Using Synthetic Division

For Exercises 23–26, consider the division of two polynomials: $f(x)\div(x-c)$. The result of the synthetic division process is shown here. Write the polynomials representing the

a. Dividend. **b.** Divisor.

c. Quotient. **d.** Remainder.

23.
$$\begin{array}{r|rrrrr} 3 & 2 & -5 & -5 & -4 & 29 \\ & & 6 & 3 & -6 & -30 \\ \hline & 2 & 1 & -2 & -10 & \boxed{-1} \end{array}$$

24.
$$\begin{array}{r|rrrrr} 2 & 1 & -5 & 2 & -1 & 20 \\ & & 2 & -6 & -8 & -18 \\ \hline & 1 & -3 & -4 & -8 & \boxed{2} \end{array}$$

25. $$\begin{array}{r|rrrr} \underline{-4} & 1 & -2 & -25 & -4 \\ & & -4 & 24 & 4 \\ \hline & 1 & -6 & -1 & \underline{|0} \end{array}$$

26. $$\begin{array}{r|rrrr} \underline{-5} & 3 & 13 & -14 & -20 \\ & & -15 & 10 & 20 \\ \hline & 3 & -2 & -4 & \underline{|0} \end{array}$$

For Exercises 27–38, use synthetic division to divide the polynomials. (See Examples 4–5 on pp. 214–215)

27. $(4x^2 + 15x + 1) \div (x + 6)$

28. $(6x^2 + 25x - 19) \div (x + 5)$

29. $(5x^2 - 17x - 12) \div (x - 4)$

30. $(2x^2 + x - 21) \div (x - 3)$

31. $(4 - 8x - 3x^2 - 5x^4) \div (x + 2)$

32. $(-5 + 2x + 5x^3 - 2x^4) \div (x + 1)$

33. $\dfrac{4x^5 - 25x^4 - 58x^3 + 232x^2 + 198x - 63}{x - 3}$

34. $\dfrac{2x^5 + 13x^4 - 3x^3 - 58x^2 - 20x + 24}{x - 2}$

35. $\dfrac{x^5 + 32}{x + 2}$

36. $\dfrac{x^4 - 81}{x + 3}$

37. $(2x^4 - 7x^3 - 56x^2 + 37x + 84) \div \left(x - \frac{3}{2}\right)$

38. $(-5x^4 - 18x^3 + 63x^2 + 128x - 60) \div \left(x - \frac{2}{5}\right)$

Learn: Apply the Remainder and Factor Theorems

39. The value $f(-6) = 39$ for a polynomial $f(x)$. What can be concluded about the remainder or quotient of $\frac{f(x)}{x+6}$?

40. Given a polynomial $f(x)$, the quotient $\frac{f(x)}{x-2}$ has a remainder of 12. What is the value of $f(2)$?

41. Given $f(x) = 2x^4 - 5x^3 + x^2 - 7$,

a. Evaluate $f(4)$.

b. Determine the remainder when $f(x)$ is divided by $(x - 4)$.

42. Given $g(x) = -3x^5 + 2x^4 + 6x^2 - x + 4$,

a. Evaluate $g(2)$.

b. Determine the remainder when $g(x)$ is divided by $(x - 2)$.

For Exercises 43–46, use the remainder theorem to evaluate the polynomial for the given values of *x*. (See Example 6 on p. 216)

43. $f(x) = 2x^4 + x^3 - 49x^2 + 79x + 15$

a. $f(-1)$ **b.** $f(3)$

c. $f(4)$ **d.** $f\left(\frac{5}{2}\right)$

44. $g(x) = 3x^4 - 22x^3 + 51x^2 + 42x + 8$

a. $g(-1)$ **b.** $g(2)$

c. $g(1)$ **d.** $g\left(\frac{4}{3}\right)$

45. $h(x) = 5x^3 - 4x^2 - 15x + 12$

a. $h(1)$ **b.** $h\left(\frac{4}{5}\right)$

c. $h(\sqrt{3})$ **d.** $h(-1)$

46. $k(x) = 2x^3 - x^2 - 14x + 7$

a. $k(2)$ **b.** $k\left(\frac{1}{2}\right)$

c. $k(\sqrt{7})$ **d.** $k(-2)$

For Exercises 47–54, use the remainder theorem to determine if the given number *c* is a zero of the polynomial. (See Example 7 on p. 218)

47. $f(x) = x^4 + 3x^3 - 7x^2 + 13x - 10$

a. $c = 2$ **b.** $c = -5$

48. $g(x) = 2x^4 + 13x^3 - 10x^2 - 19x + 14$

a. $c = -2$ **b.** $c = -7$

49. $p(x) = 2x^3 + 3x^2 - 22x - 33$

a. $c = -2$ **b.** $c = -\sqrt{11}$

50. $q(x) = 3x^3 + x^2 - 30x - 10$

a. $c = -3$ **b.** $c = -\sqrt{10}$

51. $m(x) = x^3 - 2x^2 + 25x - 50$

a. $c = 5i$ **b.** $c = -5i$

52. $n(x) = x^3 + 4x^2 + 9x + 36$

a. $c = 3i$ **b.** $c = -3i$

53. $g(x) = x^3 - 11x^2 + 25x + 37$

a. $c = 6 + i$ **b.** $c = 6 - i$

54. $f(x) = 2x^3 - 5x^2 + 54x - 26$

a. $c = 1 + 5i$ **b.** $c = 1 - 5i$

For Exercises 55–60, use the factor theorem to determine if the given binomial is a factor of $f(x)$. (See Example 8 on p. 219)

55. $f(x) = x^4 + 11x^3 + 41x^2 + 61x - 30$

a. $x + 5$ **b.** $x - 2$

56. $g(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$

a. $x - 4$ **b.** $x + 1$

57. $f(x) = x^3 + 64$

a. $x - 4$ **b.** $x + 4$

58. $f(x) = x^4 - 81$

a. $x - 3$ **b.** $x + 3$

59. $f(x) = 2x^3 + x^2 - 16x - 8$

a. $x - 1$ **b.** $x - 2\sqrt{2}$

60. $f(x) = 3x^3 - x^2 - 54x + 18$

a. $x - 2$ **b.** $x - 3\sqrt{2}$

61. Given $g(x) = x^4 - 14x^2 + 45$,

a. Evaluate $g(\sqrt{5})$.

b. Evaluate $g(-\sqrt{5})$.

c. Solve $g(x) = 0$.

62. Given $h(x) = x^4 - 15x^2 + 44$,

a. Evaluate $h(\sqrt{11})$.

b. Evaluate $h(-\sqrt{11})$.

c. Solve $h(x) = 0$.

63. a. Use synthetic division and the factor theorem to determine if $[x - (2 + 5i)]$ is a factor of $f(x) = x^2 - 4x + 29$.

b. Use synthetic division and the factor theorem to determine if $[x - (2 - 5i)]$ is a factor of $f(x) = x^2 - 4x + 29$.

c. Use the quadratic formula to solve the equation. $x^2 - 4x + 29 = 0$

d. Find the zeros of the polynomial $f(x) = x^2 - 4x + 29$.

64. a. Use synthetic division and the factor theorem to determine if $[x - (3 + 4i)]$ is a factor of $f(x) = x^2 - 6x + 25$.

b. Use synthetic division and the factor theorem to determine if $[x - (3 - 4i)]$ is a factor of $f(x) = x^2 - 6x + 25$.

c. Use the quadratic formula to solve the equation. $x^2 - 6x + 25 = 0$

d. Find the zeros of the polynomial $f(x) = x^2 - 6x + 25$.

65. a. Factor $f(x) = 2x^3 + x^2 - 37x - 36$, given that -1 is a zero. **(See Example 9 on p. 220)**

b. Solve. $2x^3 + x^2 - 37x - 36 = 0$

66. a. Factor $f(x) = 3x^3 + 16x^2 - 5x - 50$, given that -2 is a zero.

b. Solve. $3x^3 + 16x^2 - 5x - 50 = 0$

67. a. Factor $f(x) = 20x^3 + 39x^2 - 3x - 2$, given that $\frac{1}{4}$ is a zero.

b. Solve. $20x^3 + 39x^2 - 3x - 2 = 0$

68. a. Factor $f(x) = 8x^3 - 18x^2 - 11x + 15$, given that $\frac{3}{4}$ is a zero.

b. Solve. $8x^3 - 18x^2 - 11x + 15 = 0$

69. a. Factor $f(x) = 9x^3 - 33x^2 + 19x - 3$, given that 3 is a zero.

b. Solve. $9x^3 - 33x^2 + 19x - 3 = 0$

70. a. Factor $f(x) = 4x^3 - 20x^2 + 33x - 18$, given that 2 is a zero.

b. Solve. $4x^3 - 20x^2 + 33x - 18 = 0$

For Exercises 71–82, write a polynomial $f(x)$ that meets the given conditions. Answers may vary. (See Example 10 on p. 220)

71. Degree 3 polynomial with zeros 2, 3, and -4.

72. Degree 3 polynomial with zeros 2, -6, and 3.

73. Degree 4 polynomial with zeros 1, $\frac{3}{2}$ (each with multiplicity 1), and 0 (with multiplicity 2).

74. Degree 5 polynomial with zeros 2, $\frac{5}{2}$ (each with multiplicity 1), and 0 (with multiplicity 3).

75. Degree 2 polynomial with zeros $2\sqrt{11}$ and $-2\sqrt{11}$.

76. Degree 2 polynomial with zeros $5\sqrt{2}$ and $-5\sqrt{2}$.

77. Degree 3 polynomial with zeros -2, $3i$, and $-3i$.

78. Degree 3 polynomial with zeros 4, $2i$, and $-2i$.

79. Degree 3 polynomial with integer coefficients and zeros of $-\frac{2}{3}$, $\frac{1}{2}$, and 4.

80. Degree 3 polynomial with integer coefficients and zeros of $-\frac{2}{5}$, $\frac{3}{2}$, and 6.

81. Degree 2 polynomial with zeros of $7 + 8i$ and $7 - 8i$.

82. Degree 2 polynomial with zeros of $5 + 6i$ and $5 - 6i$.

Mixed Exercises

83. Given $p(x) = 2x^{452} - 4x^{92}$, is it easier to evaluate $p(1)$ by using synthetic division or by direct substitution? Find the value of $p(1)$.

84. Given $q(x) = 5x^{721} - 2x^{450}$, is it easier to evaluate $q(-1)$ by using synthetic division or by direct substitution? Find the value of $q(-1)$.

85. a. Is $(x - 1)$ a factor of $x^{100} - 1$?

b. Is $(x + 1)$ a factor of $x^{100} - 1$?

c. Is $(x - 1)$ a factor of $x^{99} - 1$?

d. Is $(x + 1)$ a factor of $x^{99} - 1$?

e. If n is a positive even integer, is $(x - 1)$ a factor of $x^n - 1$?

f. If n is a positive odd integer, is $(x + 1)$ a factor of $x^n - 1$?

86. If a fifth-degree polynomial is divided by a second-degree polynomial, the quotient is a _________-degree polynomial.

87. Determine if the statement is true or false: Zero is a zero of the polynomial $3x^5 - 7x^4 - 2x^3 - 14$.

88. Determine if the statement is true or false: Zero is a zero of the polynomial $-2x^4 + 5x^3 + 6x$.

89. Find m so that $x + 4$ is a factor of $4x^3 + 13x^2 - 5x + m$.

90. Find m so that $x + 5$ is a factor of $-3x^4 - 10x^3 + 20x^2 - 22x + m$.

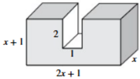
91. Find m so that $x + 2$ is a factor of $4x^3 + 5x^2 + mx + 2$.

92. Find m so that $x - 3$ is a factor of $2x^3 - 7x^2 + mx + 6$.

93. For what value of r is the statement an identity? $\frac{x^2 - x - 12}{x - 4} = x + 3 + \frac{r}{x - 4}$ provided that $x \neq 4$

94. For what value of r is the statement an identity? $\frac{x^2 - 5x - 8}{x - 2} = x - 3 + \frac{r}{x - 2}$ provided that $x \neq 2$

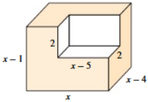
95. A metal block is formed from a rectangular solid with a rectangular piece cut out.



a. Write a polynomial $V(x)$ that represents the volume of the block. All distances in the figure are in centimeters.

b. Use synthetic division to evaluate the volume if x is 6 cm.

96. A wedge is cut from a rectangular solid.



a. Write a polynomial $V(x)$ that represents the volume of the remaining part of the solid. All distances in the figure are in feet.

b. Use synthetic division to evaluate the volume if x is 10 ft.

Write About It

97. Under what circumstances can synthetic division be used to divide polynomials?

98. How can the division algorithm be used to check the result of polynomial division?

99. Given a polynomial $f(x)$ and a constant c, state two methods by which the value $f(c)$ can be computed.

100. Write an informal explanation of the factor theorem.

Expanding Your Skills

101. **a.** Factor $f(x) = x^3 - 5x^2 + x - 5$ into factors of the form $(x - c)$, given that 5 is a zero.

b. Solve. $x^3 - 5x^2 + x - 5 = 0$

102. **a.** Factor $f(x) = x^3 - 3x^2 + 100x - 300$ into factors of the form $(x - c)$, given that 3 is a zero.

b. Solve. $x^3 - 3x^2 + 100x - 300 = 0$

103. **a.** Factor $f(x) = x^4 + 2x^3 - 2x^2 - 6x - 3$ into factors of the form $(x - c)$, given that -1 is a zero.

b. Solve. $x^4 + 2x^3 - 2x^2 - 6x - 3 = 0$

104. **a.** Factor $f(x) = x^4 + 4x^3 - x^2 - 20x - 20$ into factors of the form $(x - c)$, given that -2 is a zero.

b. Solve. $x^4 + 4x^3 - x^2 - 20x - 20 = 0$

105. **a.** Factor $f(x) = 6x^4 - 23x^3 - 29x^2 + 147x - 45$, given that 3 is a zero of multiplicity 2.

b. Solve. $6x^4 - 23x^3 - 29x^2 + 147x - 45 = 0$

106. **a.** Factor $f(x) = x^5 - 3x^4 - 6x^3 + 26x^2 - 27x + 9$, given that 1 is a zero of multiplicity 3.

b. Solve. $x^5 - 3x^4 - 6x^3 + 26x^2 - 27x + 9 = 0$

107. Factor $f(x) = 2x^3 + (9 - 4i)x^2 + (4 - 18i)x - 8i$, given that $2i$ is a zero.

108. Factor $f(x) = x^4 + 3ix^3 - 3x^2 + (2 - 9i)x + 6i$, given that $-3i$ and 1 are zeros.

109. Factor $f(x) = 3x^3 + (3\sqrt{5} - 14)x^2 - (14\sqrt{5} + 5)x - 5\sqrt{5}$, given that $-\sqrt{5}$ is a zero.

110. Factor $f(x) = x^3 + 2\sqrt{2}\,x^2 - 4x + 8\sqrt{2}$, given that $2\sqrt{2}$ is a zero.

For Exercises 111–114, use the remainder theorem to determine which numbers in the given list are zeros of the polynomial.

Possible zeros: $\left\{-3, \frac{1}{3}, 2, -\frac{1}{2}, 2 + i, -2i\right\}$

111. $f(x) = x^3 - x^2 - 7x + 15$

112. $f(x) = 6x^3 + 19x^2 + 2x - 3$

113. $f(x) = 2x^3 - 4x^2 - 9x + 18$

114. $f(x) = 9x^4 - 6x^3 + 37x^2 - 24x + 4$

Technology Connections

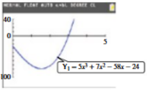
For Exercises 115–116,

a. Use the graph to determine a solution to the given equation.

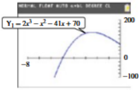
b. Verify your answer from part (a) using the remainder theorem.

c. Find the remaining solutions to the equation.

115. $5x^3 + 7x^2 - 58x - 24 = 0$



116. $2x^3 - x^2 - 41x + 70 = 0$



GO ONLINE Go online for more practice problems.

Lesson 2-4

Zeros of Polynomials

What Will You Learn?

After completing this lesson, you should be able to:

- Apply the Rational Zero Theorem
- Apply the Fundamental Theorem of Algebra
- Apply Descartes' Rule of Signs
- Find Upper and Lower Bounds

Learn: Apply the Rational Zero Theorem

The zeros of a polynomial $f(x)$ are the solutions (roots) to the corresponding polynomial equation $f(x) = 0$. For a polynomial function defined by $y = f(x)$, the real zeros of $f(x)$ are the x-intercepts of the graph of the function. Applications of polynomials and polynomial functions arise throughout mathematics. For this reason, it is important to learn techniques to find or approximate the zeros of a polynomial.

The zeros of a polynomial may be real numbers or imaginary numbers. The real zeros can be further categorized as rational or irrational numbers. For example, consider

$$f(x) = 2x^6 - 3x^5 - 7x^4 + 102x^3 - 88x^2 - 279x + 273$$

In factored form this is:

$$f(x) = (x - 1)(2x + 7)(x - \sqrt{3})(x + \sqrt{3})[x - (2 + 3i)][x - (2 - 3i)]$$

The zeros are: $\underbrace{\overbrace{1, -\tfrac{7}{2},}^{\text{rational zeros}}\ \overbrace{\sqrt{3}, -\sqrt{3},}^{\text{irrational zeros}}}_{\text{real zeros}}\ \underbrace{\overbrace{2 + 3i, 2 - 3i}}_{\text{nonreal zeros}}$

In this lesson, we develop tools to search for the zeros of polynomials. First we will consider polynomials with integer coefficients. For example, the factored form of $f(x) = 6x^2 + 13x - 5$ is $f(x) = (2x + 5)(3x - 1)$, which leads to zeros of $-\frac{5}{2}$ and $\frac{1}{3}$. The polynomial $f(x) = 6x^2 + 13x - 5$ is in the form $f(x) = ax^2 + bx + c$, where $a = 6$, $b = 13$, and $c = -5$. Notice that the numerator of each zero is a factor of c, and the denominator of each zero is a factor of a. This observation is consistent with the following theorem to search for zeros that are rational numbers.

Recall that a rational number is a number that can be expressed as a ratio of two integers.

Rational Zero Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ has integer coefficients and $a_n \neq 0$, and if $\frac{p}{q}$ (written in lowest terms) is a rational zero of f, then

- p is a factor of the constant term a_0.
- q is a factor of the leading coefficient a_n.

The rational zero theorem does not guarantee the existence of rational zeros. Rather, it indicates that *if* a rational zero exists for a polynomial, then it must be of the form

$$\frac{p}{q} = \frac{\text{Factors of } a_0 \text{ (constant term)}}{\text{Factors of } a_n \text{ (leading coefficient)}}$$

The rational zero theorem is important because it limits our search to find rational zeros (if they exist) to a finite number of choices.

Example 1: Listing All Possible Rational Zeros

List all possible rational zeros. $f(x) = -2x^5 + 3x^2 - 2x + 10$

Solution:

First note that the polynomial has integer coefficients.

$f(x) = -2x^5 + 3x^2 - 2x + 10$

The constant term is 10. $\pm 1, \pm 2, \pm 5, \pm 10$ Factors of 10

The leading coefficient is -2. $\pm 1, \pm 2$ Factors of -2

$$\frac{\text{Factors of } 10}{\text{Factors of } -2} = \frac{\pm 1, \pm 2, \pm 5, \pm 10}{\pm 1, \pm 2} = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{5}{1}, \pm\frac{10}{1}, \pm\frac{1}{2}, \cancel{\pm\frac{2}{2}}, \pm\frac{5}{2}, \cancel{\pm\frac{10}{2}}$$

The values $\pm\frac{2}{2}$ and $\pm\frac{10}{2}$ are redundant. They equal ± 1 and ± 5, respectively. The possible rational zeros are $\pm 1, \pm 2, \pm 5, \pm 10, \pm\frac{1}{2}$, and $\pm\frac{5}{2}$.

Apply the Skills

1. List all possible rational zeros of $f(x) = -4x^4 + 5x^3 - 7x^2 + 8$.

Example 2: Finding the Zeros of a Polynomial

Find the zeros. $f(x) = x^3 - 4x^2 + 3x + 2$

Solution:

We begin by first looking for rational zeros. We can apply the rational zero theorem because the polynomial has integer coefficients.

$$f(x) = 1x^3 - 4x^2 + 3x + 2$$

Possible rational zeros: $\frac{\text{Factors of } 2}{\text{Factors of } 1} = \frac{\pm1, \pm2}{\pm1} = \pm1, \pm2$

Next, use synthetic division and the remainder theorem to determine if any of the numbers in the list is a zero of *f*.

Test $x = 1$:

$$\begin{array}{r|rrrr} 1 & 1 & -4 & 3 & 2 \\ & & 1 & -3 & 0 \\ \hline & 1 & -3 & 0 & \boxed{2} \end{array}$$

The remainder is not zero. Therefore, 1 is not a zero of $f(x)$.

Test $x = -1$:

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 3 & 2 \\ & & -1 & 5 & -8 \\ \hline & 1 & -5 & 8 & \boxed{-6} \end{array}$$

The remainder is not zero. Therefore, -1 is not a zero of $f(x)$.

Test $x = 2$:

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 3 & 2 \\ & & 2 & -4 & -2 \\ \hline & 1 & -2 & -1 & \boxed{0} \end{array}$$

The remainder *is* zero. Therefore, 2 is a zero of $f(x)$.

By the factor theorem, since $f(2) = 0$, then $(x - 2)$ is a factor of $f(x)$. The quotient $(x^2 - 2x - 1)$ is also a factor of $f(x)$. We have

$$f(x) = x^3 - 4x^2 + 3x + 2 = (x - 2)(x^2 - 2x - 1)$$

We now have a third-degree polynomial written as the product of a first-degree polynomial and a quadratic polynomial. The quotient $x^2 - 2x - 1$ is called a **reduced polynomial** (or a **depressed polynomial**) of $f(x)$. It has degree 1 less than the degree of $f(x)$, and the remaining zeros of $f(x)$ are the zeros of the reduced polynomial.

At this point, we no longer need to test for more rational zeros. The reason is that any remaining zeros (whether they be rational, irrational, or imaginary) are the solutions to the quadratic equation $x^2 - 2x - 1 = 0$. There is no guesswork because the equation can be solved by using the quadratic formula.

$$x^2 - 2x - 1 = 0$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2}$$

$$= 1 \pm \sqrt{2}$$

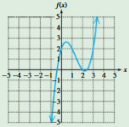
The zeros of $f(x)$ are 2, $1 + \sqrt{2}$, and $1 - \sqrt{2}$.

Insights

Since $f(1) = 2$ and $f(-1) = -6$, we know from the intermediate value theorem that $f(x)$ must have a zero between -1 and 1.

The graph of $f(x) = x^3 - 4x^2 + 3x + 2$ is shown here.

The zeros are $1 + \sqrt{2} \approx 2.41$, $1 - \sqrt{2} \approx -0.41$, and 2.



Apply the Skills

2. Find the zeros of $f(x) = x^3 - x^2 - 4x - 2$.

The polynomial in Example 2 has one rational zero. If we had continued testing for more rational zeros, we would not have found others because the remaining two zeros are irrational numbers. In Example 3, we illustrate the case where a function has multiple rational zeros.

Example 3: Finding the Zeros of a Polynomial

Find the zeros and their multiplicities. $f(x) = 2x^4 + 5x^3 - 2x^2 - 11x - 6$

Solution:

Begin by searching for rational zeros.

$$f(x) = 2x^4 + 5x^3 - 2x^2 - 11x - 6$$

Possible rational zeros:

$$\frac{\text{Factors of } -6}{\text{Factors of } 2} = \frac{\pm1, \pm2, \pm3, \pm6}{\pm1, \pm2} = \pm1, \pm2, \pm3, \pm6, \pm\frac{1}{2}, \pm\frac{3}{2}$$

We can work methodically through the list of possible rational zeros to determine which if any are actual zeros of $f(x)$. After trying several possibilities, we find that -1 is a zero of $f(x)$.

Insights It does not matter in which order you test the potential rational zeros.

$$\begin{array}{r|rrrrr} -1 & 2 & 5 & -2 & -11 & -6 \\ & & -2 & -3 & 5 & 6 \\ \hline & 2 & 3 & -5 & -6 & \boxed{0} \end{array} \qquad \text{The quotient is } 2x^3 + 3x^2 - 5x - 6.$$

Since $f(-1) = 0$, then $x + 1$ is a factor of $f(x)$.

$$f(x) = 2x^4 + 5x^3 - 2x^2 - 11x - 6 = (x + 1)\underbrace{(2x^3 + 3x^2 - 5x - 6)}_{\text{Now find the zeros of the quotient.}}$$

The zeros of $f(x)$ are -1 along with the roots of the equation $2x^3 + 3x^2 - 5x - 6 = 0$. Therefore, we need to find the zeros of $g(x) = 2x^3 + 3x^2 - 5x - 6$.

The possible rational zeros are $\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{2}, \pm\frac{3}{2}$

We will test -1 again because it may have multiplicity greater than 1.

$$\begin{array}{r|rrrr} -1 & 2 & 3 & -5 & -6 \\ & & -2 & -1 & 6 \\ \hline & 2 & 1 & -6 & \boxed{0} \end{array} \qquad \text{The quotient is } 2x^2 + x - 6.$$

The value -1 is a repeated zero. We have:

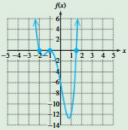
$$\begin{aligned} f(x) = 2x^4 + 5x^3 - 2x^2 - 11x - 6 &= (x + 1)(x + 1)(2x^2 + x - 6) \\ &= (x + 1)^2\underbrace{(2x - 3)(x + 2)}_{2x^2 + x - 6 \text{ factors as } (2x - 3)(x + 2).} \end{aligned}$$

The zeros of $f(x)$ are

-1 (multiplicity 2), $\frac{3}{2}$ (multiplicity 1), and -2 (multiplicity 1).

Insights

The zeros of the polynomial in Example 3 are rational numbers (and therefore real numbers). They correspond to x-intercepts on the graph of $y = f(x)$. Also notice that the graph has a touch point at $(-1, 0)$ because -1 has an even multiplicity.



Apply the Skills

3. Find the zeros and their multiplicities.

$$f(x) = 2x^4 + 3x^3 - 15x^2 - 32x - 12$$

In Example 4, we have a polynomial with no rational zeros.

Example 4: Finding the Zeros of a Polynomial

Find the zeros. $f(x) = x^4 - 2x^2 - 3$

Solution:

$f(x) = 1x^4 - 2x^2 - 3$ The possible rational zeros are $\dfrac{\pm 1, \pm 3}{\pm 1} = \pm 1, \pm 3$.

If we apply the rational zero theorem, we see that $f(x)$ has no *rational* zeros.

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & -2 & 0 & -3 \\ & & 1 & 1 & -1 & -1 \\ \hline & 1 & 1 & -1 & -1 & \boxed{-4} \end{array} \qquad \begin{array}{r|rrrrr} -1 & 1 & 0 & -2 & 0 & -3 \\ & & -1 & 1 & 1 & -1 \\ \hline & 1 & -1 & -1 & 1 & \boxed{-4} \end{array}$$

$$\begin{array}{r|rrrrr} 3 & 1 & 0 & -2 & 0 & -3 \\ & & 3 & 9 & 21 & 63 \\ \hline & 1 & 3 & 7 & 21 & \boxed{60} \end{array} \qquad \begin{array}{r|rrrrr} -3 & 1 & 0 & -2 & 0 & -3 \\ & & -3 & 9 & -21 & 63 \\ \hline & 1 & -3 & 7 & -21 & \boxed{60} \end{array}$$

However, finding the zeros of $f(x)$ is equivalent to finding the roots of the equation $x^4 - 2x^2 - 3 = 0$.

$x^4 - 2x^2 - 3 = 0$

$(x^2 - 3)(x^2 + 1) = 0$ Factor the trinomial.

$x^2 - 3 = 0$ or $x^2 + 1 = 0$ Set each factor equal to zero.

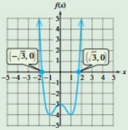
$x = \pm\sqrt{3}$ or $x = \pm i$ Apply the square root property

The zeros are $\sqrt{3}$, $-\sqrt{3}$, i, and $-i$.

Insights

From the factor theorem,
$f(x) = (x - \sqrt{3})(x + \sqrt{3})(x - i)(x + i)$.

The graph of the function defined by $f(x) = x^4 - 2x^2 - 3$ shows the real zeros of $f(x)$ as x-intercepts.



Apply the Skills

4. Find the zeros of $f(x) = x^4 - x^2 - 20$.

Learn: Apply the Fundamental Theorem of Algebra

From Examples 2–4, we see that the zeros of a polynomial may be real numbers (either rational or irrational) or nonreal numbers such as $2 + 3i$ or $5i$. In any case, the zeros are all complex numbers.

To find the zeros of a polynomial, it is important to know how many zeros to expect. This is answered by the following three theorems. The first is called the fundamental theorem of algebra because it is so basic to the foundation of algebra.

Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $f(x)$ has at least one complex zero.

The fundamental theorem of algebra, first proved by German mathematician Carl Friedrich Gauss (1777–1855), guarantees that every polynomial of degree $n \geq 1$ has at least one zero.

Now suppose that $f(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients. The fundamental theorem of algebra guarantees the existence of at least one complex zero, call this c_1. By the factor theorem, we have

$$f(x) = (x - c_1) \cdot q_1(x) \qquad \text{where } q_1(x) \text{ is a polynomial of degree } n - 1.$$

If $q_1(x)$ is of degree 1 or more, then the fundamental theorem of algebra guarantees that $q_1(x)$ must have at least one complex zero, call this c_2. Then,

$$f(x) = (x - c_1) \cdot (x - c_2) \cdot q_2(x) \qquad \text{where } q_2(x) \text{ is a polynomial of degree } n - 2.$$

We can continue with this reasoning until the quotient polynomial, $q_n(x)$, is a constant equal to the leading coefficient of $f(x)$.

Linear Factorization Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$, where $n \geq 1$ and $a_n \neq 0$, then

$f(x) = a_n(x - c_1)(x - c_2) \ldots (x - c_n)$, where $c_1, c_2, \ldots, c_n$ are complex numbers.

Note: The complex numbers $c_1, c_2, \ldots, c_n$ are not necessarily unique.

Insights

The set of complex numbers includes the set of real numbers. Therefore, theorems relating to polynomials with complex coefficients also apply to polynomials with real coefficients.

The linear factorization theorem tells us that a polynomial of degree $n \geq 1$ with complex coefficients has exactly n linear factors of the form $(x - c)$, where some of the factors may be repeated. The value of c in each factor is a zero of the function, so the function must also have n zeros provided that the zeros are counted according to their multiplicities.

Number of Zeros of a Polynomial

If $f(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $f(x)$ has exactly n complex zeros provided that each zero is counted by its multiplicity.

Insights

Refer to Examples 2–4. In each case, the number of zeros (including multiplicities) is the same as the degree of the polynomial.

Now consider the polynomial from Example 4.

$f(x) = x^4 - 2x^2 - 3$ Zeros: $\sqrt{3}, -\sqrt{3}, i, -i$

Notice that the polynomial has real coefficients. Furthermore, the zeros i and $-i$ appear as a pair. This is not a coincidence. For a polynomial with real coefficients, if $a + bi$ is a zero, then $a - bi$ is a zero.

Conjugate Zeros Theorem

If $f(x)$ is a polynomial with real coefficients and if $a + bi$ $(b \neq 0)$ is a zero of $f(x)$, then its conjugate $a - bi$ is also a zero of $f(x)$.

Example 5: Finding Zeros and Factoring a Polynomial

Given $f(x) = x^4 - 6x^3 + 28x^2 - 18x + 75$, and that $3 - 4i$ is a zero of $f(x)$,

a. Find the remaining zeros.

b. Factor $f(x)$ as a product of linear factors.

c. Solve the equation. $x^4 - 6x^3 + 28x^2 - 18x + 75 = 0$

Solution:

$f(x)$ is a fourth-degree polynomial, so we expect to find four zeros (including multiplicities). Further note that because $f(x)$ has real coefficients and because $3 - 4i$ is a zero, then the conjugate $3 + 4i$ must also be a zero. This leaves only two remaining zeros to find.

Divide by $[x-(3-4i)]$.

$$\begin{array}{r|rrrrr} 3-4i & 1 & -6 & 28 & -18 & 75 \\ & & 3-4i & -25 & 9-12i & -75 \\ \hline & 1 & -3-4i & 3 & -9-12i & \boxed{0} \end{array}$$

coefficients of the quotient

One strategy is to use synthetic division twice using the two known zeros.

Note: $(3-4i)(-3-4i)$
$=-9-12i+12i+16i^2=-25$

Note: $(3-4i)(-9-12i)$
$=-27-36i+36i+48i^2=-75$

Since $3+4i$ is a zero of $f(x)$ it must also be a zero of the quotient.

Divide by $[x-(3+4i)]$.

$$\begin{array}{r|rrrr} 3+4i & 1 & -3-4i & 3 & -9-12i \\ & & 3+4i & 0 & 9+12i \\ \hline & 1 & 0 & 3 & \boxed{0} \end{array}$$

Divide the quotient by $[x-(3+4i)]$. The resulting quotient is quadratic: x^2+3.

Now we have $f(x)=[x-(3-4i)][x-(3+4i)](x^2+3)$.

The remaining two zeros are found by solving $x^2+3=0$.

$$\begin{aligned} x^2+3&=0 \\ x^2&=-3 \\ x&=\pm i\sqrt{3} \end{aligned}$$

a. The zeros of $f(x)$ are: $3-4i$, $3+4i$, $i\sqrt{3}$, and $-i\sqrt{3}$.

b. $f(x)$ factors as four linear factors.

$$f(x)=[x-(3-4i)][x-(3+4i)](x-i\sqrt{3})(x+i\sqrt{3})$$

c. The solution set for $x^4-6x^3+28x^2-18x+75=0$ is $\{3\pm 4i, \pm i\sqrt{3}\}$.

Apply the Skills

5. Given $f(x)=x^4-2x^3+28x^2-4x+52$, and that $1+5i$ is a zero of $f(x)$,

a. Find the zeros.

b. Factor $f(x)$ as a product of linear factors.

c. Solve the equation. $x^4-2x^3+28x^2-4x+52=0$

Example 6: Building a Polynomial with Specified Conditions

a. Find a third-degree polynomial $f(x)$ with integer coefficients and with zeros of $\frac{2}{3}$ and $4+2i$.

b. Find a polynomial $g(x)$ of lowest degree with zeros of -2 (multiplicity 1) and 4 (multiplicity 3), and satisfying the condition that $g(0)=256$.

Solution:

a. $f(x)$ is to be a polynomial with integer coefficients (and therefore real coefficients). If $4+2i$ is a zero, then $4-2i$ must also be a zero. By the linear factorization theorem we have:

$$\begin{aligned} f(x) &= a\left(x - \tfrac{2}{3}\right)[x - (4 + 2i)][x - (4 - 2i)] \\ &= a\left(x - \tfrac{2}{3}\right)[x^2 - x(4 - 2i) - x(4 + 2i) + (4 + 2i)(4 - 2i)] \\ &= a\left(x - \tfrac{2}{3}\right)(x^2 - 4x + 2xi - 4x - 2xi + 16 - 4i^2) \\ &= 3\left(x - \tfrac{2}{3}\right)(x^2 - 8x + 20) \\ &= (3x - 2)(x^2 - 8x + 20) \\ &= 3x^3 - 24x^2 + 60x - 2x^2 + 16x - 40 \\ &= 3x^2 - 26x^2 + 76x - 40 \end{aligned}$$

a is a nonzero number.

To give $f(x)$ integer coefficients, choose a to be any multiple of 3. We have chosen $a = 3$.

b. $$\begin{aligned} g(x) &= a(x + 2)^1(x - 4)^3 \\ &= a(x + 2)(x^3 - 12x^2 + 48x - 64) \\ &= a(x^4 - 10x^3 + 24x^2 + 32x - 128) \end{aligned}$$

-2 is a zero of multiplicity 1.

4 is a zero of multiplicity 3.

Note:

$$\begin{aligned} (x - 4)^3 &= (x - 4)^2(x - 4) \\ &= (x^2 - 8x + 16)(x - 4) \\ &= x^3 - 12x^2 + 48x - 64 \end{aligned}$$

We also have the condition that $g(0) = 256$.

$$\begin{aligned} g(0) = a[(0)^4 - 10(0)^3 + 24(0)^2 + 32(0) - 128] &= 256 \\ -128a &= 256 \\ a &= -2 \end{aligned}$$

Therefore, $g(x) = -2(x^4 - 10x^3 + 24x^2 + 32x - 128)$

$g(x) = -2x^4 + 20x^3 - 48x^2 - 64x + 256$

Apply the Skills

6. **a.** Find a third-degree polynomial $f(x)$ with integer coefficients and with zeros of $2 + i$ and $\frac{4}{3}$.

b. Find a polynomial $g(x)$ of lowest degree with zeros of -3 (multiplicity 2) and 5 (multiplicity 2), and satisfying the condition that $g(0) = 450$.

Learn: Apply Descartes' Rule of Signs

Finding the zeros of a polynomial analytically can be a difficult (or impossible) task. For example, consider

$$f(x) = x^5 - 18x^4 + 128x^3 - 450x^2 + 783x - 540$$

Applying the rational zero theorem gives us the following possible rational zeros.

$\pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm9, \pm10, \pm12, \pm15, \pm18, \pm20,$
$\pm27, \pm30, \pm36, \pm45, \pm54, \pm60, \pm90, \pm108, \pm135, \pm180, \pm270, \pm540$

However, we can use the upper and lower bound theorem to show that the real zeros of $f(x)$ are between -1 and 18. Furthermore, we can use a tool called Descartes' rule of signs to show that none of the zeros is negative. This eliminates all possible rational zeros except for 1, 2, 3, 4, 5, 6, 9, 10, 12, and 15.

To study Descartes' rule of signs, we need to establish what is meant by "sign changes" between consecutive terms in a polynomial. For example, the following polynomial is written in descending order and has three changes in sign between consecutive coefficients.

$$2x^6 \underbrace{-\,3x^4}_{\text{positive to negative}} - x^3 \underbrace{+\,5x^2}_{\text{negative to positive}} \underbrace{-\,6x}_{\text{positive to negative}} - 4 \qquad \text{(3 sign changes)}$$

Descartes' Rule of Signs

Let $f(x)$ be a polynomial with real coefficients and a nonzero constant term. Then,

1. The number of *positive* real zeros is either
   - the same as the number of sign changes in $f(x)$ or
   - less than the number of sign changes in $f(x)$ by a positive even integer.
2. The number of *negative* real zeros is either
   - the same as the number of sign changes in $f(-x)$ or
   - less than the number of sign changes in $f(-x)$ by a positive even integer.

Insights

For a polynomial with real coefficients, the reason we reduce the number of possible real zeros in increments of 2 is because the alternative, nonreal zeros occur in conjugate pairs.

Nonreal zeros are numbers that contain the imaginary number i such as $-3i$, $3i$, $4 + 7i$, $4 - 7i$ and so on.

Descartes' rule of signs is demonstrated in Examples 7 and 8.

Example 7: Applying Descartes' Rule of Signs

Determine the number of possible positive and negative real zeros.

$$f(x) = x^5 - 6x^4 + 12x^3 - 12x^2 + 11x - 6$$

Solution:

$f(x)$ has real coefficients and the constant term is nonzero.

To determine the number of possible positive real zeros, determine the number of sign changes in $f(x)$.

$$f(x) = x^5 - 6x^4 + 12x^3 - 12x^2 + 11x - 6 \qquad \text{(5 sign changes)}$$

The number of possible positive real zeros is either 5, 3, or 1.

To determine the number of possible negative real zeros, determine the number of sign changes in $f(-x)$.

$$\begin{aligned} f(-x) &= (-x)^5 - 6(-x)^4 + 12(-x)^3 - 12(-x)^2 + 11(-x) - 6 \\ &= -x^5 - 6x^4 - 12x^3 - 12x^2 - 11x - 6 \qquad \text{(0 sign changes)} \end{aligned}$$

There are no sign changes in $f(-x)$. Therefore, $f(x)$ has no negative real zeros.

The number of possible number of nonreal zeros is the difference between the total and the number of possible (positive and negative) real zeros. Since $f(x)$ is a polynomial of degree 5 with complex coefficients, $f(x)$ has exactly 5 complex zeros provided that each zero is counted by its multiplicity.

In this case, we determined the number of possible positive real zeros is 5, 3, or 1 and there are 0 negative real zeros.

If there are 5 real zeros, there are $5 - (5 + 0) = 0$ nonreal zeros.

If there are 3 real zeros, there are $5 - (3 + 0) = 2$ nonreal zeros.

If there is 1 real zero, there are $5 - (1 + 0) = 4$ nonreal zeros.

Number of possible positive real zeros	5	3	1
Number of possible negative real zeros	0	0	0
Number of nonreal zeros	0	2	4
Total (including multiplicities)	5	5	5

The graph of $f(x) = x^5 - 6x^4 + 12x^3 - 12x^2 + 11x - 6$ is shown in Figure 2-16. Notice that there are three positive x-intercepts and therefore, three positive real zeros. There are no negative x-intercepts as expected. The remaining zeros of the polynomial $x^5 - 6x^4 + 12x^3 - 12x^2 + 11x - 6$ are not real numbers.

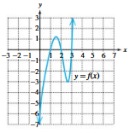


Figure 2-16

Apply the Skills

7. Determine the number of possible positive and negative real zeros for $f(x) = 4x^5 + 6x^3 + 2x^2 + 6$.

Example 8: Applying Descartes' Rule of Signs

Determine the number of possible positive and negative real zeros.

$$g(x) = 2x^6 - 5x^4 - 3x^3 + 7x^2 + 2x + 5$$

Solution:

$g(x)$ has real coefficients and the constant term is nonzero.

$g(x) = 2x^6 \underset{\smile}{-} 5x^4 - 3x^3 \underset{\smile}{+} 7x^2 + 2x + 5$ 2 sign changes in $g(x)$

The number of possible positive real zeros is either 2 or 0.

$$\begin{aligned} g(-x) &= 2(-x)^6 - 5(-x)^4 - 3(-x)^3 + 7(-x)^2 + 2(-x) + 5 \\ &= 2x^6 \underset{\smile}{-} 5x^4 \underset{\smile}{+} 3x^3 + 7x^2 \underset{\smile}{-} 2x \underset{\smile}{+} 5 \end{aligned}$$

4 sign changes in $g(-x)$

The number of possible negative real zeros is either 4, 2, or 0.

Since $f(x)$ is a polynomial of degree 6 with complex coefficients, $f(x)$ has 6 complex zeros provided that each zero is counted by its multiplicity. The number of nonreal zeros will be the difference between the total and the total number of possible real zeros. For example, if $f(x)$ has 2 positive real zeros and 2 negative real zeros, there will be $6 - (2 + 2) = 2$ nonreal zeros.

Number of possible positive real zeros	2	2	2	0	0	0
Number of possible negative real zeros	4	2	0	4	2	0
Number of nonreal zeros	0	2	4	2	4	6
Total (including multiplicities)	6	6	6	6	6	6

Apply the Skills

8. Determine the number of possible positive and negative real zeros for $g(x) = 8x^6 - 5x^7 + 3x^5 - x^2 - 3x + 1$.

Descartes' rule of signs stipulates that the constant term of the polynomial $f(x)$ is nonzero. If the constant term is 0, we can factor out the lowest power of x and apply Descartes' rule of signs to the resulting factor.

$f(x) = x^7 - 8x^6 + 15x^5$

$f(x) = x^5(x^2 - 8x + 15)$

Descartes' rule of signs can be applied to $x^2 - 8x + 15$ to show that there may be 2 or 0 remaining positive real zeros.

The value 0 is a zero of $f(x)$ of multiplicity 5.

Learn: Find Upper and Lower Bounds

The next theorem helps us limit our search for the real zeros of a polynomial. First we define two key terms.

- A real number b is called an **upper bound** of the real zeros of a polynomial if all real zeros are less than or equal to b.
- A real number a is called a **lower bound** of the real zeros of a polynomial if all real zeros are greater than or equal to a.

Check Your Work

It is important to note that upper and lower bounds are not unique. Any number greater than b is also an upper bound for the zeros of the polynomial. Likewise any number less than a is also a lower bound.

Upper and Lower Bound Theorem for the Real Zeros of a Polynomial

Let $f(x)$ be a polynomial of degree $n \geq 1$ with real coefficients and a positive leading coefficient. Further suppose that $f(x)$ is divided by $(x - c)$.

1. If $c > 0$ and if both the remainder and the coefficients of the quotient are nonnegative, then c is an upper bound for the real zeros of $f(x)$.
2. If $c < 0$ and the coefficients of the quotient and the remainder alternate in sign (with 0 being considered either positive or negative as needed), then c is a lower bound for the real zeros of $f(x)$.

The rules for finding upper and lower bounds are stated for polynomial functions having a positive leading coefficient. However, $f(x) = 0$ and $-f(x) = 0$ are equivalent equations. Therefore, if $f(x)$ has a negative leading coefficient, we can factor out -1 from $f(x)$ and apply the rule for upper and lower bounds accordingly.

Example 9: Applying the Upper and Lower Bound Theorem

Given $f(x) = 2x^5 + x^4 + 9x^2 - 32x + 20$,

a. Determine if the upper bound theorem identifies 2 as an upper bound for the real zeros of $f(x)$.

b. Determine if the lower bound theorem identifies -2 as a lower bound for the real zeros of $f(x)$.

Solution:

a. $f(x) = 2x^5 + x^4 + 9x^2 - 32x + 20$

First note that the leading coefficient of the polynomial is positive.

Divide $f(x)$ by $(x - 2)$.

$$\begin{array}{l r|rrrrrr} & 2 & 2 & 1 & 0 & 9 & -32 & 20 \\ & & & 4 & 10 & 20 & 58 & 52 \\ \cline{3-8} \text{This row nonnegative} \rightarrow & & 2 & 5 & 10 & 29 & 26 & \boxed{72} \end{array}$$

The remainder and all coefficients of the quotient are nonnegative.

2 is an upper bound for the real zeros of $f(x)$.

b. Divide $f(x)$ by $(x + 2)$.

$$\begin{array}{r|rrrrrr} -2 & 2 & 1 & 0 & 9 & -32 & 20 \\ & & -4 & 6 & -12 & 6 & 52 \\ \hline & 2 & -3 & 6 & -3 & -26 & \boxed{72} \\ & + & - & + & \underbrace{- \quad\quad -}_{\text{No sign change}} & & + \end{array}$$

The signs of the quotient do not alternate. Therefore, we cannot conclude that -2 is a lower bound for the real zeros of $f(x)$.

Apply the Skills

9. Given $f(x) = x^4 - 2x^3 - 13x^2 - 4x - 30$,

- **a.** Determine if the upper bound theorem identifies 4 as an upper bound for the real zeros of $f(x)$.
- **b.** Determine if the lower bound theorem identifies -4 as a lower bound for the real zeros of $f(x)$.

Insights

From Example 9, although we cannot conclude that -2 is a lower bound for the real zeros of $f(x)$, we can try other negative real numbers. For example, -3 is a lower bound for the real zeros of $f(x)$.

$$\begin{array}{r|rrrrrr} \underline{-3} & 2 & 1 & 0 & 9 & -32 & 20 \\ & & -6 & 15 & -45 & 108 & -228 \\ \hline & 2 & -5 & 15 & -36 & 76 & \boxed{-208} \end{array}$$

signs all alternate

Therefore, -3 is a lower bound.

In Example 10, we will use the tools presented in Lessons 2.2–2.4 to find all zeros of a polynomial.

Example 10: Finding the Zeros of a Polynomial

Find the zeros and their multiplicities. $f(x) = 2x^5 + x^4 + 9x^2 - 32x + 20$

Solution:

$f(x)$ is a fifth-degree polynomial and must have five zeros (including multiplicities). We begin by finding the rational zeros (if any exist). By the rational zero theorem, the possible rational zeros are

$$\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20, \pm\frac{1}{2}, \pm\frac{5}{2}$$

However, we also know from Example 9 that 2 is not a zero of $f(x)$, but is an upper bound for the real zeros. From the discussion in the *Insights* box following Example 9, we know that -3 is not a zero of $f(x)$, but is a lower bound for the real zeros. Therefore, we can restrict the list of possible rational zeros to those on the interval $(-3, 2)$.

$$-\frac{5}{2}, -2, \pm\frac{1}{2}, \pm 1$$

After testing several possible rational zeros, we find that 1 is a zero.

$$\begin{array}{r|rrrrrr} \underline{1} & 2 & 1 & 0 & 9 & -32 & 20 \\ & & 2 & 3 & 3 & 12 & -20 \\ \hline & 2 & 3 & 3 & 12 & -20 & \boxed{0} \end{array}$$

We have $f(x) = (x - 1)(2x^4 + 3x^3 + 3x^2 + 12x - 20)$. Now look for the zeros of the reduced polynomial. We will try 1 again because it may be a repeated zero.

$$\begin{array}{r|rrrrr} 1 & 2 & 3 & 3 & 12 & -20 \\ & & 2 & 5 & 8 & 20 \\ \hline & 2 & 5 & 8 & 20 & \boxed{0} \end{array}$$

The value 1 is a repeated zero.

We have $f(x) = (x - 1)^2(2x^3 + 5x^2 + 8x + 20)$.

Because the polynomial $2x^3 + 5x^2 + 8x + 20$ has no sign changes, Descartes' rule of signs indicates that there are no other positive real zeros. Now the list of possible rational zeros is restricted to $-\frac{5}{2}$, -2, $-\frac{1}{2}$, and -1. We find that $-\frac{5}{2}$ is a zero of $f(x)$.

$$\begin{array}{r|rrrr} -\frac{5}{2} & 2 & 5 & 8 & 20 \\ & & -5 & 0 & -20 \\ \hline & 2 & 0 & 8 & \boxed{0} \end{array}$$

So, $f(x) = (x - 1)^2\left(x + \frac{5}{2}\right)(2x^2 + 8)$.

$$\begin{aligned} 2x^2 + 8 &= 0 \\ x^2 &= -4 \\ x &= \pm 2i \end{aligned}$$

The remaining two zeros are found by solving the equation $2x^2 + 8 = 0$.

The zeros are: 1 (multiplicity of 2), $-\frac{5}{2}$, $2i$, and $-2i$ (each with multiplicity of 1).

Apply the Skills

10. Find the zeros and their multiplicities.

$$f(x) = x^5 + 6x^3 - 2x^2 - 27x - 18$$

The graph of $f(x) = 2x^5 + x^4 + 9x^2 - 32x + 20$ from Example 10 is shown in Figure 2-17. The end behavior is down to the left and up to the right as expected. The real zeros of $f(x)$ correspond to the x-intercepts $\left(-\frac{5}{2}, 0\right)$ and $(1, 0)$. The point $(1, 0)$ is a touch point because 1 is a zero with an even multiplicity. The graph crosses the x-axis at $-\frac{5}{2}$ because $-\frac{5}{2}$ is a zero with an odd multiplicity.

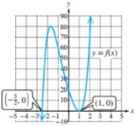
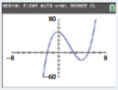


Figure 2-17

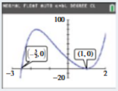
TECHNOLOGY CONNECTIONS

Applications of Graphing Utilities to Polynomials

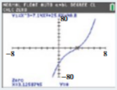
A graphing utility can help us analyze a polynomial. For example, given $f(x) = 2x^3 - 11x^2 - 5x + 50$, the possible rational zeros are: $\pm1, \pm2, \pm5, \pm10, \pm25, \pm50, \pm\frac{1}{2}, \pm\frac{5}{2}$ and $\pm\frac{25}{2}$. By graphing the function f on a graphing utility, it appears that the function may cross the x-axis at -2, 2.5, and 5. So we might consider testing these values first.


NORMAL FLOAT AUTO a+bi DEGREE CL
80
−8
8
−60

Our knowledge of algebra can also help us use a graphing device effectively. Consider $f(x) = 2x^5 + x^4 + 9x^2 - 32x + 20$ from Examples 9 and 10. We know that -3 is a lower bound for the real zeros of $f(x)$ and that 2 is an upper bound. Therefore, we can set a viewing window showing x between -3 and 2 and be guaranteed to see all the real zeros of $f(x)$.


NORMAL FLOAT AUTO a+bi DEGREE CL
100
$\left(-\frac{3}{2}, 0\right)$
(1, 0)
−3
−20
2

Finally, when analytical methods fail, we can use the Zero feature on a graphing utility to approximate the real zeros of a polynomial. Given $f(x) = x^3 - 7.14x^2 + 25.6x - 40.8$, the calculator approximates a zero of 3.1258745. The real zeros of a function can also be approximated by repeated use of the intermediate value theorem. (See the online group activity "Investigating the Bisection Method for Finding Zeros.")


NORMAL FLOAT AUTO a+bi DEGREE CL
CALC ZERO
80
−8
8
−80
Zero
X=3.1258745
Y=0

Practice 2-4

Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, perform the indicated operations. Write the answers in standard form *a* + *bi*.

R.1. $(9i)(-9i)$

R.2. $(6 + \sqrt{13}i)(6 - \sqrt{13}i)$

R.3. $(5 - 6i)^2 + (5 + 6i)^2$

Concept Connections

1. The ___________ of a polynomial $f(x)$ are the solutions (or roots) of the equation $f(x) = 0$.

2. If $f(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $f(x)$ has exactly ___________ complex zeros, provided that each zero is counted by its multiplicity.

3. The conjugate zeros theorem states that if $f(x)$ is a polynomial with real coefficients, and if $a + bi$ is a zero of $f(x)$, then ___________ is also a zero of $f(x)$.

4. A real number b is called a/an ___________ bound of the real zeros of a polynomial $f(x)$ if all real zeros of $f(x)$ are less than or equal to b.

5. A real number a is called a lower bound of the real zeros of a polynomial $f(x)$ if all real zeros of $f(x)$ are ___________ or equal to a.

6. Explain why the number 7 cannot be a rational zero of the polynomial $f(x) = 2x^3 + 5x^2 - x + 6$.

Learn: Apply the Rational Zero Theorem

For Exercises 7–12, list the possible rational zeros. (See Example 1 on p. 228)

7. $f(x) = x^5 - 2x^3 + 7x^2 + 4$

8. $g(x) = x^3 - 5x^2 + 2x - 9$

9. $h(x) = 4x^4 + 9x^3 + 2x - 6$

10. $k(x) = 25x^7 + 22x^4 - 3x^2 + 10$

11. $m(x) = -12x^6 + 4x^3 - 3x^2 + 8$

12. $n(x) = -16x^4 - 7x^3 + 2x + 6$

13. Which of the following is *not* a possible zero of $f(x) = 2x^3 - 5x^2 + 12$?

$$1,\ 7,\ \frac{5}{3},\ \frac{3}{2}$$

14. Which of the following is *not* a possible zero of $f(x) = 4x^5 - 2x^3 + 10$?

$$3,\ 5,\ \frac{5}{2},\ \frac{3}{2}$$

For Exercises 15–16, find all the rational zeros.

15. $p(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$

16. $q(x) = x^4 + x^3 - 7x^2 - 5x + 10$

For Exercises 17–28, find all the zeros. (See Examples 2–4 on pp. 228–231)

17. $c(x) = 2x^4 - 7x^3 - 17x^2 + 58x - 24$

18. $d(x) = 3x^4 - 2x^3 - 21x^2 - 4x + 12$

19. $f(x) = x^3 - 7x^2 + 6x + 20$

20. $g(x) = x^3 - 7x^2 + 14x - 6$

21. $h(x) = 5x^3 - x^2 - 35x + 7$

22. $k(x) = 7x^3 - x^2 - 21x + 3$

23. $m(x) = 3x^4 - x^3 - 36x^2 + 60x - 16$

24. $n(x) = 2x^4 + 9x^3 - 5x^2 - 57x - 45$

25. $q(x) = x^3 - 4x^2 - 2x + 20$

26. $p(x) = x^3 - 8x^2 + 29x - 52$

27. $t(x) = x^4 - x^2 - 90$

28. $v(x) = x^4 - 12x^2 - 13$

Learn: Apply the Fundamental Theorem of Algebra

29. Given a polynomial $f(x)$ of degree $n \geq 1$, the fundamental theorem of algebra guarantees at least ___________ complex zero(s).

30. The number of zeros of $f(x) = 4x^3 - 5x^2 + 6x - 3$ is ___________, provided that each zero is counted according to its multiplicity.

31. If $f(x)$ is a polynomial with real coefficients and zeros of 5 (multiplicity 2), -1 (multiplicity 1), $2i$, and $3 + 4i$, what is the minimum degree of $f(x)$?

32. If $g(x)$ is a polynomial with real coefficients and zeros of -4 (multiplicity 3), 6 (multiplicity 2), $1 + i$, and $2 - 7i$, what is the minimum degree of $g(x)$?

For Exercises 33–38, a polynomial *f*(*x*) and one or more of its zeros is given.

a. Find all the zeros.

b. Factor $f(x)$ as a product of linear factors.

c. Solve the equation $f(x) = 0$.
(See Example 5 on p. 233)

33. $f(x) = x^4 - 4x^3 + 22x^2 + 28x - 203$; $2 - 5i$ is a zero

34. $f(x) = x^4 - 6x^3 + 5x^2 + 30x - 50$; $3 - i$ is a zero

35. $f(x) = 3x^3 - 28x^2 + 83x - 68$; $4 + i$ is a zero

36. $f(x) = 5x^3 - 54x^2 + 170x - 104$; $5 + i$ is a zero

37. $f(x) = 4x^5 + 37x^4 + 117x^3 + 87x^2 - 193x - 52$; $-3 + 2i$ and $-\frac{1}{4}$ are zeros

38. $f(x) = 2x^5 - 5x^4 - 4x^3 - 22x^2 + 50x + 75$; $-1 - 2i$ and $\frac{5}{2}$ are zeros

For Exercises 39–48, write a polynomial *f*(*x*) that satisfies the given conditions. (See Example 6 on p. 234)

39. Degree 3 polynomial with integer coefficients with zeros $6i$ and $\frac{4}{5}$

40. Degree 3 polynomial with integer coefficients with zeros $-4i$ and $\frac{3}{2}$

41. Polynomial of lowest degree with zeros of -4 (multiplicity 1), 2 (multiplicity 3) and with $f(0) = 160$

42. Polynomial of lowest degree with zeros of 5 (multiplicity 2) and -3 (multiplicity 2) and with $f(0) = -450$

43. Polynomial of lowest degree with zeros of $-\frac{4}{3}$ (multiplicity 2) and $\frac{1}{2}$ (multiplicity 1) and with $f(0) = -16$

44. Polynomial of lowest degree with zeros of $-\frac{5}{6}$ (multiplicity 2) and $\frac{1}{3}$ (multiplicity 1) and with $f(0) = -25$

45. Polynomial of lowest degree with real coefficients and with zeros $7 - 4i$ and 0 (multiplicity 4)

46. Polynomial of lowest degree with real coefficients and with zeros $5 - 10i$ and 0 (multiplicity 3)

47. Polynomial of lowest degree with real coefficients and zeros of $5i$ and $6 - i$.

48. Polynomial of lowest degree with real coefficients and zeros of $-3i$ and $5 + 2i$.

Learn: Apply Descartes' Rule of Signs

For Exercises 49–56, determine the number of possible positive and negative real zeros for the given function. (See Examples 7–8 on pp. 236–238)

49. $f(x) = x^6 - 2x^4 + 4x^3 - 2x^2 - 5x - 6$

50. $g(x) = 3x^7 + 4x^4 - 6x^3 + 5x^2 - 6x + 1$

51. $k(x) = -8x^7 + 5x^6 - 3x^4 + 2x^3 - 11x^2 + 4x - 3$

52. $h(x) = -4x^9 + 6x^8 - 5x^5 - 2x^4 + 3x^2 - x + 8$

53. $p(x) = 0.11x^4 + 0.04x^3 + 0.31x^2 + 0.27x + 1.1$

54. $q(x) = -0.6x^4 + 0.8x^3 - 0.6x^2 + 0.1x - 0.4$

55. $v(x) = \frac{1}{8}x^6 + \frac{1}{6}x^4 + \frac{1}{3}x^2 + \frac{1}{10}$

56. $t(x) = \frac{1}{1000}x^6 + \frac{1}{100}x^4 + \frac{1}{10}x^2 + 1$

For Exercises 57–58, use Descartes' rule of signs to determine the total number of real zeros and the number of positive and negative real zeros. (*Hint*: First factor out *x* to its lowest power.)

57. $f(x) = x^8 + 5x^6 + 6x^4 - x^3$

58. $f(x) = -5x^8 - 3x^6 - 4x^2 + x$

Learn: Find Upper and Lower Bounds

For Exercises 59–64, (See Example 9 on p. 239)

a. Determine if the upper bound theorem identifies the given number as an upper bound for the real zeros of $f(x)$.

b. Determine if the lower bound theorem identifies the given number as a lower bound for the real zeros of $f(x)$.

59. $f(x) = x^5 + 6x^4 + 5x^2 + x - 3$

a. 2 **b.** -5

60. $f(x) = x^4 + 8x^3 - 4x^2 + 7x - 3$

a. 3 **b.** -4

61. $f(x) = 8x^3 - 42x^2 + 33x + 28$

a. 6 **b.** -1

62. $f(x) = 6x^3 - x^2 - 57x + 70$

a. 4 **b.** -4

63. $f(x) = 2x^5 + 11x^4 - 63x^2 - 50x + 40$

a. 3 **b.** -6

64. $f(x) = 3x^5 - 16x^4 + 5x^3 + 90x^2 - 138x + 36$

a. 6 **b.** -3

For Exercises 65–68, determine if the statement is true or false. If a statement is false, explain why.

65. If 5 is an upper bound for the real zeros of $f(x)$, then 6 is also an upper bound.

66. If 5 is an upper bound for the real zeros of $f(x)$, then 4 is also an upper bound.

67. If -3 is a lower bound for the real zeros of $f(x)$, then -2 is also a lower bound.

68. If -3 is a lower bound for the real zeros of $f(x)$, then -4 is also a lower bound.

For Exercises 69–84, find the zeros and their multiplicities. Consider using Descartes' rule of signs and the upper and lower bound theorem to limit your search for rational zeros. (See Example 10 on p. 240)

69. $f(x) = 8x^3 - 42x^2 + 33x + 28$

(*Hint*: **See Exercise 61.**)

70. $f(x) = 6x^3 - x^2 - 57x + 70$

(*Hint*: See **Exercise 62.**)

71. $f(x) = 2x^5 + 11x^4 - 63x^2 - 50x + 40$

(*Hint*: **See Exercise 63.**)

72. $f(x) = 3x^5 - 16x^4 + 5x^3 + 90x^2 - 138x + 36$

(*Hint*: **See Exercise 64.**)

73. $f(x) = 4x^4 + 20x^3 + 13x^2 - 30x + 9$

74. $f(x) = 9x^4 + 30x^3 + 13x^2 - 20x + 4$

75. $f(x) = 2x^4 - 11x^3 + 27x^2 - 41x + 15$

76. $g(x) = 3x^4 - 20x^3 + 51x^2 - 56x + 20$

77. $h(x) = 4x^4 - 28x^3 + 73x^2 - 90x + 50$

78. $k(x) = 9x^4 - 42x^3 + 70x^2 - 34x + 5$

79. $f(x) = x^6 + 2x^5 + 11x^4 + 20x^3 + 10x^2$

80. $f(x) = x^6 + 6x^5 + 12x^4 + 18x^3 + 27x^2$

81. $f(x) = x^5 - 10x^4 + 34x^3$

82. $f(x) = x^6 - 12x^5 + 40x^4$

83. $f(x) = -x^3 + 3x^2 - 9x - 13$

84. $f(x) = -x^3 + 5x^2 - 11x + 15$

Mixed Exercises

For Exercises 85–90, determine if the statement is true or false. If a statement is false, explain why.

85. A polynomial with real coefficients of degree 4 must have at least one real zero.

86. Given $f(x) = 2ix^4 - (3 + 6i)x^3 + 5x^2 + 7$, if $a + bi$ is a zero of $f(x)$, then $a - bi$ must also be a zero.

87. The graph of a 10th-degree polynomial must cross the x-axis exactly once.

88. Suppose that $f(x)$ is a polynomial, and that a and b are real numbers where $a < b$. If $f(a) < 0$ and $f(b) < 0$, then $f(x)$ has no real zeros on the interval $[a, b]$.

89. If c is a zero of a polynomial $f(x)$, with degree $n \geq 2$ then all other zeros of $f(x)$ are zeros of $\frac{f(x)}{x - c}$.

90. If b is an upper bound for the real zeros of a polynomial, then $-b$ is a lower bound for the real zeros of the polynomial.

91. Given that $x - c$ divides evenly into a polynomial $f(x)$, which statements are true?

- **a.** $x - c$ is a factor of $f(x)$.
- **b.** c is a zero of $f(x)$.
- **c.** The remainder of $f(x) \div (x - c)$ is 0.
- **d.** c is a solution (root) of the equation $f(x) = 0$.

92. a. Use the quadratic formula to solve $x^2 - 7x + 5 = 0$.

b. Write $x^2 - 7x + 5$ as a product of linear factors.

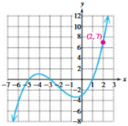
93. a. Use the intermediate value theorem to show that $f(x) = 2x^2 - 7x + 4$ has a real zero on the interval $[2, 3]$.

b. Find the zeros.

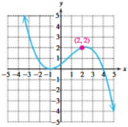
94. Show that $x - a$ is a factor of $x^n - a^n$ for any positive integer n and constant a.

For Exercises 95–98, use the graph of the polynomial function and the given point to write the factored form of a polynomial function f of lowest degree with only real roots.

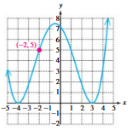
95.



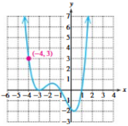
96.



97.



98.



Write About It

99. Explain why a polynomial with real coefficients of degree 3 must have at least one real zero.

100. Why is it not necessary to apply the rational zero theorem, Descartes' rule of signs, or the upper and lower bound theorem to find the zeros of a second-degree polynomial?

101. Explain why $f(x) = 5x^6 + 7x^4 + x^2 + 9$ has no real zeros.

102. Explain why the fundamental theorem of algebra does not apply to $f(x) = \sqrt{x} + 3$. That is, no complex number c exists such that $f(c) = 0$.

Expanding Your Skills

103. Suppose that $3i$ is a zero of $f(x) = x^3 - 3ix^2 - 9x + 27i$ and $g(x) = x^3 - 3x^2 + 9x - 27$.

- **a.** Factor $f(x)$ into linear factors.
- **b.** What are the zeros of $f(x)$?
- **c.** Factor $g(x)$ into linear factors.
- **d.** What are the zeros of $g(x)$?
- **e.** The value $3i$ is an imaginary zero of $f(x)$, but its conjugate $-3i$ is not. Why does this not contradict the conjugate zeros theorem?

104. Suppose that $\sqrt{2}$ is a zero of $p(x) = x^3 - \sqrt{2}x^2 - 4x + 4\sqrt{2}$ and $q(x) = x^3 - 2x^2 - 2x + 4$.

a. Factor $p(x)$ into linear factors.

b. What are the zeros of $p(x)$?

c. Factor $q(x)$ into linear factors.

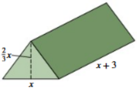
d. What are the zeros of $q(x)$?

e. Given a polynomial with rational coefficients, if an irrational zero exists, then its conjugate must also be a zero. The value $\sqrt{2}$ is an irrational zero of $p(x)$, but its conjugate $-\sqrt{2}$ is not. Why does this not contradict the given statement?

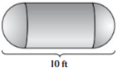
105. Let n be a positive even integer. Determine the greatest number of possible nonreal zeros of $f(x) = x^n - 1$.

106. Let n be a positive odd integer. Determine the greatest number of possible nonreal zeros of $f(x) = x^n - 1$.

107. The front face of a tent is triangular and the height of the triangle is two-thirds of the base. The length of the tent is 3 ft more than the base of the triangular face. If the tent holds a volume of 108 ft³, determine its dimensions.



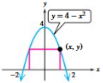
108. An underground storage tank for gasoline is in the shape of a right circular cylinder with hemispheres on each end. If the total volume of the tank is $\frac{104\pi}{3}$ ft³, find the radius of the tank.



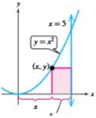
109. A food company originally sells cereal in boxes with dimensions 10 in. by 7 in. by 2.5 in. To make more profit, the company decreases each dimension of the box by x inches but keeps the price the same. If the new volume is 81 in.³ by how much was each dimension decreased?

110. A truck rental company rents a 12-ft by 8-ft by 6-ft truck for $69.95 per day plus mileage. A customer prefers to rent a less expensive smaller truck whose dimensions are x ft smaller on each side. If the volume of the smaller truck is 240 ft³, determine the dimensions of the smaller truck.

111. A rectangle is bounded by the x-axis and a parabola defined by $y = 4 - x^2$. What are the dimensions of the rectangle if the area is 6 cm²? Assume that all units of length are in centimeters.



112. A rectangle is bounded by the parabola defined by $y = x^2$, the x-axis, and the line $x = 5$ as shown in the figure. If the area of the rectangle is 12 in.² determine the dimensions of the rectangle.



Point of Interest

Early in the sixteenth century, Italian mathematicians Niccolo Tartaglia and Gerolamo Cardano solved a general cubic equation in terms of the constants appearing in the equation. Cardano's pupil, Ludovico Ferrari, then solved a general equation of fourth degree. Despite decades of work, no general solution to a fifth-degree equation was found. Finally, Norwegian mathematician Niels Abel and French mathematician Evariste Galois proved that no such solution exists.

The linear factorization theorem tells us that a polynomial of degree $n \geq 1$ factors into n linear factors over the complex numbers. If we do not factor over the set of complex numbers, then a polynomial with real coefficients can be factored into linear factors and irreducible quadratic factors. An *irreducible quadratic factor* is a quadratic polynomial that does not factor further over the set of real numbers.

For example, consider the polynomial $f(x) = x^4 - 5x^3 + 5x^2 + 25x - 26$.

Factoring over the real numbers, we have two linear factors and one irreducible quadratic factor:

$$x^4 - 5x^3 + 5x^2 + 25x - 26 = \overbrace{(x+2)(x-1)}^{\text{2 linear factors}}\overbrace{(x^2 - 6x + 13)}^{\text{irreducible quadratic factor}}$$

Factoring over the complex numbers, we have four linear factors as guaranteed by the linear factorization theorem.

$$x^4 - 5x^3 + 5x^2 + 25x - 26 = (x+2)(x-1)[x - (3+2i)][x - (3-2i)]$$

For Exercises 113–116,

a. Factor the polynomial over the set of real numbers.

b. Factor the polynomial over the set of complex numbers.

113. $f(x) = x^4 + 2x^3 + x^2 + 8x - 12$

114. $f(x) = x^4 - 6x^3 + 9x^2 - 6x + 8$

115. $f(x) = x^4 + 2x^2 - 35$

116. $f(x) = x^4 + 8x^2 - 33$

117. Find all fourth roots of 1 by solving the equation $x^4 = 1$. (*Hint*: Find the zeros of the polynomial $f(x) = x^4 - 1$.)

118. Find all sixth roots of 1 by solving the equation $x^6 = 1$. [*Hint*: Find the zeros of the polynomial $f(x) = x^6 - 1$. Begin by factoring $x^6 - 1$ as $(x^3 - 1)(x^3 + 1)$.]

119. Use the rational zero theorem to show that $\sqrt{5}$ is an irrational number. (*Hint*: Show that $f(x) = x^2 - 5$ has no rational zeros.)

120. a. Given a linear equation $ax + b = 0$ $(a \neq 0)$, the solution is given by $x =$ ____________.

b. Given a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$, the solutions are given by $x =$ ______________.

From Exercise 120, we see that linear and quadratic equations have generic formulas that can be used to find the solution sets. But what about a cubic polynomial equation? Mathematicians struggled for centuries to find such a formula. Finally, Italian mathematician Niccolo Tartaglia (1500–1557) developed a method to solve a cubic equation of the form

$$x^3 + mx = n$$

The result was later published in *Ars Magna*, by Gerolamo Cardano (1501–1576).

For Exercises 121–122, use the formula

$$x = \sqrt[3]{\sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3} + \frac{n}{2}} - \sqrt[3]{\sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3} - \frac{n}{2}}$$

to find a solution to the equation $x^3 + mx = n$.

121. $x^3 - 3x = -2$

122. $x^3 + 9x = 26$

Go online for more practice problems.

Lesson 2-5

Introduction to Rational Functions

What Will You Learn?

After completing this lesson, you should be able to:

- Apply Notation Describing Infinite Behavior of a Function
- Identify Vertical Asymptotes
- Identify Horizontal Asymptotes
- Identify Slant and Nonlinear Asymptotes

Learn: Apply Notation Describing Infinite Behavior of a Function

A group of friends go for pizza after a college football game. The total bill, including tax, comes to \$120, and the group decides to split the bill evenly.

- If there are 12 people in the group, then each person would pay $\frac{\$120}{12} = \$10$.
- If there are 15 people in the group, then each person would pay $\frac{\$120}{15} = \$8$.
- If there are 20 people in the group, then each person would pay $\frac{\$120}{20} = \$6$.

In general, for a group of x people, the cost per person $C(x)$ (in dollars) for the \$120 bill is given by $C(x) = \frac{120}{x}$. The value $C(x)$ is defined by the ratio of two polynomials. For this reason, C is called a *rational function*. Rational functions are used in a variety of applications, including in business to model average cost.

Definition of a Rational Function

Let $p(x)$ and $q(x)$ be polynomials where $q(x) \neq 0$. A function f defined by

$f(x) = \frac{p(x)}{q(x)}$ is called a **rational function**.

Note: The domain of a rational function is all real numbers excluding the real zeros of $q(x)$.

Examples of Domains for Rational Functions:

- $f(x) = \frac{1}{x}$ Domain $= \{x \mid x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$
- $g(x) = \frac{5x^2}{2x^2 + 5x - 12}$ Domain $= \left\{x \mid x \neq -4, x \neq \frac{3}{2}\right\}$ or
  $= \frac{5x^2}{(2x-3)(x+4)}$ $(-\infty, -4) \cup \left(-4, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$
- $k(x) = \frac{x+3}{x^2+4}$ Domain $= \mathbb{R}$ or $(-\infty, \infty)$ since $x^2 + 4 \neq 0$ for any real x.

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In this lesson, we will analyze rational functions. To do so, we want to determine the behavior of the function as x approaches ∞ or $-\infty$ and as x approaches values for which the function is undefined. We will use the following notation (Table 2-1).

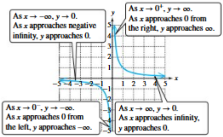
TABLE 2-1 Notation Used When Exploring Behavior of a Function

Notation	Meaning
$x \to c^+$	x approaches c from the right (but will not equal c).
$x \to c^-$	x approaches c from the left (but will not equal c).
$x \to \infty$	x approaches infinity (x increases without bound).
$x \to -\infty$	x approaches negative infinity (x decreases without bound).

For example, review the behavior of the reciprocal function $f(x) = \frac{1}{x}$ near the horizontal and vertical asymptotes (Figure 2-18).

x	$f(x) = \frac{1}{x}$
−1	−1
−10	−0.1
−100	−0.01
−1000	−0.001

x	$f(x) = \frac{1}{x}$
−1	−1
−0.1	−10
−0.01	−100
−0.001	−100



As $x \to -\infty$, $y \to 0$.
As x approaches negative infinity, y approaches 0.

As $x \to 0^+$, $y \to \infty$.
As x approaches 0 from the right, y approaches ∞.

As $x \to 0^-$, $y \to -\infty$.
As x approaches 0 from the left, y approaches $-\infty$.

As $x \to \infty$, $y \to 0$.
As x approaches infinity, y approaches 0.

Figure 2-18

x	$f(x) = \frac{1}{x}$
1	1
0.1	10
0.01	100
0.001	1000

x	$f(x) = \frac{1}{x}$
1	1
10	0.1
100	0.01
1000	0.001

In Example 1, we study the graph of another basic rational function, $f(x) = \frac{1}{x^2}$. From the definition of the function, we make the following observations.

- The domain of $f(x) = \frac{1}{x^2}$ is all real numbers excluding zero.
- f is an even function, and the graph is symmetric to the y-axis.
- The values of $f(x)$ are positive over the domain of f.

Example 1: Investigating the Behavior of a Rational Function

The graph of $f(x) = \frac{1}{x^2}$ is given.

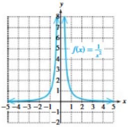
Complete the statements.

a. As $x \to -\infty$, $f(x) \to$ __________.

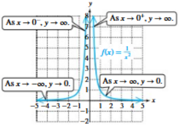
b. As $x \to 0^-$, $f(x) \to$ __________.

c. As $x \to 0^+$, $f(x) \to$ __________.

d. As $x \to \infty$, $f(x) \to$ __________.



Solution:



a. As $x \to -\infty$, $f(x) \to 0$.

b. As $x \to 0^-$, $f(x) \to \infty$.

c. As $x \to 0^+$, $f(x) \to \infty$.

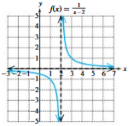
d. As $x \to \infty$, $f(x) \to 0$.

Insights

The answers to Example 1 are identified from the graph of $f(x) = \frac{1}{x^2}$. To reinforce your understanding, you can substitute values of x that simulate the given condition. For Example 1(a), try substituting $x = -10$, $x = -100$, and $x = -1000$. The value of $f(x)$ gets progressively closer to 0.

$$f(-10) = 0.01$$
$$f(-100) = 0.0001$$
$$f(-1000) = 0.000001$$

Apply the Skills



1. The graph of $f(x) = \frac{1}{x-2}$ is given. Complete the statements.

a. As $x \to -\infty$, $f(x) \to$ ______________.

b. As $x \to 2^-$, $f(x) \to$ ______________.

c. As $x \to 2^+$, $f(x) \to$ ______________.

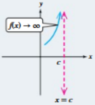
d. As $x \to \infty$, $f(x) \to$ ______________.

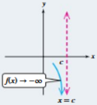
Learn: Identify Vertical Asymptotes

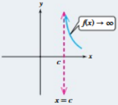
The graphs of $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$ both approach the y-axis, but do not touch the y-axis. The y-axis is called a vertical asymptote of the graphs of the functions.

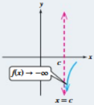
Definition of a Vertical Asymptote

The line $x = c$ is a **vertical asymptote** of the graph of a function f if $f(x)$ approaches infinity or negative infinity as x approaches c from either side.









A function may have no vertical asymptotes, one vertical asymptote, or many vertical asymptotes. To locate the vertical asymptotes of a function, determine the real numbers x where the denominator is zero, but the numerator is nonzero.

Identifying Vertical Asymptotes of a Rational Function

Consider a rational function f defined by $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ have no common factors other than 1. If c is a real zero of $q(x)$, then $x = c$ is a vertical asymptote of the graph of f.

Example 2: Identifying Vertical Asymptotes

Identify the vertical asymptotes.

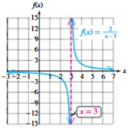
a. $f(x) = \frac{2}{x-3}$ **b.** $g(x)\frac{x-4}{3x^2+5x-2}$ **c.** $k(x) = \frac{4x^2}{x^2+4}$

Solution:

a. $f(x) = \frac{2}{x-3}$

The expression $\frac{2}{x-3}$ is written in lowest terms. The denominator is zero for $x = 3$.

f has a vertical asymptote of $x = 3$.



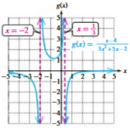
b. $g(x) = \frac{x-4}{3x^2+5x-2}$

$= \frac{x-4}{(3x-1)(x+2)}$

Factor the numerator and denominator.

The numerator and denominator share no common factors other than 1. The zeros of the denominator are $\frac{1}{3}$ and -2.

The vertical asymptotes are $x = \frac{1}{3}$ and $x = -2$.

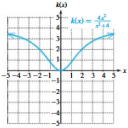


Insights The zeros of the denominator occur where $3x - 1 = 0$ and $x + 2 = 0$.

c. $k(x) = \frac{4x^2}{x^2 + 4}$

The numerator and denominator are already factored over the real numbers and the rational expression is in lowest terms.

For any real number x, the expression $x^2 + 4$ is positive. Therefore, the denominator of $k(x) = \frac{4x^2}{x^2 + 4}$ has no real zeros, and the graph of k has no vertical asymptotes.



Apply the Skills

2. Identify the vertical asymptotes.

a. $f(x) = \frac{3}{x + 1}$ **b.** $h(x)\frac{x + 7}{2x^2 - x - 10}$ **c.** $m(x) = \frac{5x}{x^4 + 1}$

The procedure to find the vertical asymptotes of a rational function is given under the condition that the numerator and denominator share no common factors. If this condition is not met, the graph of a rational function shows a different type of behavior. We investigate this situation in Example 3.

Example 3: Identifying Vertical Asymptotes

Given $f(x) = \frac{3x - 6}{x^2 - x - 2}$,

a. For what values is the function undefined?

b. Write the domain of f in interval notation.

c. Identify the vertical asymptote(s), if any.

Solution:

a. $f(x) = \frac{3x - 6}{x^2 - x - 2}$

$= \frac{3(x - 2)}{(x - 2)(x + 1)}$ Factor the numerator and denominator, and determine where the denominator is zero.

Function f is undefined for $x = 2$ and $x = -1$.

b. The domain is $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$.

c. The vertical asymptotes occur where the denominator is zero, but the numerator is nonzero. This occurs for $x = -1$. (Note that at $x = 2$, *both* the numerator and denominator are zero. As such, $x = 2$ is *not* a vertical asymptote.)

Apply the Skills

3. Repeat Example 3 with $k(x) = \frac{2x+6}{x^2+2x-3}$.

The function defined by $f(x) = \frac{3x-6}{x^2-x-2}$, or equivalently $f(x) = \frac{3(x-2)}{(x-2)(x+1)}$, is not defined for $x = 2$ or $x = -1$ because the denominator is zero. At $x = -1$, the numerator is nonzero, indicating that f has a vertical asymptote. However, at $x = 2$, *both* the numerator and denominator are zero. This means that $f(x)$ can be written as $f(x) = \frac{(x-2)}{(x-2)} \cdot \frac{3}{x+1}$ or equivalently, $f(x) = \frac{3}{x+1}$, provided that $x \neq 2$. Because 2 is not in the domain of f, the graph of f has a "hole" at $x = 2$ (Figure 2-19).

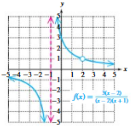


Figure 2-19

Learn: Identify Horizontal Asymptotes

Refer to the graph of $f(x) = \frac{1}{x}$ (Figure 2-20). Toward the far left and far right of the graph, $f(x)$ approaches the line $y = 0$ (the x-axis). The x-axis is called a horizontal asymptote of the graph of f.

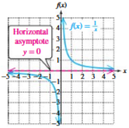
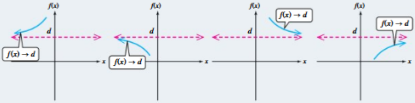


Figure 2-20

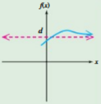
Definition of a Horizontal Asymptote

The line $y = d$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches d as x approaches infinity or negative infinity.



Insights

While the graph of a function may not cross a vertical asymptote, it may cross a horizontal asymptote.



Recall that the leading term determines the far left and far right behavior of the graph of a polynomial function. Since a rational function is the ratio of two polynomials, it seems reasonable that the leading terms of the numerator and denominator determine the end behavior of a rational function.

Identifying Horizontal Asymptotes of a Rational Function

Let f be a rational function defined by

$$f(x) = \frac{a_n x^n + a_{n-1}x^{x-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \ldots + b_1 x + b_0}$$

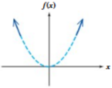
The definition of $f(x)$ indicates that n is the degree of the numerator and m is the degree of the denominator.

1. If $n > m$, then f has no horizontal asymptote.
2. If $n < m$, then the line $y = 0$ (the x-axis) is the horizontal asymptote of f.
3. If $n = m$, then the line $y = \frac{a_n}{b_m}$ is the horizontal asymptote of f.

Insights

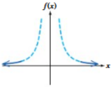
A rational function may have many vertical asymptotes, but at most one horizontal asymptote.

To illustrate these three cases, consider the following examples with numerical values given to m and n.



1. If the degree of the numerator is greater than the degree of the denominator ($n > m$), then the numerator will "dominate" the quotient. For example:

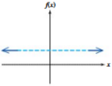
   $f(x) = \frac{x^4 + \cdots}{x^2 + \cdots}$ will behave like $y = x^2$ as $|x|$ becomes large. Therefore, f has no horizontal asymptote.



2. If the degree of the numerator is less than the degree of the denominator ($n < m$), then the denominator will "dominate" the quotient. For example:

   $f(x) = \frac{x^2 + \cdots}{x^4 + \cdots}$ will behave like $y = \frac{1}{x^2}$. The ratio $\frac{1}{x^2}$ tends toward 0 as $|x|$ becomes large. Therefore, f has a horizontal asymptote of $y = 0$.

3. If the degree of the numerator is equal to the degree of the denominator ($n = m$), then the magnitude of the numerator and denominator somewhat "offset" each other. As a result, the function tends toward a constant value equal to the ratio of the leading coefficients. For example:



$f(x) = \frac{4x^2 + \cdots}{3x^2 + \cdots}$ will behave like $y = \frac{4}{3}$ as $|x|$ becomes large. Therefore, f has a horizontal asymptote of $y = \frac{4}{3}$.

Example 4: Identifying Horizontal Asymptotes

Find the horizontal asymptotes (if any) for the given functions.

a. $f(x) = \frac{8x^2 + 1}{x^4 + 1}$ **b.** $g(x)\frac{2x^3 - 6x}{x^2 + 4}$ **c.** $h(x) = \frac{8x^2 + 9x - 5}{2x^2 + 1}$

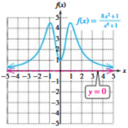
Solution:

a. $f(x) = \frac{8x^2 + 1}{x^4 + 1}$

The degree of the numerator is 2 ($n = 2$).

The degree of the denominator is 4 ($m = 4$).

Since $n < m$, then the line $y = 0$ (the x-axis) is a horizontal asymptote of f.

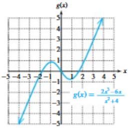


b. $g(x) = \frac{2x^3 - 6x}{x^2 + 4}$

The degree of the numerator is 3 ($n = 3$).

The degree of the denominator is 2 ($m = 2$).

Since $n > m$, then the function has no horizontal asymptotes.

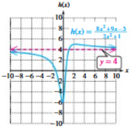


c. $h(x) = \frac{8x^2 + 9x - 5}{2x^2 + 1}$

The degree of the numerator is 2 ($n = 2$).

The degree of the denominator is 2 ($m = 2$).

Since $n = m$, then the line $y = \frac{8}{2}$ or equivalently $y = 4$ is a horizontal asymptote of the graph of f.



Apply the Skills

4. Find the horizontal asymptotes (if any) for the given functions.

a. $f(x) = \frac{7x^2 + 2x}{4x^2 - 3}$ **b.** $m(x)\frac{4x^3 + 2}{2x - 1}$ **c.** $n(x) = \frac{5}{4x^2 + 9}$

The graph of a rational function may not cross a vertical asymptote. However, as demonstrated in Example 4(c), the graph may cross a horizontal asymptote. For the purpose of graphing a rational function, it is helpful to determine where a graph crosses a horizontal asymptote.

Suppose that the line $y = d$ is a horizontal asymptote of a rational function $y = f(x)$. The solutions to the equation $f(x) = d$ are the values of x where the graph of f crosses its horizontal asymptote. If the equation has no real solution, then the graph does not cross its horizontal asymptote.

Example 5: Determining Where a Graph Crosses a Horizontal Asymptote

Given $h(x) = \frac{8x^2 + 9x - 5}{2x^2 + 1}$ determine the point where the graph of h crosses its horizontal asymptote.

Solution:

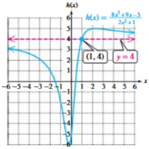
$h(x) = \frac{8x^2 + 9x - 5}{2x^2 + 1}$ From Example 4(c), the horizontal asymptote is $y = 4$.

$\frac{8x^2 + 9x - 5}{2x^2 + 1} = 4$ Set $h(x)$ equal to 4.

$\frac{8x^2 + 9x - 5}{2x^2 + 1} \cdot (2x^2 + 1) = 4 \cdot (2x^2 + 1)$ Clear fractions by multiplying by the LCD.

$$\begin{aligned} 8x^2 + 9x - 5 &= 8x^2 + 4 \\ 9x - 5 &= 4 \\ 9x &= 9 \\ x &= 1 \end{aligned}$$

The function crosses its horizontal asymptote at (1, 4).



Apply the Skills

5. Given $g(x) = \frac{3x^2 + 4x - 3}{x^2 + 3}$ determine the horizontal asymptote and the point where the graph crosses the horizontal asymptote.

Learn: Identify Slant and Nonlinear Asymptotes

In Example 4(b) we determined that the function defined by $f(x) = \frac{2x^3 - 6x}{x^2 + 4}$ has no horizontal asymptotes because the degree of the numerator is greater than the degree of the denominator.

However, as x approaches infinity and negative infinity, the graph approaches the slanted line $y = 2x$ (Figure 2-21). The line $y = 2x$ is called a *slant asymptote* of the function. A rational function has a slant asymptote if the degree of the numerator is exactly one greater than the degree of the denominator. In this case, the degree of the numerator is 3 and the degree of the denominator is 2.

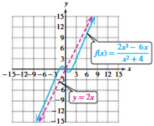


Figure 2-21

Now suppose the degree of the numerator is *more than one* greater than the degree of the denominator. In this case, the far left and far right behavior of the function will follow a nonlinear pattern. It will resemble the end behavior of the power function given by the ratio of the leading terms. For example, given $g(x) = \frac{x^4 + x^2 + 1}{x^2}$, the degree of the numerator (4) is two more than the degree of the denominator (2). Notice that the end behavior of g resembles the end behavior of $y = \frac{x^4}{x^2} = x^2$ (see Figure 2-22).

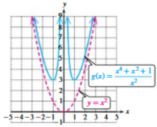


Figure 2-22

Slant and Nonlinear Asymptotes

Let f be a rational function defined by

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \cdots + b_1 x + b_0}$$

Note that n is the degree of the numerator and m is the degree of the denominator.

1. If $n = m + 1$ (that is, the degree of the numerator is exactly 1 greater than the degree of the denominator), the function has a **slant asymptote.**
2. If $n > m + 1$ (that is, the degree of the numerator is more than 1 greater than the degree of the denominator), the end behavior of the function resembles the end behavior of the power function $y = \frac{a_n x^n}{b_m x^m}$.

To find an equation of a slant or nonlinear asymptote, divide the numerator of $f(x)$ by the denominator and determine the quotient $Q(x)$. The equation $y = Q(x)$ is an equation of the asymptote.

Insights

A rational function cannot have both a horizontal asymptote and a slant or nonlinear asymptote.

Example 6: Identifying the Asymptotes of a Rational Function

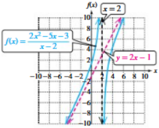
Determine the asymptotes. $f(x) = \dfrac{2x^2 - 5x - 3}{x - 2}$

Solution:

$f(x) = \dfrac{2x^2 - 5x - 3}{x - 2}$	The expression $\dfrac{2x^2 - 5x - 3}{x - 2} = \dfrac{(2x + 1)(x - 3)}{x - 2}$ is in lowest terms, and the denominator is
f has a vertical asymptote of $x = 2$.	zero at $x = 2$.
f has no horizontal asymptote.	The degree of the numerator is exactly one greater than the degree of the denominator. Therefore, f has no horizontal asymptote, but it does have a slant asymptote.

To find the slant asymptote, divide $(2x^2 - 5x - 3)$ by $(x - 2)$.

$$\begin{array}{r l}
 & \phantom{2x^2 +{}} 2x - 1 \\
x - 2 \,\big) & \overline{2x^2 + 5x - 3} \\
 & \underline{-(2x^2 - 4x)} \\
 & -x - 3 \\
 & \underline{-(-x + 2)} \\
 & -5
\end{array}$$



The quotient is $2x - 1$.

The slant asymptote is $y = 2x - 1$.

Insights

In Example 6, the divisor is of the form $x - c$. Therefore, we can also use synthetic division.

$$\begin{array}{c|ccc}
2 & 2 & -5 & -3 \\
 & & 4 & -2 \\
\hline
 & 2 & -1 & \boxed{-5}
\end{array}$$

Apply the Skills

6. Determine the asymptotes. $g(x) = \dfrac{2x^2 - 9}{x + 1}$

Example 7: Finding an Equation of a Nonlinear Asymptote

Find an equation of the nonlinear asymptote. $f(x) = \dfrac{x^5 - 3x^3 + 1}{x^3 + x}$

Solution:

The degree of the numerator (5) is exactly two more than the degree of the denominator (3). Therefore, we expect the far left and far right behavior of f to be similar to the end behavior of $y = \frac{x^5}{x^3} = x^2$. That is, f has a nonlinear asymptote that is quadratic. To find an equation of the asymptote, divide the numerator by the denominator.

$$\begin{array}{r}
x^2 \qquad -4 \\
x^3 + 0x^2 + x\overline{)x^5 + 0x^4 - 3x^3 + 0x^2 + 0x + 1} \\
\underline{-(x^5 + 0x^4 + x^3)} \qquad\qquad\qquad \\
-4x^3 + 0x^2 + 0x \quad\;\; \\
\underline{-(-4x^3 + 0x^2 - 4x)} \quad\; \\
4x + 1
\end{array}$$

The quotient is $x^2 - 4$.

The nonlinear asymptote is $y = x^2 - 4$ (shown in red in Figure 2-23).

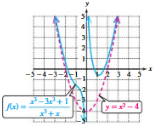


Figure 2-23

Apply the Skills

7. Find an equation of the nonlinear asymptote.

$$f(x) = \frac{-2x^4 + 8x^2 - 6}{x^2 - x}$$

Practice 2-5

Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, simplify the expression and state the restrictions on the variable.

R.1. $\frac{q^2 - 36}{q^2 - 4q - 12}$

R.2. $\frac{16 - v^2}{5v^2 - 20v}$

For Exercises R.3–R.4, divide the polynomials. Use either synthetic division or long division as appropriate.

R.3. $(3x^4 - 2x^3 + x - 4) \div (x^2 - 3x)$

R.4. $(6x^3 - 3x^2 + x - 2) \div (x + 3)$

For Exercises R.5–R.6, find the real solutions to the equation.

R.5. $\frac{2x - 3}{x^2 - x - 12} = 0$

R.6. $\frac{3x^2 - 5}{3x^2 + 2x + 1} = 1$

Concept Connections

1. The domain of a rational function defined by $f(x) = \frac{p(x)}{q(x)}$ is all real numbers excluding the zeros of __________.

2. The notation $x \to \infty$ is read as __________.

3. The notation $x \to 5^-$ is read as __________.

4. The line $x = c$ is a __________ asymptote of the graph of a function f if $f(x)$ approaches infinity or negative infinity as x approaches __________ from either the left or right.

5. To locate the vertical asymptotes of a function, determine the real numbers x where the denominator is zero, but the numerator is __________.

6. Consider a rational function in which the degree of the numerator is n and the degree of the denominator is m. If n __________ m, then the x-axis is the horizontal asymptote. If n __________ m, then the function has no horizontal asymptote.

Learn: Apply Notation Describing Infinite Behavior of a Function

For Exercises 7–12, write the domain of the function in interval notation.

7. $f(x) = \frac{x^2 - 25}{x - 5}$

8. $g(x) = \frac{x^2 - 9}{x - 3}$

9. $r(x) = \frac{2x - 3}{4x^2 + 3x - 1}$

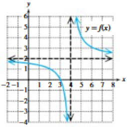
10. $p(x) = \frac{3x - 5}{2x^2 + 5x - 7}$

11. $h(x) = \frac{18x}{x^2 + 100}$

12. $k(x) = \frac{14}{x^2 + 49}$

For Exercises 13–16, refer to the graph of the function and complete the statement. (See Example 1 on p. 250)

13.



a. As $x \to -\infty$, $f(x) \to$ __________.

b. As $x \to 4^-$, $f(x) \to$ __________.

c. As $x \to 4^+$, $f(x) \to$ __________.

d. As $x \to \infty$, $f(x) \to$ __________.

e. The graph is increasing over the interval(s) __________.

f. The graph is decreasing over the interval(s) __________.

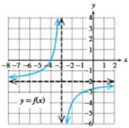
g. The domain is __________.

h. The range is __________.

i. The vertical asymptote is the line __________.

j. The horizontal asymptote is the line __________.

14.



a. As $x \to -\infty$, $f(x) \to$ __________.

b. As $x \to -3^-$, $f(x) \to$ __________.

c. As $x \to -3^+$, $f(x) \to$ __________.

d. As $x \to \infty$, $f(x) \to$ __________.

e. The graph is increasing over the interval(s) __________________.

f. The graph is decreasing over the interval(s) __________________.

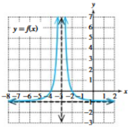
g. The domain is __________________.

h. The range is __________________.

i. The vertical asymptote is the line __________________.

j. The horizontal asymptote is the line __________________.

15.



a. As $x \to -\infty$, $f(x) \to$ __________________.

b. As $x \to -3^-$, $f(x) \to$ __________________.

c. As $x \to -3^+$, $f(x) \to$ __________________.

d. As $x \to \infty$, $f(x) \to$ __________________.

e. The graph is increasing over the interval(s) __________________.

f. The graph is decreasing over the interval(s) __________________.

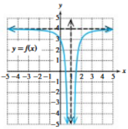
g. The domain is __________________.

h. The range is __________________.

i. The vertical asymptote is the line __________________.

j. The horizontal asymptote is the line __________________.

16.



a. As $x \to {}^{-}\infty$, $f(x) \to$ __________________.

b. As $x \to 1^-$, $f(x) \to$ __________________.

c. As $x \to 1^+$, $f(x) \to$ __________________.

d. As $x \to \infty$, $f(x) \to$ __________________.

e. The graph is increasing over the interval(s) __________________.

f. The graph is decreasing over the interval(s) __________________.

g. The domain is __________________.

h. The range is __________________.

i. The vertical asymptote is the line __________________.

j. The horizontal asymptote is the line __________________.

Learn: Identify Vertical Asymptotes

For Exercises 17–24, determine the vertical asymptotes of the graph of the function. (See Example 2 on p. 252)

17. $f(x) = \dfrac{8}{x - 4}$

18. $g(x) = \dfrac{2}{x + 7}$

19. $h(x) = \dfrac{x - 3}{2x^2 - 9x - 5}$

20. $k(x) = \dfrac{x + 2}{3x^2 + 8x - 3}$

21. $m(x) = \dfrac{x}{x^2 + 5}$

22. $n(x) = \dfrac{6}{x^4 + 1}$

23. $f(t) = \dfrac{t^2 + 2}{2t^2 + 4t - 3}$

24. $k(a) = \dfrac{5 + a^4}{3a^2 + 4a - 1}$

25. Consider $f(x) = 2x - 4$.

a. Describe the graph of f.

b. Suppose that function g is defined similar to f, but with the added restriction that $x \neq 1$. So, $g(x) = 2x - 4$ for $x \neq 1$. How would the graph of g differ from the graph of f?

26. Consider $p(x) = 3x + 2$.

a. Describe the graph of p.

b. Suppose that function q is defined similar to p, but with the added restriction that $x \neq -4$. So, $q(x) = 3x + 2$ for $x \neq -4$. How would the graph of q differ from the graph of p?

For Exercises 27–30,

a. Write the domain of f in interval notation. **(See Example 3 on p. 253)**

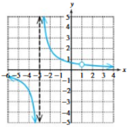
b. Simplify the rational expression defining the function.

c. Identify any vertical asymptotes.

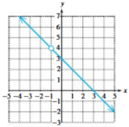
d. Identify any other values of x (other than those corresponding to vertical asymptotes) for which the function is discontinuous.

e. Identify the graph of the function from i–iv.

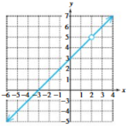
i.



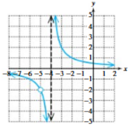
ii.



iii.



iv.



27. $f(x) = \dfrac{x^2 + x - 6}{x - 2}$

28. $f(x) = \dfrac{-x^2 + 2x + 3}{x + 1}$

29. $f(x) = \dfrac{2x + 10}{x^2 + 9x + 20}$

30. $f(x) = \dfrac{2x - 2}{x^2 + 2x - 3}$

Learn: Identify Horizontal Asymptotes

31. The graph of $f(x) = \dfrac{-x^2 + 8}{2x^2 - 3}$ will behave like which function for large values of $|x|$?

a. $y = -\frac{1}{2}$ **b.** $y = -\frac{x}{2}$

c. $y = -\frac{8}{3}$ **d.** $y = -\frac{1}{2}x - \frac{8}{3}$

32. The graph of $f(x) = \dfrac{2x^3 + 7}{5x^3}$ will behave like which function for large values of $|x|$?

a. $y = \frac{2}{5x}$ **b.** $y = \frac{2x}{5}$

c. $y = \frac{2}{5}$ **d.** $y = \frac{2}{5}x^3$

33. The graph of $f(x) = \dfrac{-3x^4 - 2x + 5}{x^5 + x^2 - 2}$ will behave like which function for large values of $|x|$?

a. $y = -3$ **b.** $y = -3x$

c. $y = -\frac{5}{2}$ **d.** $y = 0$

34. The graph of $f(x) = \dfrac{x^2 + 7x - 3}{6x^4 + 2}$ will behave like which function for large values of $|x|$?

a. $y = \frac{x^2}{6}$ **b.** $y = 0$

c. $y = -\frac{3}{2}$ **d.** $y = \frac{1}{6}x - \frac{3}{2}$

For Exercises 35–42,

a. Identify the horizontal asymptotes (if any). **(See Example 4 on p. 256)**

b. If the graph of the function has a horizontal asymptote, determine the point (if any) where the graph crosses the horizontal asymptote. **(See Example 5 on p. 257)**

35. $p(x) = \frac{5}{x^2 + 2x + 1}$

36. $q(x) = \frac{8}{x^2 + 4x + 4}$

37. $h(x) = \frac{3x^2 + 8x - 5}{x^2 + 3}$

38. $r(x) = \frac{-4x^2 + 5x - 1}{x^2 + 2}$

39. $m(x) = \frac{x^4 + 2x + 1}{5x + 2}$

40. $n(x) = \frac{x^3 - x^2 + 1}{2x - 3}$

41. $t(x) = \frac{2x + 4}{x^2 + 7x - 4}$

42. $s(x) = \frac{x + 3}{2x^2 - 3x - 5}$

43. Consider the expression $\frac{x^2 + 3x + 1}{2x^2 + 5}$.

a. Divide the numerator and denominator by the greatest power of x that appears in the denominator. That is, divide numerator and denominator by x^2.

b. As $|x| \to \infty$, what value will $\frac{3}{x}$, $\frac{1}{x^2}$, and $\frac{5}{x^2}$ approach?
(*Hint*: Substitute large values of x such as 100, 1000, 10,000, and so on to help you understand the behavior of each expression.)

c. Use the results from parts (a) and (b) to identify the horizontal asymptote for the graph of $f(x) = \frac{x^2 + 3x + 1}{2x^2 + 5}$.

44. Consider the expression $\frac{3x^3 - 2x^2 + 7x}{5x^3 + 1}$.

a. Divide the numerator and denominator by the greatest power of x that appears in the denominator.

b. As $|x| \to \infty$, what value will $-\frac{2}{x}$, $\frac{7}{x^2}$, and $\frac{1}{x^3}$ approach?

c. Use the results from parts (a) and (b) to identify the horizontal asymptote for the graph of $f(x) = \frac{3x^3 - 2x^2 + 7x}{5x^3 + 1}$.

Learn: Identify Slant and Nonlinear Asymptotes

For Exercises 45–52, write an equation of the slant or nonlinear asymptote. (See Examples 6–7)

45. $f(x) = \frac{x^2 - 3x + 2}{x - 4}$

46. $g(x) = \frac{x^2 + 5x + 2}{x + 3}$

47. $k(x) = \frac{2x^4 - x^2 + 1}{x^2 + 4}$

48. $m(x) = \frac{3x^4 + 2x^2 - 4}{x^2 + 2}$

49. $n(x) = \frac{12x^3 - 13x^2 + 6x - 1}{4x^2 - 3x + 1}$

50. $p(x) = \frac{10x^3 - 7x^2 - 2x + 8}{2x^2 - 3x + 2}$

51. $f(x) = \frac{-x^3 + 2x^2 - 4x + 7}{x - 2}$

52. $r(x) = \frac{-x^3 - 6x^2 - 2x + 2}{x + 5}$

Mixed Exercises

For Exercises 53–72, identify the asymptotes. (See Examples 4, 6, and 7 on pp. 256, 259, 260)

53. $s(x) = \frac{2x^2 - 8}{x^2}$

54. $w(x) = \frac{3x^2 - 3}{6x^2}$

55. $f(x) = \frac{2x^2 + 3}{x}$

56. $g(x) = \frac{3x^2 + 2}{x}$

57. $m(x) = \frac{1 + 6x + 3x^2}{x}$

58. $n(x) = \frac{2 + 4x - 2x^2}{2x}$

59. $w(x) = \frac{x + 7}{x^2 + 6x - 7}$

60. $v(x) = \frac{2x - 3}{2x^2 + x - 3}$

61. $h(x) = \frac{-3x^2 + 4x - 5}{x + 6}$

62. $k(x) = \frac{-2x^2 - 3x + 7}{x + 3}$

63. $s(x) = \frac{x + 1}{x^2 - 2x - 3}$

64. $t(x) = \frac{2 - x}{x^2 + 3x - 10}$

65. $p(x) = \frac{x^3 + 5x^2 - 4x + 1}{x^2 - 5}$

66. $q(x) = \frac{x^3 + 3x^2 - 2x - 4}{x^2 - 7}$

67. $r(x) = \frac{2x + 1}{x^3 + x^2 - 4x - 4}$

68. $t(x) = \frac{3x - 4}{x^3 + 2x^2 - 9x - 18}$

69. $f(x) = \dfrac{4x^3 - 2x^2 + 7x - 3}{2x^2 + 4x + 3}$

70. $a(x) = \dfrac{9x^3 - 5x + 4}{3x^2 + 2x + 1}$

71. $u(x) = \dfrac{x^4 - 2x^3 - x - 3}{x - 2}$

72. $v(x) = \dfrac{x^3 + 5x^2 + 2x + 7}{x + 5}$

73. The distance between two cities is 120 mi.

- **a.** How long will the trip take if the average speed is 30 mph?
- **b.** How long will the trip take if the average speed is 40 mph?
- **c.** How long with the trip take if the average speed is 60 mph?
- **d.** Write a function representing the duration of the trip $D(x)$ (in hours) if the average speed is x miles per hour.
- **e.** Will the duration of the trip ever equal zero?

74. A dance instructor charges $60 per hour and the charge is to be split evenly among the students in the class.

- **a.** How much would each student pay if there were 6 students in the class?
- **b.** How much would each student pay if there were 10 students in the class?
- **c.** How much would each student pay if there were 12 students in the class?
- **d.** Write a function representing the amount each student would pay $A(x)$ (in dollars) for x students in the class.
- **e.** Will the amount that each student pays ever reach zero?

75. A power company burns coal to generate electricity. The cost $C(x)$ (in \$1000) to remove x% of the air pollutants is given by

$$C(x) = \frac{600x}{100 - x}$$

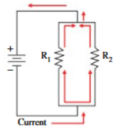
- **a.** Compute the cost to remove 25% of the air pollutants. (*Hint*: $x = 25$.)
- **b.** Determine the cost to remove 50%, 75%, and 90% of the air pollutants.
- **c.** If the power company budgets $1.4 million for pollution control, what percentage of the air pollutants can be removed?

76. The cost $C(x)$ (in \$1000) for a city to remove x% of the waste from a polluted river is given by

$$C(x) = \frac{80x}{100 - x}$$

- **a.** Determine the cost to remove 20%, 40%, and 90% of the waste. Round to the nearest thousand dollars.
- **b.** If the city has $320,000 budgeted for river cleanup, what percentage of the waste can be removed?

77. A parallel circuit is one with several paths through which electricity can travel. The total resistance in a parallel circuit is always less than the resistance in any single branch. The total resistance R for the circuit shown can be computed from the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$, where R_1 and R_2 are the resistances in the individual branches. Suppose that a resistor with a fixed resistance of 6 Ω (ohms) is placed in parallel with a variable resistor of resistance x.



- **a.** Write R as a function of x.
- **b.** Complete the table.

x	6	12	18	30
$R(x)$				

- **c.** What value does $R(x)$ approach as $x \to \infty$? Discuss the significance of this result.

78. The total resistance R of three resistors in parallel is given by

$$R = \frac{R_1R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}$$

Suppose that an 8-Ω and a 12-Ω resistor are placed in parallel with a variable resistor of resistance x.

a. Write R as a function of x.

b. What value does $R(x)$ approach as $x \to \infty$? Write the value in decimal form.

Write About It

79. Explain why $x = -2$ is not a vertical asymptote of the graph of $f(x) = \frac{x^2 + 7x + 10}{x + 2}$.

80. Write an informal definition of a horizontal asymptote of a rational function.

Expanding Your Skills

81. A chemist has a tank with 20 L of water into which 240 g of salt has been added. She adds water at a constant rate of 2 L per minute. At the same time, salt is added at a rate of 40 g per minute.

a. Write a function that expresses the amount of water $w(t)$ (in liters) after t minutes.

b. Write a function that expresses the amount of salt $s(t)$ (in grams) after t minutes.

c. Write a function that gives the concentration of salt in the water $C(t)$ (in grams per liter) after t minutes.

d. What was the concentration of salt in the water initially?

e. What is the concentration after 10 min, 20 min, 1 hr? Round to 1 decimal place.

f. What is the horizontal asymptote of function C, and what does it mean in the context of this problem?

82. In a certain city, a government official estimated the land area used for housing at 60 mi². At that time, the number of people in the city was 69,000. The amount of land used for housing increases at 3 mi² per year and the number of people increases by 1400 per year.

a. Write a function that expresses the amount of land area used for housing $a(t)$ (in square miles) after t years.

b. Write a function that expresses the number of people $n(t)$ after t years.

c. Write a function that gives the population density $D(t)$ (in people per square mile) after t years.

d. What was the population density initially?

e. What is the population density after 5 years, 10 years, and 20 years? Round to the nearest whole unit.

f. What is the horizontal asymptote of function D, and what does it mean in the context of this problem?

For Exercises 83–86, write an equation of a function that meets the given conditions. Answers may vary.

83. x-intercepts: $(-3, 0)$ and $(-1, 0)$
Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 1$
y-intercept: $\left(0, \frac{3}{4}\right)$

84. x-intercepts: $(4, 0)$ and $(2, 0)$
Vertical asymptote: $x = 1$
Horizontal asymptote: $y = 1$
y-intercept: $(0, 8)$

85. x-intercept: $\left(\frac{3}{2}, 0\right)$
Vertical asymptotes: $x = -2$ and $x = 5$
Horizontal asymptote: $y = 0$
y-intercept: $(0, 3)$

86. x-intercept: $\left(\frac{4}{3}, 0\right)$
Vertical asymptotes: $x = -3$ and $x = -4$
Horizontal asymptote: $y = 0$
y-intercept: $(0, -1)$

Technology Connections

For Exercises 87–94, graph the functions and the horizontal or slant asymptotes from the given exercises. Use the recommended window settings.

87. Exercise 55; $f(x) = \frac{2x^2 + 3}{x}$
$[-10, 10, 1]$ by $[-10, 10, 1]$

88. Exercise 56; $g(x) = \frac{3x^2 + 2}{x}$
$[-10, 10, 1]$ by $[-10, 10, 1]$

89. Exercise 61; $h(x) = \frac{-3x^2 + 4x - 5}{x + 6}$
$[-100, 100, 10]$ by $[-80, 120, 10]$

90. Exercise 62; $k(x) = \frac{-2x^2 - 3x + 7}{x + 3}$
$[-15, 15, 3]$ by $[-6, 24, 3]$

91. Exercise 67; $r(x) = \frac{2x + 1}{x^3 + x^2 - 4x - 4}$
$[-10, 10, 1]$ by $[-10, 10, 1]$

92. Exercise 68; $t(x) = \frac{3x - 4}{x^3 + 2x^2 - 9x - 18}$
$[-15, 15, 3]$ by $[-15, 15, 3]$

93. Exercise 47; $k(x) = \frac{2x^4 - x^2 + 1}{x^2 + 4}$
$[-10, 10, 1]$ by $[-10, 10, 1]$

94. Exercise 48; $m(x) = \frac{3x^4 + 2x^2 - 4}{x^2 + 2}$
$[-10, 10, 1]$ by $[-10, 10, 1]$

Go online for more practice problems.

Lesson 2-6

Graphs of Rational Functions

What Will You Learn?

After completing this lesson, you should be able to:

- Graph Rational Functions
- Use Rational Functions in Applications

Learn: Graph Rational Functions

In Lesson 2.5, we learned how to identify the vertical, horizontal, slant, and nonlinear asymptotes of a rational function. We now turn our attention to graphing rational functions. To begin, note that some rational functions can be graphed using transformations of the parent functions $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ shown in Figures 2-24 and 2-25.

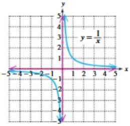


Figure 2-24

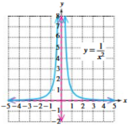


Figure 2-25

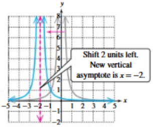
Example 1: Using Transformations to Graph a Rational Function

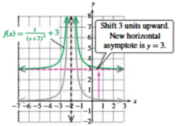
Use transformations to graph $f(x) = \frac{1}{(x+2)^2} + 3$.

Solution:

$f(x) = \frac{1}{(x+2)^2} + 3$

The graph of f is the graph of $y = \frac{1}{x^2}$ with a shift to the left 2 units and a shift upward 3 units.





Apply the Skills

1. Use transformations to graph $g(x) = \frac{1}{x-3} - 2$.

To graph a rational function that is not a simple transformation of $y = \frac{1}{x}$ or $y = \frac{1}{x^2}$, more steps must be employed. Our strategy is to find all asymptotes and key points (intercepts and points where the function crosses a horizontal asymptote). Then determine the behavior of the function on the intervals defined by these key points and the vertical asymptotes.

Graphing a Rational Function

Consider a rational function f defined by $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with no common factors.

1. Determine the y-intercept by evaluating $f(0)$.
2. Determine the x-intercept(s) by finding the real solutions of $f(x) = 0$. The value $f(x)$ equals zero when the numerator $p(x) = 0$.
3. Identify any vertical asymptotes and graph them as dashed lines.
4. Determine whether the function has a horizontal asymptote or a slant asymptote (or neither), and graph the asymptote as a dashed line.
5. Determine where the function crosses the horizontal or slant asymptote (if applicable).
6. If a test for symmetry is easy to apply, use symmetry to plot additional points. Recall:
   - f is an even function (symmetric to the y-axis) if $f(-x) = f(x)$.
   - f is an odd function (symmetric to the origin) if $f(-x) = -f(x)$.
7. Plot at least one point on the intervals defined by the x-intercepts, vertical asymptotes, and points where the function crosses a horizontal or slant asymptote.
8. Sketch the function based on the information found in steps 1–7.

Example 2: Graphing a Rational Function

Graph $f(x) = \frac{x+3}{x-2}$

Solution:

1. **Determine the y-intercept.** Evaluate $f(0)$.

   $f(0) = \frac{(0)+3}{(0)-2} = -\frac{3}{2}$

   The y-intercept is $\left(0, -\frac{3}{2}\right)$.

2. **Determine the x-intercept(s).** Find the real solutions of $f(x) = 0$.

   $\frac{x+3}{x-2} = 0$ when $x + 3 = 0$ or $x = -3$.

   The x-intercept is $(-3, 0)$.

Insights A fraction will equal zero if its numerator equals zero.

3. **Identify the vertical asymptotes.**

   The denominator $x - 2$ is zero at $x = 2$, and the numerator $x + 3$ is nonzero for $x = 2$.

   The graph has one vertical asymptote, $x = 2$.

   The vertical asymptotes occur at the values of x for which the denominator is zero and the numerator is nonzero.

4. **Determine whether f has a horizontal or slant asymptote.**

   The degree of the numerator is equal to the degree of the denominator. Therefore, the graph has a horizontal asymptote given by the ratio of leading coefficients of the numerator and denominator.

   $y = \frac{1}{1} = 1$ is the horizontal asymptote.

5. **Determine where f crosses its horizontal asymptote (if at all).**

   $$\frac{x+3}{x-2} = 1 \Rightarrow \cancel{(x-2)} \cdot \frac{x+3}{\cancel{x-2}} = (x-2) \cdot 1$$

   $$x + 3 = x - 2$$

   $$3 = -2$$

   Since $y = 1$ is the horizontal asymptote, solve the equation $f(x) = 1$.

   (Contradiction) The graph of f does not cross its horizontal asymptote.

6. **Test for symmetry.**

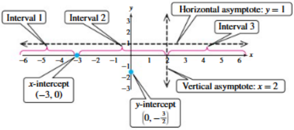
   $f(-x) = \frac{-x+3}{-x-2}$ does not equal $f(x)$ or $-f(x)$.

   The function is neither even nor odd and is not symmetric with respect to the y-axis or origin.

   Check whether $f(-x) = f(x)$ or $f(-x) = -f(x)$.

7. **Determine the behavior of f on each interval.**

   Determine the sign of the function on the intervals (shown in red) defined by the x-intercept at $x = -3$ and the vertical asymptote $x = 2$.



Interval	Test Point	Comments
$(-\infty, -3)$	$\left(-4, \frac{1}{6}\right)$	- $f(-4) = \frac{-4+3}{-4-2} = \frac{-1}{-6} = \frac{1}{6} > 0$. - Since $f(x)$ is positive on this interval, $f(x)$ must approach the horizontal asymptote $y = 1$ from below as $x \to -\infty$.
$(-3, 2)$	$\left(0, -\frac{3}{2}\right)$	- $f(0) = \frac{0+3}{0-2} = \frac{3}{-2} = -\frac{3}{2} < 0$. - Since $f(x)$ is negative on this interval, the graph crosses the x-axis at the intercept $(-3, 0)$ and continues downward (through the y-intercept). As x approaches the vertical asymptote $x = 2$ from the left, $f(x) \to -\infty$.
$(2, \infty)$	$(3, 6)$	- $f(3) = \frac{3+3}{3-2} = \frac{6}{1} = 6 > 0$. - Since $f(x)$ is positive on this interval, as x approaches the vertical asymptote $x = 2$ from the right, $f(x) \to \infty$. - Since $f(x)$ is positive on this interval, $f(x)$ must approach the horizontal asymptote from above as $x \to \infty$.

8. Sketch the function (Figure 2-26).

Plot the x- and y-intercepts $(-3, 0)$ and $\left(0, -\frac{3}{2}\right)$, and the additional points $\left(-4, -\frac{1}{6}\right)$ and $(3, 6)$. Graph the horizontal asymptote $(y = 1)$ and vertical asymptote $x = 2$ as dashed lines.

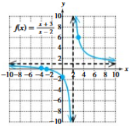


Figure 2-26

Insights

In Example 2, we can rewrite $f(x) = \frac{x+3}{x-2}$ as $f(x) = 1 + \frac{5}{x-2}$ by dividing $(x + 3) \div (x - 2)$.

$$\begin{array}{r} 1 \\ x-2\overline{)x+3} \\ \underline{-(x-2)} \\ 5 \end{array} \qquad \text{Thus, } \frac{x+3}{x-2} = 1 + \frac{5}{x-2}$$

When written in the form $f(x) = 1 + \frac{5}{x-2}$, we can graph f by shifting the graph of $y = \frac{1}{x}$ two units to the right, stretching by a factor of 5, and shifting the graph upward 1 unit.

Apply the Skills

2. Graph $f(x) = \frac{x-1}{x+4}$.

Example 3: Graphing a Rational Function

Graph $f(x) = \frac{4x}{x^2 - 4}$.

Solution:

1. **Determine the *y*-intercept.**

   $f(0) = \frac{4(0)}{(0)^2 - 4} = 0$

   The *y*-intercept is (0, 0).

   Evaluate $f(0)$.

2. **Determine the *x*-intercept(s).**

   $\frac{4x}{x^2 - 4} = 0$ for $x = 0$

   The *x*-intercept is (0, 0).

   Find the real solutions of $f(x) = 0$.

   The expression $\frac{4x}{x^2 - 4}$ equals zero when the numerator equals zero. So, solve $4x = 0$.

3. **Identify the vertical asymptotes.**

   The zeros of $x^2 - 4$ are 2 and -2.

   Vertical asymptotes: $x = 2$ and $x = -2$.

   The vertical asymptotes occur at the values of *x* for which the denominator is zero and the numerator is nonzero.

4. **Determine whether *f* has a horizontal or slant asymptote.**

   The degree of the numerator is less than the degree of the denominator.

   The horizontal asymptote is $y = 0$.

5. **Determine where *f* crosses its horizontal asymptote (if at all).**

   Set $f(x) = 0$. We have $\frac{4x}{x^2 - 4} = 0$ for $x = 0$.

   Therefore, *f* crosses its horizontal asymptote at (0, 0).

   Since $y = 0$ is the horizontal asymptote, solve the equation $f(x) = 0$.

6. **Test for symmetry.**

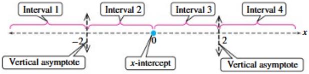
   f is an odd function because $f(-x) = \frac{4(-x)}{(-x)^2 - 4} = -\frac{4x}{x^2 - 4} = -f(x)$.

   The graph of *f* is symmetric to the origin.

   Check whether $f(-x) = f(x)$ or $f(-x) = -f(x)$.

7. **Determine the behavior of *f* on each interval.**

   Determine the sign of the function on the intervals (shown in red) defined by the *x*-intercept at $x = 0$ and the vertical asymptotes $x = -2$ and $x = 2$.



Interval	Test Point	Comments
$(-\infty, -2)$	$\left(-3, -\frac{12}{5}\right)$	• $f(-3) = \frac{4x}{x^2-4} = \frac{4(-3)}{(-3)^2-4} = \frac{-12}{5} = -\frac{12}{5} < 0$. • Since $f(x)$ is negative on this interval, $f(x)$ must approach the horizontal asymptote $y = 0$ from below as $x \to -\infty$. • Since $f(x)$ is negative on this interval, as x approaches the vertical asymptote $x = -2$ from the left, $f(x) \to -\infty$.
$(-2, 0)$	$\left(-1, \frac{4}{3}\right)$	• $f(-1) = \frac{4x}{x^2-4} = \frac{4(-1)}{(-1)^2-4} = \frac{-4}{-3} = \frac{4}{3} > 0$. • Since $f(x)$ is positive on this interval, as x approaches the vertical asymptote $x = -2$ from the right, $f(x) \to \infty$.
$(0, 2)$	$\left(1, -\frac{4}{3}\right)$	• $f(1) = \frac{4x}{x^2-4} = \frac{4(1)}{(1)^2-4} = \frac{4}{-3} = -\frac{4}{3} < 0$. • Since $f(x)$ is negative on this interval, as x approaches the vertical asymptote $x = 2$ from the left, $f(x) \to -\infty$.
$(2, \infty)$	$\left(3, \frac{12}{5}\right)$	• $f(3) = \frac{4x}{x^2-4} = \frac{4(3)}{(3)^2-4} = \frac{12}{5} > 0$. • Since $f(x)$ is positive on this interval, as x approaches the vertical asymptote $x = 2$ from the right, $f(x) \to \infty$. • Since $f(x)$ is positive on this interval, $f(x)$ must approach the horizontal asymptote from above as $x \to \infty$.

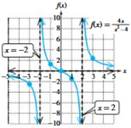
Insights

The graph of f is symmetric to the origin because $f(-x) = -f(x)$. Therefore, if $\left(-3, -\frac{12}{5}\right)$ and $\left(-1, \frac{4}{3}\right)$ are points on the graph of f, then $\left(3, \frac{12}{5}\right)$ and $\left(1, -\frac{4}{3}\right)$ are also points on the graph.

8. Sketch the function.

Plot the x- and y-intercept (0, 0). Graph the asymptotes as dashed lines. Plot the points. $\left(-3, -\frac{12}{5}\right)$, $\left(-1, \frac{4}{3}\right)$, $\left(1, -\frac{4}{3}\right)$, and $\left(3, \frac{12}{5}\right)$

Sketch the curve.



Apply the Skills

3. Graph $g(x) = \frac{-5x}{x^2-9}$.

Example 4: Graphing a Rational Function

Graph $g(x) = \frac{2x^2 - 3x - 5}{x^2 + 1}$.

Solution:

1. **Determine the y-intercept.**

   $g(0) = \frac{2(0)^2 - 3(0) - 5}{(0)^2 + 1} = -5$

   The y-intercept is $(0, -5)$.

   Evaluate $g(0)$.

2. **Determine the x-intercept(s).**

   $\frac{2x^2 - 3x - 5}{x^2 + 1} = 0$

   $2x^2 - 3x - 5 = 0$

   $(2x - 5)(x + 1) = 0$

   $x = \frac{5}{2}, x = -1$

   The x-intercepts are $\left(\frac{5}{2}, 0\right)$ and $(-1, 0)$

   Find the real solutions of $g(x) = 0$.

3. **Identify the vertical asymptotes.**

   $x^2 + 1$ is nonzero for all real numbers.

   The graph of g has no vertical asymptotes.

   The vertical asymptotes occur at the values of x for which the denominator is zero and the numerator is nonzero.

4. **Determine whether g has a horizontal or slant asymptote.**

   The degree of the numerator is equal to the degree of the denominator.

   The horizontal asymptote is $y = \frac{2}{1}$ or simply $y = 2$.

5. **Determine where g crosses its horizontal asymptote (if at all).**

   Find the real solutions to the equation $\frac{2x^2 - 3x - 5}{x^2 + 1} = 2$.

   $2x^2 - 3x - 5 = 5 = 2(x^2 + 1)$

   $2x^2 - 3x - 5 = 2x^2 + 2$

   $-3x - 5 = 2$

   $x = -\frac{7}{3}$

   g crosses its horizontal asymptote at $\left(-\frac{7}{3}, 2\right)$.

   Since $y = 2$ is the horizontal asymptote, solve the equation $g(x) = 2$.

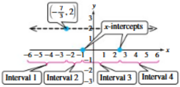
6. **Test for symmetry.**

   g is neither even nor odd because $g(-x) \neq g(x)$ and $g(x)$ and $g(-x) \neq -g(x)$.

   Check whether $g(-x) = g(x)$ or $g(-x) = -g(x)$.

7. Determine the behavior of *g* on each interval.

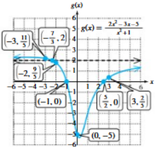
Determine the sign of the function on the intervals (shown in red) defined by the *x*-intercepts at $x = -1$ and $x = \frac{5}{2}$. Plot the point where *g* crosses its horizontal asymptote, $\left(-\frac{7}{3}, 2\right)$.



$\left(-\frac{7}{3}, 2\right)$
y
x-intercepts
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
3 2 1 −1 −2 −3
Interval 1
Interval 2
Interval 3
Interval 4

Interval	Test Point	Comments
$\left(-\infty, -\frac{7}{3}\right)$	$\left(-3, \frac{11}{5}\right)$	• Since $g(-3) = \frac{11}{5}$ is above the horizontal asymptote $y = 2$, $g(x)$ must approach the horizontal asymptote from above as $x \to -\infty$.
$\left(-\frac{7}{3}, -1\right)$	$\left(-2, \frac{9}{5}\right)$	• Plot the point $\left(-2, \frac{9}{5}\right)$ between the horizontal asymptote and the *x*-intercept of $(-1, 0)$.
$\left(-1, \frac{5}{2}\right)$	$(0, -5)$	• The point $(0, -5)$ is the *y*-intercept.
$\left(\frac{5}{2}, \infty\right)$	$\left(3, \frac{2}{5}\right)$	• Since $g(3) = \frac{2}{5}$ is below the horizontal asymptote $y = 2$, $g(x)$ must approach the horizontal asymptote from below as $x \to \infty$.

8. Sketch the function.



Apply the Skills

4. Graph $g(x) = \frac{4x^2 + 7x - 2}{x^2 + 4}$.

In Example 5, we graph a rational function with a slant asymptote.

Example 5: Graphing a Rational Function

Graph. $h(x) = \dfrac{2x^2 + 9x + 4}{x + 3}$

Solution:

1. **Determine the *y*-intercept.**

   $h(0) = \dfrac{2(0)^2 + 9(0) + 4}{(0) + 3} = \dfrac{4}{3}$

   The *y*-intercept is $\left(0, \frac{4}{3}\right)$.

   Evaluate $h(0)$.

2. **Determine the *x*-intercept(s).**

   $$\frac{2x^2 + 9x + 4}{x + 3} = 0$$
   $$2x^2 + 9x + 4 = 0$$
   $$(2x + 1)(x + 4) = 0$$
   $$x = -\frac{1}{2}, x = -4$$

   The *x*-intercepts are $\left(-\frac{1}{2}, 0\right)$ and $(-4, 0)$.

   Find the real solutions of $h(x) = 0$.

3. **Identify the vertical asymptotes.**

   $\dfrac{2x^2 + 9x + 4}{x + 3}$ is in lowest terms.

   $x + 3$ is zero for $x = -3$.

   The vertical asymptote is $x = -3$.

   The vertical asymptotes occur at the values of x for which the denominator is zero and the numerator is nonzero.

4. **Determine whether *h* has a horizontal or slant asymptote.**

   The degree of the numerator is one greater than the degree of the denominator. To find the slant asymptote, divide $(2x^2 + 9x + 4)$ by $(x + 3)$.

   $$\begin{array}{r|rrr} -3 & 2 & 9 & 4 \\ & & -6 & -9 \\ \hline & 2 & 3 & -5 \end{array}$$

   The quotient is $2x + 3$.

   The slant asymptote is $y = 2x + 3$.

5. **Determine where *h* crosses its slant asymptote (if at all).**

   Set $h(x) = 2x + 3$. We have $\dfrac{2x^2 + 9x + 4}{x + 3} = 2x + 3$.

   $$2x^2 + 9x + 4 = (2x + 3)(x + 3)$$
   $$2x^2 + 9x + 4 = 2x^2 + 9x + 9$$
   $$4 = 9 \quad \text{(No solution)}$$

   The equation has no solution. Therefore, the graph does not cross the slant asymptote.

   Since $y = 2x + 3$ is the slant asymptote, solve the equation $h(x) = 2x + 3$.

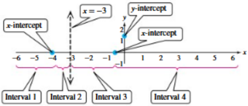
6. **Test for symmetry.**

   h is neither even nor odd because $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$.

   Check whether $h(-x) = h(x)$ or $h(-x) = -h(x)$.

7. Plot test points.

Pick values of x on the intervals defined by the x-intercepts, $(-4, 0)$ and $\left(-\frac{1}{2}, 0\right)$, and the vertical asymptote, $x = -3$.



Select test points from each interval.

The graph of $y = h(x)$ passes through $(-5, -4.5)$, $(-3.5, 6)$, $(-2, -6)$, and $(2, 6)$.

8. Sketch the graph.

Plot the x-intercepts: $(-4, 0)$ and $\left(-\frac{1}{2}, 0\right)$.

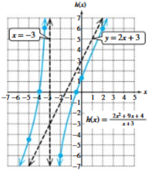
Plot the y-intercept: $\left(0, \frac{4}{3}\right)$.

Graph the asymptotes as dashed lines.

Plot the points:

$(-5, -4.5)$, $(-3.5, 6)$, $(-2, -6)$, and $(2, 6)$.

Sketch the graph.



Apply the Skills

5. Graph $k(x) = \frac{2x^2 - 7x + 3}{x - 2}$.

Learn: Use Rational Functions in Applications

In Lesson 1.5, we presented a linear model for the cost for a business to manufacture x items. The model is $C(x) = mx + b$, where m is the variable cost to produce an individual item, and b is the fixed cost.

The average cost $C(x)$ per item manufactured is the sum of all costs (variable and fixed) divided by the total number of items produced x. This is given by

$$C(x) = \frac{C(x)}{x}$$

The average cost per item will decrease as more items are produced because the fixed cost will be distributed over a greater number of items. This is demonstrated in Example 6.

Example 6: Investigating Average Cost

A cleaning service cleans homes. For each house call, the cost to the company is approximately $40 for cleaning supplies, gasoline, and labor. The business also has fixed monthly costs of $300 from phone service, advertising, and depreciation on the vehicles.

a. Write a cost function to represent the cost $C(x)$ (in dollars) for x house calls per month.

b. Write the average cost function that represents the average cost $C(x)$(in $) for x house calls per month.

c. Evaluate $C(5), C(20), C(30)$, and $C(100)$.

d. The cleaning service can realistically make a maximum of 160 calls per month. However, if the number of calls were unlimited, what value would the average cost approach? What does this mean in the context of the problem?

Solution:

a. $C(x) = 40x + 300$ The variable cost is $40 per call ($m = 40$), and the fixed cost is $300 ($b = 300$).

$C(x) = mx + b$.

b. $C(x) = \frac{40x + 300}{x}$ The average cost per item is the total cost divided by the total number of items produced.

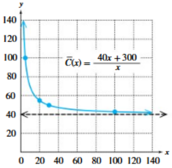
c. $C(5) = 100$ The average cost per house call is $100 if 5 calls are made.

$C(20) = 55$ The average cost per house call is $55 if 20 calls are made.

$C(30) = 50$ The average cost per house call is $50 if 30 calls are made.

$C(100) = 43$ The average cost per house call is $43 if 100 calls are made.

d. As x approaches infinity, $C(x)$ will approach its horizontal asymptote $y = 40$. This is the cost per house call in the absence of other fixed costs.



Apply the Skills

6. Repeat Example 6 under the assumption that the company cuts its fixed costs to $200 per month and pays its employees more, leading to a variable cost per house call of $50.

Practice 2-6

Practice Exercises

Prerequisite Review

R.1. Determine the x- and y-intercepts for the function. $f(x) = \frac{2x^2 - 5x - 7}{x - 1}$

R.2. Divide the polynomials. $\frac{2x^2 - 5x - 7}{x - 1}$

For Exercises R.3–R.5,

a. Determine if the function is even, odd, or neither.

b. State whether the function is symmetric with respect to the y-axis, origin, or neither.

R.3 $f(x) = \frac{3x^5 - 5x^3 + 8x}{4x^2 + 6}$

R.4. $h(x) = \frac{2x^2 - 5x - 7}{x - 1}$

R.5. $f(x) = \frac{2x^5 - 6x^3 + 8x}{2x^3 - 4x}$

For Exercises R.6–R.7, find the real solutions to the equations.

R.6. **a.** $\frac{2x^2 - 8}{x^2 - x + 1} = 0$ **b.** $\frac{2x^2 - 8}{x^2 - x + 1} = 2$

R.7. $\frac{2x - 5}{x^2 + 2x + 1} = 0$

For Exercises R.8–R.9, determine the real values of *x* for which *f*(*x*) is undefined.

R.8. $f(x) = \frac{2x^5 - 6x^3 + 8x}{2x^3 - 4x}$

R.9. $f(x) = \frac{3x + 11}{4x^2 + 28x + 49}$

Concept Connections

1. Can a rational function cross a vertical asymptote?

2. Can a rational function cross a horizontal asymptote?

3. Describe the relationship between the graph of $f(x) = \frac{1}{x}$ and the graph of $g(x) = \frac{1}{x + 4} - 5$.

4. Describe the relationship between the graph of $h(x) = \frac{1}{x^2}$ and the graph of $k(x) = -\frac{1}{(x - 3)^2}$.

5. The expression $\frac{x - 4}{x + 2}$ will equal zero when the (numerator/denominator) equals zero.

6. The expression $\frac{x - 4}{x + 2}$ is undefined when the (numerator/denominator) equals zero.

Learn: Graph Rational Functions

For Exercises 7–14, graph the functions by using transformations of the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$. (See Example 1 on p. 268)

7. $f(x) = \frac{1}{x - 3}$

8. $g(x) = \frac{1}{x + 4}$

9. $h(x) = \frac{1}{x^2} + 2$

10. $k(x) = \frac{1}{x^2} - 3$

11. $m(x) = \frac{1}{(x + 4)^2} - 3$

12. $n(x) = \frac{1}{(x - 1)^2} + 2$

13. $p(x) = -\frac{1}{x}$

14. $q(x) = -\frac{1}{x^2}$

For Exercises 15–20, for the graph of $y = f(x)$,

a. Identify the x-intercepts.

b. Identify any vertical asymptotes.

c. Identify the horizontal asymptote or slant asymptote if applicable.

d. Identify the y-intercept.

15. $f(x) = \frac{(x + 3)(2x - 7)}{(x + 2)(4x + 1)}$

16. $f(x) = \frac{(3x - 4)(x - 6)}{(2x - 3)(x + 5)}$

17. $f(x) = \frac{4x - 9}{x^2 - 9}$

18. $f(x) = \frac{5x - 8}{x^2 - 4}$

19. $f(x) = \frac{(5x - 1)(x + 3)}{x + 2}$

20. $f(x) = \frac{(4x + 3)(x + 2)}{x + 3}$

For Exercises 21–24, sketch a rational function subject to the given conditions. Answers may vary.

21. Horizontal asymptote: $y = 2$

Vertical asymptote: $x = 3$

y-intercept: $\left(0, \frac{8}{3}\right)$

x-intercept: $(4, 0)$

22. Horizontal asymptote: $y = 0$

Vertical asymptote: $x = -1$

y-intercept: $(0, 1)$

No x-intercepts

Range: $(0, \infty)$

23. Horizontal asymptote: $y = 0$

Vertical asymptotes: $x = -2$ and $x = 2$

y-intercept: $(0, 1)$

No x-intercepts

Symmetric to the y-axis

Passes through the point $\left(3, -\frac{4}{5}\right)$

24. Horizontal asymptote: $y = 3$

Vertical asymptotes: $x = -1$ and $x = 1$

y-intercept: $(0, 0)$

x-intercept: $(0, 0)$

Symmetric to the y-axis

Passes through the point $(2, 4)$

For Exercises 25–48, graph the function. (See Examples 2–5 on pp. 269–276)

25. $n(x) = \frac{-3}{2x + 7}$

26. $m(x) = \frac{-4}{2x - 5}$

27. $f(x) = \frac{x - 4}{x - 2}$

28. $g(x) = \frac{x - 3}{x - 1}$

29. $h(x) = \frac{2x - 4}{x + 3}$

30. $k(x) = \frac{3x - 9}{x + 2}$

31. $p(x) = \frac{6}{x^2 - 9}$

32. $q(x) = \frac{4}{x^2 - 16}$

33. $r(x) = \frac{5x}{x^2 - x - 6}$

34. $t(x) = \frac{4x}{x^2 - 2x - 3}$

35. $k(x) = \frac{5x - 3}{2x - 7}$

36. $h(x) = \frac{4x + 3}{3x - 5}$

37. $g(x) = \frac{3x^2 - 5x - 2}{x^2 + 1}$

38. $c(x) = \frac{2x^2 - 5x - 3}{x^2 + 1}$

39. $n(x) = \frac{x^2 + 2x + 1}{x}$

40. $m(x) = \frac{x^2 - 4x + 4}{x}$

41. $f(x) = \frac{x^2 + 7x + 10}{x + 3}$

42. $d(x) = \frac{x^2 - x - 12}{x - 2}$

43. $w(x) = \frac{-4x^2}{x^2 + 4}$

44. $u(x) = \frac{-3x^2}{x^2 + 1}$

45. $f(x) = \frac{x^3 + x^2 - 4x - 4}{x^2 + 3x}$

46. $g(x) = \frac{x^3 + 3x^2 - x - 3}{x^2 - 2x}$

47. $v(x) = \frac{2x^4}{x^2 + 9}$

48. $g(x) = \frac{4x^4}{x^2 + 8}$

Learn: Use Rational Functions in Applications

49. A sports trainer has monthly costs of \$69.95 for phone service and \$39.99 for a website and advertising. In addition, a \$20 fee is paid to the gym for each session in which the trainer works with a client.
(See Example 6 on p. 278)

a. Write a cost function to represent the cost $C(x)$ for x training sessions for a given month.

b. Write a function representing the average cost $C(x)$ for x sessions.

c. Evaluate $C(5), C(30),$ and $C(120)$.

d. The trainer can realistically have 120 sessions per month. However, if the number of sessions were unlimited, what value would the average cost approach? What does this mean in the context of the problem?

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50. An on-demand printing company has monthly overhead costs of \$1200 in rent, \$420 in electricity, \$100 for phone service, and \$200 for advertising and marketing. The printing cost is \$40 per thousand pages for paper and ink.

a. Write a cost function to represent the cost $C(x)$ for printing x thousand pages for a given month.

b. Write a function representing the average cost $C(x)$ for printing x thousand pages for a given month.

c. Evaluate $C(20), C(50), C(100)$, and $C(200)$.

d. Interpret the meaning of $C(200)$.

e. For a given month, if the printing company could print an unlimited number of pages, what value would the average cost per thousand pages approach? What does this mean in the context of the problem?

When medications are taken orally, it often takes time for the drug to be absorbed into the bloodstream. Then after more time elapses, the drug is gradually eliminated from the body, often through respiration and urine. The graph of the concentration of the drug in the bloodstream as a function of time shows a maximum value shortly after ingestion and then a gradual dissipation. This is illustrated in Exercises 51–54.

51. The concentration of a pain medication in the bloodstream is given by $C(t) = \frac{4t}{0.01t^2 + 3}$ where $C(t)$ is the concentration in micrograms (mcg) per milliliter (mL) and t is the time in minutes after ingestion.

a. What is the concentration after 10 min, 1 hr, and 2 hr? Round to one decimal place.

b. Use a graphing utility to graph the function on the window [0, 300, 30] by [0, 12, 1].

c. Use the *Maximum* feature on the graphing utility to approximate the maximum concentration in the bloodstream and the time at which this occurs. Round to one decimal place.

d. The drug can be taken every 4 hr. What is the concentration in the bloodstream at that time?

52. The concentration of an antibiotic in the bloodstream is given by $C(t) = \frac{75t}{4t^2 + 25}$ where $C(t)$ is the concentration in micrograms per milliliter and t is the time in hours after ingestion.

a. What is the concentration after 1 hr, 2 hr, and 4 hr? Round to two decimal places.

b. Use a graphing utility to graph the function on the window [0, 12, 1] by [0, 5, 1].

c. Use the *Maximum* feature on the graphing utility to approximate the maximum concentration in the bloodstream and the time at which this occurs.

d. Doctors generally do not want the concentration of an antibiotic in the bloodstream to drop significantly until the full course of the medication has been taken. The drug can be taken every 6 hr. What is the concentration in the bloodstream at that time?

53. The concentration $C(t)$ (in milligrams per liter, mg/L) of a drug in the bloodstream t hours after the drug is administered is modeled by

$$C(t) = \frac{10t}{2t^2 + 1}$$

a. Use a graphing utility to graph the function.

b. What are the domain restrictions on the function?

c. Use the graph to approximate the maximum concentration. Round to the nearest mg/L.

d. What is the limiting concentration?

54. A certain drug is designed to delay the administration of the active ingredient for several hours. The concentration $C(t)$ (in mg/L) of the active ingredient in the bloodstream t hours after taking the pill is modeled by

$$C(t) = \frac{3t}{2t^2 - 20t + 51}$$

a. Use a graphing utility to graph the function.

b. What are the domain restrictions on the function?

c. Use the graph to approximate the maximum concentration. Round to the nearest mg/L.

d. What is the limiting concentration?

The Doppler effect is a change in the observed frequency of a wave (such as a sound wave or light wave) when the source of the wave and observer are in motion relative to each other. The Doppler effect explains why an observer hears a change in pitch of an ambulance siren as the ambulance passes by the observer. The frequency $F(v)$ of a sound relative to an observer is given by $F(v) = f_a\left(\frac{s_0}{s_0 - v}\right)$, where f_a is the actual frequency of the sound at the source, s_0 is the speed of sound in air (772.4 mph), and v is the speed at which the source of sound is moving toward the observer. Use this relationship for Exercises 55–56.

55. Suppose that an ambulance moves toward an observer.

a. Write F as a function of v if the actual frequency of sound emitted by the ambulance is 560 Hz.

b. Use a graphing utility to graph the function from part (a) on the window [0, 1000, 100] by [0, 5000, 1000].

c. As the speed of the ambulance increases, what is the effect of the frequency of sound?

56. Suppose the frequency of sound emitted by a police car siren is 600 Hz.

a. Write F as a function of v if the police car is moving toward an observer.

b. Suppose that the frequency of the siren as heard by an observer is 664 Hz. Determine the velocity of the police car. Round to the nearest tenth of a mph.

c. Although a police car cannot travel close to the speed of sound, interpret the meaning of the vertical asymptote.

Mixed Exercises

57. For functions f, g, and h, check all that apply.

$$f(x) = \frac{x^2 - 5x + 6}{x - 2}$$

$$g(x) = \frac{x^2 - 5x + 6}{x - 2}$$

$$h(x) = \frac{x + 2}{x^2 - 4}$$

f	*g*	*h*	
			Domain is $(-\infty, 2) \cup (2, \infty)$.
			Domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.
			The graph has no vertical asymptote.
			Vertical asymptote: $x = 2$
			Horizontal asymptote: $y = 0$
			Slant asymptote: $y = x + 7$

58. For functions r, s, and t, check all that apply.

$$r(x) = \frac{x^2 - 25}{x^2 + 5x} \quad s(x) = \frac{x^3 - 5x^2}{x + 5} \quad t(x) = \frac{x}{x^2 - 25}$$

r	*s*	*t*	
			The function is not defined for $x = -5$
			Domain is $(-\infty, -5) \cup (-5, \infty)$.
			The function is defined for $x = 0$
			The graph has a vertical asymptote.
			The graph has a horizontal asymptote.
			The graph has a nonlinear asymptote.

For Exercises 59–60, list the *x*-values where the function is undefined, identify the intercepts, identify the asymptotes, and graph the function.

59. a. $f(x) = \dfrac{x+3}{x^2+2x-3}$

b. $g(x) = \dfrac{(x+3)^2}{x^2+2x-3}$

c. $h(x) = \dfrac{(x+3)^3}{x^2+2x-3}$

60. a. $r(x) = \dfrac{5x}{x^2-25}$

b. $s(x) = \dfrac{5x^2}{x^2-25}$

c. $t(x) = \dfrac{5x^3}{x^2-25}$

61. a. Write an equation for a rational function *f* whose graph is the same as the graph of $y = \frac{1}{x^2}$ shifted up 3 units and to the left 1 unit.

b. Write the domain and range of the function in interval notation.

62. a. Write an equation for a rational function *f* whose graph is the same as the graph of $y = \frac{1}{x}$ shifted to the right 4 units and down 3 units.

b. Write the domain and range of the function in interval notation.

For Exercises 63–64, given $y = f(x)$,

a. Divide the numerator by the denominator to write $f(x)$ in the form $f(x) = \text{quotient} + \dfrac{\text{remainder}}{\text{divisor}}$.

b. Use transformations of $y = \frac{1}{x}$ to graph the function.

63. $f(x) = \dfrac{2x+7}{x+3}$

64. $f(x) = \dfrac{5x+11}{x+2}$

Expanding Your Skills

For Exercises 65–68, sketch the graph of a function *f* under the given conditions. Answers may vary.

65. $f(3) = f(-1) = 0, f(0) = 1$

f undefined at $x = -3, x = 1$

As $x \to \pm\infty, f(x) \to 1$

As $x \to 1^+, f(x) \to -\infty$

As $x \to -3^+, f(x) \to -\infty$

As $x \to -3^-, f(x) \to \infty$

66. $f(0) = 1$

f undefined at $x = -3, x = 3$

$f(x) < 0$ for $x < -3$ and $x > 3$

As $x \to \pm\infty, f(x) \to 0$

As $x \to 3^+, f(x) \to -\infty$

As $x \to 3^-, f(x) \to \infty$

As $x \to 3^+, f(x) \to \infty$

As $x \to -3^-, f(x) \to -\infty$

67. *f* has no intercepts.

$f(2) = f(1) = 3, f(-2) = f(-1) = -3$

Asymptotes are $y = x$ and $x = 0$.

f is concave up on $(0, \infty)$.

f is concave down on $(-\infty, 0)$.

68. *f* is always positive on its domain.

f is undefined at $x = 3$.

$f(0) = f(6) = 1$

f is always concave up.

Problem Recognition Exercises

Polynomial and Rational Functions

For Exercises 1–8, refer to $p(x) = x^3 + 3x^2 - 6x - 8$ and $q(x) = x^3 - 2x^2 - 5x + 6$.

1. Find the zeros of $p(x)$.

2. Find the zeros of $q(x)$.

3. Find the x-intercept(s) of the graph of $y = q(x)$.

4. Find the x-intercept(s) of the graph of $y = p(x)$.

5. Find the x-intercepts of the graph of $f(x) = \frac{p(x)}{q(x)} = \frac{x^3 + 3x^2 - 6x - 8}{x^3 - 2x^2 - 5x + 6}$.

6. Find the vertical asymptotes of the graph of $f(x) = \frac{p(x)}{q(x)} = \frac{x^3 + 3x^2 - 6x - 8}{x^3 - 2x^2 - 5x + 6}$.

7. Find the horizontal asymptote or slant asymptote of the graph of $f(x) = \frac{p(x)}{q(x)} = \frac{x^3 + 3x^2 - 6x - 8}{x^3 - 2x^2 - 5x + 6}$

8. Determine where the graph of $f(x) = \frac{x^3 + 3x^2 - 6x - 8}{x^3 - 2x^2 - 5x + 6}$ crosses its horizontal or slant asymptote.

For Exercises 9–16, refer to $c(x) = x^3 - 4x^2 - 2x + 8$ and $d(x) = x^3 + 3x^2 - 4$.

9. Find the zeros of $c(x)$.

10. Find the zeros of $d(x)$.

11. Find the x-intercept(s) of the graph of $y = d(x)$.

12. Find the x-intercept(s) of the graph of $y = c(x)$.

13. Find the x-intercepts of the graph of $g(x) = \frac{c(x)}{d(x)} = \frac{x^3 - 4x^2 - 2x + 8}{x^3 + 3x^2 - 4}$. $(4, 0)$, $(\sqrt{2}, 0)$, and $(-\sqrt{2}, 0)$

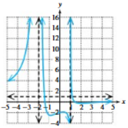
14. Find the vertical asymptotes of the graph of $g(x) = \frac{c(x)}{d(x)} = \frac{x^3 - 4x^2 - 2x + 8}{x^3 + 3x^2 - 4}$

15. Find the horizontal asymptote or slant asymptote of the graph of $g(x) = \frac{c(x)}{d(x)} = \frac{x^3 - 4x^2 - 2x + 8}{x^3 + 3x^2 - 4}$.

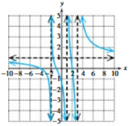
16. Determine where the graph of $g(x) = \frac{x^3 - 4x^2 - 2x + 8}{x^3 + 3x^2 - 4}$ crosses its horizontal or slant asymptote.

For Exercises 17–18, use the results from Exercises 5–8 and 13–16 to match the function with its graph.

a.



b.



17. $f(x) = \frac{x^3 + 3x^2 - 6x - 8}{x^3 - 2x^2 - 5x + 6}$

18. $g(x) = \frac{x^3 - 4x^2 - 2x + 8}{x^3 + 3x^2 - 4}$

19. Divide $(2x^3 - 4x^2 - 10x + 12) \div (x^2 - 11)$ by using an appropriate method.

- **a.** Identify the quotient $q(x)$.
- **b.** Identify the remainder $r(x)$.

20. Identify the slant asymptote of $f(x) = \frac{2x^3 - 4x^2 - 10x + 12}{x^2 - 11}$

21. Identify the point where the graph of $f(x) = \frac{2x^3 - 4x^2 - 10x + 12}{x^2 - 11}$ crosses its slant asymptote.

22. Refer to Exercise 19. Solve the equation $r(x) = 0$. How does the solution to the equation $r(x) = 0$ relate to the point where the graph of f crosses its slant asymptote?

GO ONLINE **Go online for more practice problems.**

Lesson 2-7

Polynomial and Rational Inequalities

What Will You Learn?

After completing this lesson, you should be able to:

- Solve Polynomial Inequalities
- Solve Rational Inequalities
- Solve Applications Involving Polynomial and Rational Inequalities

Learn: Solve Polynomial Inequalities

An engineer for a food manufacturer must design an aluminum container for a hot drink mix. The container is to be a right circular cylinder 5.5 in. in height. The surface area represents the amount of aluminum used and is given by

$$S(r) = 2\pi r^2 + 11\pi r \qquad \text{where } r \text{ is the radius of the can.}$$

Premium
Hot
Chocolate

The engineer wants to limit the surface area so that at most 90 in.^{2} of aluminum is used. To determine the restrictions on the radius, the engineer must solve the inequality $2\pi r^2 + 11\pi r \leq 90$ (see Exercise 131). This inequality is a quadratic inequality in the variable r. It is also categorized as a polynomial inequality of degree 2.

Definition of a Polynomial Inequality

Let $f(x)$ be a polynomial. Then an inequality of the form

$$f(x) < 0, \;\; f(x) > 0, \;\; f(x) \leq 0, \quad \text{or} \quad f(x) \geq 0$$

is called a **polynomial inequality.**

Note: A polynomial inequality is nonlinear if $f(x)$ is a polynomial of degree greater than 1.

Insights

If $f(x)$ is a quadratic polynomial, then the inequalities $f(x) < 0$, $f(x) > 0$, $f(x) \leq 0$, and $f(x) \geq 0$ are called quadratic inequalities.

Consider the polynomial inequalities $f(x) < 0$ and $f(x) > 0$. We need to determine the intervals over which $f(x)$ is negative or positive.

For example, consider the graph of $f(x) = x^2 - 6x + 5$ (Figure 2-27).

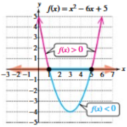


Figure 2-27

The graph shows the solution sets for the following equation and inequalities.

- $f(x) = 0$ for $\{1,5\}$.
- $f(x) < 0$ on the interval $(1, 5)$. (shown in blue)
- $f(x) > 0$ on the interval $(-\infty, 1) \cup (5, \infty)$. (shown in red)

Notice that the *x*-intercepts define the endpoints (or "boundary" points) for the solution sets of the inequalities. We can solve a polynomial inequality if we can identify the *sign* of the polynomial for each interval defined by the boundary points. This is the basis on which we solve any nonlinear inequality.

Procedure to Solve a Nonlinear Inequality

1. Express the inequality as $f(x) < 0$, $f(x) > 0$, $f(x) \leq 0$, or $f(x) \geq 0$. That is, rearrange the terms of the inequality so that one side is set to zero.
2. Find the real solutions of the related equation $f(x) = 0$ and any values of x that make $f(x)$ undefined. These are the "boundary" points for the solution set to the inequality.
3. Determine the sign of $f(x)$ on the intervals defined by the boundary points.
   - If $f(x)$ is positive, then the values of x on the interval are solutions to $f(x) > 0$.
   - If $f(x)$ is negative, then the values of x on the interval are solutions to $f(x) < 0$.
4. Determine whether the boundary points are included in the solution set.
5. Write the solution set in interval notation or set-builder notation.

Example 1: Solving a Quadratic Inequality

Solve the inequality. $3x(x - 1) > 10 - 2x$

Solution:

$3x(x - 1) > 10 - 2x$ **Step 1:** Write the inequality in the form $f(x) > 0$.

$3x^2 - 3x > 10 - 2x$

$\overbrace{3x^2 - x - 10}^{f(x)} > 0$

Step 2: Find the real solutions to the related equation $f(x) = 0$.

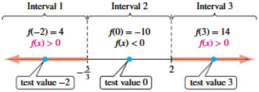
$3x^2 - x - 10 = 0$

$(3x + 5)(x - 2) = 0$

$x = -\frac{5}{3}$ and $x = 2$

The boundary points are $-\frac{5}{3}$ and 2.

Step 3: Divide the x-axis into intervals defined by the boundary points.



Determine the sign of $f(x) = 3x^2 - x - 10$ on each interval. One method is to evaluate $f(x)$ for a test value x on each interval.

Insights

To evaluate the polynomial $f(x) = 3x^2 - x - 10$ at the test points, we can perform direct substitution, such as $f(3) = 3(3)^2 - (3) - 10 = 14$, or use synthetic division and the remainder theorem:

$$\begin{array}{r|rrr} 3 & 3 & -1 & -10 \\ & & 9 & 24 \\ \hline & 3 & 8 & \boxed{14} \end{array}$$

Step 4: The solution set does not include the boundary points because the inequality is strict.

Step 5: Write the solution set.

The solution set is $\left(-\infty, -\frac{5}{3}\right) \cup (2, \infty)$ or equivalently in set-builder notation $\left\{x \middle| x < -\frac{5}{3} \text{ or } x > 2\right\}$.

Apply the Skills

1. Solve the inequality. $2x(x - 1) < 21 - x$

From Example 1, the key step is to determine the sign of $f(x)$ on the intervals $\left(-\infty, -\frac{5}{3}\right)$, $\left(-\frac{5}{3}, 2\right)$, and $(2, \infty)$. We can avoid the arithmetic from evaluating $f(x)$ at the test points by creating a sign chart. The inequality $3x^2 - x - 10 > 0$ is equivalent to $(3x + 5)(x - 2) > 0$. We have

Sign of $(3x + 5)$:	−	+	+
Sign of $(x - 2)$:	−	−	+
Sign of $(3x + 5)(x - 2)$:	+	−	+

$-\frac{5}{3}$ 2

Insights

The sign of the product in the bottom row of the sign chart is determined by the signs of the individual factors from the rows above.

The sign chart organizes the signs of each factor on the given intervals. Then the sign of the product of factors is given in the bottom row. We see that $f(x) = (3x + 5)(x - 2) > 0$ for $\left(-\infty, -\frac{5}{3}\right) \cup (2, \infty)$.

The result of Example 1 can also be viewed graphically. From Lesson 2.1, the graph of $f(x) = 3x^2 - x - 10$ is a parabola opening upward (Figure 2-28).

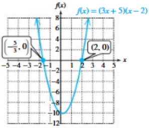


Figure 2-28

From the factored form $f(x) = (3x + 5)(x - 2)$, the x-intercepts $\left(-\frac{5}{3}, 0\right)$ and $(2, 0)$ mark the points of transition between the intervals where $f(x)$ potentially changes sign.

Example 2: Solving a Polynomial Inequality

Solve the inequality. $x^4 - 12x \geq 8x^2 - x^3$

Solution:

$x^4 - 12x \geq 8x^2 - x^3$ **Step 1:** Write the inequality in the form $f(x) \geq 0$.

$$\overbrace{x^4 + x^3 - 8x^2 - 12x}^{f(x)} \geq 0$$

$x^4 + x^3 - 8x^2 - 12x = 0$ **Step 2:** Find the real solutions to the related equation $f(x) = 0$.

$x(x^3 + x^2 - 8x - 12) = 0$ Factor the left side of the equation.

The possible rational zeros of $x^3 + x^2 - 8x - 12$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

$$\begin{array}{r|rrrr} 3 & 1 & 1 & -8 & -12 \\ & & 3 & 12 & 12 \\ \hline & 1 & 4 & 4 & \boxed{0} \end{array}$$

After testing several potential rational zeros, we find that 3 is a zero of $f(x)$.

$x(x - 3)(x^2 + 4x + 4) = 0$ Now factor the quadratic polynomial.

$x(x - 3)(x + 2)^2 = 0$

$x = 0, x = 3, x = -2$ The boundary points are 0, 3, and −2.

Step 3: The inequality $x^4 + x^3 - 8x^2 - 12x \geq 0$ is equivalent to $x(x-3)(x+2)^2 \geq 0$. Divide the x-axis into intervals defined by the boundary points and determine the sign of $f(x)$ on each interval.

	Interval 1	Interval 2	Interval 3	Interval 4
Evaluate: $f(x) = x(x-3)(x+2)^2$	$f(-3) = 18$ $f(x) > 0$	$f(-1) = 4$ $f(x) > 0$	$f(1) = -18$ $f(x) < 0$	$f(4) = 144$ $f(x) > 0$
Sign of x	−	−	+	+
Sign of $(x-3)$	−	−	−	+
Sign of $(x+2)^2$	+	+	+	+
Sign of $x(x-3)(x+2)^2$	+	+	−	+

−2 0 3

Step 4: The solution set includes the boundary points because the inequality sign includes equality. Therefore, the union of intervals 1 and 2 becomes $(-\infty, 0]$.

Step 5: Write the solution set.

The solution set is $(-\infty, 0] \cup [3, \infty)$.

In set-builder notation this is $\{x \mid x \leq 0 \text{ or } x \geq 3\}$.

Apply the Skills

2. Solve the inequality. $x^4 - 18x \geq 3x^2 - 4x^3$

The result of Example 2 can also be interpreted graphically. From Lesson 2.2, the graph of $f(x) = x^4 + x^3 - 8x^2 - 12x$ is up to the far left and up to the far right.

In factored form $f(x) = x(x-3)(x+2)^2$. The x-intercepts are $(0, 0)$, $(3, 0)$, and $(-2, 0)$. Furthermore, the factors x and $(x-3)$ have odd exponents. This means that the corresponding zeros have odd multiplicities, and that the graph will cross the x-axis at $(0, 0)$ and $(3, 0)$ and change sign. The factor $(x+2)$ has an even exponent, meaning that the corresponding zero has an even multiplicity. The graph will touch the x-axis at $(-2, 0)$ but will *not* change sign. From the sketch in Figure 2-29, we see that $f(x) \geq 0$ on the intervals $(-\infty, 0]$ and $[3, \infty)$.

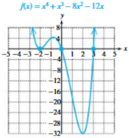


Figure 2-29

Insights

From the graph, the solution set to the related inequality $x^4 + x^3 - 8x^2 - 12x \leq 0$ would be $\{-2\} \cup [0,3]$. The boundary point −2 is included because $x^4 + x^3 - 8x^2 - 12x = 0$ at −2.

In some situations, the sign of a polynomial may be easily determined by inspection. In such a case, we can abbreviate the procedure to solve a polynomial inequality. This is demonstrated in Example 3.

Example 3: Solving Polynomial Inequalities

Solve the inequalities.

a. $4x^2 - 12x + 9 < 0$ **b.** $4x^2 - 12x + 9 \leq 0$

c. $4x^2 - 12x + 9 > 0$ **d.** $4x^2 - 12x + 9 \geq 0$

Solution:

a. $4x^2 - 12x + 9 < 0$	Factor $4x^2 - 12x + 9$ as $(2x - 3)^2$
$(2x - 3)^2 < 0$	The square of any real number is nonnegative.
The solution set is { }.	Therefore, this inequality has no solution.
b. $4x^2 - 12x + 9 \leq 0$ $(2x - 3)^2 \leq 0$	The inequality in part (b) is the same as the inequality in part (a) except that equality is included.
The solution set is $\left\{\frac{3}{2}\right\}$	The expression $(2x - 3)^2 = 0$ for $x = \frac{3}{2}$.
c. $4x^2 - 12x + 9 > 0$ $(2x - 3)^2 > 0$	The expression $(2x - 3)^2 > 0$ for all real numbers except where $(2x - 3)^2 = 0$.
The solution set is $\left(-\infty, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$. In set-builder notation $\left\{x \middle| x < \frac{3}{2} \text{ or } x > \frac{3}{2}\right\}$.	Therefore, the solution set is all real numbers except $\frac{3}{2}$.
d. $4x^2 - 12x + 9 \geq 0$ $(2x - 3)^2 \geq 0$	The square of any real number is greater than or equal to zero.
The solution set is $(-\infty, \infty)$.	Therefore, the solution set is all real numbers.

Apply the Skills

3. Solve the inequalities.

a. $25x^2 - 10x + 1 < 0$ **b.** $25x^2 - 10x + 1 \leq 0$

c. $25x^2 - 10x + 1 > 0$ **d.** $25x^2 - 10x + 1 \geq 0$

Learn: Solve Rational Inequalities

We now turn our attention to solving rational inequalities.

Definition of a Rational Inequality

Let $f(x)$ be a rational expression. Then an inequality of the form $f(x) < 0$, $f(x) > 0$, $f(x) \leq 0$, or $f(x) \geq 0$ is called a **rational inequality**.

We solve polynomial and rational inequalities in the same way with one exception. With a rational inequality such as $f(x) < 0$, the list of boundary points must include the real solutions to the related equation $f(x) = 0$ along with the values of x that make $f(x)$ undefined.

The graph of $f(x) = \frac{2x-5}{x+2}$ is shown in Figure 2-30. Notice that the function changes sign to the left and right of the vertical asymptote $x = -2$ and to the left and right of the x-intercept $\left(\frac{5}{2}, 0\right)$.

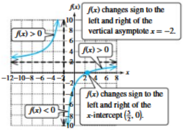


Figure 2-30

Check Your Work

The graph of a rational function will not always change sign to the left and right of a vertical asymptote or x-intercept. However, since the possibility exists, we must test each interval defined by these values of x.

From the graph of $f(x) = \frac{2x-5}{x+2}$ we can determine the solution sets for the following inequalities.

$\frac{2x-5}{x+2} < 0$ on the interval $\left(-2, \frac{5}{2}\right)$

$\frac{2x-5}{x+2} > 0$ on the intervals $(-\infty, -2) \cup \left(\frac{5}{2}, \infty\right)$

$\frac{2x-5}{x+2} > 0$ on the interval $\left(-2, \frac{5}{2}\right]$

$\frac{2x-5}{x+2} \geq 0$ on the intervals $(-\infty, -2) \cup \left[\frac{5}{2}, \infty\right)$

Check Your Work

The value -2 is excluded from the solution set because -2 makes the expression $\frac{2x-5}{x+2}$ undefined.

Example 4: Solving a Rational Inequality

Solve the inequality. $\frac{4x-5}{x-2} \le 3$

Solution:

$$\frac{4x-5}{x-2} \le 3$$

Step 1: First write the inequality in the form $f(x) \le 0$. That is, set one side to 0.

$$\overbrace{\frac{4x-5}{x-2} - 3}^{f(x)} \le 0$$

$$\frac{4x-5}{x-2} - 3 \cdot \frac{x-2}{x-2} \le 0$$

Write each term with a common denominator.

$$\frac{4x-5-3(x-2)}{x-2} \le 0$$

Simplify.

$$\frac{4x-5-3x+6}{x-2} \le 0$$

$$\frac{x+1}{x-2} \le 0$$

The expression $\frac{x+1}{x-2}$ is undefined for $x = 2$. Therefore, the value $x = 2$ is *not* part of the solution set.

However, 2 *is* a boundary point for the solution set.

$$\frac{x+1}{x-2} = 0$$

$$x = -1$$

Step 2: Solve for the real solutions to the equation $f(x) = 0$. The solution is –1, and this is another boundary point. The boundary points are $x = 2$ and $x = -1$.

Step 3: Divide the *x*-axis into intervals defined by the boundary points and determine the sign of $f(x)$ on each interval.

	Interval 1	Interval 2	Interval 3
Evaluate: $f(x) = \frac{x+1}{x-2}$	$f(-2) = \frac{1}{4}$ $f(x) > 0$	$f(1) = -2$ $f(x) < 0$	$f(3) = 4$ $f(x) > 0$
Sign of $(x+1)$	–	+	+
Sign of $(x-2)$	–	–	+
Sign of $\frac{(x+1)}{(x-2)}$	+	–	+

$-1 \qquad 2$

Step 4: Substituting the boundary point $x = -1$ into the inequality $\frac{x+1}{x-2} \le 0$ makes a true statement, $0 \le 0$. So, $x = -1$ is part of the solution set. The boundary point 2 is excluded because $\frac{x+1}{x-2}$ is undefined for $x = 2$.

Step 5: Write the solution set.

The solution set is $[-1, 2)$. In set-builder notation this is $\{x \mid -1 \le x < 2\}$.

Apply the Skills

4. Solve the inequality. $\frac{5-x}{x-1} \ge -2$

Insights

A rational expression is equal to zero where the numerator is equal to zero.

A rational expression is undefined where the denominator is equal to zero.

Example 5: Solving a Rational Inequality

Solve the inequality. $\frac{1}{x-2} \geq \frac{1}{x+3}$

Solution:

$$\frac{1}{x-2} \geq \frac{1}{x+3}$$

Step 1: First write the inequality in the form $f(x) \geq 0$. That is, set one side to 0.

$$\overbrace{\frac{1}{x-2} - \frac{1}{x+3}}^{f(x)} \geq 0$$

$$\frac{1}{x-2} \cdot \frac{x+3}{x+3} - \frac{1}{x+3} \cdot \frac{x-2}{x-2} \geq 0$$

Write each term with a common denominator.

$$\frac{x+3}{(x-2)(x+3)} - \frac{x-2}{(x-2)(x+3)} \geq 0$$

Simplify and write the numerator and denominator in factored form.

$$\frac{x+3-x+2}{(x-2)(x+3)} \geq 0$$

$$\frac{5}{(x-2)(x+3)} \geq 0$$

The expression $\frac{5}{(x-2)(x+3)}$ is undefined for $x = 2$ and $x = -3$. The values 2 and -3 are not included in the solution set, but they are boundary points for the solution set.

$$\frac{5}{(x-2)(x+3)} = 0$$

Step 2: Solve the related equation to determine any additional boundary points.

$$\frac{5}{(x-2)(x+3)} \cdot (x-2)(x+3) = 0 \cdot (x-2)(x+3)$$

$5 = 0$ (contradiction)

Clearing fractions results in the contradiction $5 = 0$. There are no solutions to the related equation.

Step 3: Divide the x-axis into intervals defined by the boundary points and determine the sign of $f(x) = \frac{5}{(x-2)(x+3)}$ on each interval.

	Interval 1	Interval 2	Interval 3
Evaluate: $f(x) = \frac{5}{(x-2)(x+3)}$	$f(-4) = \frac{5}{6}$ $f(x) > 0$	$f(0) = \frac{5}{6}$ $f(x) < 0$	$f(3) = \frac{5}{6}$ $f(x) > 0$
Sign of 5	+	+	+
Sign of $(x-2)$	−	−	+
Sign of $(x+3)$	−	+	+
Sign of $f(x) = \frac{5}{(x-2)(x+3)}$	+	−	+

-3 $\quad$ 2

Now interpret the results to solve $\frac{5}{(x-2)(x+3)} \geq 0$.

Step 4: Neither boundary point is included because $\frac{5}{(x-2)(x+3)}$ is undefined for $x = 2$ and $x = -3$.

Step 5: Write the solution set. The expression $\frac{5}{(x-2)(x+3)}$ is greater than zero for $x < -3$ and $x > 2$.

The solution set is $(-\infty, -3) \cup (2, \infty)$.

In set-builder notation, this is $\{x \mid x < -3 \text{ or } x > 2\}$.

Apply the Skills

5. Solve the inequality. $\frac{1}{x+4} \geq \frac{1}{x+2}$

In Example 6 we encounter an inequality in which the signs of the numerator and denominator of the rational expression can be determined by inspection.

Example 6: Solving Rational Inequalities

a. $\frac{x^2}{x^2+4} \geq 0$ **b.** $\frac{x^2}{x^2+4} > 0$

c. $\frac{x^2}{x^2+4} \leq 0$ **d.** $\frac{x^2}{x^2+4} < 0$

Solution:

The solution to the related equation $\frac{x^2}{x^2+4} = 0$ is $x = 0$.

The denominator is nonzero for all real numbers.

Therefore, the only boundary point is $x = 0$.

Sign of x^2	+	+
Sign of $x^2 + 4$	+	+
Sign of $\frac{x^2}{x^2+4}$	+	+

0

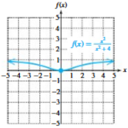
Therefore, $\frac{x^2}{x^2+4} = 0$ at $x = 0$ and is positive for all other real numbers.

a. $\frac{x^2}{x^2+4} \geq 0$ Solution set: $(-\infty, \infty)$

b. $\frac{x^2}{x^2+4} > 0$ Solution set: $(-\infty, 0) \cup (0, \infty)$

c. $\frac{x^2}{x^2+4} \leq 0$ Solution set: $\{0\}$

d. $\frac{x^2}{x^2+4} < 0$ Solution set: $\{\ \}$



Apply the Skills

6. Solve the inequalities.

a. $\frac{x^2}{x^4+1} \geq 0$ **b.** $\frac{x^2}{x^4+1} > 0$ **c.** $\frac{x^2}{x^4+1} \leq 0$ **d.** $\frac{x^2}{x^4+1} < 0$

Learn: Solve Applications Involving Polynomial and Rational Inequalities

In Lesson R.7, we studied the vertical position $s(t)$ of an object moving upward or downward under the influence of gravity. We use this model to solve the application in Example 7.

$$s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$$

Where

g is the acceleration due to gravity (32 ft/sec² or 9.8 m/sec²).

t is the time of travel.

v_0 is the initial speed.

s_0 is the initial vertical position.

Example 7: Solving an Application of a Polynomial Inequality

A toy rocket is shot straight upward from a launch pad 1 ft above ground level with an initial speed of 64 ft/sec.

a. Write a model to express the vertical position $s(t)$ (in ft) of the rocket t seconds after launch.

b. Determine the times at which the rocket is above a height of 50 ft.

Insights

Choose $g = 32$ ft/sec² because the height is given in feet and speed is given in feet per second.

Solution:

a. $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$ In this example,

$s(t) = -\frac{1}{2}(32)t^2 + (64)t + (1)$ $g = 32$ ft/sec², $v_0 = 64$ ft/sec, $s_0 = 1$ ft

$s(t) = -16t^2 + 64t + 1$

b. $-16t^2 + 64t + 1 > 50$ Write the inequality in the form $f(t) > 0$.

$\overbrace{-16t^2 + 64t - 49}^{f(t)} > 0$

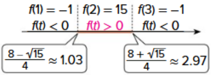
$-16t^2 + 64t - 49 = 0$ Use the quadratic formula to solve the related equation $f(t) = 0$.

$t = \frac{-64 \pm \sqrt{(64)^2 - 4(-16)(-49)}}{2(-16)}$ Evaluate $f(t) = -16t^2 + 64t - 49$ for test points in each interval.

$= \frac{-64 \pm \sqrt{960}}{-32}$

$= \frac{-64 \pm 8\sqrt{15}}{-32}$

$= \frac{8 \pm \sqrt{15}}{4} \begin{array}{l} \nearrow \approx 2.97 \\ \searrow \approx 1.03 \end{array}$


$f(1) = -1$
$f(t) < 0$
$f(2) = 15$
$f(t) > 0$
$f(3) = -1$
$f(t) < 0$
$\frac{8-\sqrt{15}}{4} \approx 1.03$
$\frac{8+\sqrt{15}}{4} \approx 2.97$

The solution set is $\left(\frac{8-\sqrt{15}}{4}, \frac{8+\sqrt{15}}{4}\right)$ or approximately (1.03, 2.97).

The rocket will be above 50 ft high between 1.03 sec and 2.97 sec after launch.

The graph of $s(t) = -16t^2 + 64t + 1$ is a parabola opening downward (Figure 2-31). We see that $s(t) > 50$ for t between 1.03 and 2.97 as expected.

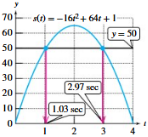


Figure 2-31

Apply the Skills

7. Repeat Example 7 under the assumption that the rocket is launched with an initial speed of 80 ft/sec from a height of 5 ft.

Practice 2-7

Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, solve the inequality and write the solution in interval notation when possible.

R.1. $1 - 3(z - 5) \geq 3 + 3(2z + 7)$

R.2. $-4(3x + 2) < 1 - (x - 4) - 9x$

R.3. $2w - 3 \geq 9$ or $w < -1$

For Exercises R.4–R.5, add the rational expressions.

R.4. $\dfrac{2}{w+2} + \dfrac{1}{3w+2}$

R.5. $y + 5 + \dfrac{1}{y-5}$

Concept Connections

1. Let $f(x)$ be a polynomial. An inequality of the form $f(x) < 0$, $f(x) > 0$, $f(x) \geq 0$, or $f(x) \leq 0$ is called a ____________ inequality. If the polynomial is of degree, ____________ then the inequality is also called a quadratic inequality.
2. Let $f(x)$ be a rational expression. An inequality of the form $f(x) < 0$, $f(x) > 0$, $f(x) \geq 0$, or $f(x) \leq 0$ is called a ____________ inequality
3. The solution set for the inequality $(x + 10)^2 \geq -4$ is ____________ whereas the solution set for the inequality $(x + 10)^2 \leq -4$ is ____________.
4. The solutions to an inequality $f(x) < 0$ are the values of x on the intervals where $f(x)$ is (positive/negative).

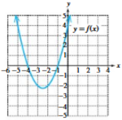
Learn: Solve Polynomial Inequalities

For Exercises 5–14, the graph of $y = f(x)$ is given. Solve the inequalities.

a. $f(x) < 0$ **b.** $f(x) \leq 0$

c. $f(x) > 0$ **d.** $f(x) \geq 0$

5.



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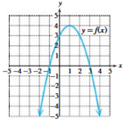

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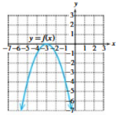

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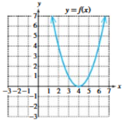

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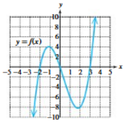

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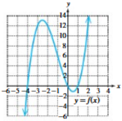

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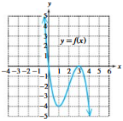

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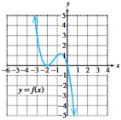

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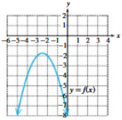

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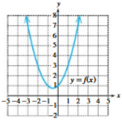


For Exercises 15–20, solve the equations and inequalities. (See Example 3 on p. 290)

15. a. $(5x - 3)(x - 5) = 0$

b. $(5x - 3)(x - 5) < 0$

c. $(5x - 3)(x - 5) \leq 0$

d. $(5x - 3)(x - 5) > 0$

e. $(5x - 3)(x - 5) \geq 0$

16. a. $(3x + 7)(x - 2) = 0$

b. $(3x + 7)(x - 2) < 0$

c. $(3x + 7)(x - 2) \leq 0$

d. $(3x + 7)(x - 2) > 0$

e. $(3x + 7)(x - 2) \geq 0$

17. a. $-x^2 + x + 12 = 0$

b. $-x^2 + x + 12 < 0$

c. $-x^2 + x + 12 \leq 0$

d. $-x^2 + x + 12 > 0$

e. $-x^2 + x + 12 \geq 0$

18. a. $-x^2 - 10x - 9 = 0$

b. $-x^2 - 10x - 9 < 0$

c. $-x^2 - 10x - 9 \leq 0$

d. $-x^2 - 10x - 9 > 0$

e. $-x^2 - 10x - 9 \geq 0$

19. a. $a^2 + 12a + 36 = 0$

b. $a^2 + 12a + 36 < 0$

c. $a^2 + 12a + 36 \leq 0$

d. $a^2 + 12a + 36 > 0$

e. $a^2 + 12a + 36 \geq 0$

20. a. $t^2 - 14t + 49 = 0$

b. $t^2 - 14t + 49 < 0$

c. $t^2 - 14t + 49 \leq 0$

d. $t^2 - 14t + 49 > 0$

e. $t^2 - 14t + 49 \geq 0$

For Exercises 21–54, solve the inequalities. (See Examples 1–2 on pp. 286–288)

21. $4w^2 - 9 \geq 0$

22. $16z^2 - 25 < 0$

23. $3w^2 + w < 2(w + 2)$

24. $5y^2 + 7y < 3(y + 4)$

25. $a^2 \geq 3a$

26. $d^2 \geq 6d$

27. $10 - 6x > 5x^2$

28. $6 - 4x > 3x^2$

29. $m^2 < 49$

30. $y^2 \geq 9$

31. $16p^2 \geq 2$

32. $54q^2 \leq 50$

33. $(x + 4)(x - 1)(x - 3) \geq 0$

34. $(x + 2)(x + 5)(x - 4) \geq 0$

35. $-5c(c + 2)^2(4 - c) > 0$

36. $-6u(u + 1)^2(3 - u) > 0$

37. $t^4 - 10t^2 + 9 \leq 0$

38. $w^4 - 20w^2 + 64 \leq 0$

39 $2x^3 + 5x^2 < 8x + 20$

40. $3x^3 - 3x < 4x^2 - 4$

41. $-2x^4 + 10x^3 - 6x^2 - 18x \geq 0$

42. $-4x^4 + 4x^3 + 64x^2 + 80x \geq 0$

43. $-5u^6 + 28u^5 - 15u^4 \leq 0$

44. $-3w^6 + 8w^5 - 4w^4 \leq 0$

45. $6x(2x - 5)^4(3x + 1)^5(x - 4) < 0$

46. $5x(3x - 2)^2(4x + 1)^3(x - 3)^4 < 0$

47. $(5x - 3)^2 > -2$

48. $(4x + 1)^2 > -6$

49. $-4 \geq (x - 7)^2$

50. $-1 \geq (x + 2)^2$

51. $16y^2 > 24y - 9$

52. $4w^2 > 20w - 25$

53. $(x + 3)(x + 1) \leq -1$

54. $(x + 2)(x + 4) \leq -1$

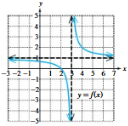
Learn: Solve Rational Inequalities

For Exercises 55–58, the graph of $y = f(x)$ is given. Solve the inequalities.

a. $f(x) < 0$ **b.** $f(x) \leq 0$

c. $f(x) > 0$ **d.** $f(x) \geq 0$

55.



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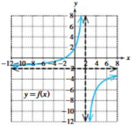

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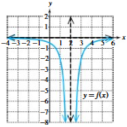
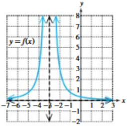


58.



For Exercises 59–62, solve the inequalities. (See Example 6 on p. 294)

59. a. $\dfrac{x + 2}{x - 3} \leq 0$

b. $\dfrac{x + 2}{x - 3} < 0$

c. $\dfrac{x + 2}{x - 3} \geq 0$

d. $\dfrac{x + 2}{x - 3} > 0$

60. a. $\dfrac{x + 4}{x - 1} \leq 0$

b. $\dfrac{x + 4}{x - 1} < 0$

c. $\dfrac{x + 4}{x - 1} \geq 0$

d. $\dfrac{x + 4}{x - 1} > 0$

61. **a.** $\dfrac{x^4}{x^2+9} \le 0$

b. $\dfrac{x^4}{x^2+9} < 0$

c. $\dfrac{x^4}{x^2+9} \ge 0$

d. $\dfrac{x^4}{x^2+9} > 0$

62. **a.** $\dfrac{-x^2}{x^4+16} \le 0$

b. $\dfrac{-x^2}{x^4+16} < 0$

c. $\dfrac{-x^2}{x^4+16} \ge 0$

d. $\dfrac{-x^2}{x^4+16} > 0$

For Exercises 63–84, solve the inequalities. (See Examples 4–5 on pp. 292–293)

63. $\dfrac{5-x}{x+1} \ge 0$

64. $\dfrac{2-x}{x+6} \ge 0$

65. $\dfrac{4-2x}{x^2} \le 0$

66. $\dfrac{9-3x}{x^2} \le 0$

67. $\dfrac{w^2-w-2}{w+3} \ge 0$

68. $\dfrac{p^2-2p-8}{p-1} \ge 0$

69. $\dfrac{5}{2t-7} > 1$

70. $\dfrac{4}{3c-8} > 1$

71. $\dfrac{2x}{x-2} \le 2$

72. $\dfrac{3x}{3x-7} \le 1$

73. $\dfrac{4-x}{x+5} \ge 2$

74. $\dfrac{3-x}{x+2} \ge 4$

75. $\dfrac{a-2}{a^2+4} \le 0$

76. $\dfrac{d-3}{d^2+1} \le 0$

77. $\dfrac{10}{x+2} \ge \dfrac{2}{x+2}$

78. $\dfrac{4}{x-x} \le \dfrac{6}{1-x}$

79. $\dfrac{4}{y-3} > -\dfrac{2}{y}$

80. $\dfrac{2}{z-1} > -\dfrac{4}{z}$

81. $\dfrac{3}{4-x} \le \dfrac{6}{1-x}$

82. $\dfrac{5}{2-x} \le \dfrac{3}{3-x}$

83. $\dfrac{(2-x)(2x+1)^2}{(x-4)^4} \le 0$

84. $\dfrac{(3-x)(4x-1)^4}{(x+2)^2} \le 0$

Learn: Solve Applications Involving Polynomial and Rational Inequalities

85. A professional fireworks team shoots an 8-in. mortar straight upwards from ground level with an initial speed of 216 ft/sec. (**See Example 7 on p. 295**)

a. Write a function modeling the vertical position $s(t)$(in ft) of the shell at a time t seconds after launch.

b. The mortar is designed to explode when the shell is at its maximum height. How long after launch will the shell explode? (*Hint*: Consider the vertex formula from Lesson 2.1.)

c. The spectators can see the shell rising once it clears a 200-ft tree line. For what period of time after launch is the shell visible before it explodes?

86. Suppose that a basketball player jumps straight up for a rebound.

a. If the initial speed leaving the ground is 16 ft/sec, write a function modeling the player's vertical position $s(t)$ (in ft) at a time t seconds after leaving the ground.

b. Find the times after leaving the ground when the player will be at a height of more than 3 ft in the air.

87. For a certain stretch of road, the distance d (in ft) required to stop a car that is traveling at speed v (in mph) before the brakes are applied can be approximated by $d(v) = 0.06v^2 + 2v$. Find the speeds for which the car can be stopped within 250 ft.

88. The population $P(t)$ of a bacteria culture is given by $P(t) = -1500t^2 + 60{,}000t + 10{,}000$, where t is the time in hours after the culture is started. Determine the time(s) at which the population will be greater than 460,000 organisms.

89. Suppose that an object that is originally at room temperature of 32°C is placed in a freezer. The temperature $T(x)$ (in °C) of the object can be approximated by the model $T(x) = \frac{320}{x^2 + 3x + 10}$, where x is the time in hours after the object is placed in the freezer.

a. What is the horizontal asymptote of the graph of this function and what does it represent in the context of this problem?

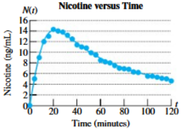
b. A chemist needs a compound cooled to less than 5°C. Determine the amount of time required for the compound to cool so that its temperature is less than 5°C.

90. The average round trip speed S (in mph) of a vehicle traveling a distance of d miles in each direction is given by $S = \frac{2d}{\frac{d}{r_1} + \frac{d}{r_2}}$ where r_1 and r_2 are the rates of speed for the initial trip and the return trip, respectively.

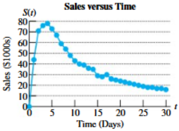
a. Suppose that a motorist travels 200 mi from home to an athletic event and averages 50 mph for the trip to the event. Determine the speeds necessary if the motorist wants the average speed for the round trip to be at least 60 mph.

b. Would the motorist be traveling within the speed limit of 70 mph?

91. A scientist measures the nicotine in the bloodstream after a volunteer begins smoking a cigarette. The blood was tested at 4-min intervals for 2 hr. The scientist fitted the data with the model $N(t) = \frac{600t}{t^2 + 450}$, where $N(t)$ is measured in nanograms per milliliter (ng/mL) and t is the time in minutes. If the smoker goes into withdrawal when the nicotine level falls below 10 ng/mL, when would the smoker need another cigarette? Round to the nearest minute.



92. A small company runs a television commercial at the beginning of the month to test its effectiveness. The director of marketing monitors sales each day for a period of one month. From the graph, he notes that there is an initial "spike" in sales followed by a gradual decrease. From the data, he fits the model $s(t) = \frac{500t}{t^2 + 10}$, where $S(t)$ represents daily sales (in \$1000s) and t represents the number of days after the commercial airs. For future commercials, the marketing manager wants to run a follow-up commercial before the sales drop below \$40,000 per day. When should the follow-up commercial be run? Round to the nearest day.



93. A rectangular quilt is to be made so that the length is 1.2 times the width. The quilt must be between 72 ft² and 96 ft² to cover the bed. Determine the restrictions on the width so that the dimensions of the quilt will meet the required area. Give exact values and the approximated values to the nearest tenth of a foot.

94. A landscaping team plans to build a rectangular garden that is between 480 yd² and 720 yd² in area. For aesthetic reasons, they also want the length to be 1.5 times the width. Determine the restrictions on the width so that the dimensions of the garden will meet the required area. Give exact values and the approximated values to the nearest tenth of a yard.

Mixed Exercises

For Exercises 95–98, solve the inequality. Write the solution set in interval notation.

95. a. $x^2 - 5x - 6 \geq 0$

b. $\frac{x+1}{x-6} \geq 0$

c. $\frac{x-6}{x+1} \leq 0$

96. a. $x^2 - 6x + 8 \le 0$

b. $\frac{x-4}{x-2} \le 0$

c. $\frac{x-2}{x-4} \ge 0$

97. a. $(x^2 + 4)(x - 1) \le 0$

b. $(x^2 - 4)(x - 1) \le 0$

c. $\frac{x^2-4}{x-1} \ge 0$

98. a. $x^2(x + 9) > 0$

b. $\frac{x}{x^2+9} \ge 0$

c. $x(x^2 + 9) < 0$

For Exercises 99–108, write the domain of the function in interval notation.

99. $f(x) = \sqrt{9 - x^2}$

100. $g(t) = \sqrt{1 - t^2}$

101. $h(a) = \sqrt{a^2 - 5}$

102. $f(u) = \sqrt{u^2 - 7}$

103. $p(x) = \sqrt{2x^2 + 9x - 18}$

104. $q(x) = \sqrt{4x^2 + 7x - 2}$

105. $r(x) = \frac{1}{\sqrt{2x^2 + 9x - 18}}$

106. $s(x) = \frac{1}{\sqrt{4x^2 + 7x - 2}}$

107. $h(x) = \sqrt{\frac{3x}{x+2}}$

108. $k(x) = \sqrt{\frac{2x}{x+1}}$

109. Let a, b, and c represent positive real numbers, where $a < b < c$, and let $a < b < c$, and let $f(x) = (x - a)^2(b - x)(x - c)^3$.

a. Complete the sign chart.

Sign of $(x - a)^2$:			
Sign of $(b - x)$:			
Sign of $(x - c)^3$:			
Sign of $(x - a)^2(b - x)(x - c)^3$:			
	a	b	c

b. Solve $f(x) > 0$.

c. Solve $f(x) < 0$.

110. Let a, b, and c represent positive real numbers, where $a < b < c$, and let $g(x) = \frac{(a-x)(x-b)^2}{(c-x)^5}$.

a. Complete the sign chart.

Sign of $(a - x)$:			
Sign of $(x - b)^2$:			
Sign of $(c - x)^5$:			
Sign of $\frac{(a-x)(x-b)^2}{(c-x)^5}$			
	a	b	c

b. Solve $g(x) > 0$.

c. Solve $g(x) < 0$.

Write About It

111. Explain how the solution set to the inequality $f(x) < 0$ is related to the graph of $y = f(x)$.

112. Explain how the solution set to the inequality $f(x) \ge 0$ is related to the graph of $y = f(x)$.

113. Explain why $\frac{x^2+2}{x^2+1} < 0$ has no solution.

114. Given $\frac{x-3}{x-1} \le 0$, explain why the solution set includes 3, but does not include 1.

Expanding Your Skills

For Exercises 115–116, write the domain of the function in interval notation.

115. $a(x) = \sqrt{1 - |x^2 - 1|}$

116. $b(x) = \sqrt{|x^2 - 2| - 2}$

The procedure to solve a polynomial or rational inequality may be applied to all inequalities of the form $f(x) > 0$, $f(x) < 0$, $f(x) \ge 0$, and $f(x) \le 0$. That is, find the real solutions to the related equation and determine restricted values of x. Then determine the sign of $f(x)$ on each interval defined by the boundary points. Use this process to solve the inequalities in Exercises 117–128.

117. $\sqrt{2x - 6} - 2 < 0$

118. $\sqrt{3x - 5} - 4 < 0$

119. $\sqrt{4 - x} - 6 \ge 0$

120. $\sqrt{5-x} - 7 \geq 0$

121. $\dfrac{1}{\sqrt{x-2}-4} \leq 0$

122. $\dfrac{1}{\sqrt{x-3}-5} \leq 0$

123. $-3 < x^2 - 6x + 5 \leq 5$

124. $8 \leq x^2 + 4x + 3 < 15$

125. $|x^2 - 4| < 5$

126. $|x^2 - 1| < 17$

127. $|x^2 - 18| > 2$

128. $|x^2 - 6| > 3$

Technology Connections

129. Given the inequality $0.55|x^3 + 4.13x^2 - 1.84x - 3.5 < 6.7$,

a. Write the inequality in the form $f(x) < 0$.

b. Graph $y = f(x)$ on a suitable viewing window.

c. Use the Zero feature to approximate the real zeros of $f(x)$. Round to 1 decimal place.

d. Use the graph to approximate the solution set for the inequality $f(x) < 0$.

130. Given the inequality $0.24x^4 + 1.8x^3 + 3.3x^2 + 2.84x - 1.8 > 4.5$,

a. Write the inequality in the form $f(x) > 0$.

b. Graph $y = f(x)$ on a suitable viewing window.

c. Use the Zero feature to approximate the real zeros of $f(x)$. Round to 1 decimal place.

d. Use the graph to approximate the solution set for the inequality $f(x) > 0$.

131. An engineer for a food manufacturer designs an aluminum container for a hot drink mix. The container is to be a right circular cylinder 5.5 in. in height. The surface area represents the amount of aluminum used and is given by $S(r) = 2\pi r^2 + 11\pi r$, where r is the radius of the can.

a. Graph the function $y = S(r)$ and the line $y = 90$ on the viewing window [0, 3, 1] by [0, 150, 10].

b. Use the Intersect feature to approximate the point of intersection of $y = S(r)$ and $y = 90$. Round to 1 decimal place if necessary.

c. Determine the restrictions on r so that the amount of aluminum used is at most 90 in.^{2}. Round to 1 decimal place.

132. The concentration $C(t)$ (in ng/mL) of a drug in the bloodstream t hours after ingestion is modeled by

$$C(t) = \frac{500t}{t^3 + 100}$$

a. Graph the function $y = C(t)$ and the line $y = 4$ on the window [0, 32, 4] by [0, 15, 3].

b. Use the Intersect feature to approximate the point(s) of intersection of $y = C(t)$ and $y = 4$. Round to 1 decimal place if necessary.

c. To avoid toxicity, a physician may give a second dose of the medicine once the concentration falls below 4 ng/mL for increasing values of t. Determine the times at which it is safe to give a second dose. Round to 1 decimal place.

Problem Recognition Exercises

Solving Equations and Inequalities

At this point in the text, we have studied several categories of equations and related inequalities. These include

- linear equations and inequalities
- quadratic equations and inequalities
- polynomial equations and inequalities
- rational equations and inequalities
- radical equations and inequalities
- absolute value equations and inequalities
- compound inequalities

For Exercises 1–30, solve the equations and inequalities. Write the solution sets to the inequalities in interval notation if possible.

1. $-\frac{1}{2} \leq -\frac{1}{4}x - 5 < 2$

2. $2x^2 - 6x = 5$

3. $50x^3 - 25x^2 - 2x + 1 = 0$

4. $\frac{-5x(x-3)^2}{2+x} \leq 0$

5. $\sqrt[4]{m+4} - 5 = -2$

6. $-5 < y$ and $-3y + 4 \geq 7$

7. $|5t - 4| + 2 = 7$

8. $3 - 4\{x - 5[x + 2(3 - 2x)]\} = -2[4 - (x - 1)]$

9. $10x(2x - 14) = -29x^2 - 100$

10. $\frac{5}{y-4} = \frac{3y}{y+2} - \frac{2y^2 - 14y}{y^2 - 2y - 8}$

11. $x(x - 14) \leq -40$

12. $\frac{1}{x^2 - 14x + 40} \leq 0$

13. $|x - 0.15| = |x + 0.05|$

14. $\sqrt{t-1} - 5 \leq 1$

15. $n^{1/2} + 7 = 10$

16. $-4x(3 - x)(x + 2)^2(x - 5)^3 \geq 0$

17. $-2x - 5(x + 3) = -4(x + 2) - 3x$

18. $\sqrt{7x + 29} - 3 = x$

19. $(x^2 - 9)^2 - 5(x^2 - 9) - 14 = 0$

20. $2 + 7x^{-1} - 15x^{-2} = 0$

21. $|8x - 3| + 10 \leq 7$

22. $2(x - 1)^{3/4} = 16$

23. $x^3 - 3x^2 < 6x - 8$

24. $\frac{3-x}{x+5} \geq 1$

25. $15 - 3(x - 1) = -2x - (x - 18)$

26. $25x^2 + 70x > -49$

27. $2 < |3 - x| - 9$

28. $-4(x - 3) < 8$ or $-7 > x - 3$

29. $\frac{1}{3}x + \frac{2}{5} > \frac{5}{6}x - 1$

30. $2|2x + 1| - 2 \leq 8$

Go online for more practice problems.

Lesson 2-8

Variation

What Will You Learn?

After completing this lesson, you should be able to:

- Write Models Involving Direct, Inverse, and Joint Variation
- Solve Applications Involving Variation

Learn: Write Models Involving Direct, Inverse, and Joint Variation

The familiar relationship $d = rt$ tells us that distance traveled equals the rate of speed times the time of travel. For a car traveling 60 mph, we have $d = 60t$. From Table 2-2, notice that as the time of travel increases, the distance increases proportionally. We say that d is directly proportional to t, or that d varies directly as t. This is shown graphically in Figure 2-32.

TABLE 2-2 Distance Traveled at Time t

t (hr)	d (mi)
1	60
2	120
3	180
4	240
5	300
6	360

$d = 60t$

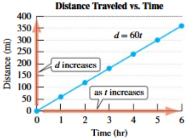


Figure 2-32

Now suppose that a motorist travels a fixed distance of 240 mi. We have

$$d = rt$$
$$240 = rt \rightarrow t = \frac{240}{r}$$

The time of travel t varies *inversely* as the rate of speed. As the rate r increases, the time of travel will decrease proportionally. Likewise, for slower rates, the time of travel is greater. See Table 2-3 and Figure 2-33.

TABLE 2-3 Travel Time for Rate r

r (mph)	t (hr)
10	24
20	12
30	8
40	6
50	4.8
60	4

$t = \frac{240}{r}$

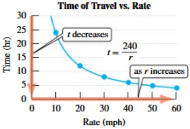


Figure 2-33

Direct and Inverse Variation

Let k be a nonzero constant real number. The statements on the left are equivalent to the equation on the right.

1. y varies **directly** as x.
   y is **directly** proportional to x. $\Big\}\; y = kx$
2. y varies **inversely** as x.
   y is **inversely** proportional to x. $\Big\}\; y = \frac{k}{x}$

Note: The value of k is called the **constant of variation**.

The first step in using a variation model is to write a statement as an equivalent mathematical equation.

Example 1: Writing a Variation Model

Write a variation model using k as the constant of variation.

Solution:

a. The amount of medicine A prescribed by a physician varies directly as the weight of the patient w.	Since the variables are directly related, set up the *product* of k and w. $A = kw$
b. The frequency f in a vibrating string is inversely proportional to the length L of the string.	Since the variables are inversely related, set up the *quotient* of k and L. $f = \frac{k}{L}$
c. The variable y varies directly as the square of x and inversely as the square root of z.	Since the square of the variable x is directly related to y, set up the *product* of k and x^2. And since the square root of z is inversely related to y, set up the *quotient* of k and $\sqrt{z}$. $y = \frac{kx^2}{\sqrt{z}}$

Insights

Notice that in each variation model, the constant of variation, k, is always in the numerator.

Apply the Skills

1. Write a variation model using k as the constant of variation.
   - **a.** The distance d that a spring stretches varies directly as the force F applied to the spring.
   - **b.** The force F required to keep a car from skidding on a curved road varies inversely as the radius r of the curve.
   - **c.** The variable a varies directly as b and inversely as the cube root of c.

Sometimes a variable varies directly as the product of two or more other variables. In such a case we have joint variation.

Joint Variation

Let k be a nonzero constant real number. The statements on the left are equivalent to the equation on the right.

$$\left.\begin{array}{l} y \text{ varies } \textbf{jointly} \text{ as } w \text{ and } x. \\ y \text{ is } \textbf{jointly} \text{ proportional to } w \text{ and } x. \end{array}\right\} y = kwx$$

Example 2: Writing a Joint Variation Model

Write a variation model using k as the constant of variation.

Solution:

a. y varies jointly as t and the cube root of u.	The variable t and the quantity $\sqrt[3]{u}$ are jointly related to y. Set up the product of k, t, and $\sqrt[3]{u}$. $y = kt\sqrt[3]{u}$
b. The gravitational force of attraction between two planets varies jointly as the product of their masses and inversely as the square of the distance between them.	Let m_1 and m_2 represent the masses of the planets, let d represent the distance between the planets, and let F represent the gravitational force between the planets. $F = \frac{km_1m_2}{d^2}$

Apply the Skills

2. Write a variation model using k as the constant of variation.

- **a.** The kinetic energy E of an object varies jointly as the object's mass m and the square of its velocity v.
- **b.** z varies jointly as x and y and inversely as the square root of w.

Learn: Solve Applications Involving Variation

Consider the variation models $y = kx$ and $y = \frac{k}{x}$. In either case, if values for x and y are known, we can solve for k. Once k is known, we can write a variation model and use it to find y if x is known, or to find x if y is known. This concept is the basis for solving many applications involving variation.

Procedure to Solve an Application Involving Variation

Step 1 Write a general variation model that relates the variables given in the problem. Let k represent the constant of variation.

Step 2 Solve for k by substituting known values of the variables into the model from step 1.

Step 3 Substitute the value of k into the original variation model from step 1.

Step 4 Use the variation model from step 3 to solve the application.

Example 3: Solving an Application Involving Direct Variation

The amount of an allergy medicine that a physician prescribes for a child varies directly as the weight of the child. Clinical research suggests that 13.5 mg of the drug should be given for a 30-lb child.

a. How much should be prescribed for a 50-lb child?

b. How much should be prescribed for a 60-lb child?

c. A nurse wants to double-check the dosage on a doctor's order of 18 mg. For a child of what weight is this dosage appropriate?

Solution:

Let A represent the amount of medicine. Label the variables.

Let w represent the weight of the child.

$A = kw$	**Step 1:** Write a general variation model.
$13.5 = k(30)$	**Step 2:** Substitute known values of A and w into the variation model.
$\frac{13.5}{30} = k$	Solve for k by dividing both sides by 30.
$k = 0.45$	
$A = 0.45w$	**Step 3:** Substitute the value of k into the original variation model.
a. $A = 0.45(50)$	**Step 4:** Solve the application by substituting 50 for w.
$A = 22.5$	A 50-lb child would require 22.5 mg of the drug.
b. $A = 0.45(60)$	Substitute 60 for w.
$A = 27$	A 60-lb child would require 27 mg of the drug.
c. $A = 0.45w$	
$18 = 0.45w$	Substitute 18 mg for the amount of medicine. Solve for the weight w.
$40 = w$	A 40-lb child would receive 18 mg.

Apply the Skills

3. The amount of the medicine ampicillin that a physician prescribes for a child varies directly as the weight of the child. A physician prescribes 420 mg for a 35-lb child.
   - **a.** How much should be prescribed for a 30-lb child?
   - **b.** How much should be prescribed for a 40-lb child?

Insights

Notice from Example 3 that more medicine is given in proportion to a patient's weight.

A 60-lb patient weighs twice as much as a 30-lb patient and the amount of medicine given is also twice as much. This is consistent with direct variation.

Example 4: Solving an Application Involving Inverse Variation

The loudness of sound measured in decibels (dB) varies inversely as the square of the distance between the listener and the source of the sound. If the loudness of sound is 17.92 dB at a distance of 10 ft from a stereo speaker, what is the decibel level 20 ft from the speaker?

Solution:

Let L represent the loudness of sound in decibels and d represent the distance in feet. The inverse relationship between decibel level and the square of the distance is modeled by:

$$\begin{aligned} L &= \frac{k}{d^2} & \\ 17.92 &= \frac{k}{(10)^2} & \text{Substitute } L = 17.92 \text{ dB and } d = 10 \text{ ft.} \\ 17.92 &= \frac{k}{100} & \\ (17.92)100 &= \frac{k}{100} \cdot 100 & \text{Solve for } k \text{ (clear fractions).} \\ k &= 1792 & \\ L &= \frac{1792}{d^2} & \text{Substitute } k = 1792 \text{ into the original model } L = \frac{k}{d^2}. \end{aligned}$$

With the value of k known, we can find L for any value of d.

$$\begin{aligned} L &= \frac{1792}{(20)^2} & \\ &= 4.48 \text{ dB} & \text{Find the loudness when } d = 20 \text{ ft.} \end{aligned}$$

Notice that the loudness of sound is 17.92 dB at a distance 10 ft from the speaker. When the distance from the speaker is increased to 20 ft, the decibel level decreases to 4.48 dB. This is consistent with an inverse relationship. For $k > 0$, as one variable is increased, the other is decreased. It also seems reasonable that the farther one moves away from the source of a sound, the softer the sound becomes.

Apply the Skills

4. The yield on a bond varies inversely as the price. The yield on a particular bond is 4% when the price is \$100. Find the yield when the price is \$80.

Example 5: Solving an Application Involving Joint Variation

Meteorologists define hurricane season for the northern Atlantic Ocean to be from June 1 to November 30. During this time, residents in the Caribbean Islands, Central America, and the Eastern Seaboard and Gulf Coast of the United States watch tropical weather forecasts with intense interest. These fearsome storms bring deadly storm surge, damaging winds, and physical and economic destruction to the areas they hit.

The winds of a hurricane are strong enough to send a piece of plywood through a tree. Furthermore, the kinetic energy of an object (energy that a body possesses by virtue of being in motion) varies jointly as the weight of the object at sea level and as the square of its velocity. During a hurricane, a 0.5-lb stone traveling 60 mph has 81 joules (J) of kinetic energy. Suppose the wind speed doubles to 120 mph. Find the kinetic energy.

Solution:

Let E represent the kinetic energy, let w represent the weight, and let v represent the velocity of the stone. The variation model is

$$E = kwv^2$$

$81 = k(0.5)(60)^2$ Substitute $E = 81$ J, $w = 0.5$ lb, and $v = 60$ mph.

$81 = k(0.5)(3600)$ Simplify the exponent.

$81 = k(1800)$

$\frac{81}{1800} = \frac{k(1800)}{1800}$ Divide by 1800.

$0.045 = k$ Solve for k.

With the value of k known, the model $E = kwv^2$ can be written as $E = 0.045wv^2$. We now find the kinetic energy of a 0.5-lb stone traveling 120 mph.

$$E = 0.045(0.5)(120)^2$$
$$= 324$$

The kinetic energy of a 0.5-lb stone traveling 120 mph is 324 J.

Apply the Skills

5. The amount of simple interest earned in an account varies jointly as the interest rate and time of the investment. An account earns \$200 in 2 yr at 4% interest. How much interest would be earned in 3 yr at a rate of 5%?

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Practice 2-8

Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, solve for the indicated variable.

R.1. $A = Ptr$ for r

R.2. $V = \frac{1}{3}\pi r^2 h$ for $h = \frac{3V}{\pi r^2}$

R.3. $k = \frac{IR}{E}$ for E

R.4. Solve for x. $\frac{16}{12} = \frac{24}{x}$

Concept Connections

1. If k is a nonzero constant real number, then the statement $y = kx$ implies that y varies ________________ as x.

2. If k is a nonzero constant real number, then the statement $y = \frac{k}{x}$ implies that y varies ________________ as x.

3. The value of k in the variation models $y = kx$ and $y = \frac{k}{x}$ is called the ________________ of ________________.

4. If y varies directly as two or more other variables such as x and w, then $y = kxw$, and we say that y varies ________________ as x and w.

5. **a.** Given $y = 2x$, evaluate y for the given values of x: $x = 1$, $x = 2$, $x = 3$, $x = 4$, and $x = 5$.

b. How does y change when x is doubled?

c. How does y change when x is tripled?

d. Complete the statement. Given $y = 2x$, when x increases, y (increases/decreases) proportionally.

e. Complete the statement. Given $y = 2x$, when x decreases, y (increases/decreases) proportionally.

6. **a.** Given $y = \frac{24}{x}$, evaluate y for the given values of x: $x = 1$, $x = 2$, $x = 3$, $x = 4$, and $x = 6$.

b. How does y change when x is doubled?

c. How does y change when x is tripled?

d. Complete the statement. Given $y = \frac{24}{x}$, when x increases, y (increases/decreases) proportionally.

e. Complete the statement. Given $y = \frac{24}{x}$, when x decreases, y (increases/decreases) proportionally.

7. The time required to drive from Atlanta, Georgia, to Nashville, Tennessee, varies ________________ as the average speed at which a vehicle travels.

8. The amount of a person's paycheck varies ________________ as the number of hours worked.

9. The volume of a right circular cone varies ________________ as the square of the radius of the cylinder and as the height of the cylinder.

10. A student's grade on a test varies ________________ as the number of hours the student spends studying for the test.

Learn: Write Models Involving Direct, Inverse, and Joint Variation

For Exercises 11–20, write a variation model using k as the constant of variation. (See Examples 1–2 on pp. 306–307)

11. The circumference C of a circle varies directly as its radius r.

12. Simple interest I on a loan or investment varies directly as the amount A of the loan.

13. The average cost per minute C for a flat rate cell phone plan is inversely proportional to the number of minutes used n.

14. The time of travel t is inversely proportional to the rate of travel r.

15. The volume V of a right circular cylinder varies jointly as the height h of the cylinder and as the square of the radius r of the cylinder.

16. The volume V of a rectangular solid varies jointly as the length l and width w of the solid.

17. The variable E is directly proportional to s and inversely proportional to the square root of n.

18. The variable n is directly proportional to the square of σ and inversely proportional to the square of E.

19. The variable c varies jointly as m and n and inversely as the cube of t.

20. The variable d varies jointly as u and v and inversely as the cube root of T.

Learn: Solve Applications Involving Variation

For Exercises 21–26, find the constant of variation *k*.

21. y varies directly as x. When x is 8, y is 20.

22. m varies directly as x. When x is 10, m is 42.

23. p is inversely proportional to q. When q is 18, p is 54.

24. T is inversely proportional to x. When x is 50, T is 200.

25. y varies jointly as w and v. When w is 40 and v is 0.2, y is 40.

26. N varies jointly as t and p. When t is 2 and p is 2.5, N is 15.

27. The value of y equals 4 when $x = 10$.

Find y when $x = 5$ if

- **a.** y varies directly as x.
- **b.** y varies inversely as x.

28. The value of y equals 24 when x is $\frac{1}{2}$. Find y when $x = 3$ if

- **a.** y varies directly as x.
- **b.** y varies inversely as x.

For Exercises 29–48, use a variation model to solve for the unknown value.

29. The amount of a pain reliever that a physician prescribes for a child varies directly as the weight of the child. A physician prescribes 180 mg of the medicine for a 40-lb child. (**See Example 3 on p. 308**)

- **a.** How much medicine would be prescribed for a 50-lb child?
- **b.** How much would be prescribed for a 60-lb child?
- **c.** How much would be prescribed for a 70-lb child?
- **d.** If 135 mg of medicine is prescribed, what is the weight of the child?

30. The number of people that a ham can serve varies directly as the weight of the ham. An 8-lb ham feeds 20 people.

- **a.** How many people will a 10-lb ham serve?
- **b.** How many people will a 15-lb ham serve?
- **c.** How many people will an 18-lb ham serve?
- **d.** If a ham feeds 30 people, what is the weight of the ham?

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31. A rental car company charges a fixed amount to rent a car per day. Therefore, the cost per mile to rent a car for a given day is inversely proportional to the number of miles driven. If 100 mi is driven, the average daily cost is \$0.80 per mile.

- **a.** Find the cost per mile if 200 mi is driven.
- **b.** Find the cost per mile if 300 mi is driven.
- **c.** Find the cost per mile if 400 mi is driven.
- **d.** If the cost per mile is \$0.16, how many miles were driven?

32. A chef self-publishes a cookbook and finds that the number of books sold per month varies inversely as the price of the book. The chef can sell 1500 books per month when the price is set at \$8 per book.

a. How many books would be sold per month if the price were \$12?

b. How many books would be sold per month if the price were \$15?

c. How many books would be sold per month if the price were \$6?

d. If the chef sells 1200 books, what price was set?

33. The distance that a bicycle travels in 1 min varies directly as the number of revolutions per minute (rpm) that the wheels are turning. A bicycle with a 14-in. radius travels approximately 440 ft in 1 min if the wheels turn at 60 rpm. How far will the bicycle travel in 1 min if the wheels turn at 87 rpm?

34. The amount of pollution entering the atmosphere varies directly as the number of people living in an area. If 100,000 people create 71,000 tons of pollutants, how many tons enter the atmosphere in a city with 750,000 people?

35. The stopping distance of a car is directly proportional to the square of the speed of the car.

a. If a car traveling 50 mph has a stopping distance of 170 ft, find the stopping distance of a car that is traveling 70 mph.

b. If it takes 244.8 ft for a car to stop, how fast was it traveling before the brakes were applied?

36. The area of a picture projected on a wall varies directly as the square of the distance from the projector to the wall.

a. If a 15-ft distance produces a 36 ft^2 picture, what is the area of the picture when the projection unit is moved to a distance of 25 ft from the wall?

b. If the projected image is 144 ft^2, how far is the projector from the wall?

37. The time required to complete a job varies inversely as the number of people working on the job. It takes 8 people 12 days to do a job. **(See Example 4 on p. 309)**

a. How many days will it take if 15 people work on the job?

b. If the contractor wants to complete the job in 8 days, how many people should work on the job?

38. The yield on a bond varies inversely as the price. The yield on a particular bond is 5% when the price is \$120.

a. Find the yield when the price is \$100.

b. What price is necessary for a yield of 7.5%?

39. The current in a wire varies directly as the voltage and inversely as the resistance. If the current is 9 amperes (A) when the voltage is 90 volts (V) and the resistance is 10 ohms (Ω), find the current when the voltage is 160 V and the resistance is 5 Ω.

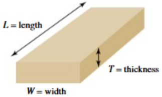
40. The resistance of a wire varies directly as its length and inversely as the square of its diameter. A 50-ft wire with a 0.2-in. diameter has a resistance of 0.0125 Ω. Find the resistance of a 40-ft wire with a diameter of 0.1 in.

41. The amount of simple interest owed on a loan varies jointly as the amount of principal borrowed and the amount of time the money is borrowed. If \$4000 in principal results in \$480 in interest in 2 yr, determine how much interest will be owed on \$6000 in 4 yr.

42. The amount of simple interest earned in an account varies jointly as the amount of principal invested and the amount of time the money is invested. If \$5000 in principal earns \$750 in 6 yr, determine how much interest will be earned on \$8000 in 4 yr.

43. The body mass index (BMI) of an individual varies directly as the weight of the individual and inversely as the square of the height of the individual. The body mass index for a 150-lb person who is 70 in. tall is 21.52. Determine the BMI for an individual who is 68 in. tall and 180 lb. Round to 2 decimal places. **(See Example 5 on p. 310)**

44. The strength of a wooden beam varies jointly as the width of the beam and the square of the thickness of the beam, and inversely as the length of the beam. A beam that is 48 in. long, 6 in. wide, and 2 in. thick can support a load of 417 lb. Find the maximum load that can be safely supported by a board that is 12 in. wide, 72 in. long, and 4 in. thick.



45. The speed of a racing canoe in still water varies directly as the square root of the length of the canoe. A 16-ft canoe can travel 6.2 mph in still water. Find the speed of a 25-ft canoe.

46. The period of a pendulum is the length of time required to complete one swing back and forth. The period varies directly as the square root of the length of the pendulum. If it takes 1.8 sec for a 0.81-m pendulum to complete one period, what is the period of a 1-m pendulum?

47. The cost to carpet a rectangular room varies jointly as the length of the room and the width of the room. A 10-yd by 15-yd room costs \$3870 to carpet. What is the cost to carpet a room that is 18 yd by 24 yd?

48. The cost to tile a rectangular kitchen varies jointly as the length of the kitchen and the width of the kitchen. A 10-ft by 12-ft kitchen costs \$1104 to tile. How much will it cost to tile a kitchen that is 20 ft by 14 ft?

Mixed Exercises

For Exercises 49–50, write a variation model using *k* as the constant of variation. Then find *k* based on the given values of the other variables.

49. $x = 128$, $y = 8$, and $z = 4$

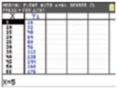
- **a.** x varies directly as the cube of y and inversely as the square root of z.
- **b.** x varies directly as the square of z and inversely as the cube root of y.

50. $r = 48$, $p = 2$, $w = 3$, and $t = 4$

- **a.** r varies jointly as w and the square of p and inversely as t.
- **b.** r varies jointly as the product of p and w and inversely as the square of t.

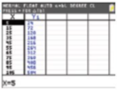
For Exercises 51–54, use the given data to find a variation model relating *y* to *x*.

51.



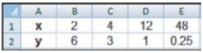
X=5

52.



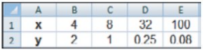
X=5

53.



	A	B	C	D	E
1	x	2	4	12	48
2	y	6	3	1	0.25

54.



	A	B	C	D	E
1	x	4	8	32	100
2	y	2	1	0.25	0.08

55. Which formula(s) can represent a variation model?

- **a.** $y = kxyz$
- **b.** $y = kx + yz$
- **c.** $y = \frac{kx}{yz}$
- **d.** $y = kx - yz$

56. Which formula(s) can represent a variation model?

- **a.** $y = k\sqrt{x} - z^2$
- **b.** $y = \frac{k\sqrt{x}}{z^2}$
- **c.** $y = k\sqrt{xz^2}$
- **d.** $y = k + \sqrt{xz^2}$

Write About It

For Exercises 57–58, write a statement in words that describes the variation model given. Use *k* as the constant of variation.

57. $P = \frac{kv^2}{t}$

58. $E = \frac{kc^2}{\sqrt{b}}$

Expanding Your Skills

59. The light from a lightbulb radiates outward in all directions.

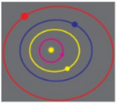
a. Consider the interior of an imaginary sphere on which the light shines. The surface area of the sphere is directly proportional to the square of the radius. If the surface area of a sphere with a 10-m radius is 400π m², determine the exact surface area of a sphere with a 20-m radius.

b. Explain how the surface area changed when the radius of the sphere increased from 10 m to 20 m.

c. Based on your answer from part (b), how would you expect the intensity of light to change from a point 10 m from the lightbulb to a point 20 m from the lightbulb?

d. The intensity of light from a light source varies inversely as the square of the distance from the source. If the intensity of a lightbulb is 200 lumen/m² (lux) at a distance of 10 m, determine the intensity at 20 m.

60. Kepler's third law states that the square of the time T required for a planet to complete one orbit around the Sun is directly proportional to the cube of the average distance d of the planet to the Sun. For the Earth assume that $d = 9.3 \times 10^7$ mi and $T = 365$ days.



a. Find the period of Mars, given that the distance between Mars and the Sun is 1.5 times the distance from the Earth to the Sun. Round to the nearest day.

b. Find the average distance of Venus to the Sun, given that Venus revolves around the Sun in 223 days. Round to the nearest million miles.

61. The intensity of radiation varies inversely as the square of the distance from the source to the receiver. If the distance is increased to 10 times its original value, what is the effect on the intensity to the receiver?

62. Suppose that y varies inversely as the cube of x. If the value of x is decreased to $\frac{1}{4}$ of its original value, what is the effect on y?

63. Suppose that y varies directly as x^2 and inversely as w^4. If both x and w are doubled, what is the effect on y?

64. Suppose that y varies directly as x^5 and inversely as w^2. If both x and w are doubled, what is the effect on y?

65. Suppose that y varies jointly as x and w^3. If x is replaced by $\frac{1}{3}x$ and w is replaced by $3w$, what is the effect on y?

66. Suppose that y varies jointly as x^4 and w. If x is replaced by $\frac{1}{4}x$ and w is replaced by $4w$, what is the effect on y?

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Chapter 2 Review

Chapter 2 Key Concepts

Lesson 2-1 Quadratic Functions and Applications

Key Concepts	Reference
Quadratic function: The function defined by $f(x) = ax^2 + bx + c$ ($a \neq 0$) is called a **quadratic function**. A quadratic function can be written in **vertex form:** $f(x) = a(x - h)^2 + k$ by completing the square.	p. 165
• The graph of a quadratic function $f(x) = a(x - h)^2 + k$ is a parabola. • The vertex is $(\boldsymbol{h}, \boldsymbol{k})$. • If $a > 0$, the parabola opens upward, and the minimum value of the function is k. • If $a < 0$, the parabola opens downward, and the maximum value of the function is k. • The axis of symmetry is the line $x = h$. • The x-intercepts are determined by the real solutions to the equation $f(x) = 0$. • The y-intercept is determined by $f(0)$.	p. 165
• Given $f(x) = ax^2 + bx + c$ ($a \neq 0$), the vertex of the parabola is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.	p. 169

Lesson 2-2 Introduction to Polynomial Functions

Key Concepts	Reference
Let n be a whole number and $a_n, a_{n-1}, a_{n-2}, \dots, a_1, a_0$ be real numbers, where $a_n \neq 0$. Then a function defined by $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ is called a **polynomial function of degree *n*.**	p. 185
The far left and far right behavior of the graph of a polynomial function is determined by the leading term of the polynomial, a_nx^n. n is even and $a_n > 0$ ↖ ↗ n is even and $a_n < 0$ ↙ ↘ n is odd and $a_n > 0$ ↙ ↗ n is odd and $a_n < 0$ ↖ ↘	p. 187
The **zeros** of a polynomial function defined by $y = f(x)$ are the values of x in the domain of f for which $f(x) = 0$. These are the real solutions (or **roots**) of the equation $f(x) = 0$.	p. 189

If a polynomial function f has a factor $(x - c)$ that appears exactly k times, then c is a **zero of multiplicity k**. - If c is a real zero of odd multiplicity, then the graph of $y = f(x)$ *crosses* the x-axis at c. - If c is a real zero of even multiplicity, then the graph of $y = f(x)$ *touches* the x-axis (but does not cross) at c.	p. 191
Intermediate value theorem: Let f be a polynomial function. For $a < b$, if $f(a)$ and $f(b)$ have opposite signs, then f has at least one zero on the interval $[a, b]$.	p. 193
The graph of a polynomial function of degree n will have at most $n - 1$ turning points.	p. 194

Lesson 2-3 Division of Polynomials and the Remainder and Factor Theorems

Key Concepts	Reference
Long division can be used to divide two polynomials.	p. 210
Synthetic division can be used to divide polynomials if the divisor is of the form $x - c$.	p. 214
Remainder theorem: If a polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.	p. 216
Factor theorem: Let $f(x)$ be a polynomial. **1.** If $f(c) = 0$, then $(x - c)$ is a factor of $f(x)$. **2.** If $(x - c)$ is a factor of $f(x)$, then $f(c) = 0$.	p. 219

Lesson 2-4 Zeros of Polynomials

Key Concepts	Reference
Rational zero theorem: If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients and $a_n \neq 0$, and if $\frac{p}{q}$ (written in lowest terms) is a rational zero of f, then - p is a factor of the constant term, a_0. - q is a factor of the leading coefficient a_n.	p. 228
Fundamental theorem of algebra: If $f(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $f(x)$ has at least one complex zero.	p. 232
Linear factorization theorem: If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $n \geq 1$ and $an \neq 0$, then $f(x) = a_n(x - c_1)(x - c_2) \ldots (x - c_n)$, where $c_1, c_2, \ldots, c_n$ are complex numbers.	p. 232

If $f(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $f(x)$ has exactly n complex zeros provided that each zero is counted by its multiplicity.	p. 233
Conjugate zeros theorem: If $f(x)$ is a polynomial with real coefficients and if $a + bi$ is a zero of $f(x)$, then its conjugate $a - bi$ is also a zero of $f(x)$.	p. 233
Descartes' rule of signs: Let $f(x)$ be a polynomial with real coefficients and a nonzero constant term. Then, **1.** The number of *positive* real zeros is either the same as the number of sign changes in $f(x)$ or less than the number of sign changes in $f(x)$ by a positive even integer. **2.** The number of *negative* real zeros is either the same as the number of sign changes in $f(-x)$ or less than the number of sign changes in $f(-x)$ by a positive even integer.	p. 236
Upper and lower bounds: Let $f(x)$ be a polynomial of degree $n \geq 1$ with real coefficients and a positive leading coefficient. Further suppose that $f(x)$ is divided by $(x - c)$. **1.** If $c > 0$ and if both the remainder and the coefficients of the quotient are nonnegative, then c is an upper bound for the real zeros of $f(x)$. **2.** If $c < 0$ and the coefficients of the quotient and the remainder alternate in sign (with 0 being considered either positive or negative as needed), then c is a lower bound for the real zeros of $f(x)$.	p. 239

Lesson 2-5 Introduction to Rational Functions

Key Concepts	Reference
Let $p(x)$ and $q(x)$ be polynomials where $q(x) \neq 0$. A function f defined by $f(x) = \frac{p(x)}{q(x)}$ is called a **rational function**.	p. 249
The line $x = c$ is a **vertical asymptote** of the graph of $y = f(x)$ if $f(x)$ approaches infinity or negative infinity as x approaches c from either side. To locate the vertical asymptotes of a function, determine the real numbers x where the denominator is zero, but the numerator is nonzero.	p. 251
The line $y = d$ is a **horizontal asymptote** of the graph of $y = f(x)$ if $f(x)$ approaches d as x approaches infinity or negative infinity.	p. 254
Let f be a rational function defined by $f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \cdots + b_1 x + b_0}$ **1.** If $n > m$, then f has no horizontal asymptote. **2.** If $n < m$, then the line $y = 0$ (the x-axis) is the horizontal asymptote of f. **3.** If $n = m$, then the line $y = \frac{a_n}{b_m}$ is the horizontal asymptote of f.	p. 255

Slant and nonlinear asymptotes: Let *f* be a rational function defined by $f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \cdots + b_1 x + b_0}$ Note that *n* is the degree of the numerator and *m* is the degree of the denominator. **1.** If $n = m + 1$ (that is, the degree of the numerator is exactly 1 greater than the degree of the denominator), the function has a **slant asymptote**. **2.** If $n > m + 1$ (that is, the degree of the numerator is more than 1 greater than the degree of the denominator), the end behavior of the function resembles the end behavior of the power function $y = \frac{a_n x^n}{b_m x^m}$. To find an equation of a slant or nonlinear asymptote, divide the numerator of $f(x)$ by the denominator and determine the quotient $Q(x)$. The equation $y = Q(x)$ is an equation of the asymptote.	p. 258

Lesson 2-6 Graphs of Rational Functions

Key Concepts	Reference
Graphing a Rational Function: Consider a rational function *f* defined by $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with no common factors. **1.** Determine the *y*-intercept by evaluating $f(0)$. **2.** Determine the *x*-intercept(s) by finding the real solutions of $f(x) = 0$. The value $f(x)$ equals zero when the numerator $p(x) = 0$. **3.** Identify any vertical asymptotes and graph them as dashed lines. **4.** Determine whether the function has a horizontal asymptote or a slant asymptote (or neither), and graph the asymptote as a dashed line. **5.** Determine where the function crosses the horizontal or slant asymptote (if applicable). **6.** If a test for symmetry is easy to apply, use symmetry to plot additional points. Recall: • *f* is an even function (symmetric to the *y*-axis) if $f(-x) = f(x)$. • *f* is an odd function (symmetric to the origin) if $f(-x) = -f(x)$. **7.** Plot at least one point on the intervals defined by the *x*-intercepts, vertical asymptotes, and points where the function crosses a horizontal or slant asymptote. **8.** Sketch the function based on the information found in steps 1–7.	p. 269

Lesson 2-7 Polynomial and Rational Inequalities

Key Concepts	Reference
Let $f(x)$ be a polynomial. Then an inequality of the form $f(x) < 0$, $f(x) > 0$, $f(x) \leq 0$, or $f(x) \geq 0$ is called a **polynomial inequality**.	p. 285
Let $f(x)$ be a rational expression. Then an inequality of the form $f(x) < 0$, $f(x) > 0$, $f(x) \leq 0$, or $f(x) \geq 0$ is called a **rational inequality**.	p. 291
Solving nonlinear inequalities: **1.** Express the inequality as $f(x) < 0$, $f(x) > 0$, $f(x) \leq 0$, or $f(x) \geq 0$. **2.** Find the real solutions of the related equation $f(x) = 0$ and the values of x where $f(x)$ is undefined. These are the "boundary" points for the solution set to the inequality. **3.** Determine the sign of $f(x)$ on the intervals defined by the boundary points. • If $f(x)$ is positive, then the values of x on the interval are solutions to $f(x) > 0$. • If $f(x)$ is negative, then the values of x on the interval are solutions to $f(x) < 0$. **4.** Determine whether the boundary points are included in the solution set. **5.** Write the solution set.	p. 286

Lesson 2-8 Variation

Key Concepts	Reference
Let k be a nonzero constant real number. Then the statements on the left are equivalent to the equations on the right. **1.** $\left.\begin{array}{l}y \text{ varies directly as } x.\\ y \text{ is directly proportional to } x.\end{array}\right\} y = kx$ **2.** $\left.\begin{array}{l}y \text{ varies inversely as } x.\\ y \text{ is inversely proportional to } x.\end{array}\right\} y = \frac{k}{x}$ **3.** $\left.\begin{array}{l}y \text{ varies jointly as } w \text{ and } x.\\ y \text{ is jointly proportional to } w \text{ and } x.\end{array}\right\} y = kwx$	p. 306
The value of k is called the **constant of variation**.	p. 307

Go online for a detailed chapter summary.

Chapter 2 Test

1. Given $f(x) = 2x^2 - 12x + 16$,

- **a.** Write the equation in vertex form: $f(x) = a(x - h)^2 + k$.
- **b.** Determine whether the parabola opens upward or downward.
- **c.** Identify the vertex.
- **d.** Identify the x-intercepts.
- **e.** Identify the y-intercept.
- **f.** Sketch the function.
- **g.** Determine the axis of symmetry.
- **h.** Determine the minimum or maximum value of the function.
- **i.** State the domain and range.

2. Given $f(x) = 2x^4 - 5x^3 - 17x^2 + 41x - 21$,

- **a.** Determine the end behavior of the graph of the function.
- **b.** List all possible rational zeros.
- **c.** Find all the zeros of the function and state their multiplicities.
- **d.** Determine the x-intercepts.
- **e.** Determine the y-intercept.
- **f.** Is the function even, odd, or neither?
- **g.** Graph the function.

3. Given $f(x) = -0.25x^3(x - 2)^2(x + 1)^4$,

- **a.** Identify the leading term.
- **b.** Determine the end behavior of the graph of the function.
- **c.** Find all the zeros of the function and state their multiplicities.

4. Given $f(x) = x^4 + 5x^2 - 36$,

- **a** How many zeros does $f(x)$ have (including multiplicities)?
- **b.** Find the zeros of $f(x)$.
- **c.** Identify the x-intercepts of the graph of f.
- **d.** Is the function even, odd, or neither?

5. Determine whether the intermediate value theorem guarantees that the function has a zero on the given interval.

$$f(x) = x^3 - 5x^2 + 2x + 5$$

- **a.** $[-2, -1]$
- **b.** $[-1, 0]$
- **c.** $[0, 1]$
- **d.** $[1, 2]$

6. **a.** Divide the polynomials. $\frac{2x^4 - 4x^3 + x - 5}{x^2 - 3x + 1}$

- **b.** Identify the dividend, divisor, quotient, and remainder.

7 Given $f(x) = 5x^4 + 47x^3 + 80x^2 - 51x - 9$,

- **a.** Is $\frac{3}{5}$ a zero of $f(x)$?
- **b.** Is -1 a zero of $f(x)$?
- **c.** Is $(x + 1)$ a factor of $f(x)$?
- **d.** Is $(x + 3)$ a factor of $f(x)$?
- **e.** Use the remainder theorem to evaluate $f(-2)$.

8. Given $f(x) = x^4 - 8x^3 + 21x^2 - 32x + 68$ and that $2i$ is a zero of $f(x)$,

- **a.** Find all zeros of $f(x)$.
- **b.** Factor $f(x)$ as a product of linear factors.
- **c.** Solve the equation $f(x) = 0$.

9. Given $f(x) = 3x^4 + 7x^3 - 12x^2 - 14x + 12$,

- **a.** How many zeros does $f(x)$ have (including multiplicities)?
- **b.** List the possible rational zeros.
- **c.** Determine if the upper bound theorem identifies 2 as an upper bound for the real zeros of $f(x)$.
- **d.** Determine if the lower bound theorem identifies -4 as a lower bound for the real zeros of $f(x)$.
- **e.** Revise the list of possible rational zeros based on the answers to parts (c) and (d).
- **f.** Find the rational zeros.
- **g.** Find all the zeros.
- **h.** Graph the function.

10. Write a third-degree polynomial $f(x)$ with integer coefficients and zeros of $\frac{1}{5}, -\frac{2}{3}$, and 4.

11. Determine the number of possible positive and negative real zeros for $f(x) = -6x^7 - 4x^5 + 2x^4 - 3x^2 + 1$.

For Exercises 12–15, determine the asymptotes (vertical, horizontal, slant, and nonlinear).

12. $r(x) = \frac{2x^2 - 3x + 5}{x - 7}$

13. $p(x) = \frac{-3x + 1}{4x^2 - 1}$

14. $n(x) = \frac{5x^2 - 2x + 1}{3x^2 + 4}$

15. $m(x) = \frac{x^4 - 4x^2 + 1}{x^2 + 2}$

For Exercises 16–18, graph the function.

16. $m(x) = \frac{1}{x^2} + 3$

17. $h(x) = \frac{-4}{x^2 - 4}$

18. $k(x) = \frac{x^2 - 2x + 1}{x}$

For Exercises 19–20, the graph of $y = f(x)$ is given. Solve the inequalities.

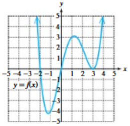
a. $f(x) < 0$

b. $f(x) \leq 0$

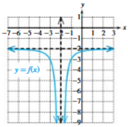
c. $f(x) > 0$

d. $f(x) \geq 0$

19.



20.



For Exercises 21–27, solve the inequality.

21. $c^2 < c + 20$

22. $y^3 > 13y - 12$

23. $-2x(x - 4)^2(x + 1)^3 \leq 0$

24. $9x^2 + 42x + 49 > 0$

25. $\frac{x + 3}{2 - x} \leq 0$

26. $\frac{-4}{x^2 - 9} \geq 0$

27. $\frac{-4}{x - 1} < \frac{3}{x}$

28. Solve the inequalities.

- **a.** $\frac{x + 1}{x - 5} \leq 0$
- **b.** $\frac{x + 1}{x - 5} < 0$
- **c.** $\frac{x + 1}{x - 5} \geq 0$
- **d.** $\frac{x + 1}{x - 5} > 0$

29. Write a variation model using k as the constant of variation: Energy E varies directly as the square of the velocity v of the wind.

30. Solve for the constant of variation k: The variable w varies jointly as y and the square root of x, and inversely as z. The value of w is 7.2 when x is 4, y is 6, and z is 7.

31. The surface area of a cube varies directly as the square of the length of an edge. The surface area is 24 ft² when the length of an edge is 2 ft. Find the surface area of a cube with an edge that is 7 ft.

32. The weight of a body varies inversely as the square of its distance from the center of the Earth. The radius of the Earth is approximately 4000 mi. How much would a 180-lb person weigh 20 mi above the surface of the Earth? Round to the nearest pound.

33. The pressure of wind on a wall varies jointly as the area of the wall and the square of the velocity of the wind. If the velocity of the wind is tripled, what is the effect on the pressure on the wall?

34. The population $P(t)$ of rabbits in a wildlife area t years after being introduced to the area is given by

$$p(t) = \frac{2000t}{t+1}$$

a. Determine the number of rabbits after 1 yr, 5 yr, and 10 yr. Round to the nearest whole unit.

b. What will the rabbit population approach as t approaches infinity?

35. An agricultural school wants to determine the number of corn plants per acre that will produce the maximum yield. The model $y(n) = -0.103n^2 + 8.32n + 15.1$ represents the yield $y(n)$ (in bushels per acre) based on n thousand plants per acre.

a. Evaluate $y(20)$, $y(30)$, and $y(60)$ and interpret their meaning in the context of this problem.

b. Determine the number of plants per acre that will maximize yield. Round to the nearest hundred plants.

c. What is the maximum yield? Round to the nearest bushel per acre.

36. Suppose that a rocket is shot straight upward from ground level with an initial speed of 98 m/sec.

a. Write a model that represents the height of the rocket $s(t)$ (in meters) t seconds after launch. (*Hint*: Use the model $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$ with $g = 9.8\text{m/sec}^2$).

b. When will the rocket reach its maximum height?

c. What is the maximum height?

d. Determine the time interval for which the rocket will be more than 200 m high. Round to the nearest tenth of a second.

37. The number of yearly visits to physicians' offices varies in part by the age of the patient. For the data shown in the table, a represents the age of patients (in yr) and $n(a)$ represents the corresponding number of visits to physicians' offices per year. (*Source*: Centers for Disease Control, www.cdc.gov)

a	8	20	35	55	65	85
$n(a)$	2.7	2.0	2.5	3.7	6.7	7.6

a. Use regression to find a quadratic function to model the data.

b. At what age is the number of yearly visits to physicians' offices the least? Round to the nearest year of age.

c. What is the minimum number of yearly visits? Round to 1 decimal place.

GO ONLINE **Go online for more practice problems.**

Chapter 2 Cumulative Review Exercises

1. Given $r(x) = \frac{2x^2 - 3}{x^2 - 16}$,

- **a.** Find the vertical asymptotes.
- **b.** Find the horizontal asymptote or slant asymptote.

2. Find a polynomial $f(x)$ of lowest degree with real coefficients and with zeros of $3 + 2i$ and 2.

3. Given $f(x) = 2x^3 - x^2 - 8x - 5$,

- **a.** Determine the end behavior of the graph of f.
- **b.** Find the zeros and their multiplicities.
- **c.** Find the x-intercepts.
- **d.** Find the y-intercept.
- **e.** Graph the function.

4. Divide and write the answer in the form $a + bi$. $\frac{3 + 2i}{4 - i}$

5. Determine the center and radius of the circle given by $x^2 + y^2 + 8x - 14y + 56 = 0$

6. Write an equation of the line passing through the points $(4, -8)$ and $(2, -3)$. Write the answer in slope-intercept form.

7. Determine the x– and y-intercepts of the graph of $x = y^2 - 9$.

8. Graph $f(x) = \begin{cases} -x - 1 & \text{for } x < 1 \\ \sqrt{x - 1} & \text{for } x \geq 1 \end{cases}$

9. Solve the equation for m. $v_0 t = \sqrt{m - t}$

10. Given $f(x) = 2x^2 - 6x + 1$,

- **a.** Find the x-intercepts.
- **b.** Find the y-intercept.
- **c.** Find the vertex of the parabola.

11. Factor. $125x^6 - y^9$

For Exercises 12–14, simplify the expression.

12. $\left(\frac{4x^3y^{-5}}{z^{-2}}\right)^{-3}\left(\frac{4y^{-6}}{x^{-12}}\right)^{1/2}$

13. $\sqrt[3]{250z^5xy^{21}}$

14. $\dfrac{\frac{1}{3x} - \frac{1}{x^2}}{\frac{1}{3} - \frac{3}{x^2}}$

For Exercises 15–20, solve the equations and inequalities. Write the solutions to the inequalities in interval notation if possible.

15. $-5 \leq -\frac{1}{4}x + 3 < \frac{1}{2}$

16. $|x - 3| + 4 \leq 10$

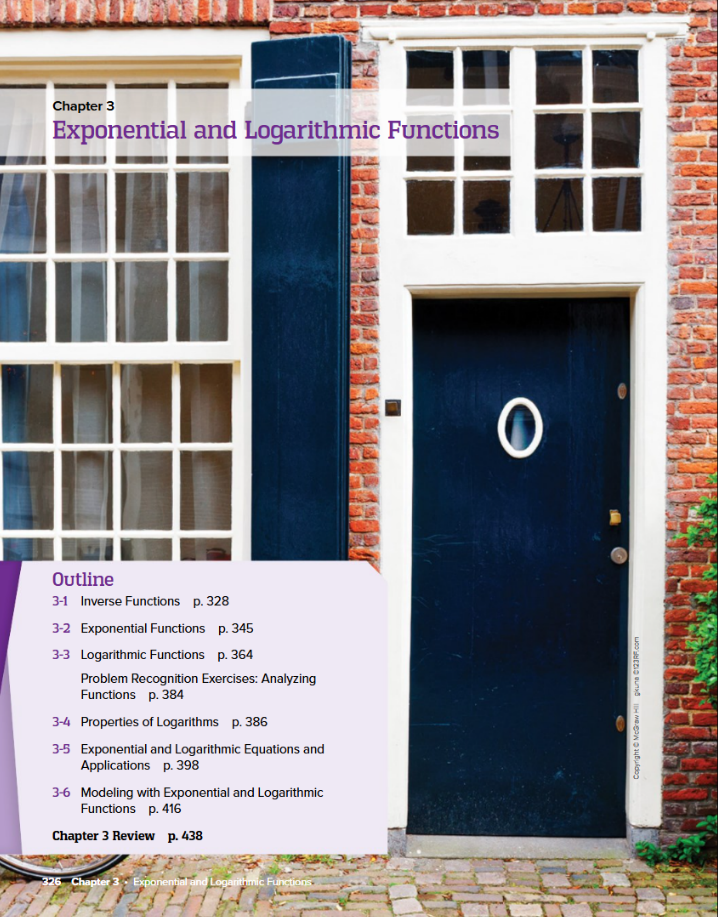
17. $|2x + 1| = |x - 4|$

18. $c^2 - 5c + 9 < c(c + 3)$

19. $\frac{49x^2 + 14x + 1}{x} > 0$

20. $\sqrt{4x - 3} - \sqrt{x + 12} = 0$

GO ONLINE **Go online for more practice problems.**



Chapter 3

Exponential and Logarithmic Functions

Outline

3-1 Inverse Functions p. 328

3-2 Exponential Functions p. 345

3-3 Logarithmic Functions p. 364

Problem Recognition Exercises: Analyzing Functions p. 384

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Launch Activity: Reason Quantitatively

The Best Rate

Maria wants to invest her $20,000 savings for a time to use as a down payment on a home. Maria has the option to choose between two different investment firms. Firm X offers a growth rate of 8.25% compounded monthly. Firm Y has the same expected yield but will compound Maria's investment quarterly.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer this question: What is the shortest amount of time it will take Maria to earn $25,000 for a down payment?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

GO ONLINE to answer these questions

Lesson 3-1

Inverse Functions

What Will You Learn?

After completing this lesson, you should be able to:

- Identify One-to-One Functions
- Determine Whether Two Functions Are Inverses
- Find the Inverse of a Function

Learn: Identify One-to-One Functions

Throughout our study of algebra, we have made use of the fact that the operations of addition and subtraction are inverse operations. For example, adding 5 to a number and then subtracting 5 from the result gives us the original number. Likewise, multiplication and division are inverse operations. We now look at the concept of an inverse function.

Changing currency is an important consideration when traveling abroad. For example, traveling between the United States and several countries in Europe would involve changing American dollars to euros and then changing back for the return trip. Fortunately, we can use a function to change from one currency to the other, and then use the function's *inverse* to change back again.

Suppose that $1 (American dollar) can be exchanged for 0.8 € (euro). Then,

$f(x) = 0.8x$ gives the number of euros $f(x)$ that can be bought from x dollars.

$g(x) = \frac{x}{0.8}$ gives the number of dollars $g(x)$ that can be bought from x euros.

Tables 3-1 and 3-2 show the values of $f(x)$ and $g(x)$ for several values of x.

TABLE 3-1 Values of $f(x)$ for Selected Values of x

x (Dollars)	$f(x) = 0.8x$ (Euros)
100	80
150	120
200	160
250	200

TABLE 3-2 Values of $g(x)$ for Selected Values of x

x (Euros)	$g(x) = \frac{x}{0.8}$ (Dollars)
80	100
120	150
160	200
200	250

In this example, functions *f* and *g* are inverses of each other, and we observe several interesting characteristics about inverse functions.

- By listing the ordered pairs from Tables 3-1 and 3-2, notice that the *x* and *y* values are reversed.

$$f\!: \{(100, 80), (150, 120), (200, 160), (250, 200)\}$$

$$g\!: \{(80, 100), (120, 150), (160, 200), (200, 250)\}$$

- For a function and its inverse, the values of *x* and *y* are interchanged. This tells us that the domain of a function is the same as the range of its inverse and vice versa.
- From the graphs of *f* and *g* (Figure 3-1), we see that the corresponding points on *f* and *g* are symmetric with respect to the line $y = x$.

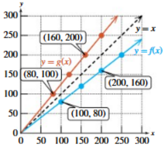


Figure 3-1

- When we compose functions *f* and *g* in both directions, the result is the input value *x*. In a sense, what function *f* does to *x*, function *g* "undoes" and vice versa.

$$(f \circ g)(x) = f[g(x)] = 0.8\left(\frac{x}{0.8}\right) = x$$
$$(g \circ f)(x) = g[f(x)] = \frac{(0.8x)}{0.8} = x$$

The inverse of any relation is found by interchanging the values of *x* and *y* in the relation. However, the inverse of a function may itself not be a function. For example, consider $f = \{(1, 3), (2, 3)\}$ shown in blue in Figure 3-2. The inverse is the set of ordered pairs $\{(3, 1), (3, 2)\}$ shown in red. Notice that the relation defining the inverse of *f* is not a function because it fails the vertical line test.

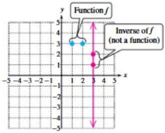


Figure 3-2

The function $f = \{(1, 3), (2, 3)\}$ has two points that are aligned horizontally. When the *x* and *y* values are reversed to form the inverse, the resulting points will be aligned vertically and will fail the vertical line test. So, a function will have an inverse function only if no points in the original function are aligned horizontally.

That is, no two distinct points on the function may have the same *y* value. In such a case, the function is said to be one-to-one.

Definition of a One-to-One Function

A function *f* is a **one-to-one function,** if for *a* and *b* in the domain of *f*,

if $a \neq b$, then $f(a) \neq f(b)$, or equivalently, if $f(a) = f(b)$, then $a = b$.

The definition of a one-to-one function tells us that each *y* value in the range is associated with only one *x* value in the domain. This implies that the graph of a one-to-one function will have no two points aligned horizontally.

Horizontal Line Test for a One-to-One Function

A function defined by $y = f(x)$ is a one-to-one function if no horizontal line intersects the graph in more than one point.

Example 1: Determining Whether a Function is One-to-One

Determine whether the function is one-to-one.

a. $f = \{(1, 4), (2, 3), (-2, 4)\}$

b. $g = \{(-3, 4), (1, -1), (2, 0)\}$

Solution:

same *y* value

a. $f = \{(1, 4), (2, 3), (-2, 4)\}$

different *x* value

f is not a one-to-one function.

The ordered pairs (1, 4) and (−2, 4) have the same *y* value but different *x* values. That is, $f(1) = f(-2)$, but $1 \neq -2$.

A horizontal line passes through the function in more than one point (Figure 3-3).

All points have different *y*-values.

b. $g = \{(-3, 4), (1, -1), (2, 0)\}$

g is a one-to-one function.

Each unique ordered pair has a different *y* value, so the function is one-to-one.

No horizontal line passes through the function in more than one point (Figure 3-4).

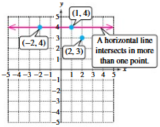


Figure 3-3

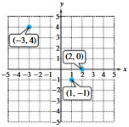


Figure 3-4

Apply the Skills

1. Determine whether the function is one-to-one.

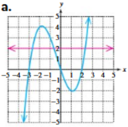
   a. $h = \{(4, -5), (6, 1), (2, 4), (0, -3)\}$ b. $k = \{(1, 0), (3, 0), (4, -5)\}$

Example 2: Using the Horizontal Line Test

Use the horizontal line test to determine if the graph in blue defines y as a one-to-one function of x.

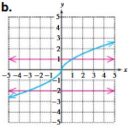
Solution:

a.



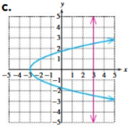
The graph does not define y as a one-to-one function of x because a horizontal line intersects the graph in more than one point.

b.



The graph does define y as a one-to-one function of x because no horizontal line intersects the graph in more than one point.

c.

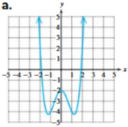


The relation does not define y as a function of x, because it fails the vertical line test. If the relation is not a function, it is not a one-to-one function.

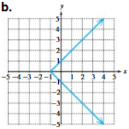
Apply the Skills

2. Use the horizontal line test to determine if the graph defines y as a one-to-one function of x.

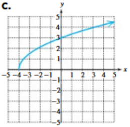
a.



b.



c.



In Example 3, we use algebraic methods to determine whether a function is one-to-one.

Example 3: Determining Whether a Function Is One-to-One

Use the definition of a one-to-one function to determine whether the function is one-to-one.

a. $f(x) = 2x - 3$ **b.** $f(x) = x^2 + 1$

Solution:

a. We must show that if $f(a) = f(b)$, then $a = b$.

Assume that $f(a) = f(b)$. That is,

$$2a - 3 = 2b - 3$$
$$2a - 3 + 3 = 2b - 3 + 3$$
$$\frac{2a}{2} = \frac{2b}{2}$$
$$a = b$$

$f(a) = f(b)$ implies that $a = b$.

Therefore, f is one-to-one.

The logic of this algebraic proof begins with the assumption that $f(a) = f(b)$, that is, that two y values are equal. For a one-to-one function, this can happen only if the x values (in this case a and b) are the same.

Otherwise, if $a \neq b$, we would have the same y value with two different x values and f would not be one-to-one.

b. $f(x) = x^2 + 1$

Assume that $f(a) = f(b)$.

$$a^2 + 1 = b^2 + 1$$
$$a^2 = b^2$$
$$a = \pm b$$

For nonzero values of b, $f(a) = f(b)$ does not necessarily imply that $a = b$. Therefore, f is not one-to-one.

From the graph of $f(x) = x^2 + 1$, we see that f is not one-to-one (Figure 3-5). We can also show this algebraically by finding two ordered pairs with the same y value but different x values. From the graph, we have arbitrarily selected $(-2, 5)$ and $(2, 5)$.

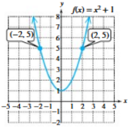


Figure 3-5

If $a = 2$ and $b = -2$, we have:

$$\left.\begin{array}{l} f(a) = f(2) = (2)^2 + 1 = 5 \\ f(b) = f(-2) = (-2)^2 + 1 = 5 \end{array}\right\} \text{Same } y \text{ value but differed } x \text{ values}$$

We have that $f(a) = f(b)$, but $a \neq b$.

Therefore, f fails to be a one-to-one function.

Insights

To show that a function is not one-to-one, we only need one counterexample. That is, any pair of distinct points on the function that have different y values is sufficient to show that the function is not one-to-one.

Apply the Skills

3. Determine whether the function is one-to-one.

a. $f(x) = -4x + 1$

b. $f(x) = |x| - 3$

Learn: Determine Whether Two Functions Are Inverses

We now have enough background to define an inverse function.

Definition of an Inverse Function

Let f be a one-to-one function. Then g is the **inverse of** f if the following conditions are both true.

1. $(f \circ g)(x) = x$ for all x in the domain of g.
2. $(g \circ f)(x) = x$ for all x in the domain of f.

We should also note that if g is the inverse of f, then f is the inverse of g. Furthermore, given a function f, we often denote its inverse as f^{-1}. So given a function f and its inverse f^{-1}, the definition implies that

$$(f \circ f^{-1})(x) = x \quad \text{and} \quad (f^{-1} \circ f)(x) = x$$

Check Your Work

Do not confuse inverse notation f^{-1} with exponential notation. The notation f^{-1} does not mean $\frac{1}{f}$.

Example 4: Determining Whether Two Functions Are Inverses

Determine whether the functions are inverses.

a. $f(x) = 100 + 12x$ and $g(x) = \frac{x - 100}{12}$

b. $h(x) = \sqrt[3]{x - 1}$ and $k(x) = -1 + x^3$

Solution:

a.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{x-100}{12}\right) \\ &= 100 + 12\left(\frac{x-100}{12}\right) \\ &= 100 + (x - 100) \\ &= x \checkmark\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(100 + 12x) \\ &= \frac{(100 + 12x) - 100}{12} \\ &= \frac{12x}{12} \\ &= x \checkmark\end{aligned}$$

Since $(f \circ g)(x) = (g \circ f)(x) = x$, f and g are inverses.

b.

$$\begin{aligned}(h \circ k)(x) &= h(k(x)) \\ &= \sqrt[3]{(-1 + x^3) - 1} \\ &= \sqrt[3]{x^3 - 2} \neq x\end{aligned}$$

If either $(h \circ k)(x) \neq x$ or $(k \circ h)(x) \neq x$, then h and k are *not* inverses.

Since $(h \circ k)(x) \neq x$, h and k are not inverses.

Apply the Skills

4. Determine whether the functions are inverses.

a. $f(x) = \frac{x+6}{2}$ and $g(x) = 2(x - 6)$

b. $m(x) = \frac{5}{x-2}$ and $n(x) = \frac{2x+5}{x}$

Learn: Find the Inverse of a Function

For a one-to-one function defined by $y = f(x)$, the inverse is a function $y = f^{-1}(x)$ that performs the inverse operations in the reverse order. The function given by $f(x) = 100 + 12x$ multiplies x by 12 first, and then adds 100 to the result. Therefore, the inverse function must *subtract* 100 from x first and then *divide* by 12.

$$f^{-1}(x) = \frac{x - 100}{12}$$

To facilitate the process of finding an equation of the inverse of a one-to-one function, we offer the following steps.

Procedure to Find an Equation of an Inverse of a Function

For a one-to-one function defined by $y = f(x)$, the equation of the inverse can be found as follows:

Step 1 Replace $f(x)$ with y.

Step 2 Interchange x and y.

Step 3 Solve for y.

Step 4 Replace y with $f^{-1}(x)$.

Example 5: Finding an Equation of an Inverse Function

Write an equation for the inverse function for $f(x) = 3x - 1$.

Solution:

Function f is a linear function, and its graph is a nonvertical line. Therefore, f is a one-to-one function.

$f(x) = 3x - 1$

$y = 3x - 1$ **Step 1:** Replace $f(x)$ with y.

$x = 3y - 1$ **Step 2:** Interchange x and y.

$x + 1 = 3y$ **Step 3:** Solve for y. Add 1 to both sides and divide by 3.

$\frac{x+1}{3} = y$

$f^{-1}(x) = \frac{x+1}{3}$ **Step 4:** Replace y with $f^{-1}(x)$.

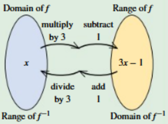
Check Your Work

To check the result, verify that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

$(f \circ f^{-1})(x) = 3\left(\frac{x+1}{3}\right) - 1 = x$ ✓ and $(f^{-1} \circ f)(x) = \frac{(3x-1)+1}{3} = x$ ✓

Insights

We can sometimes find an equation of an inverse function by mentally reversing the operations given in the original function. In Example 5, the function f multiplies x by 3 and then subtracts 1. Therefore, f^{-1} must add 1 to x and then divide by 3.



Apply the Skills

5. Write an equation for the inverse function for $f(x) = 4x + 3$.

The key step in determining the equation of the inverse of a function is to interchange x and y. By so doing, a point (a, b) on f corresponds to a point (b, a) on f^{-1}. This is why the graphs of f and f^{-1} are symmetric with respect to the line $y = x$. From Example 5, notice that the point (2, 5) on the graph of f corresponds to the point (5, 2) on the graph of f^{-1} (Figure 3-6).

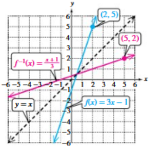


Figure 3-6

Example 6: Finding an Equation of an Inverse Function

Write an equation for the inverse function for the one-to-one function defined by $f(x) = \frac{3 - x}{x + 3}$.

Solution:

$f(x) = \frac{3 - x}{x + 3}$

$y = \frac{3 - x}{x + 3}$ **Step 1:** Replace $f(x)$ with y.

$x = \frac{3 - y}{y + 3}$ **Step 2:** Interchange x and y.

Step 3: Solve for y.

$x(y + 3) = 3 - y$ Clear fractions (multiply both sides by $y + 3$).

$xy + 3x = 3 - y$ Apply the distributive property.

$xy + y = 3 - 3x$ Collect the y terms on one side.

$y(x + 1) = 3 - 3x$ Factor out y as the greatest common factor.

$y = \frac{3 - 3x}{x + 1}$ Divide both sides by $x + 1$.

$f^{-1}(x) = \frac{3 - 3x}{x + 1}$ **Step 4:** Replace y with $f^{-1}(x)$.

Apply the Skills

6. Write an equation for the inverse function for the one-to-one function defined by $f(x) = \frac{x - 2}{x + 2}$.

Insights

In Example 6, we can show that f is a one-to-one function by graphing the function (see Lesson 2.6). Or we can show that $f(a) = f(b)$ implies that $a = b$ by solving the equation $\frac{3 - a}{a + 3} = \frac{3 - b}{b + 3}$ for a or b to show that $a = b$.

For a function that is not one-to-one, sometimes we restrict its domain to create a new function that is one-to-one. This is demonstrated in Example 7.

Example 7: Finding an Equation of an Inverse Function

Given $m(x) = x^2 + 4$ for $x \geq 0$, write an equation of the inverse.

Solution:

The graph of $y = x^2 + 4$ is a parabola with vertex (0, 4). See Figure 3-7. The function is not one-to-one. However, with the restriction on the domain that $x \geq 0$, the graph consists of only the right branch of the parabola (Figure 3-8). This *is* a one-to-one function.

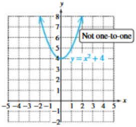


Figure 3-7

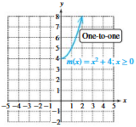


Figure 3-8

To find the inverse, we have

$m(x) = x^2 + 4; \quad x \geq 0$

$y = x^2 + 4; \quad x \geq 0$ **Step 1:** Replace $m(x)$ with y.

$x = y^2 + 4 \quad y \geq 0$ **Step 2:** Interchange x and y. Notice that the restriction $x \geq 0$ becomes $y \geq 0$.

$x - 4 = y^2$ **Step 3:** Solve for y by subtracting 4 from both sides.

$y = \pm\sqrt{x - 4}$ Apply the square root property.

$y = +\sqrt{x - 4}$ Choose the positive square root of $(x - 4)$ because of the restriction $y \geq 0$.

$m^{-1}(x) = \sqrt{x - 4}$ **Step 4:** Replace y with $m^{-1}(x)$.

The graphs of m and m^{-1} are symmetric with respect to the line $y = x$ as expected (Figure 3-9).

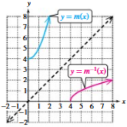


Figure 3-9

Apply the Skills

7. Given $n(x) = x^2 + 1$ for $x \leq 0$, write an equation of the inverse.

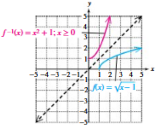
Example 8: Finding an Equation of an Inverse Function

Given $f(x) = \sqrt{x - 1}$, find an equation of the inverse.

Solution:

The function f is a one-to-one function and the graph is the same as the graph of $y = \sqrt{x}$ with a shift 1 unit to the right. The domain of f is $\{x | x \geq 1\}$ and the range is $\{y | y \geq 0\}$. When defining the inverse, we will have the conditions that $x \geq 0$ and $y \geq 1$.

$f(x) = \sqrt{x-1}$	Note that $x \geq 1$ and $y \geq 0$.
$y = \sqrt{x-1}$	
$x = \sqrt{y-1}$	Interchange x and y. Note that $y \geq 1$ and $x \geq 0$.
$x^2 = y - 1$	Square both sides.
$y = x^2 + 1$	
$f^{-1}(x) = x^2 + 1, \quad x \geq 0$	The restriction $x \geq 0$ on f^{-1} is necessary because f has the restriction that $y \geq 0$. Furthermore, $y = x^2 + 1$ is not a one-to-one function without a restricted domain.



Insights

When finding the inverse of a function, the key step of interchanging x and y has the effect of interchanging the domain and range between the function and its inverse.

Apply the Skills

8. Given $g(x) = \sqrt{x+2}$, find an equation of the inverse.

Practice 3-1

Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, find the domain. Write the answer in interval notation.

R.1. $f(x) = \frac{x+4}{x+1}$

R.2. $k(p) = \frac{p+1}{p^2+4}$

R.3. $m(x) = \sqrt{2-8x}$

For Exercises R.4–R.6, refer to functions *n* and *p* to evaluate the function.

$n(x) = x + 1 \qquad p(x) = x^2 - 3x$

R.4. $(n \circ p)(x)$

R.5. $(p \circ n)(x)$

R.6. $(p \circ p)(x)$

Concept Connections

1. Given the function $f = \{(1, 2), (2, 3), (3, 4)\}$ write the set of ordered pairs representing f^{-1}.
2. The graphs of a function and its inverse are symmetric with respect to the line __________.
3. If no horizontal line intersects the graph of a function *f* in more than one point, then *f* is a _____-_____-_____ function.
4. Given a one-to-one function *f*, if $f(a) = f(b)$, then *a* __________ *b*.
5. Let *f* be a one-to-one function and let *g* be the inverse of *f*. Then $(f \circ g)(x) =$ __________ and $(g \circ f)(x) =$ __________.
6. If (a, b) is a point on the graph of a one-to-one function *f*, then the corresponding ordered pair __________ is a point on the graph of f^{-1}.

Learn: Identify One-to-One Functions

For Exercises 7–12, a relation in *x* and *y* is given. Determine if the relation defines *y* as a one-to-one function of *x*. (See Example 1, p. 330)

7. $\{(6, -5), (4, 2), (3, 1), (8, 4)\}$

8. $\{(-14, 1), (-2, 3), (7, 4), (-9, -2)\}$

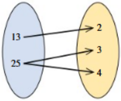
9.

x	*y*
0.6	1.8
1	–1.1
0.5	1.8
2.4	0.7

10.

x	*y*
12.5	3.21
5.75	–4.5
2.34	7.25
–12.7	3.21

11.



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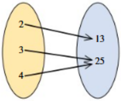
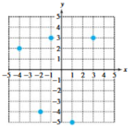


For Exercises 13–22, determine if the relation defines *y* as a one-to-one function of *x*. (See Example 2, on p. 331)

13.



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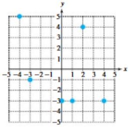

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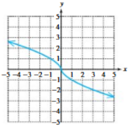

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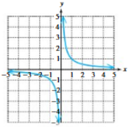

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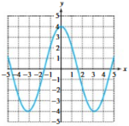

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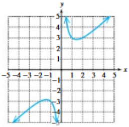

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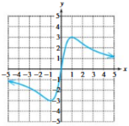

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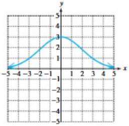

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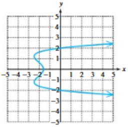

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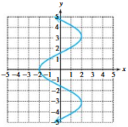
22.



No

For Exercises 23–30, use the definition of a one-to-one function to determine if the function is one-to-one. (See Example 3, p. 332)

23. $f(x) = 4x - 7$

24. $h(x) = -3x + 2$

25. $g(x) = x^3 + 8$

26. $k(x) = x^3 - 27$

27. $m(x) = x^2 - 4$

28. $n(x) = x^2 + 1$

29. $p(x) = |x + 1|$

30. $q(x) = |x - 3|$

Learn: Determine Whether Two Functions Are Inverses

For Exercises 31–36, determine if the two functions are inverses. (See Example 4, p. 333)

31. $f(x) = 5x + 4$ and $g(x) = \frac{x - 4}{5}$

32. $h(x) = 7x - 3$ and $k(x) = \frac{x + 3}{7}$

33. $m(x) = \frac{-2 + x}{6}$ and $n(x) = 6x - 2$

34. $p(x) = \frac{-3 + x}{4}$ and $q(x) = 4x - 3$

35. $t(x) = \frac{4}{x - 1}$ and $v(x) = \frac{x + 4}{x}$

36. $w(x) = \frac{6}{x + 2}$ and $z(x) = \frac{6 - 2x}{x}$

37. There were 2000 applicants for enrollment to the freshman class at a small college in the year 2010. The number of applications has risen linearly by roughly 150 per year. The number of applications $f(x)$ is given by $f(x) = 2000 + 150x$, where x is the number of years since 2010.

a. Determine if the function $g(x) = \frac{x - 2000}{150}$ is the inverse of f.

b. Interpret the meaning of function g in the context of this problem.

38. The monthly sales for January for a whole foods market was \$60,000 and has increased linearly by \$2500 per month. The amount in sales $f(x)$ (in \$) is given by $f(x) = 60{,}000 + 2500x$, where x is the number of months since January.

a. Determine if the function $g(x) = \frac{x - 60{,}000}{2500}$ is the inverse of f.

b. Interpret the meaning of function g in the context of this problem.

Learn: Find the Inverse of a Function

39. a. Show that $f(x) = 2x - 3$ defines a one-to-one function.

b. Write an equation for $f^{-1}(x)$.

c. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.

40. a. Show that $f(x) = 4x + 4$ defines a one-to-one function.

b. Write an equation for $f^{-1}(x)$.

c. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.

For Exercises 41–52, a one-to-one function is given. Write an equation for the inverse function. (See Examples 5–6, pp. 335–336)

41. $f(x) = \frac{4 - x}{9}$

42. $g(x) = \frac{8 - x}{3}$

43. $h(x) = \sqrt[3]{x - 5}$

44. $k(x) = \sqrt[3]{x + 8}$

45. $m(x) = 4x^3 + 2$

46. $n(x) = 2x^3 - 5$

47. $c(x) = \frac{5}{x + 2}$

48. $s(x) = \frac{2}{x - 3}$

49. $t(x) = \frac{x - 4}{x + 2}$

50. $v(x) = \frac{x - 5}{x + 1}$

51. $f(x) = \frac{(x - a)^3}{b} - c$

52. $g(x) = b(x + a)^3 + c$

53. a. Graph $f(x) = x^2 - 3;\ x \le 0$. **(See Example 7, p. 336)**

b. From the graph of f, is f a one-to-one function?

c. Write the domain of f in interval notation.

d. Write the range of f in interval notation.

e. Write an equation for $f^{-1}(x)$.

f. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.

g. Write the domain of f^{-1} in interval notation.

h. Write the range of f^{-1} in interval notation.

54. a. Graph $f(x) = x^2 + 1;\ x \le 0$.

b. From the graph of f, is f a one-to-one function?

c. Write the domain of f in interval notation.

d. Write the range of f in interval notation.

e. Write an equation for $f^{-1}(x)$.

f. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.

g. Write the domain of f^{-1} in interval notation.

h. Write the range of f^{-1} in interval notation.

55. a. Graph $f(x) = \sqrt{x + 1}$. **(See Example 8, p. 337)**

b. From the graph of f, is f a one-to-one function?

c. Write the domain of f in interval notation.

d. Write the range of f in interval notation.

e. Write an equation for $f^{-1}(x)$.

f. Explain why the restriction $x \ge 0$ is placed on f^{-1}.

g. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.

h. Write the domain of f^{-1} in interval notation.

i. Write the range of f^{-1} in interval notation.

56. a. Graph $f(x) = \sqrt{x - 2}$.

b. From the graph of *f*, is *f* a one-to-one function?

c. Write the domain of *f* in interval notation.

d. Write the range of *f* in interval notation.

e. Write an equation for $f^{-1}(x)$.

f. Explain why the restriction $x \geq 0$ is placed on f^{-1}.

g. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.

h. Write the domain of f^{-1} in interval notation.

i. Write the range of f^{-1} in interval notation.

57. Given that the domain of a one-to-one function *f* is $[0, \infty)$ and the range of *f* is $[0, 4)$, state the domain and range of f^{-1}.

58. Given that the domain of a one-to-one function *f* is $[-3, 5)$ and the range of *f* is $(-2, \infty)$, state the domain and range of f^{-1}.

59. Given $f(x) = |x| + 3$; $x \leq 0$, write an equation for f^{-1}. (*Hint*: Sketch $f(x)$ and note the domain and range.)

60. Given $f(x) = |x| - 3$; $x \geq 0$, write an equation for f^{-1}. (*Hint*: Sketch $f(x)$ and note the domain and range.)

For Exercises 61–66, fill in the blanks and determine an equation for $f^{-1}(x)$ mentally.

61. If function *f* adds 6 to *x*, then f^{-1} __________ 6 from *x*. Function *f* is defined by $f(x) = x + 6$, and function f^{-1} is defined by $f^{-1}(x) =$ __________.

62. If function *f* multiplies *x* by 2, then f^{-1} __________ *x* by 2. Function *f* is defined by $f(x) = 2x$, and function f^{-1} is defined by $f^{-1}(x) =$ __________.

63. Suppose that function *f* multiplies *x* by 7 and subtracts 4. Write an equation for $f^{-1}(x)$.

64. Suppose that function *f* divides *x* by 3 and adds 11. Write an equation for $f^{-1}(x)$.

65. Suppose that function *f* cubes *x* and adds 20. Write an equation for $f^{-1}(x)$.

66. Suppose that function *f* takes the cube root of *x* and subtracts 10. Write an equation for $f^{-1}(x)$.

For Exercises 67–70, find the inverse mentally.

67. $f(x) = 8x + 1$

68. $p(x) = 2x - 10$

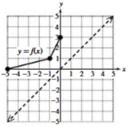
69. $q(x) = \sqrt[5]{x - 4} + 1$

70. $m(x) = \sqrt[3]{4x} + 3$

Mixed Exercises

For Exercises 71–74, the graph of a function is given. Graph the inverse function.

71.



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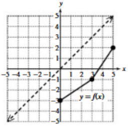

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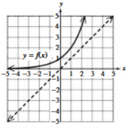

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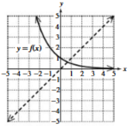
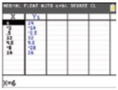


For Exercises 75–76, the table defines $Y_1 = f(x)$ as a one-to-one function of x. Find the values of f^{-1} for the selected values of x.

75. a. $f^{-1}(32)$

b. $f^{-1}(-2.5)$

c. $f^{-1}(26)$


NORMAL FLOAT AUTO a+bi DEGREE CL

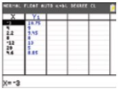
X | Y1

X=6

76. a. $f^{-1}(5)$

b. $f^{-1}(9.45)$

c. $f^{-1}(8)$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X | Y1

X=−3

For Exercises 77–80, determine if the statement is true or false. If a statement is false, explain why.

77. All linear functions with a nonzero slope have an inverse function.

78. The domain of any one-to-one function is the same as the domain of its inverse function.

79. The range of a one-to-one function is the same as the range of its inverse function.

80. No quadratic function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is one-to-one.

81. Based on data from recent hurricanes, the function defined by $w(x) = -1.17x + 1220$ gives the wind speed $w(x)$ (in mph) based on the barometric pressure x (in millibars, mb).

a. Approximate the wind speed for a hurricane with a barometric pressure of 1000 mb.

b. Write a function representing the inverse of w and interpret its meaning in context.

c. Approximate the barometric pressure for a hurricane with wind speed 100 mph. Round to the nearest mb.

82. The function defined by $F(x) = \frac{9}{5}x + 32$ gives the temperature $F(x)$ (in degrees Fahrenheit) based on the temperature x (in Celsius).

a. Determine the temperature in Fahrenheit if the temperature in Celsius is 25°C.

b. Write a function representing the inverse of F and interpret its meaning in context.

c. Determine the temperature in Celsius if the temperature in Fahrenheit is 5°F.

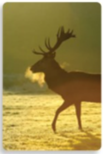
83. Suppose that during normal respiration, the volume of air inhaled per breath (called "tidal volume") by a mammal of any size is 6.33 mL per kilogram of body mass.

a. Write a function representing the tidal volume $T(x)$ (in mL) of a mammal of mass x (in kg).

b. Write an equation for $T^{-1}(x)$.

c. What does the inverse function represent in the context of this problem?

d. Find $T^{-1}(170)$ and interpret its meaning in context. Round to the nearest whole unit.



84. At a cruising altitude of 35,000 ft, a certain airplane travels 555 mph.

a. Write a function representing the distance $d(x)$ (in mi) for x hours at cruising altitude.

b. Write an equation for $d^{-1}(x)$.

c. What does the inverse function represent in the context of this problem?

d. Evaluate $d^{-1}(2553)$ and interpret its meaning in context.

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85. The millage rate is the amount of property tax per \$1000 of the taxable value of a home. For a certain county the millage rate is 24 mil (\$24 in tax per \$1000 of taxable value of the home). A city within the county also imposes a flat fee of \$108 per home.

a. Write a function representing the total amount of property tax $T(x)$ (in \$) for a home with a taxable value of x thousand dollars.

b. Write an equation for $T^{-1}(x)$.

c. What does the inverse function represent in the context of this problem?

d. Evaluate $T^{-1}(2988)$ and interpret its meaning in context.

86. Beginning on January 1, park rangers in Everglades National Park began recording the water level for one particularly dry area of the park. The water level was initially 2.5 ft and decreased by approximately 0.015 ft/day.

a. Write a function representing the water level $L(x)$ (in ft), x days after January 1.

b. Write an equation for $L^{-1}(x)$.

c. What does the inverse function represent in the context of this problem?

d. Evaluate $L^{-1}(1.9)$ and interpret its meaning in context.

Write About It

87. Explain the relationship between the domain and range of a one-to-one function f and its inverse f^{-1}.

88. Write an informal definition of a one-to-one function.

89. Explain why, if a horizontal line intersects the graph of a function in more than one point, then the function is not one-to-one.

90. Explain why the domain of $f(x) = x^2 + k$ must be restricted to find an inverse function.

Expanding Your Skills

91. Consider a function defined as follows: Given x, the value $f(x)$ is the exponent above the base of 2 that produces x. For example, $f(16) = 4$ because $2^4 = 16$. Evaluate

a. $f(8)$ **b.** $f(32)$

c. $f(2)$ **d.** $f\left(\frac{1}{8}\right)$

92. Consider a function defined as follows: Given x, the value $f(x)$ is the exponent above the base of 3 that produces x. For example, $f(9) = 2$ because $3^2 = 9$. Evaluate

a. $f(27)$ **b.** $f(81)$

c. $f(3)$ **d.** $f\left(\frac{1}{9}\right)$

93. Show that every increasing function is one-to-one.

94. A function is said to be periodic if there exists some nonzero real number p, called the period, such that $f(x + p) = f(x)$ for all real numbers x in the domain of f. Explain why no periodic function is one-to-one.

For Exercises 95–96, verify that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ for each pair of functions.

95. $f(x) = 2x - 7$, $g(x) = \dfrac{x+1}{2}$

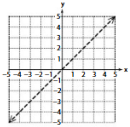
96. $f(x) = x^3$, $g(x) = x - 1$

97. Gives $f(x) = \begin{cases} x+2 & -5 \le x < 0 \\ \sqrt{x+4} & 0 \le x \le 5 \end{cases}$

a. Find the *range* for each branch of the piecewise-defined function.

b. Use the information from part (a) to write the inverse of f. (*Hint:* The range of f will become the domain of f^{-1}).

c. Graph both f and f^{-1} on the same grid.



98. Given $f(x) = \begin{cases} -\frac{1}{3}x + 1 & -3 \le x < 3 \\ -2x + 6 & 3 \le x \le 5 \end{cases}$, find the inverse function f^{-1}.

Go online for more practice problems.

Lesson 3-2

Exponential Functions

What Will You Learn?

After completing this lesson, you should be able to:

- Graph Exponential Functions
- Evaluate the Exponential Function Base *e*
- Use Exponential Functions to Compute Compound Interest
- Use Exponential Functions in Applications

Learn: Graph Exponential Functions

The concept of a function was first introduced in Lesson 1.3. Since then we have learned to recognize several categories of functions. In this lesson and the next, we will define two new types of functions called exponential functions and logarithmic functions.

To introduce exponential functions, consider two salary options for a new job. Option A pays $1 million for 1 month's work. Option B starts with 2¢ on the first day, and every day thereafter the salary is doubled. At first glance, the million-dollar option appears to be more favorable. However, Table 3-3 shows otherwise. The daily payments for 30 days are listed for Option B.

TABLE 3-3 Daily Payments for Option B

Day	Payment	Day	Payment	Day	Payment
1	2¢	11	$20.48	21	$20,971.52
2	4¢	12	$40.96	22	$41,943.04
3	8¢	13	$81.92	23	$83,886.08
4	16¢	14	$163.84	24	$167,772.16
5	32¢	15	$327.68	25	$335,544.32
6	64¢	16	$655.36	26	$671,088.64
7	$1.28	17	$1310.72	27	$1,342,177.28
8	$2.56	18	$2621.44	28	$2,684,354.56
9	$5.12	19	$5242.88	29	$5,368,709.12
10	$10.24	20	$10,485.76	30	$10,737,418.24

The salary for the 30th day for Option B is over $10 million. Taking the sum of the payments, we see that the total salary for the 30-day period is $21,474,836.46.

The daily salary $S(x)$ (in ¢) for Option B can be represented by the function $S(x) = 2^x$, where x is the number of days on the job. An interesting characteristic of this function is that for every positive 1-unit change in x, the function value doubles. The function $S(x) = 2^x$ is called an exponential function.

Insights

Consider the pattern involved for the payment for day $x = 1, 2, 3, 4, 5, \ldots$

$2^1 ¢ = 2¢$
$2^2 ¢ = 4¢$
$2^3 ¢ = 8¢$
$2^4 ¢ = 16¢$
$2^5 ¢ = 32¢$

Definition of an Exponential Function

Let b be a constant real number such that $b > 0$ and $b \neq 1$. Then for any real number x, a function of the form $f(x) = b^x$ is called an **exponential function of base *b*.**

An exponential function is recognized as a function with a constant base (positive and not equal to 1) with a variable exponent, *x*.

Exponential Functions	**Not Exponential Functions**
$f(x) = 3^x$	$m(x) = x^2$ base is not constant
$g(x) = \left(\frac{1}{3}\right)^x$	$n(x) = \left(-\frac{1}{3}\right)^x$ base is negative
$h(x) = (\sqrt{2})^x$	$p(x) = 1^x$ base is 1

Insights

- The base of an exponential function must not be negative to avoid situations where the function values are not real numbers. For example, $f(x) = (-4)^x$ is not defined for $x = \dfrac{1}{2}$ because $\sqrt{-4}$ is not a real number.
- The base of an exponential function must not equal 1 because $f(x) = 1^x = 1$ for all real numbers x. This is a constant function, not an exponential function.

At this point in the text, we have evaluated exponential expressions with integer exponents and with rational exponents. For example,

$$4^2 = 16 \qquad 4^{\frac{1}{2}} = \sqrt{4} = 2$$

$$4^{-1} = \frac{1}{4} \qquad 4^{\frac{10}{23}} = \sqrt[23]{4^{10}} \approx 1.827112184$$

However, how do we evaluate an exponential expression with an *irrational* exponent such as 4^{π}? In such a case, the exponent is a nonterminating, nonrepeating decimal. We define an exponential expression raised to an irrational exponent as a sequence of approximations using rational exponents. For example:

$$4^{3.14} \approx 77.7084726$$
$$4^{3.141} \approx 77.81627412$$
$$4^{3.1415} \approx 77.87023095$$
$$\ldots$$
$$4^{\pi} \approx 77.88023365$$

With this definition of a base raised to an irrational exponent, we can define an exponential function over the entire set of real numbers. In Example 1, we graph two exponential functions by plotting points.

Example 1: Graphing Exponential Functions

Graph the functions.

a. $f(x) = 2^x$ **b.** $g(x) = \left(\frac{1}{2}\right)^x$

Solution:

Table 3-4 shows several values of $f(x)$ and $g(x)$ for both positive and negative values of x.

TABLE 3-4 Values of $f(x)$ and $g(x)$ for Selected Values of x

x	$f(x) = 2^x$	$g(x) = \left(\frac{1}{2}\right)^x$
-3	$\frac{1}{8}$	8
-2	$\frac{1}{4}$	4
-1	$\frac{1}{2}$	2
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$

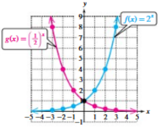


Figure 3-10

Notice that $g(x) = \left(\frac{1}{2}\right)^x$ is equivalent to $g(x) = 2^{-x}$. Therefore, the graph of $g(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$ is the same as the graph of $f(x) = 2^x$ with a reflection across the y-axis (Figure 3-10).

The graphs in Figure 3-10 illustrate several important features of exponential functions.

Insights

The values of $f(x)$ become closer and closer to 0 as $x \to -\infty$. This means that the x-axis is a horizontal asymptote.

Likewise, the values of $g(x)$ become closer to 0 as $x \to \infty$. The x-axis is a horizontal asymptote.

Apply the Skills

1. Graph the functions.

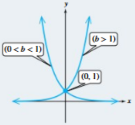
a. $f(x) = 5^x$ **b.** $g(x) = \left(\frac{1}{5}\right)^x$

Graphs of $f(x) = b^x$

The graph of an exponential function defined by $f(x) = b^x$ ($b > 0$ and $b \neq 1$) has the following properties.

1. If $b > 1$, f is an *increasing* exponential function, sometimes called an **exponential growth function**.

   If $0 < b < 1$, f is a *decreasing* exponential function, sometimes called an **exponential decay function**.
2. The domain is the set of all real numbers, $(-\infty, \infty)$.
3. The range is $(0, \infty)$.
4. The line $y = 0$ (x-axis) is a horizontal asymptote.
5. The function passes through the point $(0, 1)$ because $f(0) = b^0 = 1$.



These properties indicate that the graph of an exponential function is an increasing function if the base is greater than 1. Furthermore, the base affects the rate of increase. Consider the graphs of $f(x) = 2^x$ and $k(x) = 5^x$ (Figure 3-11). For every positive 1-unit change in x, $f(x) = 2^x$ is 2 times as great and $k(x) = 5^x$ is 5 times as great (Table 3-5).

TABLE 3-5 Values of $f(x)$ and $k(x)$ for Selected Values of x

x	$f(x) = 2^x$	$k(x) = 5^x$
-3	$\frac{1}{8}$	$\frac{1}{125}$
-2	$\frac{1}{4}$	$\frac{1}{25}$
-1	$\frac{1}{2}$	$\frac{1}{5}$
0	1	1
1	2	5
2	4	25
3	8	125

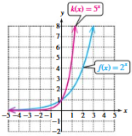


Figure 3-11

In Example 2, we use the transformations of functions learned in Lesson 1.6 to graph an exponential function.

If $h > 0$, shift to the right.
If $h < 0$, shift to the left.

$$f(x) = ab^{x-h} + k$$

If $a < 0$, reflect across the x-axis.
Shrink vertically if $0 < |a| < 1$.
Stretch vertically if $|a| > 1$.

If $k > 0$, shift upward.
If $k < 0$, shift downward.

Example 2: Graphing an Exponential Function

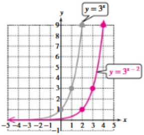
Graph. $f(x) = 3^{x-2} + 4$

Solution:

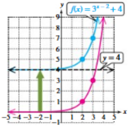
The graph of f is the graph of the parent function $y = 3^x$ shifted 2 units to the right and 4 units upward.

The parent function $y = 3^x$ is an increasing exponential function. We can plot a few points on the graph of $y = 3^x$ and use these points and the horizontal asymptote to form the outline of the transformed graph.

x	$y = 3^x$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9



Shift 2 units to the right. For example, the point (0, 1) on $y = 3^x$ corresponds to (2, 1) on $y = 3^{x-2}$.



Shift the graph of $y = 3^{x-2}$ up 4 units. Notice that with the vertical shift, the new horizontal asymptote is $y = 4$.

Apply the Skills

2. Graph. $g(x) = 2^{x+2} - 1$

Learn: Evaluate the Exponential Function Base *e*

We now introduce an important exponential function whose base is an irrational number called e. Consider the expression $\left(1 + \frac{1}{x}\right)^x$. The value of the expression for increasingly large values of x approaches a constant (Table 3-6, p. 350).

As $x \to \infty$, the expression $\left(1 + \frac{1}{x}\right)^x$ approaches a constant value that we call e. From Table 3-6, this value is approximately 2.718281828.

$$e \approx 2.718281828$$

TABLE 3-6 Values of $\left(1+\frac{1}{x}\right)^x$ for Selected Values of x

x	$\left(1+\frac{1}{x}\right)^x$
100	2.70481382942
1000	2.71692393224
10,000	2.71814592683
100,000	2.71826823717
1,000,000	2.71828046932
1,000,000,000	2.71828182710

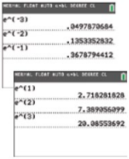
The value of *e* is an irrational number (a non-terminating, nonrepeating decimal) and like the number π, it is a universal constant. The function defined by $f(x) = e^x$ is called the exponential function base *e* or the **natural exponential function.**

Example 3: Graphing $f(x) = e^x$

Graph the function. $f(x) = e^x$

Solution:

Because the base *e* is greater than 1 ($e \approx 2.718281828$), the graph is an increasing exponential function. We can use a calculator to evaluate $f(x) = e^x$ at several values of *x*. On many calculators, the exponential function, base *e*, is invoked by selecting 2ND LN or by accessing e^x on the keyboard.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NORMAL FLOAT AUTO a+bi DEGREE CL

```
e^(-3)
                .0497870684
e^(-2)
                .1353352832
e^(-1)
                .3678794412
```

```
e^(1)
                2.718281828
e^(2)
                7.389056099
e^(3)
                20.08553692
```

x	$f(x) = e^x$
−3	0.050
−2	0.135
−1	0.368
0	1.000
1	2.718
2	7.389
3	20.086

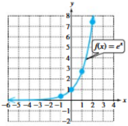

Figure 3-12

The graph of $f(x) = e^x$ is shown in Figure 3-12.

Insights

In Lesson 3.3, we will see that the exponential function base *e* is the inverse of the natural logarithmic function, $y = \ln x$. This is why the exponential function base *e* is accessed with the 2ND LN keys.

Apply the Skills

3. Explain how the graph of $f(x) = -e^{x-1}$ is related to the graph of $y = e^x$.

Learn: Use Exponential Functions to Compute Compound Interest

Recall that simple interest is interest computed on the principal amount invested (or borrowed). Compound interest is interest computed on both the original principal and the interest already accrued.

Suppose that interest is compounded annually (one time per year) on an investment of P dollars at an annual interest rate r for t years. Then the amount A (in $) in the account after 1, 2, and 3 yr is computed as follows:

After 1 yr: $\begin{pmatrix}\text{Total}\\ \text{amount}\end{pmatrix} = \begin{pmatrix}\text{Initial}\\ \text{principle}\end{pmatrix} + (\text{Interest})$

$$A = P + Pr$$

The interest is given by $I = Prt$, where $t = 1$ yr. So $I = Pr$.

$$= P(1 + r) \qquad \text{Factor out } P.$$

After 2 yr: $\begin{pmatrix}\text{Total}\\ \text{amount}\end{pmatrix} = \begin{pmatrix}\text{Year 1}\\ \text{balance}\end{pmatrix} + \begin{pmatrix}\text{Interest on}\\ \text{Year 1 balance}\end{pmatrix}$

$$A = P(1 + r) + [P(1 + r)]r$$

$$= P(1 + r)(1 + r) \qquad \text{Factor out } P(1 + r)$$

$$= P(1 + r)^2$$

After 3 yr: $\begin{pmatrix}\text{Total}\\ \text{amount}\end{pmatrix} = \begin{pmatrix}\text{Year 2}\\ \text{balance}\end{pmatrix}\begin{pmatrix}\text{Interest on}\\ \text{Year 2 balance}\end{pmatrix}$

$$A = P(1 + r)^2 + [P(1 + r)^2]r$$

$$= P(1 + r)^2\,(1 + r) \qquad \text{Factor out } P(1 + r)^2.$$

$$= P(1 + r)^3$$

$$\dots$$

After t years: $A = P(1 + r)^t$

Amount in an account with interest compounded continuously.

Compound interest is often computed more frequently during the course of 1 yr. Let n represent the number of compounding periods per year. For example:

$n = 1$	for interest compounded annually
$n = 4$	for interest compounded quarterly
$n = 12$	for interest compounded monthly
$n = 365$	for interest compounded daily

Each compounding period represents a fraction of a year, and the interest rate is scaled accordingly for each compounding period as $\frac{1}{n} \cdot r$ or $\frac{r}{n}$. The number of compounding periods over the course of the investment is nt.

Therefore, to determine the amount in an account where interest is compounded n times per year, we have

$$\overset{\text{replace } t \text{ with } nt}{A = \underset{\text{replace } r \text{ with } \frac{r}{n}}{P(1+r)^t}} \qquad A = P\left(1+\frac{r}{n}\right)^{nt}$$

Amount in an account with interest compounded n times per year.

Now suppose it were possible to compute interest continuously, that is, for $n \to \infty$. If we use the substitution $x = \frac{n}{r}$ (which implies that $n = xr$), the formula for compound interest becomes

$$A = P\left(1+\frac{r}{n}\right)^{nt} \xrightarrow{\text{Substitute } x = \frac{n}{r} \text{ (or } n = xr)} P\left(1+\frac{r}{xr}\right)^{xrt} = P\left(1+\frac{1}{x}\right)^{xrt} = P\left[\left(1+\frac{1}{x}\right)^{x}\right]^{rt}$$

For a fixed interest rate r, as n approaches infinity, x also approaches infinity. Since the expression $\left(1+\frac{1}{x}\right)^x$ approaches e as $x \to \infty$, we have

$$A = Pe^{rt}$$

Amount in an account with interest compounded continuously.

Summary of Formulas Relating to Simple and Compound Interest

Suppose that P dollars in principal is invested (or borrowed) at an annual interest rate r for t years. Then

- $I = Prt$ Amount of simple interest I (in \$).
- $A = P\left(1+\frac{r}{n}\right)^{nt}$ The future value A (in \$) of the account after t years with n compounding periods per year.
- $A = Pe^{rt}$ The future value A (in \$) of the account after t years under continuous compounding.

In Example 4, we compare the value of an investment after 10 yr under several different compounding options.

Example 4: Computing the Balance on an Account

Suppose that \$5000 is invested and pays 6.5% per year under the following compounding options.

a. Compounded annually

b. Compounded quarterly

c. Compounded monthly

d. Compounded daily

e. Compounded continuously

Determine the total amount in the account after 10 yr with each option.

Solution:

Using $A = P\left(1 + \frac{r}{n}\right)^{nt}$ and $A = Pe^{rt}$, we have

Compounding Option	n Value	Formula	Result
Annually	$n = 1$	$A = 5000\left(1 + \frac{0.065}{1}\right)^{(1 \cdot 10)}$	\$9385.69
Quarterly	$n = 4$	$A = 5000\left(1 + \frac{0.065}{4}\right)^{(4 \cdot 10)}$	\$9527.79
Monthly	$n = 12$	$A = 5000\left(1 + \frac{0.065}{12}\right)^{(12 \cdot 10)}$	\$9560.92
Daily	$n = 365$	$A = 5000\left(1 + \frac{0.065}{365}\right)^{(365 \cdot 10)}$	\$9577.15
Continuously	Not applicable	$\mathrm{A} = 5000e^{(0.065 \cdot 10)}$	\$9577.70

Notice that there is a \$192.01 difference in the account balance between annual compounding and continuous compounding. The table also supports our finding that

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \text{ converges to } A = Pe^{rt} \text{ as } n \to \infty.$$

Apply the Skills

4. Suppose that \$8000 is invested and pays 4.5% per year under the following compounding options.

- **a.** Compounded annually
- **b.** Compounded quarterly
- **c.** Compounded monthly
- **d.** Compounded daily
- **e.** Compounded continuously

Determine the total amount in the account after 5 yr with each option.

Learn: Use Exponential Functions in Applications

Increasing and decreasing exponential functions can be used in a variety of real-world applications. For example:

- Population growth can often be modeled by an exponential function.
- The growth of an investment under compound interest increases exponentially.
- The mass of a radioactive substance decreases exponentially with time.
- The temperature of a cup of coffee decreases exponentially as it approaches room temperature.

A substance that undergoes radioactive decay is said to be radioactive. The **half-life** of a radioactive substance is the amount of time it takes for one-half of the original amount of the substance to change into something else. That is, after each half-life, the amount of the original substance decreases by one-half.

Point of Interest

In 1898, Marie Curie discovered the highly radioactive element radium. She shared the 1903 Nobel Prize in physics for her research on radioactivity and was awarded the 1911 Nobel Prize in chemistry for her discovery of radium and polonium. Marie Curie died in 1934 from complications of excessive exposure to radiation.

Example 5: Using an Exponential Function in an Application

The half-life of radium 226 is 1620 yr. In a sample originally having 1 g of radium 226, the amount $A(t)$ (in grams) of radium 226 present after t years is given by $A(t) = \left(\frac{1}{2}\right)^{t/1620}$ where t is the time in years after the start of the experiment. How much radium will be present after

a. 1620 yr? **b.** 3240 yr? **c.** 4860 yr?

Solution:

a.

$$\begin{aligned} A(t) &= \left(\tfrac{1}{2}\right)^{t/1620} \\ A(1620) &= \left(\tfrac{1}{2}\right)^{1620/1620} \\ &= \left(\tfrac{1}{2}\right)^{1} \\ &= 0.5 \end{aligned}$$

b.

$$\begin{aligned} A(t) &= \left(\tfrac{1}{2}\right)^{t/1620} \\ A(3240) &= \left(\tfrac{1}{2}\right)^{3240/1620} \\ &= \left(\tfrac{1}{2}\right)^{2} \\ &= 0.25 \end{aligned}$$

c.

$$\begin{aligned} A(t) &= \left(\tfrac{1}{2}\right)^{t/1620} \\ A(4860) &= \left(\tfrac{1}{2}\right)^{4860/1620} \\ &= \left(\tfrac{1}{2}\right)^{3} \\ &= 0.125 \end{aligned}$$

The half-life of radium is 1620 yr. Therefore, we can interpret these results as follows:

After 1620 yr (1 half-life), 0.5 g remains $\left(\frac{1}{2}\text{ of the original amount remains}\right)$.

After 3240 yr (2 half-lives), 0.25 g remains $\left(\frac{1}{4}\text{ of the original amount remains}\right)$.

After 4860 yr (3 half-lives), 0.125 g remains $\left(\frac{1}{8}\text{ of the original amount remains}\right)$.

Apply the Skills

5. Cesium-137 is a radioactive metal with a short half-life of 30 yr. In a sample originally having 2g of cesium-137, the amount $A(t)$ (in grams) of cesium-137 present after t years is given by $A(t) = 2\left(\frac{1}{2}\right)^{t/30}$. How much cesium-137 will be present after

a. 30 yr? **b.** 60 yr? **c.** 90 yr?

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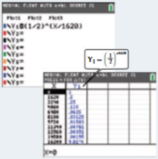
TECHNOLOGY CONNECTIONS

Graphing an Exponential Function

A graphing utility can be used to graph and analyze exponential functions.

The table shows several values of $A(t) = \left(\frac{1}{2}\right)^{x/1620}$ for selected values of x.

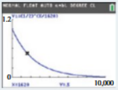

NORMAL FLOAT AUTO a+bi DEGREE CL
Plot1 Plot2 Plot3
$Y_1 = \left(\frac{1}{2}\right)^{x/1620}$
PRESS + FOR ΔTbl

```
Y1=(1/2)^(X/1620)
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=
Y9=
```

X	Y1
0	1
1620	.5
3240	.25
4860	.125
6480	.0625
8100	.03125
9720	.01563
11340	.00781
12960	.00391
14580	.00195
16200	9.8E-4

X=0

The graph of $A(t) = \left(\frac{1}{2}\right)^{x/1620}$ is shown on the viewing window [0, 10,000, 1000] by [0, 1.2, 0.2]. Notice that the graph is a decreasing exponential function because the base is between 0 and 1.

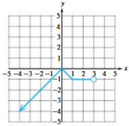


Practice 3-2

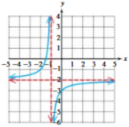
Practice Exercises

Prerequisite Review

R.1. Determine the domain and range.



R.2. Refer to the graph of the function and complete the statement.



R.3. Use transformations to graph $q(x) = -(x - 3)^2 + 4$.

- **a.** As $x \to -\infty$, $f(x) \to$ __________
- **b.** As $x \to -1^+$, $f(x) \to$__________
- **c.** As $x \to \infty$, $f(x) \to$ __________
- **d.** As $x \to -1^-$, $f(x) \to$ __________
- **e.** The graph is decreasing over the interval(s) __________.
- **f.** The graph is increasing over the interval(s) __________.

Concept Connections

1. The function defined by $y = x^3$(is/is not) an exponential function, whereas the function defined by $y = 3^x$(is/is not) an exponential function.
2. The graph of $f(x) = \left(\frac{5}{3}\right)^x$ is (increasing/decreasing) over its domain.
3. The graph of $f(x) = \left(\frac{3}{5}\right)^x$ is (increasing/decreasing) over its domain.
4. The domain of an exponential function $f(x) = b^x$ is ______________.
5. The range of an exponential function $f(x) = b^x$ is ______________.
6. All exponential functions $f(x) = b^x$ pass through the point ______________.
7. The horizontal asymptote of an exponential function $f(x) = b^x$ is the line____________.
8. As $x \to \infty$, the value of $\left(1 + \frac{1}{x}\right)^x$ approaches ______________.

Learn: Graph Exponential Functions

For Exercises 9–12, evaluate the functions at the given values of *x*. Round to 4 decimal places if necessary.

9. $f(x) = 5^x$
   - **a.** $f(-1)$
   - **b.** $f(4.8)$
   - **c.** $f(\sqrt{2})$
   - **d.** $f(\pi)$
10. $g(x) = 7^x$
    - **a.** $g(-2)$
    - **b.** $g(5.9)$
    - **c.** $g(\sqrt{11})$
    - **d.** $g(e)$
11. $h(x) = \left(\frac{1}{4}\right)^x$
    - **a.** $h(-3)$
    - **b.** $h(1.4)$
    - **c.** $h(\sqrt{3})$
    - **d.** $h(0.5e)$
12. $k(x) = \left(\frac{1}{6}\right)^x$
    - **a.** $k(-3)$
    - **b.** $k(1.4)$
    - **c.** $k(\sqrt{0.5})$
    - **d.** $k(0.5\pi)$

13. Which functions are exponential functions?

a. $f(x) = 4.2^x$

b. $g(x) = x^{4.2}$

c. $h(x) = 4.2x$

d. $k(x) = (\sqrt{4.2})^x$

e. $m(x) = (-4.2)^x$

14. Which functions are exponential functions?

a. $v(x) = (-\pi)^x$

b. $t(x) = \pi^x$

c. $w(x) = \pi x$

d. $n(x) = (\sqrt{\pi})^x$

e. $p(x) = x^\pi$

For Exercises 15–22, graph the functions and write the domain and range in interval notation. (See Example 1, p. 347)

15. $f(x) = 3^x$

16. $g(x) = 4^x$

17· $h(x) = \left(\frac{1}{3}\right)^x$

18. $k(x) = \left(\frac{1}{4}\right)^x$

19. $m(x) = \left(\frac{3}{2}\right)^x$

20. $n(x) = \left(\frac{5}{4}\right)^x$

21. $b(x) = \left(\frac{2}{3}\right)^x$

22. $c(x) = \left(\frac{4}{5}\right)^x$

For Exercises 23–32,

a. Use transformations of the graphs of $y = 3^x$ (see Exercise 15) and $y = 4^x$ (see Exercise 16) to graph the given function. **(See Example 2, p. 349)**

b. Write the domain and range in interval notation.

c. Write an equation of the asymptote.

23. $f(x) = 3^x + 2$

24. $g(x) = 4^x - 3$

25. $m(x) = 3^{x+2}$

26. $n(x) = 4^{x-3}$

27. $p(x) = 3^{x-4} - 1$

28. $q(x) = 4^{x+1} + 2$

29. $k(x) = -3^x$

30. $h(x) = -4^x$

31. $t(x) = 3^{-x}$

32. $v(x) = 4^{-x}$

For Exercises 33–36,

a. Use transformations of the graphs of $y = \left(\frac{1}{3}\right)^x$ (see Exercise 17) and $y = \left(\frac{1}{4}\right)^x$ (see Exercise 18) to graph the given function. (**See Example 2, p. 349**)

b. Write the domain and range in interval notation.

c. Write an equation of the asymptote.

33. $f(x) = \left(\frac{1}{3}\right)^{x+1} - 3$

34. $g(x) = \left(\frac{1}{4}\right)^{x-2} + 1$

35. $k(x) = -\left(\frac{1}{3}\right)^x + 2$

36. $h(x) = -\left(\frac{1}{4}\right)^{x-2} - 2$

Learn: Evaluate the Exponential Function Base *e*

For Exercises 37–38, evaluate the functions for the given values of *x*. Round to 4 decimal places.

37. $f(x) = e^x$

a. $f(4)$

b. $f(-3.2)$

c. $f(\sqrt{13})$

d. $f(\pi)$

38. $f(x) = ex$

a. $f(-3)$

b. $f(6.8)$

c. $f(\sqrt{7})$

d. $f(e)$

For Exercises 39–44,

a. Use transformations of the graph of $y = e^x$ to graph the given function. **(See Example 3, p. 350)**

b. Write the domain and range in interval notation.

c. Write an equation of the asymptote.

39. $f(x) = e^{x-4}$

40. $g(x) = e^{x-2}$

41. $h(x) = e^x + 2$

42. $k(x) = e^x - 1$

43. $m(x) = -e^x - 3$

44. $n(x) = -e^x + 4$

Learn: Use Exponential Functions to Compute Compound Interest

For Exercises 45–46, complete the table to determine the effect of the number of compounding periods when computing interest. (See Example 4, p. 352)

45. Suppose that \$10,000 is invested at 4% interest for 5 yr under the following compounding options. Complete the table.

	Compounding Option	*n* Value	Result
a.	Annually		
b.	Quarterly		
c.	Monthly		
d.	Daily		
e.	Continuously		

46. Suppose that \$8000 is invested at 3.5% interest for 20 yr under the following compounding options. Complete the table.

	Compounding Option	*n* Value	Result
a.	Annually		
b.	Quarterly		
c.	Monthly		
d.	Daily		
e.	Continuously		

For Exercises 47–48, suppose that *P* dollars in principal is invested for *t* years at the given interest rates with continuous compounding. Determine the amount that the investment is worth at the end of the given time period.

47. $P = \$20{,}000$, $t = 10$ yr

a. 3% interest

b. 4% interest

c. 5.5% interest

48. $P = \$6000$, $t = 12$ yr

a. 1% interest

b. 2% interest

c. 4.5% interest

49. Aniyah needs to borrow \$10,000. The money can be borrowed at 5.5% simple interest for 4 yr or borrowed at 5% with interest compounded continuously for 4 yr.

a. How much total interest would Aniyah pay at 5.5% simple interest?

b. How much total interest would Aniyah pay at 5% interest compounded continuously?

c. Which option results in less total interest?

50. Antonne needs to borrow \$15,000 to buy a car. The money can be borrowed at 6.7% simple interest for 5 yr or borrowed at 6.4% interest compounded continuously for 5 yr.

a. How much total interest would Antonne pay at 6.7% simple interest?

b. How much total interest would Antonne pay at 6.4% interest compounded continuously?

c. Which option results in less total interest?

51. Jerome wants to invest \$25,000 as part of a retirement plan. The money can be invested at 5.2% simple interest for 30 yr, or invested at 3.8% interest compounded continuously for 30 yr. Which option results in more total interest?

52. Skyler wants to invest \$35,000 as part of a retirement plan. The money can be invested at 4.8% simple interest for 20 yr, or invested at 3.6% interest compounded continuously for 20 yr. Which option results in more total interest?

Learn: Use Exponential Functions in Applications

53. Strontium-90 (^{90}Sr) is a by-product of nuclear fission with a half-life of approximately 28.9 yr. After the Chernobyl nuclear reactor accident in 1986, large areas surrounding the site were contaminated with ^{90}Sr. If 10 μg (micrograms) of ^{90}Sr is present in a sample, the function $A(t) = 0.1\left(\frac{1}{2}\right)^{t/138.4}$ gives the amount $A(t)$ (in μg) present after t years. Evaluate the function for the given values of t and interpret the meaning in context. Round to 3 decimal places if necessary. (**See Example 5, p. 354**)

a. $A(28.9)$ **b.** $A(57.8)$ **c.** $A(100)$

54. The murder of Alexander Litvinenko, a Russian dissident, was thought to be by poisoning from the rare and highly radioactive element polonium-210 (^{210}Po). The half-life of ^{210}Po is 138.4 yr. If 0.1 mg of ^{210}Po is present in a sample, then $A(t) = 0.1\left(\frac{1}{2}\right)^{t/138.4}$ gives the amount $A(t)$ (in mg) present after t years. Evaluate the function for the given values of t and interpret the meaning in context. Round to 3 decimal places if necessary.

a. $A(138.4)$ **b.** $A(276.8)$ **c.** $A(500)$

55. In 2021, the population of the United States was approximately 335 million with a 0.7% annual growth rate. (*Source*: www.cia.gov) At this rate, the population $P(t)$ (in millions) can be approximated by $P(t) = 335(1.007)^t$, where t is the time in years since 2021.

a. Is the graph of P an increasing or decreasing exponential function?

b. Evaluate $P(0)$ and interpret its meaning in the context of this problem.

c. Evaluate $P(5)$ and interpret its meaning in the context of this problem. Round the population value to the nearest million.

d. Evaluate $P(10)$ and $P(15)$.

e. Evaluate $P(200)$ and use this result to determine if it is reasonable to expect this model to continue indefinitely.

56. The population of Mexico in 2020 was approximately 129 million with an annual growth rate of 1.04%. At this rate, the population $P(t)$ (in millions) can be approximated by $P(t) = 129(1.0104)^t$, where t is the time in years since 2020. (*Source*: www.cia.gov)

a. Is the graph of P an increasing or decreasing exponential function?

b. Evaluate $P(0)$ and interpret its meaning in the context of this problem.

c. Evaluate $P(5)$ and interpret its meaning in the context of this problem. Round the population value to the nearest million.

d. Evaluate $P(15)$ and $P(25)$.

57. The atmospheric pressure on an object decreases as altitude increases. If a is the height (in km) above sea level, then the pressure $P(a)$ (in mmHg) is approximated by $P(a) = 760e^{-0.13a}$.

a. Find the atmospheric pressure at sea level.

b. Determine the atmospheric pressure at 8.848 km (the altitude of Mt. Everest). Round to the nearest whole unit.

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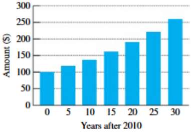
58. The function defined by $A(t) = 100e^{0.0318t}$ approximates the equivalent amount of money needed t years after the year 2010 to equal \$100 of buying power in the year 2010. The value 0.0318 is related to the average rate of inflation.

- **a.** Evaluate $A(15)$ and interpret its meaning in the context of this problem.
- **b.** Verify that by the year 2032, more than \$200 will be needed to have the same buying power as \$100 in 2010.



Newton's law of cooling indicates that the temperature of a warm object, such as a cake coming out of the oven, will decrease exponentially with time and will approach the temperature of the surrounding air. The temperature $T(t)$ is modeled by $T(t) = T_a + (T_0 - T_a)e^{-kt}$. In this model, T_a represents the temperature of the surrounding air, T_0 represents the initial temperature of the object, and *t* is the time after the object starts cooling. The value of *k* is a constant of proportion relating the temperature of the object to its rate of temperature change. Use this model for Exercises 59–60.

59. A cake comes out of the oven at 350°F and is placed on a cooling rack in a 78°F kitchen. After checking the temperature several minutes later, the value of k is measured as 0.046.

- **a.** Write a function that models the temperature $T(t)$ (in °F) of the cake t minutes after being removed from the oven.
- **b.** What is the temperature of the cake 10 min after coming out of the oven? Round to the nearest degree.
- **c.** It is recommended that the cake should not be frosted until it has cooled to under 100°F. If a baker waits 1 hr to frost the cake, will the cake be cool enough to frost?

60. Water in a water heater is originally 122°F. The water heater is shut off and the water cools to the temperature of the surrounding air, which is 60°F. The water cools slowly because of the insulation inside the heater, and the value of k is measured as 0.00351.

- **a.** Write a function that models the temperature $T(t)$ (in °F) of the water t hours after the water heater is shut off.
- **b.** What is the temperature of the water 12 hr after the heater is shut off? Round to the nearest degree.
- **c.** Dominic does not like to shower with water less than 115°F. If Dominic waits 24 hr, will the water still be warm enough for a shower?

Mixed Exercises

61. A farmer depreciates a \$120,000 tractor. The estimated resale value $V(t)$ (in \$1000) of the tractor t years after purchase is 80% of its value from the previous year. Therefore, the resale value can be approximated by $V(t) = 120(0.8)^t$.

- **a.** Find the resale value 5 yr after purchase. Round to the nearest \$1000.
- **b.** The farmer estimates that the cost to run the tractor is \$18/hr in labor, \$36/hr in fuel, and \$22/hr in overhead costs (for maintenance and repair). Estimate the farmer's cost to run the tractor for the first year if the tractor is run for a total of 800 hr. Include hourly costs and depreciation.

62. A veterinarian depreciates a \$10,000 X-ray machine. The estimated resale value $V(t)$ (in \$) after t years is 90% of its value from the previous year. Therefore, the resale value can be approximated by $V(t) = 10{,}000(0.9)^t$.

- **a.** Find the resale value after 4 yr.
- **b.** If the veterinarian wants to sell the practice 8 yr after the *X*-ray machine was purchased, how much is the machine worth? Round to the nearest \$100.

For Exercises 63–64, solve the equations in parts (a)–(c) by inspection. Then estimate the solutions to parts (d) and (e) between two consecutive integers.

63. a. $2^x = 4$ **b.** $2^x = 8$

c. $2^x = 16$ **d.** $2^x = 7$

e. $2^x = 10$

64. a. $3^x = 3$ **b.** $3^x = 9$

c. $3^x = 27$ **d.** $3^x = 7$

e. $3^x = 10$

65. a. Graph $f(x) = 2^x$. (**See Example 1, p. 347**)

b. Is *f* a one-to-one function?

c. Write the domain and range of *f* in interval notation.

d. Graph f^{-1} on the same coordinate system as *f*.

e. Write the domain and range of f^{-1} in interval notation.

f. From the graph evaluate $f^{-1}(1)$, $f^{-1}(2)$, and $f^{-1}(4)$.

66. a. Graph $g(x) = 3^x$ (see Exercise 15).

b. Is *g* a one-to-one function?

c. Write the domain and range of *g* in interval notation.

d. Graph g^{-1} on the same coordinate system as *g*.

e. Write the domain and range of g^{-1} in interval notation.

f. From the graph evaluate $g^{-1}(1)$, $g^{-1}(3)$, and $g^{-1}\left(\frac{1}{3}\right)$.

67. Refer to the graphs of $f(x) = 2^x$ and the inverse function, $y = f^{-1}(x)$ from Exercise 65. Fill in the blanks.

a. As $x \to \infty$, $f(x) \to$ ________________.

b. As $x \to -\infty$, $f(x) \to$ ________________.

c. As $x \to \infty$, $f^{-1}(x) \to$ ________________.

d. As $x \to 0+, f^{-1}(x) \to$ ________________.

68. Refer to the graphs of $g(x) = 3^x$ and the inverse function, $y = g^{-1}(x)$ from Exercise 66. Fill in the blanks.

a. As $x \to \infty$, $g(x) \to$ ________________.

b. As $x \to -\infty$, $g(x) \to$ ________________.

c. As $x \to \infty$, $g^{-1}(x) \to$ ________________.

d. As $x \to 0^+$, $g^{-1}(x) \to$ ________________.

Write About It

69. Explain why the equation $2^x = -2$ has no solution.

70. Explain why the $f(x) = x^2$ is not an exponential function.

Expanding Your Skills

For Exercises 71–72, find the real solutions to the equation.

71. $3x^2e^{-x} - 6xe^{-x} = 0$

72. $x^2e^x - e^x = 0$

73. Use the properties of exponents to simplify.

a. e^xe^h

b. $(e^x)^2$

c. $\frac{e^x}{e^h}$

d. $ex \cdot e^{-x}$

e. e^{-2x}

74. Factor.

a. $e^{x+h} - e^x$

b. $e^{4x} - e^{2x}$

75. Multiply. $(ex + e^{-x})^2$

76. Multiply. $(ex - e^{-x})^2$

77. Multiply and simplify. $(5^x - 5)^2 - (5^x + 5)^2$

78· Multiply and simplify. $(2^{-x} + 2)^2 + (2^{-x} - 2)^2$

The hyperbolic cosine function and hyperbolic sine function are defined as follows:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \text{ and } \sinh(x) = \frac{e^x - e^{-x}}{2}$$

These functions are used in Exercises 79–80.

79. Show that $\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1$.

80. Show that $2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} - e^{-2x}}{2}$.

For Exercises 81–82, find the difference quotient $\frac{f(x+h) - f(x)}{h}$. Write the answers in factored form.

81. $f(x) = e^x$

82. $f(x) = 2^x$

For Exercises 83–85, graph the function and write the domain and range in interval notation.

83. $f(x) = \begin{cases} 2^x & x > 2 \\ x & x \le 2 \end{cases}$

84. $g(x) = \begin{cases} 3^{-x} & x \le 0 \\ -\frac{1}{3}x - 1 & x > 0 \end{cases}$

85. $h(x) = \begin{cases} \frac{1}{x} & x > 0 \\ -e^{-x} & x < 0 \end{cases}$

Technology Connections

86. Graph the following functions on the window [–3, 3, 1] by [–1, 8, 1] and comment on the behavior of the graphs near $x = 0$.

$$Y_1 = e^x$$
$$Y_2 = 1 + x + \frac{x^2}{2}$$
$$Y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Point of Interest

For many medical researchers, the answer to the question, "Who is the most important individual in the history of modern medicine?" would be Henrietta Lacks.

Henrietta Lacks, born in 1920 to a poor family in Roanoke, Virginia, is in a large way responsible for advances in cancer research, gene mapping, in vitro fertilization, vaccine development, and the study of countless diseases. For more than 70 yr, she has helped scientists all over the world study the biology of human cells—study that has led to such monumental medical achievements as the vaccines for polio and COVID-19.

In 1951, at age 31, Henrietta went to Johns Hopkins Hospital complaining of abdominal pain. Although Johns Hopkins was one of the few leading hospitals that served African Americans, it was also a segregated hospital in which Black and white patients were housed in different wards. Ultimately, Lacks was diagnosed and treated for a very aggressive form of cervical cancer. As part of her care, a biopsy of cells from Henrietta's cervix was taken and then cultured (grown) in a petri dish for further study. Interestingly, earlier efforts to keep human cells alive outside the body had failed, but Henrietta's cells proved to be remarkably durable and prolific. Scientists found that her cells not only lived but thrived. Her cells grew and multiplied in a process called cell mitosis in approximately 24 hr.

Henrietta's condition deteriorated rapidly after her cancer diagnosis, and she passed away shortly after her visit to Johns Hopkins. But today, more than 70 yr later, trillions of HeLa cells (named for the first letters of her first and last names) are used in laboratories around the world for research. In a sense, Henrietta's cells have become "immortal."

Photo Researchers/Science History Images/Alamy Stock Photo

Thanks to her family, Henrietta's legacy and her profound contribution to modern medicine are memorialized through the Henrietta Lacks Foundation, symposia, and student scholarships and awards. Her story is one that bears reading for all. It is a story that has brought forth an open dialogue about the social issues prevalent in the United States at the time her cells were collected. It raises important questions about modern medicine, bioethics, race, gender, class, and intellectual property.

87. Suppose that a scientist orders a vial of HeLa cells in solution from a cell bank. It is further known that the cell density is 5.3×10^5 cells/mL (cells per milliliter), and that for the growing conditions the scientist will use to culture the cells, the doubling time is approximately 24 hr.

a. If the scientist uses 5 mL of solution from the vial to use in a culture, approximately how many cells will be present initially?

b. If the cells are allowed to grow continuously in culture and do not run out of space or nutrients, then $N(t) = N_0 \cdot 2^{t/d}$ represents the number of cells $N(t)$ at a given time t (in hours) after the culture begins. The value of d is the doubling time (in hours). Write a function that represents the number of HeLa cells the scientist can expect as a function of time.

c. How many cells can be expected after 24 hr? Round to the nearest hundred thousand.

d. How many cells can be expected after 36 hr? Round to the nearest hundred thousand.

e. Evaluate $N(168)$ and interpret the meaning in the context of this problem. Round to the nearest million.

Further Reading

Sklott, Rebecca. *The Immortal Life of Henrietta Lacks*. Crown Publishers, 2010.

GO ONLINE **Go online for more practice problems.**

Lesson 3-3

Logarithmic Functions

What Will You Learn?

After completing this lesson, you should be able to:

- Convert Between Logarithmic and Exponential Forms
- Evaluate Logarithmic Expressions
- Apply Basic Properties of Logarithms
- Graph Logarithmic Functions
- Use Logarithmic Functions in Applications

Learn: Convert Between Logarithmic and Exponential Forms

Consider the following equations in which the variable is located in the exponent of an expression. In some cases, the solution can be found by inspection.

Equation	**Solution**
$5^x = 5$	$x = 1$
$5^x = 20$	$x = ?$
$5^x = 25$	$x = 2$
$5^x = 60$	$x = ?$
$5^x = 125$	$x = 3$

The equation $5^x = 20$ cannot be solved by inspection. However, we suspect that x is between 1 and 2 because $5^1 = 5$ and $5^2 = 25$. To solve for x explicitly, we must isolate x by performing the inverse operation of 5^x. Fortunately, all exponential functions $y = b^x$ ($b > 0, b \neq 1$) are one-to-one and therefore have inverse functions. The inverse of an exponential function, base b, is the *logarithmic* function base b, which we define here.

Definition of a Logarithmic Function

If x and b are positive real numbers such that $b \neq 1$, then $y = \log_b x$ is called the **logarithmic function base *b***, where

$$y = \log_b x \text{ is equivalent to } b^y = x$$

Notes:

- Given $y = \log_b x$, the value y is the exponent to which b must be raised to obtain x.
- The value of y is called the **logarithm**, b is called the **base**, and x is called the **argument**.
- The equations $y = \log_b x$ and $b^y = x$ both define the same relationship between x and y. The expression $y = \log_b x$ is called the **logarithmic form**, and $b^y = x$ is called the **exponential form**.

The logarithmic function base *b* is defined as the inverse of the exponential function base *b*.

exponential function $f(x) = b^x$ First replace *f*(*x*) with *y*.

$y = b^x$ Next, interchange *x* and *y*.

inverse of exponential function $x = b^y$ This equation provides an implicit relationship between *x* and *y*. To solve for *y* explicitly (that is, to isolate *y*), we must use logarithmic notation.

logarithmic function $y = \log_b x$

To be able to solve equations involving logarithms, it is often advantageous to write a logarithmic expression in its exponential form.

Example 1: Writing Logarithmic Form and Exponential Form

Write each equation in exponential form.

a. $\log_2 16 = 4$ **b.** $\log_{10}\left(\frac{1}{100}\right) = -2$ **c.** $\log_7 1 = 0$

Solution:

Logarithmic form $y = \log_b x$ **Exponential form $b^y = x$**

a. $\log_2 16 = 4 \Leftrightarrow 2^4 = 16$

The logarithm is the exponent to which the base is raised to obtain *x*.

b. $\log_{10}\left(\frac{1}{100}\right) = -2 \Leftrightarrow 10^{-2} = \frac{1}{100}$

c. $\log_7 1 = 0 \Leftrightarrow 7^0 = 1$

Apply the Skills

1. Write each equation in exponential form.

a. $\log_3 9 = 2$ **b.** $\log_{10}\left(\frac{1}{1000}\right) = -3$ **c.** $\log_6 1 = 0$

In Example 2 we reverse this process and write an exponential equation in its logarithmic form.

Example 2: Writing Exponential Form and Logarithmic Form

Write each equation in logarithmic form.

a. $3^4 = 81$ **b.** $10^6 = 1{,}000{,}000$ **c.** $\left(\frac{1}{5}\right)^{-1} = 5$

Solution:

Exponential form $b^y = x$ **Logarithmic form $\log_b x = y$**

a. $\underbrace{3}_{\text{base}}^{\overset{\text{logarithm}}{4}} = \underbrace{81}_{\text{argument}} \Leftrightarrow \log_{\underbrace{3}_{\text{base}}} \underbrace{81}_{\text{argument}} = \overset{\text{logarithm (power)}}{4}$

b. $10^6 = 1{,}000{,}000 \quad \Leftrightarrow \quad \log_{10} 1{,}000{,}000 = 6$

c. $\left(\frac{1}{5}\right)^{-1} = 5 \quad \Leftrightarrow \quad \log_{1/5} 5 = -1$

Apply the Skills

1. Write each equation in logarithmic form.

a. $2^5 = 32$ **b.** $10^4 = 10{,}000$ **c.** $\left(\frac{1}{8}\right)^{-2} = 64$

Learn: Evaluate Logarithmic Expressions

To evaluate a logarithmic expression, we can write the expression in exponential form. Then we make use of the equivalence property of exponential expressions. This states that if two exponential expressions of the same base are equal, then their exponents must be equal.

Equivalence Property of Exponential Expressions

If b, x, and y are real numbers, with $b > 0$ and $b \neq 1$, then

$b^x = b^y$ implies that $x = y$.

In Example 3, we evaluate several logarithmic expressions.

Example 3: Evaluating Logarithmic Expressions

Evaluate each expression.

a. $\log_4 16$ **b.** $\log_2 8$ **c.** $\log_{1/2} 8$

Solution:

Let y represent the value of the logarithm.

a. $\log_4 16$ is the exponent to which 4 must be raised to equal 16. That is, $4^{\square} = 16$.

$\log_4 16 = y$

$4^y = 16$ or $4^y = 4^2$ Write the equivalent exponential form.

$y = 2$

Therefore, $\log_4 16 = 2$.

Check Your Work $4^2 = 16$ ✓

b. $\log_2 8$ is the exponent to which 2 must be raised to obtain 8. That is, $2^{\square} = 8$.

$\log_2 8 = y$

$2^y = 8$ or $2^y = 2^3$ Write the equivalent exponential form.

$y = 3$

Therefore, $\log_2 8 = 3$.

Check Your Work $2^3 = 8$ ✓

c. $\log_{1/2} 8 = y$

$\left(\frac{1}{2}\right)^y = 8$ or $\left(\frac{1}{2}\right)^y = \left(\frac{1}{2}\right)^{-3}$ Write the equivalent exponential form.

$y = -3$

Therefore, $\log_{1/2} 8 = -3$.

Check Your Work $\left(\frac{1}{2}\right)^{-3} = 8$ ✓

Insights

Once you become comfortable with the concept of a logarithm, you can take fewer steps to evaluate a logarithm.

To evaluate the expression $\log_4 16$ we ask $4^{\square} = 16$. The exponent is 2, so $\log_4 16 = 2$.

Likewise, to evaluate $\log_2 8$ we ask $2^{\square} = 8$. So $\log_2 8 = 3$.

Apply the Skills

3. Evaluate each expression.

a. $\log_5 125$

b. $\log_3 81$

c. $\log_4\left(\frac{1}{64}\right)$

The statement $y = \log_b x$ represents a family of logarithmic functions where the base is any positive real number except 1. Two specific logarithmic functions that come up often in applications are the logarithmic functions base 10 and base e.

Definition of Common and Natural Logarithmic Functions

- The logarithmic function base 10 is called the **common logarithmic function**. The common logarithmic function is denoted by $y = \log x$. Notice that the base 10 is not explicitly written; that is, $y = \log_{10} x$ is written simply as $y = \log x$.
- The logarithmic function base e is called the **natural logarithmic function**. The natural logarithmic function is denoted by $y = \ln x$; that is, $y = \log_e x$ is written as $y = \ln x$.

Insights To help you remember the notation $y = \ln x$, think of "ln" as "log natural."

Example 4: Evaluating Common and Natural Logarithms

Evaluate.

a. $\log 100{,}000$ **b.** $\log 0.001$ **c.** $\ln e^4$ **d.** $\ln\left(\frac{1}{e}\right)$

Solution:

Let y represent the value of the logarithm.

a. $\log 100{,}000 = y$

$10^y = 100{,}000$ or $10^y = 10^5$ Write the equivalent exponential form.

$y = 5$

So, $\log 100{,}000 = 5$ because $10^5 = 100{,}000$.

b. $\log 0.001 = y$

$10^y = 0.001$ or $10^y = 10^{-3}$ Write the equivalent exponential form.

$y = -3$

So, $\log 0.001 = -3$ because $10^{-3} = 0.001$.

c. $\ln e^4 = y$

$e^y = e^4$ Write the equivalent exponential form.

$y = 4$

Therefore, $\ln e^4 = 4$.

d. $\ln\left(\frac{1}{e}\right) = y$

$e^y = \left(\frac{1}{e}\right)$ or $e^y = e^{-1}$ Write the equivalent exponential form.

$y = -1$

Therefore, $\ln\left(\frac{1}{e}\right) = -1$.

Insights

In Example 4, to evaluate:

a. $\log 100{,}000$ we ask $10^{\square} = 100{,}000$.
b. $\log 0.001$ we ask $10^{\square} = 0.001$.
c. $\ln e^4$ we ask $e^{\square} = e^4$.
d. $\ln\left(\frac{1}{e}\right)$ we ask $e^{\square} = e^{-1}$.

Apply the Skills

4. Evaluate.

a. $\log 10{,}000{,}000$ **b.** $\log 0.1$ **c.** $\ln e^5$ **d.** $\ln e$

Most scientific calculators have a key for the common logarithmic function LOG and a key for the natural logarithmic function LN. We demonstrate their use in Example 5.

Example 5: Approximating Common and Natural Logarithms

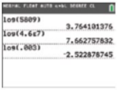
Approximate the logarithms.

a. $\log 5809$ **b.** $\log(4.6 \times 10^7)$ **c.** $\log 0.003$

d. $\ln 472$ **e.** $\ln 0.05$ **f.** $\ln\sqrt{87}$

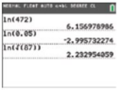
Solution:

For parts (a)–(c), use the LOG key.



```
log(5809)
                3.764101376
log(4.6ᴇ7)
                7.662757832
log(.003)
               ⁻2.522878745
```

For parts (d)–(f), use the LN key.



```
ln(472)
                6.156978986
ln(0.05)
               ⁻2.995732274
ln(√(87))
                2.232954059
```

When using a calculator, there is always potential for user-input error. Therefore, it is good practice to estimate values when possible to confirm the reasonableness of an answer from a calculator. For example,

For part (a), $10^3 < 5809 < 10^4$. Therefore, $3 < \log 5809 < 4$.

For part (b), $10^7 < 4.6 \times 10^7 < 10^8$. Therefore, $7 < \log(4.6 \times 10^7) < 8$.

For part (c), $10^{-3} < 0.003 < 10^{-2}$. Therefore, $-3 < \log 0.003 < -2$.

Apply the Skills

5. Approximate the logarithms. Round to 4 decimal places.

a. $\log 229$ **b.** $\log(3.76 \times 10^{12})$ **c.** $\log 0.0216$

d. $\ln 87$ **e.** $\ln 0.0032$ **f.** $\ln \pi$

Insights

There are an infinite number of logarithmic functions. The calculator has keys for the base 10 and base *e* logarithmic functions only.

To approximate a logarithm of different base, we need to use the change-of-base formula presented in Lesson 3.4.

Learn: Apply Basic Properties of Logarithms

From the definition of a logarithmic function, we have the following basic properties.

Basic Properties of Logarithms

Property	Example
1. $\log_b 1 = 0$ because $b^0 = 1$	$\log_5 1 = 0$ because $5^0 = 1$
2. $\log_b b = 1$ because $b^1 = b$	$\log_3 3 = 1$ because $3^1 = 3$
3. $\log_b b^x = x$ because $b^x = b^x$	$\log_2 2^x = x$ because $2^x = 2^x$
4. $b^{\log_b x} = x$ because $\log_b x = \log_b x$	$7^{\log_7 x} = x$ because $\log_7 x = \log_7 x$

Properties 3 and 4 follow from the fact that a logarithmic function is the inverse of an exponential function of the same base. Given $f(x) = b^x$ and $f^{-1}(x) = \log_b x$,

$$(f \circ f^{-1})(x) = b^{(\log_b x)} = x \qquad \text{(Property 4)}$$

$$(f^{-1} \circ f)(x) = \log_b(b^x) = x \qquad \text{(Property 3)}$$

Example 6: Applying the Properties of Logarithms

Simplify.

a. $\log_3 3^{10}$ **b.** $\ln e^2$ **c.** $\log_{11} 11$ **d.** $\log 10$

e. $\log_{\sqrt{7}} 1$ **f.** $\ln 1$ **g.** $5^{\log_5(c^2+4)}$ **h.** $10^{\log(a^2+b^2)}$

Solution:

a. $\log_3 3^{10} = 10$ Property 3 **b.** $\ln e^2 = \log_e e^2 = 2$ Property 3

c. $\log_{11} 11 = 1$ Property 2 **d.** $\log 10 = \log_{10} 10 = 1$ Property 2

e. $\log_{\sqrt{7}} 1 = 0$ Property 1 **f.** $\ln 1 = 0$ Property 1

g. $5^{\log_5(c^2+4)} = c^2 + 4$ Property 4 **h.** $10^{\log(a^2+b^2)} = a^2 + b^2$ Property 4

Apply the Skills

6. Simplify.

a. $\log_{13} 13$ **b.** $\ln e$ **c.** $a^{\log_a 3}$ **d.** $e^{\ln 6}$

e. $\log_\pi 1$ **f.** $\log 1$ **g.** $\log_9 9^{\sqrt{2}}$ **h.** $\log 10^e$

Learn: Graph Logarithmic Functions

Since a logarithmic function $y = \log_b x$ is the inverse of the corresponding exponential function $y = b^x$, their graphs must be symmetric with respect to the line $y = x$. See Figures 3-13 and 3-14.

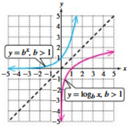


Figure 3-13

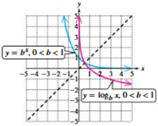


Figure 3-14

From Figures 3-13 and 3-14, the range of $y = b^x$ is the set of positive real numbers. As expected, the domain of its inverse function $y = \log_b x$ is the set of positive real numbers.

Example 7: Graphing Logarithmic Functions

Graph the functions.

a. $y = \log_2 x$ **b.** $y = \log_{1/4} x$

Solution:

To find points on a logarithmic function, we can interchange the *x*- and *y*-coordinates of the ordered pairs on the corresponding exponential function.

a. To graph $y = \log_2 x$, interchange the *x*- and *y*-coordinates of the ordered pairs from its inverse function $y = 2^x$. The graph of $y = \log_2 x$ is shown in Figure 3-15.

Exponential Function

x	$y = 2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

Logarithmic Function

x	$y = \log_2 x$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Switch *x* and *y*.

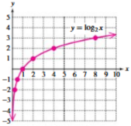


Figure 3-15

b. To graph $y = \log_{1/4} x$, interchange the x- and y-coordinates of the ordered pairs from its inverse function $y = \left(\frac{1}{4}\right)^x$. See Figure 3-16.

Exponential Function

x	$y = \left(\frac{1}{4}\right)^x$
−3	64
−2	16
−1	4
0	1
1	$\frac{1}{4}$
2	$\frac{1}{16}$
3	$\frac{1}{64}$

Logarithmic Function

x	$y = \log_{1/4} x$
64	−3
16	−2
4	−1
1	0
$\frac{1}{4}$	1
$\frac{1}{16}$	2
$\frac{1}{64}$	3

Switch x and y.

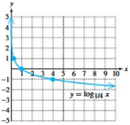


Figure 3-16

Apply the Skills

7. Graph the functions.

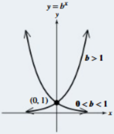
a. $y = \log_4 x$

b. $y = \log_{1/2} x$

Based on the graphs in Example 7 and our knowledge of exponential functions, we offer the following summary of the characteristics of logarithmic and exponential functions.

Graphs of Exponential and Logarithmic Functions

Exponential Functions



Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

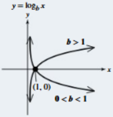
Horizontal asymptote: $y = 0$

Passes through (0, 1)

If $b > 1$, the function is increasing.

If $0 < b < 1$, the function is decreasing.

Logarithmic Functions



Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

Vertical asymptote: $x = 0$

Passes through (1, 0)

If $b > 1$, the function is increasing.

If $0 < b < 1$, the function is decreasing.

The roles of x and y are reversed between a function and its inverse. Therefore, it is not surprising that the domain and range are reversed between exponential and logarithmic functions. Furthermore, an exponential function passes through (0, 1), whereas a logarithmic function passes through (1, 0). An exponential function has a horizontal asymptote of $y = 0$, whereas a logarithmic function has a vertical asymptote of $x = 0$.

In Example 8 we use the transformations of functions learned in Lesson 1.6 to graph a logarithmic function.

If $h > 0$, shift to the right.
If $h < 0$, shift to the left.

If $k > 0$, shift upward.
If $k < 0$, shift downward.

$$f(x) = a\log_b(x - h) + k$$

If $a < 0$, reflect across the x-axis.
Shrink vertically if $0 < |a| < 1$.
Stretch vertically if $|a| > 1$.

Example 8: Using Transformations to Graph a Logarithmic Function

Graph the function. Identify the vertical asymptote and write the domain in interval notation.

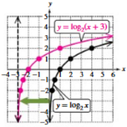
$$f(x) = \log_2(x + 3) - 2$$

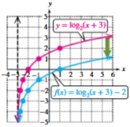
Solution:

The graph of the "parent" function $y = \log_2 x$ was presented in Example 7. The graph of $f(x) = \log_2(x + 3) - 2$ is the graph of $y = \log_2 x$ shifted to the left 3 units and down 2 units.

We can plot a few points on the graph of $y = \log_2 x$ and use these points and the vertical asymptote to form an outline of the transformed graph.

x	$y = \log_2 x$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2





The graph of $f(x) = \log_2(x + 3) - 2$ is shown in blue.

The vertical asymptote is $x = -3$. The domain is $(-3, \infty)$.

Apply the Skills

8. Graph the function. Identify the vertical asymptote and write the domain in interval notation. $g(x) = \log_3(x - 4) + 1$

The domain of $f(x) = \log_b x$ is restricted to $x > 0$. In Example 8, this graph was shifted to the left 3 units, restricting the domain of $f(x) = \log_2(x + 3) - 2$ to $x > -3$. The domain of a logarithmic function is the set of real numbers that make the argument positive.

Example 9: Identifying the Domain of a Logarithmic Function

Write the domain in interval notation.

a. $f(x) = \log_2(2x + 4)$ **b.** $g(x) = \ln(5 - x)$ **c.** $h(x) = \log(x^2 - 9)$

Solution:

a. $f(x) = \log_2(2x + 4)$

$2x + 4 > 0$ Set the argument greater than zero.

$2x > -4$ Solve for x.

$x > -2$

The domain is $(-2, \infty)$.
The graph of f is shown in Figure 3-17.
The vertical asymptote is $x = -2$.

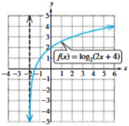


Figure 3-17

b. $g(x) = \ln(5 - x)$

$5 - x > 0$ Set the argument greater than zero.

$-x > -5$ Subtract 5 and divide by -1

$x < 5$ (reverse the inequality sign).

The domain is $(-\infty, 5)$.
The graph of g is shown in Figure 3-18.
The vertical asymptote is $x = 5$.

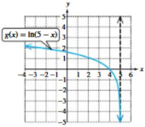


Figure 3-18

c. $h(x) = \log(x^2 - 9)$

$x^2 - 9 > 0$

Set the argument greater than zero. The result is a polynomial inequality (Lesson 2.7).

$(x - 3)(x + 3) = 0$

Solve the related equation by setting one side equal to zero and factoring the other side.

Sign of $(x - 3)$:	−	−	+
Sign of $(x + 3)$:	−	+	+
Sign of $(x - 3)(x + 3)$:	+	−	+

-3 3

The boundary points for the solution set are the solutions to the equation: $x = 3$ and $x = -3$.

The domain is $(-\infty, -3) \cup (3, \infty)$.

The graph of h is shown in Figure 3-19.

The vertical asymptotes are $x = -3$ and $x = 3$.

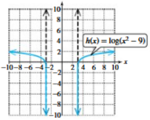


Figure 3-19

Apply the Skills

9. Write the domain in interval notation.

a. $f(x) = \log_4(1 - 3x)$

b. $g(x) = \log(2 + x)$

c. $m(x) = \ln(64 - x^2)$

Learn: Use Logarithmic Functions in Applications

When physical quantities vary over a large range, it is often convenient to take a logarithm of the quantity to have a more manageable set of numbers. For example, suppose a set of data values consists of 10, 100, 1000, and 10,000. The corresponding common logarithms are 1, 2, 3, and 4. The latter list of numbers is easier to manipulate and to visualize on a graph. For this reason, logarithmic scales are used in applications such as

- measuring pH (representing hydrogen ion concentration from 10^{-14} to 1 moles per liter).
- measuring wave energy from an earthquake (often ranging from 10^6 J to 10^{17} J).
- measuring loudness of sound on the decibel scale (representing sound intensity from 10^{-12} to 10^2 watts per square meter).

Example 10: Using a Logarithmic Function in an Application

The intensity I of an earthquake is measured by a seismograph—a device that measures amplitudes of shock waves. I_0 is a minimum reference intensity of a "zero-level" earthquake against which the intensities of other earthquakes may be compared. The magnitude M of an earthquake of intensity I is given by $M = \log\left(\frac{I}{I_0}\right)$.

a. Determine the magnitude of the earthquake that devastated Haiti on January 12, 2010, if the intensity was approximately $10^{7.0}$ times I_0.

b. Determine the magnitude of the earthquake that occurred near Washington, D.C., on August 23, 2011, if the intensity was approximately $10^{5.8}$ times I_0.

c. How many times more intense was the earthquake that hit Haiti than the earthquake that hit Washington, D.C.? Round to the nearest whole unit.

Solution:

a. $M = \log\left(\frac{I}{I_0}\right)$

$$M = \left(\frac{10^{7.0}\cdot I_0}{I_0}\right)$$
$$= \log 10^{7.0}$$
$$= 7.0$$

b. $M = \log\left(\frac{I}{I_0}\right)$

$$M = \left(\frac{10^{5.8}\cdot I_0}{I_0}\right)$$
$$= \log 10^{5.8}$$
$$= 5.8$$

c. Using the intensities given in parts (a) and (b), we have

$$\frac{10^{7.0} I_0}{10^{5.8} I_0} = 10^{7.0-5.8} = 10^{1.2} \approx 16$$

The earthquake in Haiti was approximately 16 times more intense.

Apply the Skills

10. a. Determine the magnitude of an earthquake that is $10^{5.2}$ times I_0.

b. Determine the magnitude of an earthquake that is $10^{4.2}$ times I_0.

c. How many times more intense is a 5.2-magnitude earthquake than a 4.2-magnitude earthquake?

Practice 3-3

Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, simplify each expression.

R.1. $9^{-3/2}$

R.2. $64^{2/3}$

R.3. $(-8)^{2/3}$

R.4. $\frac{1}{25^{-1/2}}$

For Exercises R.5–R.6, solve the polynomial inequality. Write the answer in interval notation.

R.5. $y^2 + 20y \geq -75$

R.6. $x^2 - 8x + 16 > 0$

Concept Connections

1. Given positive real numbers x and b such that $b \neq 1$, $y = \log_b x$ is the __________ function base b and is equivalent to $b^y = x$.

2. Given $y = \log_b x$, the value y is called the __________, b is called the __________, and x is called the __________.

3. The logarithmic function base 10 is called the __________ logarithmic function, and the logarithmic function base e is called the __________ logarithmic function.

4. Given $y = \log x$, the base is understood to be __________. Given $y = \ln x$, the base is understood to be __________.

5. $\log_b 1 =$ __________ because $b^{\square} = 1$.

6. $\log_b b =$ __________ because $b^{\square} = b$.

7. $f(x) = \log_b x$ and $g(x) = b^x$ are inverse functions. Therefore, $\log_b b^x =$ __________ and $b^{\log_b x} =$ __________.

8. The graph of $y = \log_b x$ passes through the point (1, 0) and the line __________ is a (horizontal/vertical) asymptote.

For the exercises in this set, assume that all variable expressions represent positive real numbers.

Learn: Convert Between Logarithmic and Exponential Forms

For Exercises 9–16, write the equation in exponential form. (See Example 1, p. 365)

9. $\log_8 64 = 2$

10. $\log_9 81 = 2$

11. $\log\left(\frac{1}{10{,}000}\right) = -4$

12. $\log\left(\frac{1}{1{,}000{,}000}\right) = -6$

13. $\ln 1 = 0$

14. $\log_8 1 = 0$

15. $\log_a b = c$

16. $\log_x M = N$

For Exercises 17–24, write the equation in logarithmic form. (See Example 2, p. 365)

17. $5^3 = 125$

18. $2^5 = 32$

19. $\left(\frac{1}{5}\right)^{-3} = 125$

20. $\left(\frac{1}{2}\right)^{-5} = 32$

21. $10^9 = 1{,}000{,}000{,}000$

22. $e^1 = e$

23. $a^7 = b$

24. $M^3 = N$

Learn: Evaluate Logarithmic Expressions

For Exercises 25–50, simplify the expression without using a calculator. (See Examples 3–4, pp. 366–368)

25. $\log_3 9$

26. $\log_2 16$

27. $\log_5 5$

28. $\log_6 6$

29. $\log 100{,}000{,}000$

30. $\log 10{,}000{,}000$

31. $\log_2\left(\frac{1}{16}\right)$

32. $\log_3\left(\frac{1}{9}\right)$

33. $\log\left(\frac{1}{10}\right)$

34. $\log\left(\frac{1}{10{,}000}\right)$

35. $\ln e^6$

36. $\ln e^{10}$

37. $\ln\left(\frac{1}{e^3}\right)$

38. $\ln\left(\frac{1}{e^8}\right)$

39. $\log_{1/7} 49$

40. $\log_{1/4} 16$

41. $\log_{1/2}\left(\frac{1}{32}\right)$

42. $\log_{1/6}\left(\frac{1}{36}\right)$

43. $\log 0.00001$

44. $\log 0.0001$

45. $\log_{3/2} \frac{4}{9}$

46. $\log_{3/2} \frac{9}{4}$

47. $\log_3 \sqrt[5]{3}$

48. $\log_2 \sqrt[3]{2}$

49. $\log_5 \sqrt{\frac{1}{5}}$

50. $\log \sqrt{\frac{1}{1000}}$

For Exercises 51–52, estimate the value of each logarithm between two consecutive integers. Then use a calculator to approximate the value to 4 decimal places. For example, log 8970 is between 3 and 4 because $10^3 < 8970 < 10^4$. (See Example 5, p. 369)

51. a. $\log 46{,}832$

b. $\log 1{,}247{,}310$

c. $\log 0.24$

d. $\log 0.0000032$

e. $\log(5.6 \times 10^5)$

f. $\log(5.1 \times 10^{-3})$

52. a. $\log 293{,}416$

b. $\log 897$

c. $\log 0.038$

d. $\log 0.00061$

e. $\log(9.1 \times 10^8)$

f. $\log(8.2 \times 10^{-2})$

For Exercises 53–54, approximate $f(x) = \ln x$ for the given values of x. Round to 4 decimal places. (See Example 5, p. 369)

53. a. $f(94)$

b. $f(0.182)$

c. $f(\sqrt{155})$

d. $f(4\pi)$

e. $f(3.9 \times 10^9)$

f. $f(7.1 \times 10^{-4})$

54. a. $f(1860)$

b. $f(0.0694)$

c. $f(\sqrt{87})$

d. $f(2\pi)$

e. $f(1.3 \times 10^{12})$

f. $f(8.5 \times 10^{-17})$

Learn: Apply Basic Properties of Logarithms

For Exercises 55–64, simplify the expression without using a calculator. (See Example 6, p. 370)

55. $\log_4 4^{11}$

56. $\log_6 6^7$

57. $\log_c c$

58. $\log_d d$

59. $5^{\log_5(x+y)}$

60. $4^{\log_4(a-c)}$

61. $\ln e^{a+b}$

62. $\ln e^{x^2+1}$

63. $\log_{\sqrt{5}} 1$

64. $\log_\pi 1$

Learn: Graph Logarithmic Functions

For Exercises 65–70, graph the function. (See Example 7, p. 371)

65. $y = \log_3 x$

66. $y = \log_5 x$

67. $y = \log_{1/3} x$

68. $y = \log_{1/5} x$

69. $y = \ln x$

70. $y = \log x$

For Exercises 71–78, (See Example 8, p. 373)

a. Use transformations of the graphs of $y = \log_2 x$ (see Example 7, p. 371) and $y = \log_3 x$ (see Exercise 65) to graph the given functions.

b. Write the domain and range in interval notation.

c. Write an equation of the asymptote.

71. $y = \log_3(x + 2)$

72. $y = \log_2(x + 3)$

73. $y = 2 + \log_3 x$

74. $y = 3 + \log_2 x$

75. $y = \log_3(x - 1) - 3$

76. $y = \log_2(x - 2) - 1$

77. $y = -\log_3 x$

78. $y = -\log_2 x$

For Exercises 79–92, write the domain in interval notation. (See Example 9, p. 374)

79. $f(x) = \log(8 - x)$

80. $g(x) = \log(3 - x)$

81. $h(x) = \log_2(6x + 7)$

82. $k(x) = \log_3(5x + 6)$

83. $m(x) = \ln(x^2 + 14)$

84. $n(x) = \ln(x^2 + 11)$

85. $f(x) = \log_4(x^2 - 16)$

86. $g(x) = \log_7(x^2 - 49)$

87. $m(x) = 3 + \ln\dfrac{1}{\sqrt{11 - x}}$

88. $n(x) = 4 - \log\dfrac{1}{\sqrt{x + 5}}$

89. $p(x) = \log(x^2 - x - 12)$

90. $q(x) = \log(x^2 + 10x + 9)$

91. $r(x) = \log_3(4 - x)^2$

92. $s(x) = \log_5(3 - x)^2$

Learn: Use Logarithmic Functions in Applications

93. In 1989, the Loma Prieta earthquake damaged the city of San Francisco with an intensity of approximately $10^{6.9} I_0$. Film footage of the 1989 earthquake was captured on a number of video cameras, including a broadcast of Game 3 of the World Series played at Candlestick Park. (**See Example 10, p. 375**)

a. Determine the magnitude of the Loma Prieta earthquake.

b. Smaller earthquakes occur daily in the San Francisco area, and most are not detectable without a seismograph. Determine the magnitude of an earthquake with an intensity of $10^{3.2} I_0$.

c. How many times more intense was the Loma Prieta earthquake than an earthquake with a magnitude of 3.2? Round to the nearest whole unit.

94. The intensities of earthquakes are measured with seismographs all over the world at different distances from the epicenter. Suppose that the intensity of a medium earthquake is originally reported as $10^{5.4}$ times I_0. Later this value is revised as $10^{5.8}$ times I_0.

a. Determine the magnitude of the earthquake using the original estimate for intensity.

b. Determine the magnitude using the revised estimate for intensity.

c. How many times more intense was the earthquake than originally thought? Round to 1 decimal place.

Sounds are produced when vibrating objects create pressure waves in some medium such as air. When these variations in pressure reach the human eardrum, it causes the eardrum to vibrate in a similar manner, and the ear detects sound. The intensity of sound is measured as power per unit area. The threshold for hearing (minimum sound detectable by a young, healthy ear) is defined to be $I_0 = 10^{-12}$ W/m² (watts per square meter). The sound level L, or "loudness" of sound, is measured in decibels (dB) as $L = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the given sound. Use this formula for Exercises 95–96.

95. a. Find the sound level of a jet plane taking off if its intensity is 10^{15} times the intensity of I_0.

b. Find the sound level of the noise from city traffic if its intensity is 10^9 times I_0.

c. How many times more intense is the sound of a jet plane taking off than noise from city traffic?

96. a. Find the sound level of a motorcycle if its intensity is 10^{10} times I_0.

b. Find the sound level of a vacuum cleaner if its intensity is 10^7 times I_0.

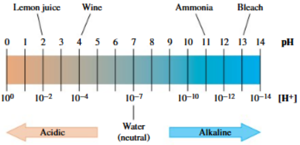
c. How many times more intense is the sound of a motorcycle than a vacuum cleaner?

Scientists use the pH scale to represent the level of acidity or alkalinity of a liquid. This is based on the molar concentration of hydrogen ions, [H⁺]. Since the values of [H⁺] vary over a large range, 1×10^0 mole per liter to 1×10^{-14} mole per liter (mol/L), a logarithmic scale is used to compute pH. The formula

$$\mathrm{pH} = -\log[\mathrm{H}^+]$$

represents the pH of a liquid as a function of its concentration of hydrogen ions, [H⁺].

The pH scale ranges from 0 to 14. Pure water is taken as neutral having a pH of 7. A pH less than 7 is acidic. A pH greater than 7 is alkaline (or basic). For Exercises 97–98, use the formula for pH. Round pH values to 1 decimal place.



97. Vinegar and lemon juice are both acids. Their [H⁺] values are 5.0×10^{-3} mol/L and 1×10^{-2} mol/L, respectively.

a. Find the pH for vinegar.

b. Find the pH for lemon juice.

c. Which substance is more acidic?

98. Bleach and milk of magnesia are both bases. Their [H⁺] values are 2.0×10^{-13} mol/L and 4.1×10^{-10} mol/L, respectively.

a. Find the pH for bleach.

b. Find the pH for milk of magnesia.

c. Which substance is more basic?

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Mixed Exercises

For Exercises 99–102,

a. Write the equation in exponential form.

b. Solve the equation from part (a).

c. Verify that the solution checks in the original equation.

99. $\log_3(x + 1) = 4$ **100.** $\log_2(x - 5) = 4$

101. $\log_4(7x - 6) = 3$ **102.** $\log_5(9x - 11) = 2$

For Exercises 103–106, evaluate the expressions.

103. $\log_3(\log_4 64)$

104. $\log_2\left[\log_{1/2}\left(\frac{1}{4}\right)\right]$

105. $\log_{16}(\log_{81} 3)$

106. $\log_4(\log_{16} 4)$

107. **a.** Evaluate $\log_2 2 + \log_2 4$

b. Evaluate $\log_2(2 \cdot 4)$

c. How do the values of the expressions in parts (a) and (b) compare?

108. **a.** Evaluate $\log_3 3 + \log_3 27$

b. Evaluate $\log_3(3 \cdot 27)$

c. How do the values of the expressions in parts (a) and (b) compare?

109. **a.** Evaluate $\log_4 64 - \log_4 4$

b. Evaluate $\log_4\left(\frac{64}{4}\right)$

c. How do the values of the expressions in parts (a) and (b) compare?

110. **a.** Evaluate $\log 100{,}000 - \log 100$

b. Evaluate $\log\left(\frac{100{,}000}{100}\right)$

c. How do the values of the expressions in parts (a) and (b) compare?

111. **a.** Evaluate $\log_2 2^5$

b. Evaluate $5 \cdot \log_2 2$

c. How do the values of the expressions in parts (a) and (b) compare?

112. **a.** Evaluate $\log_7 7^6$

b. Evaluate $6 \cdot \log_7 7$

c. How do the values of the expressions in parts (a) and (b) compare?

113. The time t (in years) required for an investment to double with interest compounded continuously depends on the interest rate r according to the function $t(r) = \frac{\ln 2}{r}$.

a. If an interest rate of 3.5% is secured, determine the length of time needed for an initial investment to double. Round to 1 decimal place.

b. Evaluate $t(0.04)$, $t(0.06)$, and $t(0.08)$.

114. The number n of monthly payments of P dollars each required to pay off a loan of A dollars in its entirety at interest rate r is given by

$$n = -\frac{\log\left(1 - \frac{Ar}{12P}\right)}{\log\left(1 + \frac{r}{12}\right)}$$

a. A college student wants to buy a car and can only afford payments of \$200 per month. If the student borrows \$3000 and pays it off at 6% interest, how many months will it take to retire the loan? Round to the nearest month.

b. Determine the number of monthly payments of \$611.09 that would be required to pay off a home loan of \$128,000 at 4% interest.

For Exercises 115–116, use a calculator to approximate the given logarithms to 4 decimal places.

115. **a.** Avogadro's number is 6.022×10^{23}. Approximate $\log(6.022 \times 10^{23})$.

b. Planck's constant is 6.626×10^{-34} J · sec. Approximate $\log(6.626 \times 10^{-34})$.

c. Compare the value of the common logarithm to the power of 10 used in scientific notation.

116. **a.** The speed of light is 2.9979×10^8 m/sec. Approximate $\log(2.9979 \times 10^8)$.

b. An elementary charge is 1.602×10^{-19} C. Approximate $\log(1.602 \times 10^{-19})$.

c. Compare the value of the common logarithm to the power of 10 used in scientific notation.

Expanding Your Skills

For Exercises 117–122, write the domain in interval notation.

117. $t(x) = \log_4\left(\frac{x-1}{x-3}\right)$

118. $r(x) = \log_5\left(\frac{x+2}{x-4}\right)$

119. $s(x) = \ln(\sqrt{x+5} - 1)$

120. $v(x) = \ln(\sqrt{x-8} - 1)$

121. $c(x) = \log\left(\frac{1}{\sqrt{x-6}}\right)$

122. $d(x) = \log\left(\frac{1}{\sqrt{x+8}}\right)$

Technology Connections

123. **a.** Graph $f(x) = \ln x$ and $g(x) = (x-1) - (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$ on the viewing window [–2, 4, 1] by [–5, 2, 1]. How do the graphs compare on the interval (0, 2)?

b. Use function g to approximate ln 1.5. Round to 4 decimal places.

124. Compare the graphs of $Y_1 = \frac{e^x - e^{-x}}{2}$, $Y_2 = \ln\left(x + \sqrt{x^2+1}\right)$ and $Y_3 = x$ on the viewing window [–16.1, 16.1, 1] by [–10, 10, 1]. Based on the graphs, how do you suspect that the functions are related?

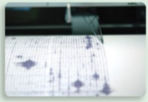
125. Compare the graphs of the functions.

$Y_1 = \ln(2x)$ and $Y_2 = \ln 2 + \ln x$

126. Compare the graphs of the functions.

$Y_1 = \ln\left(\frac{x}{2}\right)$ and $Y_2 = \ln x - \ln 2$

Point of Interest



In 1935, American geologist Charles Richter developed the local magnitude scale (M_l), or Richter scale, for measuring the intensity of moderate-sized earthquakes ($3 < M_l < 7$). Today, scientists use a different scale, called the moment magnitude scale, that is calibrated to the Richter scale for moderate-sized earthquakes but is more reliable for events of higher magnitude. However, the Richter scale and other magnitude scales use a base 10 logarithmic relationship to compare the amplitudes of waves on a seismogram.

The severity of an earthquake is measured by ground motion amplitude. One way to measure amplitude is with a seismograph—a device that has a mass attached to a fixed base. During an earthquake, the base moves, but due to inertia, the mass does not. The motion of the base relative to the mass can be transformed into electrical voltage that is recorded on paper or in digital form. The resulting data are shown visually as a zig-zag pattern, and the distance from the tip of a "spike" to the center is the amplitude of movement. The resulting data are then scaled proportionally to match the absolute motion of the ground.

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Many earthquakes are so small that they are not detectable by a seismograph. Large earthquakes can have amplitudes that are thousands, millions, or even billions times greater. For this reason, a linear scale is inconvenient for recording earthquake magnitude because it does not provide a suitable display for a wide range of values. Exercise 127 takes you through an activity to investigate the benefit of a logarithmic scale for studying and displaying data involving several orders of magnitude.

127. Consider a company in which the yearly salaries of its employees vary across several orders of magnitude. For example, the company may have a part-time employee who makes \$20,000 per year, a manager who makes \$80,000 per year, an executive who makes \$2.4 million per year, and the CEO who makes \$1 billion per year. Graphing the salaries on the vertical axis would require a scale that reaches up to at least \$1 billion (Figure 3-20). As a result, the lesser salaries appear as small blips on the graph, and the differences between the salaries cannot be seen.

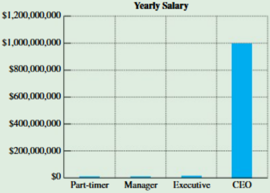
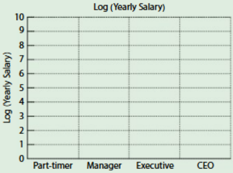


Figure 3-20

Employee	Salary	Log (salary)
Part time	\$20,000	
Manager	\$80,000	
Executive	\$2,400,000	
CEO	\$1,000,000,000	

a. Rather than graphing salary on the vertical scale, consider graphing the *logarithm* of each salary (base 10). To begin, fill in the table. Round the logarithms to 4 decimal places.

b. Create a bar graph with the logarithm of each salary shown on the vertical axis.



c. In Figure 3-20, the tick marks on the vertical axis are in equal increments of \$200 million. By contrast, the vertical scale in the graph in part (b) represents differences in powers of 10. For example, the increment from 2 to 4 on the vertical scale represents a difference of 10^2 or 100. The increment from 2 to 6 on the vertical scale represents a difference of 10^4 or 10,000. The increment from 2 to 8 represents 10^6 or 1,000,000. What is the difference in the logarithms of the salaries of the manager and the part-timer, and what does it mean as a power of 10?

d. What is the difference in the logarithms of the salaries of the executive and the part-timer, and what does it mean as a power of 10?

e. What is the difference in the logarithms of the salaries of the CEO and the part-timer, and what does it mean as a power of 10 rounded to the nearest thousand?

Problem Recognition Exercises

Analyzing Functions

For Exercises 1–14,

- **a.** Write the domain.
- **b.** Write the range.
- **c.** Find the x-intercept(s).
- **d.** Find the y-intercept.
- **e.** Determine the asymptotes if applicable.
- **f.** Determine the intervals over which the function is increasing.
- **g.** Determine the intervals over which the function is decreasing.
- **h.** Match the function with its graph from the list A–N.

1. $f(x) = 3$

2. $g(x) = 2x - 3$

3. $d(x) = (x - 3)^2 - 4$

4. $h(x) = \sqrt[3]{x - 2}$

5. $k(x) = \dfrac{2}{x - 1}$

6. $z(x) = \dfrac{3x}{x + 2}$

7. $p(x) = \left(\dfrac{4}{3}\right)^x$

8. $q(x) = -x^2 - 6x - 9$

9. $m(x) = |x - 4| - 1$

10. $n(x) = -|x| + 3$

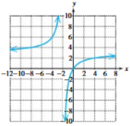
11. $r(x) = \sqrt{3 - x}$

12. $s(x) = \sqrt{x - 3}$

13. $t(x) = e^x + 2$

14. $v(x) = \ln(x + 2)$

A.



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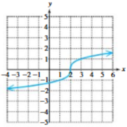
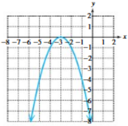
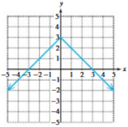


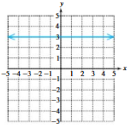
C.



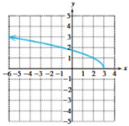
D.



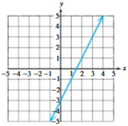
E.



F.



G.



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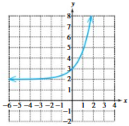
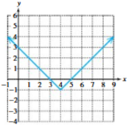
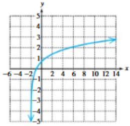


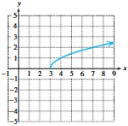
I.



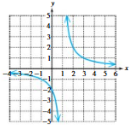
J.



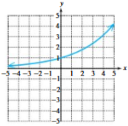
K.



L.



M.



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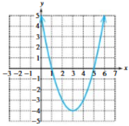


GO ONLINE **Go online for more practice problems.**

Lesson 3-4

Properties of Logarithms

What Will You Learn?

After completing this lesson, you should be able to:

- Apply the Product, Quotient, and Power Properties of Logarithms
- Write a Logarithmic Expression in Expanded Form
- Write a Logarithmic Expression as a Single Logarithm
- Apply the Change-of-Base Formula

Learn: Apply the Product, Quotient, and Power Properties of Logarithms

By definition, $y = \log_b x$ is equivalent to $b^y = x$. Because a logarithm is an exponent, the properties of exponents can be applied to logarithms. The first is called the product property of logarithms.

Product Property of Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$. Then

$$\log_b(xy) = \log_b x + \log_b y.$$

The logarithm of a product equals the sum of the logarithms of the factors.

Proof:

Let $M = \log_b x$, which implies $b^M = x$.

Let $N = \log_b y$, which implies $b^N = y$.

Then $xy = b^M b^N = b^{M+N}$.

Writing the expression $xy = b^{M+N}$ in logarithmic form, we have,

$$\log_b(xy) = M + N$$

$$\log_b(xy) = \log_b x + \log_b y \checkmark$$

Insights

When two factors of the same base are multiplied, the base is unchanged and we add the exponents. This is the underlying principle for the product property of logarithms.

To demonstrate the product property of logarithms, simplify the following expressions by using the order of operations.

$$\log_3(3 \cdot 9) \overset{?}{=} \log_3 3 + \log_3 9$$
$$\log_3 27 \overset{?}{=} 1 + 2$$
$$3 \overset{?}{=} 3 \checkmark \text{ True}$$

Example 1: Applying the Product Property of Logarithms

Write the logarithm as a sum and simplify if possible. Assume that *x* and *y* represent positive real numbers.

a. $\log_2(8x)$

b. $\ln(5xy)$

Solution:

a. $\log_2(8x) = \log_2 8 + \log_2 x$ Product property of logarithms

$= 3 + \log_2 x$ Simplify. $\log_2 8 = \log_2 2^3 = 3$

b. $\ln(5xy) = \ln 5 + \ln x + \ln y$

Apply the Skills

1. Write the logarithm as a sum and simplify if possible. Assume that *a*, *c*, and *d* represent positive real numbers.

a. $\log_4(16a)$

b. $\log(12cd)$

The quotient rule of exponents tells us that $\frac{b^M}{b^N} = b^{M-N}$ for $b \neq 0$. This property can be applied to logarithms.

Quotient Property of Logarithms

Let *b*, *x*, and *y* be positive real numbers where $b \neq 1$. Then

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y.$$

The logarithm of a quotient equals the difference of the logarithm of the numerator and the logarithm of the denominator.

The proof of the quotient property for logarithms is similar to the proof of the product property (see Exercise 117). To demonstrate the quotient property for logarithms, simplify the following expressions by using the order of operations.

$$\log\left(\frac{1{,}000{,}000}{100}\right) \overset{?}{=} \log 1{,}000{,}000 - \log 100$$
$$\log 10{,}000 \overset{?}{=} 6 - 2$$
$$4 \overset{?}{=} 4 \checkmark \text{ True}$$

Example 2: Applying the Quotient Property of Logarithms

Write the logarithm as the difference of logarithms and simplify if possible. Assume that the variables represent positive real numbers.

a. $\log_3\left(\frac{c}{d}\right)$ **b.** $\log\left(\frac{x}{1000}\right)$

Solution:

a. $\log_3\left(\frac{c}{d}\right) = \log_3 c - \log_3 d$ Quotient property of logarithms.

b. $\log\left(\frac{x}{1000}\right) = \log x - \log 1000$ Quotient property of logarithms.

$= \log x - 3$ Simplify. $\log 1000 = \log 10^3 = 3$

Apply the Skills

2. Write the logarithm as the difference of logarithms and simplify if possible. Assume that *t* represents a positive real number.

a. $\log_6\left(\frac{8}{t}\right)$ **b.** $\ln\left(\frac{e}{12}\right)$

The last property we present here is the power property of logarithms. The power property of exponents tells us that $(b^M)^N = b^{MN}$. The same principle can be applied to logarithms.

Power Property of Logarithms

Let *b* and *x* be positive real numbers where $b \neq 1$. Let *p* be any real number. Then

$$\log_b x^p = p \log_b x.$$

The power property of logarithms is proved in Exercise 118.

Example 3: Applying the Power Property of Logarithms

Apply the power property of logarithms. **a.** $\ln\sqrt[5]{x^2}$ **b.** $\log x^2$

Solution:

a. $\ln\sqrt[5]{x^2} = \ln x^{2/5}$ Write $\sqrt[5]{x^2}$ using rational exponents.

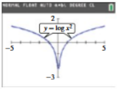
$= \frac{2}{5}\ln x$ provided that $x > 0$ Apply the power rule.

b. $\log x^2 = 2\log x$ provided that $x > 0$ Apply the power rule.

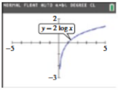
In both parts (a) and (b), the condition that $x > 0$ is mandatory. The properties of logarithms hold true only for values of the variable for which the logarithms are defined. That is, the arguments must be positive.

From the graphs of $y = \log x^2$ and $y = 2\log x$, we see that the domains are different. Therefore, the statement $\log x^2 = 2\log x$ is true only for $x > 0$.

$y = \log x^2$ Domain: $(-\infty, 0) \cup (0, \infty)$



$y = 2 \log x$ Domain: $(0, \infty)$



Apply the Skills

3. Apply the power property of logarithms.

a. $\log_5 \sqrt[5]{x^4}$ **b.** $\ln x^4$

At this point, we have learned seven properties of logarithms. The properties hold true for values of the variable for which the logarithms are defined. **Therefore, in the examples and exercises, we will assume that the variable expressions within the logarithms represent positive real numbers.**

Properties of Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$, and let p be a real number. Then the following properties of logarithms are true.

1. $\log_b 1 = 0$

2. $\log_b b = 1$

3. $\log_b b^p = p$

4. $b^{\log_b x} = x$

5. $\log_b(xy) = \log_b x + \log_b y$	**Product property**
6. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$	**Quotient property**
7. $\log_b x^p = p \log_b x$	**Power property**

Learn: Write a Logarithmic Expression in Expanded Form

Properties 5, 6, and 7 can be used in either direction. For example,

$\log\left(\frac{ab}{c}\right) = \log a + \log b - \log c$ or $\log a + \log b - \log c = \log\left(\frac{ab}{c}\right)$.

In some applications of algebra and calculus, the "condensed" form of the logarithm is preferred. In other applications, the "expanded" form is preferred. In Examples 4–6, we practice manipulating logarithmic expressions in both forms.

Example 4: Writing a Logarithmic Expression in Expanded Form

Write the expression as the sum or difference of logarithms.

a. $\log_2\left(\frac{z^3}{xy^5}\right)$ **b.** $\log\sqrt[3]{\frac{(x+y)^2}{10}}$

Solution:

a.
$$\begin{aligned} \log_2\left(\frac{z^3}{xy^5}\right) &= \log_2 z^3 - \log_2(xy^5) && \text{Apply the quotient property.}\\ &= \log_2 z^3 - (\log_2 x + \log_2 y^5) && \text{Apply the product property.}\\ &= \log_2 z^3 - \log_2 x - \log_2 y^5 && \text{Apply the distributive property.}\\ &= 3\log_2 z - \log_2 x - 5\log_2 y && \text{Apply the power property.} \end{aligned}$$

b.
$$\begin{aligned} \log\sqrt[3]{\frac{(x+y)^2}{10}} &= \log\left[\frac{(x+y)^2}{10}\right]^{1/3} && \text{Write the radical expression with rational exponents.}\\ &= \frac{1}{3}\log\left[\frac{(x+y)^2}{10}\right] && \text{Apply the power property.}\\ &= \frac{1}{3}[\log(x+y)^2 - \log 10] && \text{Apply the quotient property.}\\ &= \frac{1}{3}[2\log(x+y) - 1] && \text{Apply the power property and simplify: } \log 10 = 1.\\ &= \frac{2}{3}\log(x+y) - \frac{1}{3} && \text{Apply the distributive property.} \end{aligned}$$

Check Your Work

In Example 4(b) do not try to simplify $\log(x+y)$. The argument contains a sum, not a product.

$\log(x+y)$ cannot be simplified.

sum

Compare to the logarithm of a product which can be simplified.

$\log(xy) = \log x + \log y$

product

Apply the Skills

4. Write the expression as the sum or difference of logarithms.

a. $\ln\left(\frac{a^4b}{c^9}\right)$ **b.** $\log_5\sqrt[3]{\frac{25}{(a^2+b)^2}}$

Learn: Write a Logarithmic Expression as a Single Logarithm

In Examples 5 and 6, we demonstrate how to write a sum or difference of logarithms as a single logarithm. We apply Properties 5, 6, and 7 of logarithms in reverse.

Example 5: Writing the Sum or Difference of Logarithms as a Single Logarithm

Write the expression as a single logarithm and simplify the result if possible.

$$\log_2 560 - \log_2 7 - \log_2 5$$

Solution:

$\log_2 560 - \log_2 7 - \log_2 5$	
$= \log_2 560 - (\log_2 7 + \log_2 5)$	Factor out -1 from the last two terms.
$= \log_2 560 - \log_2(7 \cdot 5)$	Apply the product property.
$= \log_2\left(\frac{560}{7 \cdot 5}\right)$	Apply the quotient property.
$= \log_2 16$	Simplify within the argument.
$= 4$	Simplify. $\log_2 16 = \log_2 2^4 = 4$

Apply the Skills

5. Write the expression as a single logarithm and simplify the result if possible. $\log_3 54 + \log_3 10 - \log_3 20$

Example 6: Writing the Sum or Difference of Logarithms as a Single Logarithm

Write the expression as a single logarithm and simplify the result if possible.

a. $3\log a - \frac{1}{2}\log b - \frac{1}{2}\log c$

b. $\frac{1}{2}\ln x + \ln(x^2 - 1) - \ln(x + 1)$

Solution:

a. $3\log a - \frac{1}{2}\log b - \frac{1}{2}\log c$	
$= 3\log a - \frac{1}{2}(\log b + \log c)$	Factor out $-\frac{1}{2}$ from the last two terms.
$= 3\log a - \frac{1}{2}\log(bc)$	Apply the product property.
$= \log a^3 - \log(bc)^{1/2}$	Apply the power property.
$= \log a^3 - \log\sqrt{bc}$	
$= \log\left(\frac{a^3}{\sqrt{bc}}\right)$	Apply the quotient property.
b. $\frac{1}{2}\ln x + \ln(x^2 - 1) - \ln(x + 1)$	
$= \ln x^{1/2} + \ln(x^2 - 1) - \ln(x + 1)$	Apply the power property.
$= \ln[x^{1/2}(x^2 - 1)] - \ln(x + 1)$	Apply the product property.
$= \ln\left[\frac{\sqrt{x}(x^2 - 1)}{x + 1}\right]$	Apply the quotient property.
$= \ln\left[\frac{\sqrt{x}(x + 1)(x - 1)}{x + 1}\right]$	Factor the numerator of the argument.
$= \ln[\sqrt{x}(x - 1)]$	Simplify the argument.

Check Your Work

In all examples and exercises in which we manipulate logarithmic expressions, it is important to note that the equivalences are true only for the values of the variables that make the expressions defined. In Example 6(b) we have the restriction that $x > 1$.

Apply the Skills

6. Write the expression as a single logarithm and simplify the result if possible.

a. $3\log x - \frac{1}{3}\log y - \frac{2}{3}\log z$ **b.** $\frac{1}{3}\ln t + \ln(t^2 - 9) - \ln(t - 3)$

Example 7: Applying Properties of Logarithms

Given that $\log_b 2 \approx 0.356$ and $\log_b 3 \approx 0.565$, approximate the value of $\log_b 36$.

Solution:

$\log_b 36 = \log_b (2 \cdot 3)^2$ Write the argument as a product of the factors 2 and 3.

$= 2\log_b (2 \cdot 3)$ Apply the power property of logarithms.

$= 2(\log_b 2 + \log_b 3)$ Apply the product property of logarithms.

$\approx 2(0.356 + 0.565)$ Simplify.

≈ 1.842

Apply the Skills

7. Given that $\log_b 2 \approx 0.356$ and $\log_b 3 \approx 0.565$, approximate the value of $\log_b 24$.

Learn: Apply the Change-of-Base Formula

A calculator can be used to approximate the value of a logarithm base 10 or base *e* by using the LOG key or the LN key, respectively. However, to use a calculator to evaluate a logarithmic expression with a different base, we must use the change-of-base formula.

Change-of-Base Formula

Let *a* and *b* be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then for any positive real number *x*,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Note: The change-of-base formula converts a logarithm of one base to a ratio of logarithms of a different base. For the purpose of using a calculator, we often apply the change-of-base formula with base 10 or base *e*.

$$\log_b x = \frac{\log x}{\log b}$$

Original base is *b*.

Ratio of base 10 logarithms

$$\log_b x = \frac{\ln x}{\ln b}$$

Original base is *b*.

Ratio of base *e* logarithms

To derive the change-of-base formula, assume that a and b are positive real numbers with $a \neq 1$ and $b \neq 1$. Begin by letting $y = \log_b x$. If $y = \log_b x$, then

$b^y = x$	Write the original logarithm in exponential form.
$\log_a b^y = \log_a x$	Take the logarithm base a on both sides.
$y \cdot \log_a b = \log_a x$	Apply the power property of logarithms.
$y = \dfrac{\log_a x}{\log_a b}$	Solve for y.
$\log_b x = \dfrac{\log_a x}{\log_a b}$	Replace y with $\log_b x$. This is the change-of-base formula.

Example 8: Applying the Change-of-Base Formula

a. Estimate $\log_4 153$ between two consecutive integers.

b. Use the change-of-base formula to approximate $\log_4 153$ by using base 10. Round to 4 decimal places.

c. Use the change-of-base formula to approximate $\log_4 153$ by using base e.

d. Check the result by using the related exponential form.

Solution:

a. $64 < 153 < 256$

$4^3 < 153 < 4^4$

$3 < \log_4 153 < 4$ $\log_4 153$ is between 3 and 4.

b. $\log_4 153 = \dfrac{\log 153}{\log 4} \approx \dfrac{2.184691431}{0.6020599913} \approx 3.6287$

c. $\log_4 153 = \dfrac{\ln 153}{\ln 4} \approx \dfrac{5.030437921}{1.386294361} \approx 3.6287$

d. $4^{3.6287} \approx 153$ ✓

Insights

Although the numerators and denominators in parts (b) and (c) are different, their ratios are the same.

Apply the Skills

8. a. Estimate $\log_6 23$ between two consecutive integers.

b. Use the change-of-base formula to evaluate $\log_6 23$ by using base 10. Round to 4 decimal places.

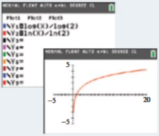
c. Use the change-of-base formula to evaluate $\log_6 23$ by using base e. Round to 4 decimal places.

d. Check the result by using the related exponential form.

TECHNOLOGY CONNECTIONS

Using the Change-of-Base Formula to Graph a Logarithmic Function

The change-of-base formula can be used to graph logarithmic functions using a graphing utility. For example, to graph $Y_1 = \log_2 x$, enter the function as $Y = \log(x)/\log(2)$ or $Y = \ln(x)/\ln(2)$.


NORMAL FLOAT AUTO a+bi DEGREE CL
Plot1 Plot2 Plot3
NORMAL FLOAT AUTO a+bi DEGREE CL
5
−5
20
−5

```
\Y1=log(X)/log(2)
\Y2=ln(X)/ln(2)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
```

Point of Interest

The slide rule, first built in England in the early 17th century, is a mechanical computing device that uses logarithmic scales to perform operations involving multiplication, division, roots, logarithms, exponentials, and trigonometry. Amazingly, slide rules were used into the space age by engineers in the 1960's to help send astronauts to the moon. It was only with the invention of the pocket calculator that slide rules were replaced by modern computing devices.

For more information about slide rules and other mathematical artifacts, see https://sliderulemuseum.com/.

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Practice 3-4

Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, use the properties of exponents to simplify the expression.

R.1. $x^{-3} \cdot x^5 \cdot x^7$

R.2. $\frac{y^{-2}y^{10}}{y^3}$

R.3. $(4w^{-3}z^4)^2$

R.4. $\left(\frac{7k^4}{n}\right)^{-3}$

Concept Connections

1. The product property of logarithms states that $\log_b(xy) =$ __________ for positive real numbers b, x, and y, where $b \neq 1$.
2. The __________ property of logarithms states that $\log_b\left(\frac{x}{y}\right) =$ __________ for positive real numbers b, x, and y, where $b \neq 1$.
3. The power property of logarithms states that for any real number p, $\log_b x^p =$ __________ for positive real numbers b, x, and y, where $b \neq 1$.
4. The change-of-base formula states that $\log_b x$ can be written as a ratio of logarithms with base a as $\log_b x = \frac{\square}{\square}$
5. The change-of-base formula is often used to convert a logarithm to a ratio of logarithms with base __________ or base __________ so that a calculator can be used to approximate the logarithm.
6. To use a graphing utility to graph the function defined by $y = \log_5 x$, use the change-of-base formula to write the function as $y =$ __________ or $y =$ __________.

For the exercises in this set, assume that all variable expressions represent positive real numbers.

Learn: Apply the Product, Quotient, and Power Properties of Logarithms

For Exercises 7–12, use the product property of logarithms to write the logarithm as a sum of logarithms. Then simplify if possible. (See Example 1, p. 387)

7. $\log_5(125z)$

8. $\log_7(49k)$

9. $\log(8cd)$

10. $\log(24vw)$

11. $\log_2[(x + y) \cdot z]$

12. $\log_3[(a + b) \cdot c]$

For Exercises 13–18, use the quotient property of logarithms to write the logarithm as a difference of logarithms. Then simplify if possible. (See Example 2, p. 388)

13. $\log_{12}\left(\frac{p}{q}\right)$

14. $\log_9\left(\frac{m}{n}\right)$

15. $\ln\left(\frac{e}{5}\right)$

16. $\ln\left(\frac{x}{e}\right)$

17. $\log\left(\frac{m^2 + n}{100}\right)$

18. $\log\left(\frac{1000}{c^2 + 1}\right)$

For Exercises 19–24, apply the power property of logarithms. (See Example 3, p. 388)

19. $\log(2x - 3)^4$

20. $\log(8t - 3)^2$

21. $\log_6 \sqrt[7]{x^3}$

22. $\log_8 \sqrt[4]{x^3}$

23. $\ln 2^{kt}$

24. $\ln(0.5)^{rt}$

Learn: Write a Logarithmic Expression in Expanded Form

For Exercises 25–44, write the logarithm as a sum or difference of logarithms. Simplify each term as much as possible. (See Example 4, p. 390)

25. $\log_4(7yz)$

26. $\log_2(5ab)$

27. $\log_7\left(\frac{1}{7}mn^2\right)$

28. $\log_4\left(\frac{1}{16}t^3v\right)$

29. $\log_2\left(\frac{x^{10}}{yz}\right)$

30. $\log_5\left(\frac{p^5}{mn}\right)$

31. $\log_6\left(\frac{p^5}{qt^3}\right)$

32. $\log_8\left(\frac{a^4}{b^9c}\right)$

33. $\log\left(\frac{10}{\sqrt{a^2 + b^2}}\right)$

34. $\log\left(\frac{\sqrt{d^2 + 1}}{10,000}\right)$

35. $\ln\left(\dfrac{\sqrt[3]{xy}}{wz^2}\right)$

36. $\ln\left(\dfrac{\sqrt[4]{pq}}{t^3m}\right)$

37. $\ln\sqrt[4]{\dfrac{a^2+4}{e^3}}$

38. $\ln\sqrt[5]{\dfrac{e^2}{c^2+5}}$

39. $\log\left[\dfrac{2x(x^2+3)^8}{\sqrt{4-3x}}\right]$

40. $\log\left[\dfrac{5y(4x+1)^7}{\sqrt[3]{2-7x}}\right]$

41. $\log_5\sqrt[3]{x\sqrt{5}}$

42. $\log_2\sqrt[4]{y\sqrt{2}}$

43. $\log_2\left[\dfrac{4a^2\sqrt{3-b}}{c(b+4)^2}\right]$

44. $\log_3\left[\dfrac{27x^3\sqrt{y^2-1}}{y(x-1)^2}\right]$

Learn: Write a Logarithmic Expression as a Single Logarithm

For Exercises 45–68, write the logarithmic expression as a single logarithm with coefficient 1, and simplify as much as possible. (See Examples 5–6, p. 391)

45. $\ln y + \ln 4$

46. $\log 5 + \log p$

47. $\log_{15} 3 + \log_{15} 5$

48. $\log_{12} 8 + \log_{12} 18$

49. $\log_7 98 - \log_7 2$

50. $\log_6 144 - \log_6 4$

51. $\log 150 - \log 3 - \log 5$

52. $\log_3 693 - \log_3 33 - \log_3 7$

53. $2\log_2 x + \log_2 t$

54. $5\log_4 y + \log_4 w$

55. $4\log_8 m - 3\log_8 n - 2\log_8 p$

56. $8\log_3 x - 2\log_3 z - 7\log_3 y$

57. $3[\ln x - \ln(x+3) - \ln(x-3)]$

58. $2[\log(p-4) - \log(p-1) - \log(p+4)]$

59. $\frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x-1)$

60. $\frac{1}{3}\ln(x^2+1) - \frac{1}{3}\ln(x+1)$

61. $6\log x - \frac{1}{3}\log y - \frac{2}{3}\log z$

62. $15\log c - \frac{1}{4}\log d - \frac{3}{4}\log k$

63. $\frac{1}{3}\log_4 p + \log_4(q^2-16) - \log_4(q-4)$

64. $\frac{1}{4}\log_2 w + \log_2(w^2-100) - \log_2(w+10)$

65. $\frac{1}{2}[6\ln(x+2) + \ln x - \ln x^2]$

66. $\frac{1}{3}[12\ln(x-5) + \ln x - \ln x^3]$

67. $\log(8y^2-7y) + \log y^{-1}$

68. $\log(9t^3-5t) + \log t^{-1}$

For Exercises 69–78, use $\log_b 2 \approx 0.356$, $\log_b 3 \approx 0.565$, and $\log_b 5 \approx 0.827$ to approximate the value of the given logarithms. (See Example 7, p. 392)

69. $\log_b 15$

70. $\log_b 10$

71. $\log_b 81$

72. $\log_b 125$

73. $\log_b 50$

74. $\log_b 12$

75. $\log_b\left(\frac{15}{2}\right)$

76. $\log_b\left(\frac{6}{5}\right)$

77. $\log_b 100$

78. $\log_b 225$

Learn: Apply the Change-of-Base Formula

For Exercises 79–84, (See Example 8, p. 393)

a. Estimate the value of the logarithm between two consecutive integers. For example, $\log_2 7$ is between 2 and 3 because $2^2 < 7 < 2^3$.

b. Use the change-of-base formula and a calculator to approximate the logarithm to 4 decimal places.

c. Check the result by using the related exponential form.

79. $\log_2 15$

80. $\log_3 15$

81. $\log_5 3$

82. $\log_8 5$

83. $\log_2 0.3$

84. $\log_2 0.2$

For Exercises 85–88, use the change-of-base formula and a calculator to approximate the given logarithms. Round to 4 decimal places. Then check the answer by using the related exponential form. (See Example 8, p. 393)

85. $\log_2(4.68 \times 10^7)$

86. $\log_2(2.54 \times 10^{10})$

87. $\log_4(5.68 \times 10^{-6})$

88. $\log_4(9.84 \times 10^{-5})$

Mixed Exercises

For Exercises 89–98, determine if the statement is true or false. For each false statement, provide a counterexample. For example, $\log(x+y) \neq \log x + \log y$ because $\log(2+8) \neq \log 2 + \log 8$ (the left side is 1 and the right side is approximately 1.204).

89. $\log e = \dfrac{1}{\ln 10}$

90. $\ln 10 = \dfrac{1}{\log e}$

91. $\log_5\left(\frac{1}{x}\right) = \frac{1}{\log_5 x}$

92. $\log_6\left(\frac{1}{t}\right) = \frac{1}{\log_6 t}$

93. $\log_4\left(\frac{1}{p}\right) = -\log_4 p$

94. $\log_8\left(\frac{1}{w}\right) = -\log_8 w$

95. $\log(xy) = (\log x)(\log y)$

96. $\log\left(\frac{x}{y}\right) = \frac{\log x}{\log y}$

97. $\log_2(7y) + \log_2 1 = \log_2(7y)$

98. $\log_4(3d) + \log_4 1 = \log_4(3d)$

Write About It

99. Explain why the product property of logarithms does not apply to the following statement.

$\log_5(-5) + \log_5(-25)$

$= \log_5[(-5)(-25)]$

$= \log_5 125 = 3$

100. Explain how to use the change-of-base formula and explain why it is important.

Expanding Your Skills

For Exercises 101–104, write the expression as a single, simplified logarithm.

101. $\log 3 - 4$

102. $2 + \ln 3$

103. $\ln 2 - 4$

104. $\log_2 12 + 3$

For Exercises 105–110, use $\ln x = a$ and $\ln y = b$ to rewrite each expression in terms of a and b and simplify, if possible.

105. $\ln(x^3y)$

106. $\ln\left(\frac{y}{x^2}\right)$

107. $\ln\left(\frac{ex^2}{\sqrt{y}}\right)$

108. $\ln\left(e^2x^3\sqrt[3]{y}\right)$

109. $\frac{\ln x^3 - \ln y^3}{\ln\left(\frac{x}{y}\right)}$

110. $\frac{\ln\left(\frac{1}{x}\right) + \ln\left(\frac{1}{y}\right)}{\ln(xy)}$

111. a. Write the difference quotient for $f(x) = \ln x$.

b. Show that the difference quotient from part (a) can be written as $\ln\left(\frac{x+h}{x}\right)^{1/h}$.

112. Show that

$-\ln\left(x - \sqrt{x^2 - 1}\right) = \ln\left(x + \sqrt{x^2 - 1}\right)$

113. Show that

$\log\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) + \log\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$

$= \log c - \log a$

114. Show that

$\ln\left(\frac{c + \sqrt{c^2 - x^2}}{c - \sqrt{c^2 - x^2}}\right) = 2\ln\left(c + \sqrt{c^2 - x^2}\right) - 2\ln x$

115. Use the change-of-base formula to write $(\log_2 5)(\log_5 9)$ as a single logarithm.

116. Use the change-of-base formula to write $(\log_3 11)(\log_{11} 4)$ as a single logarithm.

117. Prove the quotient property of logarithms:

$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$.

(*Hint*: Modify the proof of the product property given on page 386.)

118. Prove the power property of logarithms: $\log_b x^p = p\log_b x$.

Technology Connections

For Exercises 119–122, graph the function.

119. $f(x) = \log_5(x + 4)$

120. $g(x) = \log_7(x - 3)$

121. $k(x) = -3 + \log_{1/2} x$

122. $h(x) = 4 + \log_{1/3} x$

123. a. Graph $Y_1 = \log|x|$ and $Y_2 = \frac{1}{2}\log x^2$. How are the graphs related?

b. Show algebraically that $\frac{1}{2}\log x^2 = \log|x|$.

124. Graph $Y_1 = \ln(0.1x)$, $Y_2 = \ln(0.5x)$, $Y_3 = \ln x$, and $Y_4 = \ln(2x)$. How are the graphs related? Support your answer algebraically.

Go online for more practice problems.

Lesson 3-5

Exponential and Logarithmic Equations and Applications

What Will You Learn?

After completing this lesson, you should be able to:

- Solve Exponential Equations
- Solve Logarithmic Equations
- Use Exponential and Logarithmic Equations in Applications

Learn: Solve Exponential Equations

A couple invests \$8000 in a bond fund. The expected yield is 4.5% and the earnings are reinvested monthly. The growth of the investment is modeled by

$A = 8000\left(1 + \frac{0.045}{12}\right)^{12t}$ where A is the amount in the account after t years.

If the couple wants to know how long it will take for the investment to double, they would solve the equation:

$$16{,}000 = 8000\left(1 + \frac{0.045}{12}\right)^{12t} \text{(See Example 11.)}$$

This equation is called an **exponential equation** because the equation contains a variable in the exponent. To solve an exponential equation first note that all exponential functions are one-to-one. Therefore, $b^x = b^y$ implies that $x = y$. This is called the equivalence property of exponential expressions.

Equivalence Property of Exponential Expressions

If b, x, and y are real numbers with $b > 0$ and $b \neq 1$, then

$b^x = b^y$ implies that $x = y$.

Insights

The equivalence property tells us that if two exponential expressions with the same base are equal, then their exponents must be equal.

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Example 1: Solving Exponential Equations Using the Equivalence Property

Solve.

a. $3^{2x-6} = 81$ **b.** $25^{4-t} = \left(\frac{1}{5}\right)^{3t+1}$

Solution:

a. $3^{2x-6} = 81$

$3^{2x-6} = 3^4$ Write 81 as an exponential expression with a base of 3.

$2x - 6 = 4$ Equate the exponents.

$x = 5$

The solution set is {5}.

Check Your Work

$3^{2x-6} = 81$

$3^{2(5)-6} \stackrel{?}{=} 81$

$3^4 \stackrel{?}{=} 81$ ✓

b. $25^{4-t} = \left(\frac{1}{5}\right)^{3t+1}$

$(5^2)^{4-t} = (5^{-1})^{3t+1}$ Express both 25 and $\frac{1}{5}$ as integer powers of 5.

$5^{2(4-t)} = 5^{-1(3t+1)}$ Apply the power property of exponents: $(b^m)^n = b^{m \cdot n}$.

$5^{8-2t} = 5^{-3t-1}$ Apply the distributive property within the exponents.

$8 - 2t = -3t - 1$ Equate the exponents.

$t = -9$ The solution checks in the original equation.

The solution set is {−9}.

Check Your Work

When writing the expression $(5^2)^{4-t}$ as $5^{2(4-t)}$, it is important to use parentheses around the quantity $(4 - t)$. The exponent of 2 must be multiplied by the entire quantity $(4 - t)$. Likewise, parentheses are used around $(3t + 1)$ in the expression $5^{-1(3t+1)}$.

Apply the Skills

1. Solve.

a. $4^{2x-3} = 64$ **b.** $27^{2w+5} = \left(\frac{1}{3}\right)^{2-5w}$

In Example 1, we were able to write the left and right sides of the equation with a common base. However, most exponential equations cannot be written in this form by inspection. For example:

$$7^x = 60$$
$$7^x = 7^?$$

60 is not *a* recognizable power of 7.

To solve such an equation, we can take a logarithm of the same base on each side of the equation, and then apply the power property of logarithms. This is demonstrated in Examples 2–4.

Steps to Solve Exponential Equations by Using Logarithms

1. Isolate the exponential expression on one side of the equation.
2. Take a logarithm of the same base on both sides of the equation.
3. Use the power property of logarithms to "bring down" the exponent.
4. Solve the resulting equation.

Example 2: Solving an Exponential Equation Using Logarithms

Solve.

$7^x = 60$

Solution:

$7^x = 60$	The exponential expression 7^x is isolated.
$\log 7^x = \log 60$	Take a logarithm of the same base on both sides of the equation. In this case, we have chosen base 10.
$x \log 7 = \log 60$	Apply the power property of logarithms.
$x = \dfrac{\log 60}{\log 7} \approx 2.1041$	Divide both sides by $\log 7$.

This equation is now linear.

It is important to note that the exact solution to this equation is $\frac{\log 60}{\log 7}$ or equivalently by the change-of-base formula, $\log_7 60$. The value 2.1041 is merely an approximation.

The solution set is $\left\{\frac{\log 60}{\log 7}\right\}$ or $\{\log_7 60\}$.

Check Your Work

While 2.1041 is only an approximation, it is useful to check the result.

$$7^{2.1041} \approx 60$$

Apply the Skills

2. Solve. $5^x = 83$

To solve the equation from Example 2, we can take a logarithm of any base. For example:

$$7^x = 60$$
$$\log_7 7^x = \log_7 60$$
$$x = \log_7 60 \text{ (solution)}$$

Take the logarithm base 7 on both sides.

$$7^x = 60$$
$$\ln 7^x = \ln 60$$
$$x \ln 7 = \ln 60$$
$$x = \frac{\ln 60}{\ln 7} \text{ (solution)}$$

Take the natural logarithm on both sides.

The values $\log_7 60, \frac{\log 60}{\log 7}$, and $\frac{\ln 60}{\ln 7}$ are all equivalent. However, common logarithms and natural logarithms are often used to express the solution to an exponential equation so that the solution can be approximated on a calculator.

Example 3: Solving Exponential Equations Using Logarithms

Solve.

a. $10^{5+2x} + 820 = 49{,}600$

b. $2000 = 18{,}000e^{-0.4t}$

Solution:

a. $10^{5+2x} + 820 = 49{,}600$

Isolate the exponential expression on the left by subtracting 820 on both sides.

$$10^{5+2x} = 48{,}780$$

Since the exponential expression on the left has a base of 10, take the log base 10 on both sides.

$$\log 10^{5+2x} = \log 48{,}780$$

On the left, $\log 10^{5+2x} = 5 + 2x$.

$$5 + 2x = \log 48{,}780$$

Solve the linear equation by subtracting 5 and dividing by 2.

$$2x = \log 48{,}780 - 5$$

$$x = \frac{\log 48{,}780 - 5}{2} \approx -0.1559$$

The solution checks in the original equation.

The solution set is $\left\{\frac{\log 48{,}780 - 5}{2}\right\}$.

b. $2000 = 18{,}000e^{-0.4t}$

Isolate the exponential expression on the right by dividing both sides by 18,000.

$$\frac{1}{9} = e^{-0.4t}$$

Since the exponential expression on the right has a base of e, take the log base e on both sides.

$$\ln\left(\frac{1}{9}\right) = \ln e^{-0.4t}$$

On the right, $\ln e^{-0.4t} = -0.4t$.

$$\ln\left(\frac{1}{9}\right) = -0.4t$$

linear equation

Solve the linear equation by dividing by -0.4.

$$\frac{\ln\left(\frac{1}{9}\right)}{-0.4} = t$$

The exact solution to the equation can be written in a variety of forms by applying the properties of logarithms:

$$\frac{\ln\left(\frac{1}{9}\right)}{-0.4} = \frac{\ln 1 - \ln 9}{-0.4} = \frac{0 - \ln 9}{-0.4} = \frac{\ln 9}{0.4} \approx 5.4931$$

Alternatively, $\frac{\ln 9}{0.4} = \frac{\ln 9}{\frac{2}{5}} = \frac{5\ln 9}{2} \approx 5.4931$

The solution set is $\left\{\frac{\ln 9}{0.4}\right\}$ or $\left\{\frac{5\ln 9}{2}\right\}$.

Apply the Skills

3. Solve.

a. $400 + 10^{4x-1} = 63{,}000$

b. $100 = 700e^{-0.2k}$

In Example 4, we have an equation with two exponential expressions involving different bases.

Example 4: Solving an Exponential Equation

Solve. $4^{2x-7} = 5^{3x+1}$

Solution:

$$4^{2x-7} = 5^{3x+1}$$

$$\ln 4^{2x-7} = \ln 5^{3x+1}$$

Take a logarithm of the same base on both sides.

$$(2x-7)\ln 4 = (3x+1)\ln 5$$

Apply the power property of logarithms.

$$2x\ln 4 - 7\ln 4 = 3x\ln 5 + \ln 5$$

Apply the distributive property.

$$2x\ln 4 - 3x\ln 5 = \ln 5 + 7\ln 4$$

Collect x terms on one side of the equation.

$$x(2\ln 4 - 3\ln 5) = \ln 5 + 7\ln 4$$

Factor out x on the left.

$$x = \frac{\ln 5 + 7\ln 4}{2\ln 4 - 3\ln 5} \approx -5.5034$$

Divide by $(2\ln 4 - 3\ln 5)$.

The solution set is $\left\{\frac{\ln 5 + 7\ln 4}{2\ln 4 - 3\ln 5}\right\}$.

The solution checks in the original equation.

Apply the Skills

4. Solve. $3^{5x-6} = 2^{4x+1}$

In Example 5, we look at an exponential equation in quadratic form.

Example 5: Solving an Exponential Equation in Quadratic Form

Solve. $e^{2x} + 5e^{x} - 36 = 0$

Solution:

$e^{2x} + 5e^{x} - 36 = 0$	
$(e^{x})^2 + 5(e^{x}) - 36 = 0$	Note that $e^{2x} = (e^{x})^2$.
$u^2 + 5u - 36 = 0$	The equation is in quadratic form. Let $u = e^{x}$.
$(u - 4)(u + 9) = 0$	Factor.
$u = 4$ or $u = -9$	
$e^{x} = 4$ or $e^{x} = -9$	Back substitute. The second equation $e^{x} = -9$ has no solution.
$\ln e^{x} = \ln 4$	
$x = \ln 4 \approx 1.3863$	
The solution set is $\{\ln 4\}$.	The solution checks in the original equation.

No solution to this equation because $\ln(-9)$ is undefined.

Check Your Work

Recall that the range of $f(x) = e^{x}$ is the set of positive real numbers. Therefore, $e^{x} \neq -9$.

Apply the Skills

5. Solve. $e^{2x} - 5e^{x} - 14 = 0$

Learn: Solve Logarithmic Equations

An equation containing a variable within a logarithmic expression is called a **logarithmic equation.** For example:

$\log_2(3x - 4) = \log_2(x + 2)$ and $\ln(x + 4) = 7$ are logarithmic equations.

Given an equation in which two logarithms of the same base are equated, we can apply the equivalence property of logarithms. Since all logarithmic functions are one-to-one, $\log_b x = \log_b y$ implies that $x = y$.

Equivalence Property of Logarithmic Expressions

If b, x, and y are positive real numbers with $b \neq 1$, then

$$\log_b x = \log_b y \quad \text{implies that } x = y.$$

Insights

The equivalence property tells us that if two logarithmic expressions with the same base are equal, then their arguments must be equal.

Example 6: Solving a Logarithmic Equation Using the Equivalence Property

Solve. $\log_2(3x-4)=\log_2(x+2)$

Solution:

$\log_2(3x-4)=\log_2(x+2)$	Two logarithms of the same base are equated.
$3x-4=x+2$	Equate the arguments.
$2x=6$	Solve for x.
$x=3$	Because the domain of a logarithmic function is restricted, it is mandatory that we check all potential solutions to a logarithmic equation.

The solution set is $\{3\}$.

Check Your Work

$$\log_2(3x-4)=\log_2(x+2)$$

$$\log_2[3(3)-4]\overset{?}{=}\log_2[(3)+2]$$

$$\log_2 5\overset{?}{=}\log_2 5\ \checkmark$$

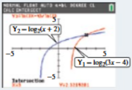
Apply the Skills

6. Solve. $\log_2(7x-4)=\log_2(2x+1)$

TECHNOLOGY CONNECTIONS

Using a Calculator to View the Potential Solutions to a Logarithmic Equation

The solution to the equation in Example 6 is the x-coordinate of the point of intersection of $Y_1=\log_2(3x-4)$ and $Y_2=\log_2(x+2)$. The domain of $Y_1=\log_2(3x-4)$ is $\left\{x \middle| x>\frac{4}{3}\right\}$ and the domain of $Y_2=\log_2(x+2)$ is $\{x|x>-2\}$. The solution to the equation $Y_1=Y_2$ may not lie outside the domain of either function. This is why it is mandatory to check all potential solutions to a logarithmic equation.



In Example 7, we encounter a logarithmic equation in which one or more solutions do not check.

Example 7: Solving a Logarithmic Equation

Solve. $\ln(x-4)=\ln(x+6)-\ln x$

Solution:

$\ln(x-4)=\ln(x+6)-\ln x$	
$\ln(x-4)=\ln\left(\frac{x+6}{x}\right)$	Combine the two logarithmic terms on the right.
$x-4=\frac{x+6}{x}$	Apply the equivalence property of logarithms.
$x^2-4x=x+6$	Clear fractions by multiplying both sides by x.
$x^2-5x-6=0$	The resulting equation is quadratic.
$(x-6)(x+1)=0$	
$x=6$ or $x=-1$	The potential solutions are 6 and -1.

Check Your Work

$\ln(x-4)=\ln(x+6)-\ln x$

$\ln(6-4)\stackrel{?}{=}\ln(6+6)-\ln 6$

$\ln 2\stackrel{?}{=}\ln 12-\ln 6$

$\ln 2\stackrel{?}{=}\ln\left(\frac{12}{6}\right)$ ✓

$\ln(x-4)=\ln(x+6)-\ln x$

$\ln(-1-4)\stackrel{?}{=}\ln(-1+6)-\ln(-1)$

$\ln(-5)\stackrel{?}{=}\ln 5-\ln(-1)$

undefined

undefined

The only solution that checks is 6.

The solution set is {6}.

Apply the Skills

7. Solve. $\ln x+\ln(x-8)=\ln(x-20)$

Many logarithmic equations, such as $4\log_3(2t-7)=8$ and $\log_2 x=3-\log_2(x-2)$, involve logarithmic terms and constant terms. In such a case, we can apply the properties of logarithms to write the equation in the form $\log_b x=k$, where k is a constant. At this point, we can solve for x by writing the equation in its equivalent exponential form $x=b^k$.

Solving Logarithmic Equations by Using Exponential Form

Step 1 Given a logarithmic equation, isolate the logarithms on one side of the equation.

Step 2 Use the properties of logarithms to write the equation in the form $\log_b x=k$, where k is a constant.

Step 3 Write the equation in exponential form.

Step 4 Solve the equation from step 3.

Step 5 Check the potential solution(s) in the original equation.

Example 8: Solving a Logarithmic Equation

Solve. $4\log_3(2t-7)=8$

Solution:

$4\log_3(2t-7)=8$	Isolate the logarithm by dividing both sides by 4.
$\log_3(2t-7)=2$	The equation is in the form $\log_b x = k$, where $x = 2t - 7$.
$2t-7=3^2$	Write the equation in exponential form.
$2t-7=9$	Solve for t.
$t=8$	

Check Your Work

$$4\log_3[2(8)-7]\overset{?}{=}8$$

$$4\log_3 9\overset{?}{=}8$$

$$4\cdot 2\overset{?}{=}8\ ✓$$

The solution set is $\{8\}$.

Apply the Skills

8. Solve. $8\log_4(w+6)=24$

Example 9: Solving a Logarithmic Equation

Solve. $\log(w+47)=2.6$

Solution:

$\log(w+47)=2.6$	The equation is in the form $\log_b x = k$ where $x = w + 47$ and $b = 10$.
$w+47=10^{2.6}$	Write the equation in exponential form.
$w=10^{2.6}-47\approx 351.1072$	Solve the resulting linear equation.

Check Your Work

$$\log(w+47)=2.6$$

$$\log[(10^{2.6}-47)+47]\overset{?}{=}2.6$$

$$\log 10^{2.6}\overset{?}{=}2.6\ ✓$$

The solution set is $\{10^{2.6}-47\}$.

Apply the Skills

9. Solve. $\log(t - 18) = 1.4$

Example 10 contains multiple logarithmic terms and a constant term. We apply the strategy of collecting the logarithmic terms on one side and the constant term on the other side. Then after combining the logarithmic terms, we write the equation in exponential form.

Example 10: Solving a Logarithmic Equation

Solve. $\log_2 x = 3 - \log_2(x - 2)$

Solution:

$$\log_2 x = 3 - \log_2(x - 2)$$

$$\log_2 x + \log_2(x - 2) = 3$$

Isolate the logarithms on one side of the equation.

$$\log_2[x(x - 2)] = 3$$

Use the product property of logarithms to write a single logarithm.

$$x(x - 2) = 2^3$$

Write the equation in exponential form.

$$x^2 - 2x = 8$$

$$x^2 - 2x - 8 = 0$$

Set one side equal to zero.

$$(x - 4)(x + 2) = 0$$

$$x = 4 \quad x = -2$$

Check Your Work

$$\log_2 x = 3 - \log_2(x - 2) \qquad \log_2 x = 3 - \log_2(x - 2)$$

$$\log_2 4 \stackrel{?}{=} 3 - \log_2(4 - 2) \qquad \log_2(-2) \stackrel{?}{=} 3 - \log_2(-2 - 2)$$

$$\log_2 4 \stackrel{?}{=} 3 - \log_2 (2) \qquad \underbrace{\log_2(-2)}_{\text{undefined}} \stackrel{?}{=} 3 - \underbrace{\log_2(-4)}_{\text{undefined}}$$

$$2 \stackrel{?}{=} 3 - 1 \checkmark$$

The only solution that checks is $x = 4$.

The solution set is $\{4\}$.

Apply the Skills

10. Solve. $2 - \log_7 x = \log_7(x - 48)$

Learn: Use Exponential and Logarithmic Equations in Applications

In Examples 11 and 12, we solve applications involving exponential and logarithmic equations.

Example 11: Using an Exponential Equation in a Finance Application

A couple invests \$8000 in a bond fund. The expected yield is 4.5% and the earnings are reinvested monthly.

a. Use $A = P\left(1+\frac{r}{n}\right)^{nt}$ to write a model representing the amount A (in \$) in the account after t years. The value r is the interest rate and n is the number of times interest is compounded per year.

b. Determine how long it will take the initial investment to double. Round to 1 decimal place.

Solution:

a. $A = P\left(1+\frac{r}{n}\right)^{nt}$

$A = 8000\left(1+\frac{0.045}{12}\right)^{12t}$ Substitute $P = 8000$, $r = 0.045$, and $n = 12$.

Insights

Recall that monthly compounding indicates that interest is computed $n = 12$ times per year.

b. $16{,}000 = 8000\left(1+\frac{0.045}{12}\right)^{12t}$ The couple wants to double their money from \$8000 to \$16,000. Substitute $A = 16{,}000$ and solve for t.

$2 = \left(1+\frac{0.045}{12}\right)^{12t}$ Isolate the exponential expression by dividing both sides by 8000.

$\ln 2 = \ln\left(1+\frac{0.045}{12}\right)^{12t}$ Take a logarithm of the same base on both sides. We have chosen to use the natural logarithm.

$\ln 2 = 12t\ln\left(1+\frac{0.045}{12}\right)$ Apply the power property of logarithms. The equation is now linear in the variable t.

$\dfrac{\ln 2}{12\ln\left(1+\frac{0.045}{12}\right)} = t$ Divide both sides by $12\ln\left(1+\frac{0.045}{12}\right)$.

$t \approx 15.4$

It will take approximately 15.4 yr for the investment to double.

Apply the Skills

11. Determine how long it will take \$8000 compounded monthly at 6% to double. Round to 1 decimal place.

In Example 12, we use a logarithmic equation in an application.

Example 12: Using a Logarithmic Equation in a Medical Application

Suppose that the sound at a rock concert measures 124 dB (decibels).

a. Use the formula $L = 10\log\left(\frac{I}{I_0}\right)$ to find the intensity of sound I (in W/m²). The variable L represents the loudness of sound (in dB) and $I_0 = 10^{-12}$ W/m².

b. If the threshold at which sounds become painful is 1 W/m², will the music at this concert be physically painful? (Ignore the quality of the music.)

Solution:

a. $L = 10\ \log\left(\frac{I}{I_0}\right)$

$124 = 10\log\left(\frac{I}{10^{-12}}\right)$	Substitute 124 for L and 10^{-12} for I_0.
$12.4 = \log\left(\frac{I}{10^{-12}}\right)$	Divide both sides by 10. The logarithm is now isolated.
$10^{12.4} = \frac{I}{10^{-12}}$	Write the equation in exponential form.
$10^{12.4} \cdot 10^{-12} = I$	Multiply both sides by 10^{-12}.
$I = 10^{0.4} \approx 2.5$ W/m²	Simplify.

b. The intensity of sound at the rock concert is approximately 2.5 W/m². This is above the threshold for pain.

Apply the Skills

12. a. Find the intensity of sound from a leaf blower if the decibel level is 115 dB.

b. Is the intensity of sound from a leaf blower above the threshold for pain?

TECHNOLOGY CONNECTIONS

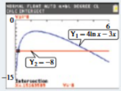
Using a Calculator to Approximate the Solutions to Exponential and Logarithmic Equations

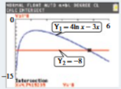
There are many situations in which analytical methods fail to give a solution to a logarithmic or exponential equation. To find solutions graphically,

Enter the left side of the equation as Y_1.

Enter the right side of the equation as Y_2.

Then determine the point(s) of intersection of the graphs.





Example: $4\ln x - 3x = -8$

$Y_1 = 4\ln x - 3x$ and $Y_2 = -8$

Solutions: $x \approx 0.1516$ and $x \approx 4.7419$

Practice 3-5

Practice Exercises

Prerequisite Review

For Exercises R.1–R.6, solve the equation.

R.1. $7 = 5 - 2(4q - 1)$

R.2. $15n(n + 3) = 14n - 10$

R.3. $y^2 - 5y - 9 = 0$

R.4. $\frac{20}{n^2 - 2n} + 5 = \frac{10}{n - 2}$

R.5. $\sqrt{a + 18} + 2 = a$

R.6. $36d^{-2} - 5d^{-1} - 1 = 0$

Concept Connections

1. An equation such as $4^x = 9$ is called an __________ equation because the equation contains a variable in the exponent.

2. The equivalence property of exponential expressions states that if $b^x = b^y$, then __________ = __________.

3. The equivalence property of logarithmic expressions states that if $\log_b x = \log_b y$, then __________ = __________.

4. An equation containing a variable within a logarithmic expression is called a __________ equation.

Learn: Solve Exponential Equations

For Exercises 5–16, solve the equation. (See Example 1, p. 399)

5. $3^x = 81$

6. $2^x = 32$

7. $\sqrt[3]{5} = 5^t$

8. $\sqrt{3} = 3^w$

9. $2^{-3y+1} = 16$

10. $5^{2z+2} = 625$

11. $11^{3c+1} = \left(\frac{1}{11}\right)^{c-5}$

12. $7^{2x-3} = \left(\frac{1}{49}\right)^{x+1}$

13. $8^{2x-5} = 32^{x-6}$

14. $27^{x-4} = 9^{2x+1}$

15. $100^{3t-5} = 1000^{3-t}$

16. $100{,}000^{2w+1} = 10{,}000^{4-w}$

For Exercises 17–34, solve the equation. Write the solution set with the exact values given in terms of common or natural logarithms. Also give approximate solutions to 4 decimal places. (See Examples 2–5, pp. 400–403)

17. $6^t = 87$

18. $2^z = 70$

19. $1024 = 19^x + 4$

20. $801 = 23^y + 6$

21. $10^{3+4x} - 8100 = 120{,}000$

22. $10^{5+8x} + 4200 = 84{,}000$

23. $21{,}000 = 63{,}000e^{-0.2t}$

24. $80 = 320e^{-0.5t}$

25. $4e^{2n-5} + 3 = 11$

26. $5e^{4m-3} - 7 = 13$

27. $3^{6x+5} = 5^{2x}$

28. $7^{4x-1} = 3^{5x}$

29. $2^{1-6x} = 7^{3x+4}$

30. $11^{1-8x} = 9^{2x+3}$

31. $e^{2x} - 9e^x - 22 = 0$

32. $e^{2x} - 6e^x - 16 = 0$

33. $e^{2x} = -9e^x$

34. $e^{2x} = -7e^x$

Learn: Solve Logarithmic Equations

For Exercises 35–36, determine if the given value of *x* is a solution to the logarithmic equation.

35. $\log_2(x - 31) = 5 - \log_2 x$

- **a.** $x = 16$
- **b.** $x = 32$
- **c.** $x = -1$

36. $\log_4 x = 3 - \log_4(x - 63)$

a. $x = 64$

b. $x = -1$

c. $x = 32$

For Exercises 37–60, solve the equation. Write the solution set with the exact solutions. Also give approximate solutions to 4 decimal places if necessary. (See Examples 6–10, pp. 404–410)

37. $\log_4(3w + 11) = \log_4(3 - w)$

38. $\log_7(12 - t) = \log_7(t + 6)$

39. $\log(x^2 + 7x) = \log 18$

40. $\log(p^2 + 6p) = \log 7$

41. $6\log_5(4p - 3) - 2 = 16$

42. $5\log_6(7w + 1) + 3 = 13$

43. $2\log_8(3y - 5) + 20 = 24$

44. $5\log_3(7 - 5z) + 2 = 17$

45. $\log(p + 17) = 4.1$

46. $\log(q - 6) = 3.5$

47. $2\ln(4 - 3t) + 1 = 7$

48. $4\ln(6 - 5t) + 2 = 22$

49. $\log_2 w - 3 = -\log_2(w + 2)$

50. $\log_3 y + \log_3(y + 6) = 3$

51. $\log_6(7x - 2) = 1 + \log_6(x + 5)$

52. $\log_4(5x - 13) = 1 + \log_4(x - 2)$

53. $\log_5 z = 3 - \log_5(z - 20)$

54. $\log_2 x = 4 - \log_2(x - 6)$

55. $\ln x + \ln(x - 4) = \ln(3x - 10)$

56. $\ln x + \ln(x - 3) = \ln(5x - 7)$

57. $\log x + \log(x - 7) = \log(x - 15)$

58. $\log x + \log(x - 10) = \log(x - 18)$

59. $\log_8(6 - m) + \log_8(-m - 1) = 1$

60. $\log_3(n - 5) + \log_3(n + 3) = 2$

Learn: Use Exponential and Logarithmic Equations in Applications

For Exercises 61–70, use the model $A = Pe^{rt}$ or $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where A is the future value of P dollars invested at interest rate r compounded continuously or n times per year for t years. (See Example 11, p. 408)

61. If $10,000 is invested in an account earning 5.5% interest compounded continuously, determine how long it will take the money to triple. Round to the nearest year.

62. If a couple has $80,000 in a retirement account, how long will it take the money to grow to $1,000,000 if it grows by 6% compounded continuously? Round to the nearest year.

63. A $2500 bond grows to $3729.56 in 10 yr under continuous compounding. Find the interest rate. Round to the nearest whole percent.

64. $5000 grows to $5438.10 in 2 yr under continuous compounding. Find the interest rate. Round to the nearest tenth of a percent.

65. An $8000 investment grows to $9289.50 at 3% interest compounded quarterly. For how long was the money invested? Round to the nearest year.

66. $20,000 is invested at 3.5% interest compounded monthly. How long will it take for the investment to double? Round to the nearest tenth of a year.

67. A $25,000 inheritance is invested for 15 yr compounded quarterly and grows to $52,680. Find the interest rate. Round to the nearest percent.

68. A $10,000 investment grows to $11,273 in 4 yr compounded monthly. Find the interest rate. Round to the nearest percent.

69. If $4000 is put aside in a money market account with interest compounded continuously at 2.2%, find the time required for the account to *earn* $1000. Round to the nearest month.

70. Angel puts aside \$10,000 in an account with interest compounded continuously at 2.7%. How long will it take to *earn* \$2000? Round to the nearest month.

71. Physicians often treat thyroid cancer with a radioactive form of iodine called iodine-131 (^{131}I). The radiological half-life of ^{131}I is approximately 8 days, but the biological half-life for most individuals is 4.2 days. The biological half-life is shorter because in addition to ^{131}I being lost to decay, the iodine is also excreted from the body in urine, sweat, and saliva.

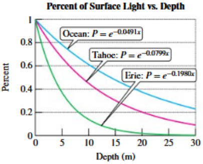
For a patient treated with 100 mCi (millicuries) of ^{131}I, the radioactivity level R (in mCi) after t days is given by $R = 100(2)^{-t/4.2}$.

- **a.** State law mandates that the patient stay in an isolated hospital room for 2 days after treatment with ^{131}I. Determine the radioactivity level at the end of 2 days. Round to the nearest whole unit.
- **b.** After the patient is released from the hospital, the patient is directed to avoid direct human contact until the radioactivity level drops below 30 mCi. For how many days *after* leaving the hospital will the patient need to stay in isolation? Round to the nearest tenth of a day.

72. Caffeine occurs naturally in a variety of food products such as coffee, tea, and chocolate. The kidneys filter the blood and remove caffeine and other drugs through urine. The biological half-life of caffeine is approximately 6 hr. If one cup of coffee has 80 mg of caffeine, then the amount of caffeine C (in mg) remaining after t hours is given by $C = 80(2)^{-t/6}$.

- **a.** How long will it take for the amount of caffeine to drop below 60 mg? Round to 1 decimal place.
- **b.** Laura has trouble sleeping if she has more than 30 mg of caffeine in her bloodstream. How many hours after drinking a cup of coffee would Laura have to wait so that the coffee would not disrupt her sleep? Round to 1 decimal place.

Sunlight is absorbed in water, and as a result the light intensity in oceans, lakes, and ponds decreases exponentially with depth. The percentage of visible light, P (in decimal form), at a depth of x meters is given by $P = e^{-kx}$, where k is a constant related to the clarity and other physical properties of the water. The graph shows models for the open ocean, Lake Tahoe, and Lake Erie for data taken under similar conditions. Use these models for Exercises 73–76.



73. Determine the depth at which the light intensity is half the value from the surface for each body of water given. Round to the nearest tenth of a meter.

74. Determine the depth at which the light intensity is 20% of the value from the surface for each body of water given. Round to the nearest tenth of a meter.

75. The *euphotic* depth is the depth at which light intensity falls to 1% of the value at the surface. This depth is of interest to scientists because no appreciable photosynthesis takes place. Find the euphotic depth for the open ocean. Round to the nearest tenth of a meter.

76. Refer to Exercise 75, and find the euphotic depth for Lake Tahoe and for Lake Erie. Round to the nearest tenth of a meter.

77. Forge welding is a process in which two pieces of steel are joined together by heating the pieces of steel and hammering them together. A welder takes a piece of steel from a forge at 1600°F and places it on an anvil where the outdoor temperature is 50°F. The temperature of the steel T (in °F) can be modeled by $T = 50 + 1550e^{-0.05t}$, where t is the time in minutes after the steel is removed from the forge. How long will it take for the steel to reach a temperature of 100°F so that it can be handled without heat protection? Round to the nearest minute.

78. A pie comes out of the oven at 325°F and is placed to cool in a 70°F kitchen. The temperature of the pie T (in °F) after t minutes is given by $T = 70 + 255e^{-0.017t}$. The pie is cool enough to cut when the temperature reaches 110°F. How long will this take? Round to the nearest minute.

For Exercises 79–80, the formula $L = 10 \log\left(\frac{I}{I_0}\right)$ gives the loudness of sound L (in dB) based on the intensity of sound I (in W/m²). The value $I_0 = 10^{-12}$ W/m² is the minimal threshold for hearing for midfrequency sounds. Hearing impairment is often measured according to the minimal sound level (in dB) detected by an individual for sounds at various frequencies. For one frequency, the table depicts the level of hearing impairment.

Category	Loudness (dB)
Mild	$26 \le L \le 40$
Moderate	$41 \le L \le 55$
Moderately severe	$56 \le L \le 70$
Severe	$71 \le L \le 90$
Profound	$L > 90$

79. a. If the minimum intensity heard by an individual is 3.4×10^{-8} W/m², determine if the individual has a hearing impairment.

b. If the minimum loudness of sound detected by an individual is 30 dB, determine the corresponding intensity of sound. (**See Example 12, p. 409**)

80. Determine the range that represents the intensity of sound that can be heard by an individual with severe hearing impairment.

For Exercises 81–82, use the formula $\text{pH} = -\log[\text{H}^+]$. The variable pH represents the level of acidity or alkalinity of a liquid on the pH scale, and H^+ is the concentration of hydronium ions in the solution. Determine the value of H^+ (in mol/L) for the following liquids, given their pH values.

81. a. Seawater pH = 8.5

b. Acid rain pH = 2.3

82. a. Milk pH = 6.2

b. Sodium bicarbonate pH = 8.4

83. A new teaching method to teach vocabulary to sixth-graders involves having students work in groups on an assignment to learn new words. After the lesson was completed, the students were tested at 1-month intervals. The average score for the class $S(t)$ can be modelled by

$$S(t) = 94 - 18 \ln(t + 1)$$

where t is the time in months after completing the assignment. If the average score is 65, how many months had passed since the students completed the assignment? Round to the nearest month.

84. A company spends x hundred dollars on an advertising campaign. The amount of money in sales $S(x)$ (in \$1000) for the 4-month period after the advertising campaign can be modelled by

$$S(x) = 5 + 7 \ln(x + 1)$$

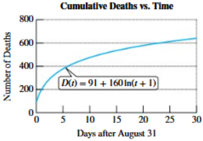
If the sales total \$19,100, how much was spent on advertising? Round to the nearest dollar.

85. Radiated seismic energy from an earthquake is estimated by $\log E = 4.4 + 1.5M$, where E is the energy in Joules (J) and M is surface wave magnitude.

a. How many times more energy does an 8.2-magnitude earthquake have than a 5.5-magnitude earthquake? Round to the nearest thousand.

b. How many times more energy does a 7-magnitude earthquake have than a 6-magnitude earthquake? Round to the nearest whole number.

86. On August 31, 1854, an epidemic of cholera was discovered in London, England, resulting from a contaminated community water pump. By the end of September, more than 600 citizens who drank water from the pump had died. The cumulative number of deaths $D(t)$ at a time t days after August 31 is given by $D(t) = 91 + 160\ln(t + 1)$.



a. Determine the cumulative number of deaths by September 15. Round to the nearest whole unit.

b. Approximately how many days after August 31 did the cumulative number of deaths reach 600?

Mixed Exercises

For Exercises 87–94, find an equation for the inverse function.

87. $f(x) = 2^x - 7$

88. $f(x) = 5^x + 6$

89. $f(x) = \ln(x + 5)$

90. $f(x) = \ln(x - 7)$

91. $f(x) = 10^{x-3} + 1$

92. $f(x) = 10^{x+2} - 4$

93. $f(x) = \log(x + 7) - 9$

94. $f(x) = \log(x - 11) + 8$

For Exercises 95–112, solve the equation. Write the solution set with exact solutions. Also give approximate solutions to 4 decimal places if necessary.

95. $5^{|x|} - 3 = 122$

96. $11^{|x|} + 9 = 130$

97. $\log x - 2\log 3 = 2$

98. $\log y - 3\log 5 = 3$

99. $6^{x^2-2} = 36$

100. $8^{y^2-7} = 64$

101. $\log_9|x + 4| = \log_9 6$

102. $\log_8|3 - x| = \log_8 5$

103. $x^2e^x = 9e^x$

104. $x^26^x = 6^x$

105. $\log_3(\log_3 x) = 0$

106. $\log_5(\log_5 x) = 1$

107. $3|\ln x| - 12 = 0$

108. $7|\ln x| - 14 = 0$

109. $\log_3 x - \log_3(2x + 6) = \frac{1}{2}\log_3 4$

110. $\log_5 x - \log_5(x + 1) = \frac{1}{3}\log_5 8$

111. $2e^x(e^x - 3) = 3e^x - 4$

112. $3e^x(e^x - 6) = 4e^x - 7$

For Exercises 113–130, write the solution set with exact answers.

113. $e^{2x} - 8e^x - 20 = 0$

114. $e^{2x} - 3e^x - 28 = 0$

115. $10^{2x} - 7 \cdot 10^x - 18 = 0$

116. $10^{2x} - 2 \cdot 10^x - 15 = 0$

117. $2e^{2x} - 7e^x + 3 = 0$

118. $3e^{2x} - 19e^x + 6 = 0$

119. $2 \cdot 10^{2x} - 11 \cdot 10^x + 12 = 0$

120. $9 \cdot 10^{2x} - 17 \cdot 10^x + 8 = 0$

121. $\log_3 8 - \log_3 (x + 5) = 2$

122. $\log_2 (x + 3) - \log_2 (x + 2) = 1$

123. $\log_2 (h - 1) + \log_2 (h + 1) = 3$

124. $\log_3 k + \log_3 (2k + 3) = 2$

125. $\log (x + 2) = \log (3x - 6)$

126. $\log x = \log (1 - x)$

127. $\ln (x + 5) - \ln x = \ln (4x)$

128. $\log (6y - 7) + \log y = \log 5$

129. $\log (4m) = \log 2 + \log (m - 3)$

130. $\log (-h) + \log 3 = \log (2h - 15)$

Write About It

131. Explain the process to solve the equation $4^x = 11$.

132. Explain the process to solve the equation $\log_b 5 + \log_b (x - 3) = 4$.

Expanding Your Skills

For Exercises 133–152, solve the equation.

133. $\dfrac{10^x - 13 \cdot 10^{-x}}{3} = 4$

134. $\dfrac{e^x - 9e^{-x}}{2} = 4$

135. $(\ln x)^2 - \ln x^5 = -4$

136. $(\ln x)^2 + \ln x^3 = -2$

137. $(\log x)^2 = \log x^2$

138. $(\log x)^2 = \log x^3$

139. $\log w + 4\sqrt{\log w} - 12 = 0$

140. $\ln x + 3\sqrt{\ln x} - 10 = 0$

141. $e^{2x} - 8e^x + 6 = 0$

142. $e^{2x} - 6e^x + 4 = 0$

143. $\log_5 \sqrt{6c + 5} + \log_5 \sqrt{c} = 1$

144. $\log_3 \sqrt{x - 8} + \log_3 \sqrt{x} = 1$

145. $\log w + 4\sqrt{\log w} - 12 = 0$

146. $\ln x + 3\sqrt{\ln x} - 10 = 0$

147. $\dfrac{e^x + e^{-x}}{2} = 4$

148. $\dfrac{e^x + e^{-x}}{2} = 3$

149. $2xe^{2x} + 2e^{2x}x^2 = 0$

150. $e^{x/2} + \frac{1}{2}xe^{x/2} = 0$

151. $3x^2e^{-x} - e^{-x}x^3 = 0$

152. $8xe^{-2x} - 8x^2e^{-2x} = 0$

Technology Connections

For Exercises 153–156, an equation is given in the form $Y_1(x) = Y_2(x)$. Graph Y_1 and Y_2 on a graphing utility on the window [10, 10, 1] by [10, 10, 1]. Then approximate the point(s) of intersection to approximate the solution(s) to the equation. Round to 4 decimal places.

153. $4x - e^x + 6 = 0$

154. $x^3 - e^{2x} + 4 = 0$

155. $x^2 + 5 \log x = 6$

156. $x^2 - 0.05 \ln x = 4$

GO ONLINE Go online for more practice problems.

Lesson 3-6

Modeling with Exponential and Logarithmic Functions

What Will You Learn?

After completing this lesson, you should be able to:

- Solve Literal Equations for a Specified Variable
- Create Models for Exponential Growth and Decay
- Apply Logistic Growth Models
- Create Exponential and Logarithmic Models Using Regression

Learn: Solve Literal Equations for a Specified Variable

A short-term model to predict the U.S. population P is $P = 335e^{0.00698t}$, where t is the number of years since 2021. If we solve this equation for t, we have

$$t = \frac{\ln\left(\frac{P}{335}\right)}{0.00698} \quad \text{or equivalently} \quad t = \frac{\ln P - \ln 335}{0.00698}.$$

This is a model that predicts the time required for the U.S. population to reach a value P. Manipulating an equation for a specified variable was first introduced in Lesson R.5. In Example 1, we revisit this skill using exponential and logarithmic equations.

Example 1: Solving an Equation for a Specified Variable

a. Given $P = 100e^{kx} - 100$, solve for x. (Used in geology)

b. Given $L = 8.8 + 5.1 \log D$, solve for D. (Used in astronomy)

Solution:

a. $P = 100e^{kx} - 100$

$P + 100 = 100e^{kx}$ Add 100 to both sides to isolate the x term.

$\frac{P + 100}{100} = e^{kx}$ Divide by 100.

$\ln\left(\frac{P + 100}{100}\right) = \ln e^{kx}$ Take the natural logarithm of both sides.

$\ln\left(\frac{P + 100}{100}\right) = kx$ Simplify: $\ln e^{kx} = kx$

$x = \frac{\ln\left(\frac{P + 100}{100}\right)}{k}$ Divide by k.

$x = \frac{\ln\left(\frac{P + 100}{100}\right)}{k}$ or equivalently $x = \frac{\ln(P + 100) - \ln 100}{k}$

b. $L = 8.8 + 5.1 \log D$

$\frac{L - 8.8}{5.1} = \log D$ Subtract 8.8 from both sides and divide by 5.1.

$D = 10^{(L-8.8)/5.1}$ Write the equation in exponential form.

Apply the Skills

1. **a.** Given $T = 78 + 272e^{-kt}$, solve for k.

   b. Given $S = 90 - 20\ \ln(t + 1)$, solve for t.

Learn: Create Models for Exponential Growth and Decay

In Lesson 3-2, we defined an exponential function as $y = b^x$, where $b > 0$ and $b \neq 1$. Throughout the chapter, we have used transformations of basic exponential functions to solve a variety of applications. The following variation of the general exponential form is used to solve applications involving exponential growth and decay.

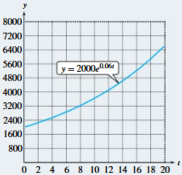
Exponential Growth and Decay Models

Let y be a variable changing exponentially with respect to t, and let y_0 represent the initial value of y when $t = 0$. Then for a constant k:

- If $k > 0$, then $y = y_0e^{kt}$ is a model for exponential growth.

  Example:

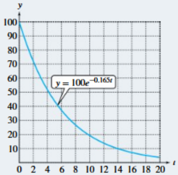
  $y = 2000e^{0.06t}$ represents the value of a \$2000 investment after t years with interest compounded continuously. (*Note*: $k = 0.06 > 0$)



- If $k < 0$, then $y = y_0e^{kt}$ is a model for exponential decay.

  Example:

  $y = 100e^{-0.165t}$ represents the radioactivity level t hours after a patient is treated for thyroid cancer with 100 mCi of radioactive iodine. (*Note*: $k = -0.165 < 0$)



The model $y = y_0e^{kt}$ is often presented with different letters or symbols in place of y, y_0, k, and t to convey their meaning in the context of the application. For example, to compute the value of an investment under continuous compounding, we have

$A = Pe^{rt}$

P (for principal) is used in place of y_0.
r (for the annual interest rate) is used in place of k.
A (for the future value of the investment) is used in place of y.

We can also use function notation when expressing a model for exponential growth or decay. For example, consider the model for population growth.

$P(t) = P_0e^{kt}$

P_0 (for initial population) is used in place of y_0.
$P(t)$ represents the population as a function of time and is used in place of y.

Example 2: Creating a Model for Growth of an Investment

Suppose that \$15,000 is invested and at the end of 3 yr, the value of the account is \$19,356.92. Use the model $A = Pe^{rt}$ to determine the average rate of return r under continuous compounding.

Solution:

$A = Pe^{rt}$	Begin with an appropriate model.
$A = 15{,}000e^{rt}$	P represents the initial value of the account (initial principal). Substitute 15,000 for P.
$19{,}356.92 = 15{,}000e^{r(3)}$	We have a known data point where $A = 19{,}356.92$ when $t = 3$. Substituting these values into the formula enables us to solve for r.
$\dfrac{19{,}356.92}{15{,}000} = e^{3r}$	Divide both sides by 15,000.
$\ln\left(\dfrac{19{,}356.92}{15{,}000}\right) = \ln(e^{3r})$	Take the natural logarithm of both sides.
$\ln\left(\dfrac{19{,}356.92}{15{,}000}\right) = 3r$	Simplify: $\ln e^{3r} = 3r$
$r = \dfrac{\ln\left(\dfrac{19{,}356.92}{15{,}000}\right)}{3}$	Divide by 3 to isolate r.
$r \approx 0.085$	

The average rate of return is approximately 8.5%.

Apply the Skills

2. Suppose that \$10,000 is invested and at the end of 5 yr, the value of the account is \$13,771.28. Use the model $A = Pe^{rt}$ to determine the average rate of return r under continuous compounding.

In Example 3, we build a model to predict short-term population growth.

Example 3: Creating a Model for Population Growth

On January 1, 2010, the population of California was approximately 37.3 million. On January 1, 2019, the population was 40.0 million. Let $t = 0$ represent the year 2010.

a. Write a function defined by $P(t) = P_0e^{kt}$ to represent the population of California $P(t)$ (in millions), t years after 2010.

b. Use the function from part (a) to predict the population in 2025. Round to 1 decimal place.

c. Use the function from part (a) to determine the year for which the population of California will be twice that of the year 2010.

Solution:

a.

$P(t) = P_0e^{kt}$	Begin with an appropriate model.
$P(t) = 37.3e^{kt}$	The initial population is $P_0 = 37.3$ million.
$40.0 = 37.3e^{k(9)}$	We have a known data point $P(9) = 40.0$. Substituting these values into the function enables us to solve for k.
$\frac{40.0}{37.3} = e^{9k}$	Divide both sides by 37.3.
$\ln\left(\frac{40.0}{37.3}\right) = \ln(e^{9k})$	Take the natural logarithm of both sides.
$\ln\left(\frac{40.0}{37.3}\right) = 9k$	
$k = \frac{\ln\left(\frac{40.0}{37.3}\right)}{9} \approx 0.00777$	Divide both sides by 9 to isolate k.
$P(t) = 37.3e^{0.00777t}$	This model gives the population as a function of time.

b.

$P(t) = 37.3e^{0.00777t}$	
$P(15) = 37.3e^{0.00777(15)}$	The year 2025 is 15 years after 2010. Substitute 15 for t.
≈ 41.9	

The population in California in 2025 will be approximately 41.9 million if this trend continues.

c.

$P(t) = 37.3e^{0.00777t}$	Two times California's population of 2010 is 2(37.3 million) which is 74.6 million.
$74.6 = 37.3e^{0.00777t}$	Substitute 74.6 for $P(t)$.
$\frac{74.6}{37.3} = e^{0.00777t}$	Divide both sides by 37.3.
$2 = e^{0.00777t}$	
$\ln 2 = 0.00777t$	Take the natural logarithm of both sides.
$t = \frac{\ln 2}{0.00777} \approx 89.2$	Divide both sides by 0.00777 to isolate t.

The population of California will reach 74.6 million in the year 2099 if this trend continues.

Insights

The value of k in the model $P(t) = P_0e^{kt}$ is called a parameter and is related to the growth rate of the population being studied. The value of k will be different for different populations.

Apply the Skills

3. On January 1, 2010, the population of Texas was 25.2 million. On January 1, 2019, the population was 29.1 million. Let $t = 0$ represent the year 2010.

- **a.** Write a function defined by $P(t) = P_0e^{kt}$ to represent the population $P(t)$ of Texas t years after 2010.
- **b.** Use the function in part (a) to predict the population in 2029. Round to 1 decimal place.
- **c.** Use the function in part (a) to determine the year for which the population of Texas will reach 40 million if this trend continues.

An exponential model can be presented with a base other than base e. For example, suppose that a culture of bacteria begins with 5000 organisms and the population doubles every 4 hr. Then the population $P(t)$ can be modeled by

$P(t) = 5000(2)^{t/4}$, where t is the time in hours after the culture was started.

Notice that this function is defined using base 2. It is important to realize that any exponential function of one base can be rewritten in terms of an exponential function of another base. In particular we are interested in expressing the function with base e.

Writing an Exponential Expression Using Base e

Let t and b be real numbers, where $b > 0$ and $b \neq 1$. Then,

$$b^t \text{ is equivalent to } e^{(\ln b)t}.$$

To show that $e^{(\ln b)t} = b^t$, use the power property of exponents; that is,

$$e^{(\ln b)t} = (e^{\ln b})^t = b^t$$

Example 4: Writing an Exponential Function with Base *e*

- **a.** The population $P(t)$ of a culture of bacteria is given by $P(t) = 5000(2)^{t/4}$, where t is the time in hours after the culture was started. Write the rule for this function using base e.
- **b.** Find the population after 12 hr using both forms of the function from part (a).

Solution:

a. $P(t) = 5000(2)^{t/4}$

Note that $2^{t/4} = (2^t)^{1/4}$

$= [e^{(\ln 2)t}]^{1/4}$ Apply the property that $e^{(\ln b)t} = b^t$.

$= e^{[(\ln 2)/4]t}$ Apply the power rule of exponents.

Therefore, $P(t) = 5000(2)^{t/4}$

$= 5000e^{[(\ln 2)/4]t}$

$\approx 5000e^{0.17329t}$

b. $P(t) = 5000(2)^{t/4}$ $P(t) \approx 5000e^{0.17329t}$

$P(12) = 5000(2)^{(12)/4}$ $P(12) \approx 5000e^{0.17329(12)}$

$= 40{,}000$ $\approx 40{,}000$

Apply the Skills

4. **a.** Given $P(t) = 10{,}000(2)^{-0.4t}$, write the rule for this function using base e.

b. Find the function value for $t = 10$ for both forms of the function from part (a).

In Example 5, we apply an exponential decay function to determine the age of a bone through radiocarbon dating. Animals ingest carbon through respiration and through the food they eat. Most of the carbon is carbon-12 (^{12}C), an abundant and stable form of carbon. However, a small percentage of carbon is the radioactive isotope, carbon-14 (^{14}C). The ratio of carbon-12 to carbon-14 is constant for all living things. When an organism dies, it no longer takes in carbon from the environment. Therefore, as the carbon-14 decays, the ratio of carbon-12 to carbon-14 changes. Scientists know that the half-life of ^{14}C is 5730 years and from this, they can build a model to represent the amount of ^{14}C remaining t years after death. This is illustrated in Example 5.

Example 5: Creating a Model for Exponential Decay

a. Carbon-14 has a half-life of 5730 yr. Write a model of the form $Q(t) = Q_0e^{-kt}$ to represent the amount $Q(t)$ of carbon-14 remaining after t years if no additional carbon is ingested.

b. An archeologist uncovers human remains at an ancient Roman burial site and finds that 76.6% of the carbon-14 still remains in the bone. How old is the bone? Round to the nearest hundred years.

Solution:

a. $Q(t) = Q_0e^{-kt}$ Begin with a general exponential decay model.

$0.5Q_0 = Q_0e^{-k(5730)}$ Substitute the known data value. One-half of the original quantity Q_0 is present after 5730 yr.

$0.5 = e^{-k(5730)}$ Divide by Q_0 on both sides.

$\ln 0.5 = -5730k$ Take the natural logarithm of both sides.

$k = \frac{\ln 0.5}{-5730}$ Divide by –5730.

≈ 0.000121

$Q(t) = Q_0e^{-0.000121t}$

b. $0.766Q_0 = Q_0e^{-0.000121t}$ The quantity $Q(t)$ of carbon-14 in the bone is 76.6% of Q_0.

$0.766 = e^{-0.000121t}$ Divide by Q_0 on both sides.

$\ln 0.766 = -0.000121t$ Take the natural logarithm of both sides.

$t = \frac{\ln 0.766}{-0.000121} \approx 2200$ Divide by –0.000121 to isolate t.

The bone is approximately 2200 years old.

Apply the Skills

5. Use the function $Q(t) = Q_0e^{-0.000121t}$ to determine the age of a piece of wood that has 42% of its carbon-14 remaining. Round to the nearest 10 yr.

Insights

Given the half-life of a radioactive substance, we can also write an exponential model using base $\frac{1}{2}$. The format is

$$Q(t) = Q_0\left(\frac{1}{2}\right)^{t/h}$$

where h is the half-life of the substance.

In Example 5, we have

$$Q(t) = Q_0\left(\frac{1}{2}\right)^{t/5730}$$

Learn: Apply Logistic Growth Models

In Examples 3 and 4, we used a model of the form $P(t) = P_0e^{kt}$ to predict population as an exponential function of time. However, unbounded population growth is not possible due to limited resources. A growth model that addresses this problem is called logistic growth. In particular, a logistic growth model imposes a limiting value on the dependent variable.

Logistic Growth Model

A logistic growth model is a function written in the form

$$y = \frac{c}{1 + ae^{-bt}}$$

where a, b, and c are positive constants.

The general logistic growth equation can be written with a complex fraction.

$$y = \frac{c}{1 + \frac{a}{e^{bt}}}$$

This term approaches 0 as t approaches ∞.

In this form, we can see that for large values of t, the term $\frac{a}{e^{bt}}$ approaches 0, and the function value y approaches $\frac{c}{1}$.

The line $y = c$ is a horizontal asymptote of the graph, and c represents the limiting value of the function (Figure 3-21).

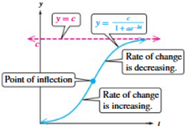


Figure 3-21

Notice that the graph of a logistic curve is increasing over its entire domain. However, the *rate* of increase begins to decrease as the function levels off and approaches the horizontal asymptote $y = c$.

Insights

The rate of increase of a logistic curve changes from increasing to decreasing to the left and right of a point called the *point of inflection*.

In Example 3 we created a function to approximate the population of California assuming unlimited growth. In Example 6, we use a logistic growth model.

Example 6: Using Logistic Growth to Model Population

The population of California $P(t)$ (in millions) can be approximated by the logistic growth function

$P(t) = \frac{93.3}{1 + 1.5e^{-0.013t}}$ where t is the number of years since the year 2010.

a. Determine the population in the year 2010.

b. Use this function to determine the time required for the population of California to double from its value of 37.3 million in 2010. Compare this with the result from Example 3(c).

c. What is the limiting value of the population of California under this model?

Solution:

a. $$P(t) = \frac{93.3}{1 + 1.5e^{-0.013t}}$$

$$P(0) = \frac{93.3}{1 + 1.5e^{-0.013(0)}}$$

To find the initial population of California with this model substitute 0 for t.

$$= \frac{93.3}{1 + 1.5e^{0}}$$

Recall that $e^0 = 1$.

$$= \frac{93.3}{1 + 1.5(1)} \approx 37.3$$

The population was approximately 37.3 million in the year 2010.

b. $P(t) = \dfrac{93.3}{1 + 1.5e^{-0.013t}}$

Two times California's population of 2010 is 2(37.3 million) which is 74.6 million.

$$74.6 = \frac{93.3}{1 + 1.5e^{-0.013t}}$$

Substitute 74.6 for $P(t)$.

$$74.6(1 + 1.5e^{-0.013t}) = 93.3$$

Multiply both sides by $(1 + 1.5e^{-0.013t})$ to clear fractions.

$$\frac{74.6(1 + 1.5e^{-0.013t})}{74.6} = \frac{93.3}{74.6}$$

Divide by 74.6 on both sides.

$$1 + 1.5e^{-0.013t} \approx 1.25$$

$$1.5e^{-0.013t} \approx 0.25$$

Subtract 1 from both sides.

$$e^{-0.013t} \approx \frac{0.25}{1.5}$$

Divide by 1.5 on both sides.

$$\ln(e^{-0.013t}) \approx \ln\left(\frac{0.25}{1.5}\right)$$

Take the natural logarithm of both sides.

$$-0.013t \approx \ln\left(\frac{0.25}{1.5}\right)$$

$$t \approx \frac{\ln\left(\frac{0.25}{1.5}\right)}{-0.013} \approx 137.8$$

Divide by -0.013 on both sides.

The population will double in approximately 138 years. This is 49 years later than the predicted value from Example 3(c).

The graphs of $P(t) = \dfrac{93.3}{1 + 1.5e^{-0.013t}}$ and $P(t) = 37.3e^{0.00777t}$ are shown in Figure 3-22.

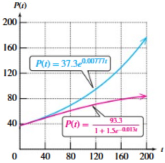


Figure 3-22

Notice that the two models agree closely for short-term population growth (out to ≈ 2040). However, in the long term, the unbounded exponential model breaks down. The logistic growth model approaches a limiting population which is reasonable due to the limited resources to sustain a large human population.

c. $P(t) = \frac{93.3}{1 + 1.5e^{-0.013t}} = \frac{93.3}{1 + \frac{1.5}{e^{0.013t}}}$

As $t \to \infty$, the term $\frac{1.5}{e^{0.013t}} \to 0$.

As *t* becomes large, the denominator of $\frac{1.5}{e^{0.013t}}$ also becomes large. This causes the quotient to approach zero. Therefore, as *t* approaches infinity, *P*(*t*) approaches 93.3. Under this model, the limiting value for the population of California is 93.3 million.

Apply the Skills

6. The score on a test of dexterity is given by $P(t) = \frac{100}{1 + 19e^{-0.354x}}$, where *x* is the number of times the test is taken.

a. Use the function to determine the minimum number of times required for the score to exceed 90.

b. What is the limiting value of the scores?

Learn: Create Exponential and Logarithmic Models Using Regression

In Examples 7 and 8, we use a graphing utility and regression techniques to find an exponential model or logarithmic model based on observed data.

Example 7: Creating an Exponential Model from Observed Data

The amount of sunlight *y* [in langleys (Ly)—a unit used to measure solar energy in calories/cm^2] is measured for six different depths *x* (in meters) in Lake Lyndon B. Johnson in Texas.

x (m)	1	3	5	7	9	11
y (Ly)	300	161	89	50	27	15

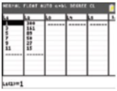
a. Graph the data.

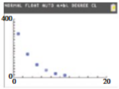
b. From visual inspection of the graph, which model would best represent the data? Choose from $y = mx + b$ (linear), $y = ab^x$ (exponential), or $y = a + b \ln x$ (logarithmic).

c. Use a graphing utility to find a regression equation that fits the data.

Solution:

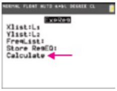
a. Enter the data in two lists.



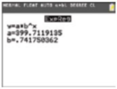


b. Note that for large depths, the amount of sunlight approaches 0. Therefore, the curve is asymptotic to the *x*-axis. This is consistent with a decreasing exponential model. The exponential model $y = ab^x$ appears to fit.

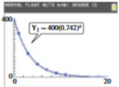
c. Under the STAT menu, choose CALC, ExpReg, and then Calculate.



```
ExpReg
Xlist:L1
Ylist:L2
FreqList:
Store RegEQ:
Calculate
```



```
ExpReg
y=a*b^x
a=399.7119135
b=.741750362
```



The equation $y = ab^x$ is $y = 400(0.742)^x$.

Apply the Skills

7. For the given data,

x	1	3	5	7	9	11
y	2.9	5.6	11.1	22.4	43.0	85.0

a. Graph the data points.

b. Use a graphing utility to find a model of the form $y = ab^x$ to fit the data.

Example 8: Creating a Logarithmic Model from Observed Data

The diameter x (in mm) of a sugar maple tree, along with the corresponding age y (in yr) of the tree is given for six different trees.

x (mm)	1	50	100	200	300	400
y (yr)	4	60	72	82	89	94

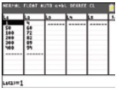
a. Graph the data.

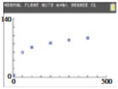
b. From visual inspection of the graph, which model would best represent the data? Choose from $y = mx + b$ (linear), $y = ab^x$ (exponential), or $y = a + b \ln x$ (logarithmic).

c. Use a graphing utility to find a regression equation that fits the data.

Solution:

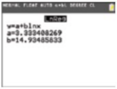
a. Enter the data into two lists.



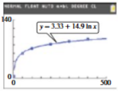


b. By inspection of the graph, the logarithmic model $y = a + b \ln x$ appears to fit.

c. Under the STAT menu, choose CALC, and then LnReg.



```
LnReg
y=a+blnx
a=3.333408269
b=14.93485833
```



Apply the Skills

8. For the given data,

x	1	5	9	13	17	21
y	11.9	19.3	21.9	23.5	24.7	25.7

a. Graph the data points.

b. Use a graphing utility to find a model of the form $y = a + b \ln x$ to fit the data.

Practice 3-6

Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, solve for the indicated variable.

R.1. $P = \frac{1}{3}Lm$ for L

R.2. $Q = \frac{3}{4}m^3$ for $m, m > 0$

R.3. $9 + \sqrt{a^2 - b^2} = c$ for b

R.4. Given $f(x) = 5x^2 + 2x$, evaluate

a. $f(-2)$ **b.** $f(-1)$ **c.** $f(0)$ **d.** $f(1)$

Concept Connections

1. If $k > 0$, the equation $y = y_0e^{kt}$ is a model for exponential (growth/decay), whereas if $k < 0$, the equation is a model for exponential (growth/decay).

2. A function defined by $y = ab^x$ can be written in terms of an exponential function base e as ______________.

3. A function defined by $y = \frac{c}{1 + ae^{-bt}}$ is called a ______________ growth model and imposes a limiting value on y.

4. Given a logistic growth function $y = \frac{c}{1 + ae^{-bt}}$, the limiting value of y is ______________.

Learn: Solve Literal Equations for a Specified Variable

For Exercises 5–14, solve for the indicated variable. (See Example 1, p. 416)

5. $Q = Q_0e^{-kt}$ for k (used in chemistry)

6. $N = N_0e^{-0.025t}$ for t (used in chemistry)

7. $M = 8.8 + 5.1 \log D$ for D (used in astronomy)

8. $\log E - 12.2 = 1.44\ M$ for E (used in geology)

9. $\text{pH} = -\log[\text{H}^+]$ for $[\text{H}^+]$ (used in chemistry)

10. $L = 10\log\left(\frac{I}{I_0}\right)$ for I (used in medicine)

11. $A = P(1 + r)^t$ for t (used in finance)

12. $A = Pe^{rt}$ for r (used in finance)

13. $\ln\left(\frac{k}{A}\right) = \frac{-E}{RT}$ for k (used in chemistry)

14. $-\frac{1}{k}\ln\left(\frac{P}{14.7}\right) = A$ for P (used in meteorology)

Learn: Create Models for Exponential Growth and Decay

15. Suppose that $12,000 is invested in a bond fund and the account grows to $14,309.26 in 4 yr. (**See Example 2, p. 418**)

a. Use the model $A = Pe^{rt}$ to determine the average rate of return under continuous compounding. Round to the nearest tenth of a percent.

b. How long will it take the investment to reach $20,000 if the rate of return continues? Round to the nearest tenth of a year.

16. Suppose that $50,000 from a retirement account is invested in a large cap stock fund. After 20 yr, the value is $194,809.67.

a. Use the model $A = Pe^{rt}$ to determine the average rate of return under continuous compounding. Round to the nearest tenth of a percent.

b. How long will it take the investment to reach one-quarter million dollars? Round to the nearest tenth of a year.

17. Suppose that P dollars in principal is invested in an account earning 3.2% interest compounded continuously. At the end of 3 yr, the amount in the account has earned $806.07 in interest.

a. Find the original principal. Round to the nearest dollar. (*Hint:* Use the model $A = Pe^{rt}$ and substitute $P + 806.07$ for A.)

b. Using the original principal from part (a) and the model $A = Pe^{rt}$, determine the time required for the investment to reach $10,000. Round to the nearest year.

18. Suppose that P dollars in principal is invested in an account earning 2.1% interest compounded continuously. At the end of 2 yr, the amount in the account has earned \$193.03 in interest.

a. Find the original principal. Round to the nearest dollar. (*Hint:* Use the model $A = Pe^{rt}$ and substitute $P + 193.03$ for A.)

b. Using the original principal from part (a) and the model $A = Pe^{rt}$, determine the time required for the investment to reach \$6000. Round to the nearest tenth of a year.

19. An initial census taken in a small Asian country found the population to be 22.9 million. Ten years later the population had increased to 23.7 million. During the same time period, a census was taken in a South Pacific nation. The initial census found a population of 19.0 million. Ten years later, the population had increased to 22.6 million. **(See Example 3, p. 419)**

Country	Population (in millions) when $t = 0$	Population (in millions) when $t = 10$	$P(t) = P_0e^{kt}$
Asian			
South Pacific			

a. Write a function of the form $P(t) = P_0e^{kt}$ to model the population $P(t)$ (in millions) t years after the initial census for the Asian country.

b. Write a function of the form $P(t) = P_0e^{kt}$ to model the population $P(t)$ (in millions) t years after the initial census for the South Pacific country.

c. Use the models from parts (a) and (b) to predict the population 20 years after the initial census was taken for each country. Round to the nearest hundred thousand.

d. The South Pacific country had fewer people than the Asian country in the initial census, yet from the result of part (c), the South Pacific country will have more people 20 years later. Why?

e. Assuming that the population growth trend continues in each country, use the models from parts (a) and (b) to predict the number of years for the population to reach 30 million in each country. Round to the nearest year.

20. An initial census taken in a Scandinavian country found the population to be 7.3 million. Ten years later, the population had increased to 7.8 million. During the same time period, a census was taken in a Middle Eastern nation. The initial census found a population of 6.7 million. Ten years later, the population had increased to 7.7 million.

Country	Population (in millions) when $t = 0$	Population (in millions) when $t = 10$	$P(t) = P_0e^{kt}$
Scandinavian			
Middle Eastern			

a. Write a function of the form $P(t) = P_0e^{kt}$ to model the population $P(t)$ (in millions) t years after the initial census for the Scandinavian country.

b. Write a function of the form $P(t) = P_0e^{kt}$ to model the population $P(t)$ (in millions) t years after the initial census for the Middle Eastern country.

c. Use the models from parts (a) and (b) to predict the population 20 years after the initial census was taken for each country. Round to the nearest hundred thousand.

d. The Middle Eastern country had fewer people than the Scandinavian country in the initial census, yet from the result of part (c), the Middle Eastern country will have more people 20 years later. Why?

e. Assuming that the population growth trend continues in each country, use the models from parts (a) and (b) to predict the number of years for the population to reach 10 million in each country. Round to the nearest year.

21. The table gives two functions of the form $P(t) = ab^t$ that represent the populations of a Central American country and a Northern European country, t years after a baseline census is taken. **(See Example 4, p. 420)**

a. For each function given in the table, write an equivalent function using base e. That is, write a function of the form $P(t) = P_0e^{kt}$. Also, determine the population of each country in the baseline census.

Country	$P(t) = ab^t$	$P(t) = P_0e^{kt}$	Baseline Population
Central American	$P(t) = 4.3(1.0135)^t$		
Northern European	$P(t) = 4.6(1.0062)^t$		

b. The populations of the two given countries are very close for the initial census, but their growth rates are different. Use the model to approximate the number of years for the population of each country to reach 5 million. Round to the nearest year.

c. The Central American nation had fewer people than the Northern European country according to the initial census. Why will the Central American country reach a population of 5 million sooner than the Northern European country?

22. The table gives two functions of the form $P(t) = ab^t$ that represent the populations of a Caribbean nation and a small European country, t years after a baseline census is taken.

a. For each function given in the table, write an equivalent function using base e. That is, write a function of the form $P(t) = P_0e^{kt}$. Also, determine the population of each country in the baseline census.

Country	$P(t) = ab^t$	$P(t) = P_0e^{kt}$	Baseline Population
Caribbean	$P(t) = 8.5(1.0158)^t$		
European	$P(t) = 9.0(1.0048)^t$		

b. The populations of the two given countries are very close for the initial census, but their growth rates are different. Use the model to approximate the number of years for the population of each country to reach 10 million. Round to the nearest year.

c. The Caribbean nation had fewer people in the initial census than the European country. Why did the Caribbean nation reach a population of 10 million sooner than the European country?

For Exercises 23–24, refer to the model $Q(t) = Q_0e^{-0.000121t}$ used in Example 5 for radiocarbon dating.

23. A sample from a mummified bull was taken from a pyramid in Dashur, Egypt. The sample shows that 78% of the carbon-14 still remains. How old is the sample? Round to the nearest year. (**See Example 5, p. 421**)

24. At the "Marmes Man" archeological site in southeastern Washington State, scientists uncovered the oldest human remains yet to be found in Washington State. A sample from a human bone taken from the site showed that 29.4% of the carbon-14 still remained. How old is the sample? Round to the nearest year.

25. The isotope of plutonium ^{238}Pu is used to make thermoelectric power sources for spacecraft. Suppose that a space probe was launched in 2012 with 2.0 kg of ^{238}Pu.

a. If the half-life of ^{238}Pu is 87.7 yr, write a function of the form $Q(t) = Q_0e^{-kt}$ to model the quantity $Q(t)$ of ^{238}Pu left after t years.

b. If 1.6 kg of ^{238}Pu is required to power the spacecraft's data transmitter, for how long after launch would scientists be able to receive data? Round to the nearest year.

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26. Technetium-99 (^{99m}Tc) is a radionuclide used widely in nuclear medicine. ^{99m}Tc is combined with another substance that is readily absorbed by a targeted body organ. Then, special cameras sensitive to the gamma rays emitted by the technetium are used to record pictures of the organ. Suppose that a technician prepares a sample of ^{99m}Tc-pyrophosphate to image the heart of a patient suspected of having had a mild heart attack.

a. At noon, the patient is given 10 mCi (millicuries) of ^{99m}Tc. If the half-life of ^{99m}Tc is 6 hr, write a function of the form $Q(t) = Q_0e^{-kt}$ to model the radioactivity level $Q(t)$ after t hours.

b. At what time will the level of radioactivity reach 3 mCi? Round to the nearest tenth of an hour.

27. Fluorodeoxyglucose is a derivative of glucose that contains the radionuclide fluorine-18 (^{18}F). A patient is given a sample of this material containing 300 MBq of ^{18}F (a megabecquerel is a unit of radioactivity). The patient then undergoes a PET scan (positron emission tomography) to detect areas of metabolic activity indicative of cancer. After 174 min, one-third of the original dose remains in the body.

a. Write a function of the form $Q(t) = Q_0e^{-kt}$ to model the radioactivity level $Q(t)$ of fluorine-18 at a time t minutes after the initial dose.

b. What is the half-life of ^{18}F? Round to the nearest minute.

28. Painful bone metastases are common in advanced prostate cancer. Physicians often order treatment with strontium-89 (^{89}Sr), a radionuclide with a strong affinity for bone tissue. A patient is given a sample containing 4 mCi of ^{89}Sr.

a. If 20% of the ^{89}Sr remains in the body after 90 days, write a function of the form $Q(t) = Q_0e^{-kt}$ to model the amount $Q(t)$ of radioactivity in the body t days after the initial dose.

b. What is the biological half-life of ^{89}Sr under this treatment? Round to the nearest tenth of a day.

29. Two million *E. coli* bacteria are present in a laboratory culture. An antibacterial agent is introduced and the population of bacteria $P(t)$ decreases by half every 6 hr. The population can be represented by $P(t) = 2{,}000{,}000\left(\frac{1}{2}\right)^{t/6}$.

a. Convert this to an exponential function using base e.

b. Verify that the original function and the result from part (a) yield the same result for $P(0)$, $P(6)$, $P(12)$, and $P(60)$. (*Note:* There may be round-off error.)

30. The half-life of radium-226 is 1620 yr. Given a sample of 1 g of radium-226, the quantity left $Q(t)$ (in g) after t years is given by

$$Q(t) = \left(\frac{1}{2}\right)^{t/1620}.$$

a. Convert this to an exponential function using base e.

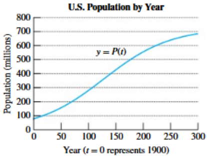
b. Verify that the original function and the result from part (a) yield the same result for $Q(0)$, $Q(1620)$, and $Q(3240)$. (*Note:* There may be round-off error.)

Learn: Apply Logistic Growth Models

31. The population of the United States $P(t)$ (in millions) since January 1, 1900, can be approximated by

$$P(t) = \frac{725}{1 + 8.295e^{-0.0165t}}$$

where t is the number of years since January 1, 1900. (**See Example 6, p. 423**)



a. Evaluate $P(0)$ and interpret its meaning in the context of this problem.

b. Use the function to approximate the U.S. population on January 1, 2020. Round to the nearest million.

c. Use the function to approximate the U.S. population on January 1, 2050.

d. From the model, during which year would the U.S. population reach 500 million?

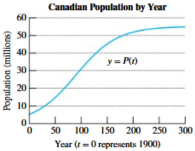
e. What value will the term $\frac{8.295}{e^{0.0165t}}$ approach as $t \to \infty$?

f. Determine the limiting value of $P(t)$.

32. The population of Canada $P(t)$ (in millions) since January 1, 1900, can be approximated by

$$P(t) = \frac{55.1}{1 + 9.6e^{-0.02515t}}$$

where t is the number of years since January 1, 1900.



a. Evaluate $P(0)$ and interpret its meaning in the context of this problem.

b. Use the function to approximate the Canadian population on January 1, 2015. Round to the nearest tenth of a million.

c. Use the function to approximate the Canadian population on January 1, 2040.

d. From the model, during which year would the Canadian population reach 45 million?

e. What value will the term $\frac{9.6}{e^{0.02515t}}$ approach as $t \to \infty$?

f. Determine the limiting value of $P(t)$.

33. The number of computers $N(t)$ (in millions) infected by a computer virus can be approximated by

$$N(t) = \frac{2.4}{1 + 15e^{-0.72t}}$$

where t is the time in months after the virus was first detected.

a. Determine the number of computers initially infected when the virus was first detected.

b. How many computers were infected after 6 months? Round to the nearest hundred thousand.

c. Determine the amount of time required after initial detection for the virus to affect 1 million computers. Round to the nearest tenth of a month.

d. What is the limiting value of the number of computers infected according to this model?

34. After a new product is launched, the cumulative sales $S(t)$ (in \$1000) t weeks after launch is given by

$$S(t) = \frac{72}{1 + 9e^{-0.36t}}$$

a. Determine the cumulative amount in sales 3 weeks after launch. Round to the nearest thousand.

b. Determine the amount of time required for the cumulative sales to reach \$70,000.

c. What is the limiting value in sales?

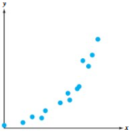
Learn: Create Exponential and Logarithmic Models Using Regression

For Exercises 35–38, a graph of data is given. From visual inspection, which model would best fit the data? Choose from

$y = mx + b$ (linear) $y = ab^x$ (exponential)

$y = a + b\ln x$ (logarithmic) $y = \frac{c}{1 + ae^{-bx}}$ (logistic)

35.



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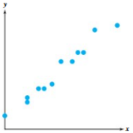

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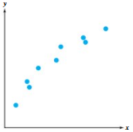

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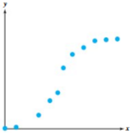


For Exercises 39–46, a table of data is given.

a. Graph the points and from visual inspection, select the model that would best fit the data. Choose from

$y = mx + b$ (linear) $y = ab^x$ (exponential)

$y = a + b\ln x$ (logarithmic) $y = \frac{c}{1 + ae^{-bx}}$ (logistic)

b. Use a graphing utility to find a function that fits the data. (*Hint:* For a logistic model, go to STAT, CALC, Logistic.)

39.

x	y
0	2.3
4	3.6
8	5.7
12	9.1
16	14
20	22

40.

x	y
0	52
1	67
2	87
3	114
4	147
5	195

41.

x	y
3	2.7
7	12.2
13	25.7
15	30
17	34
21	44.4

42.

x	y
0	640
20	530
40	430
50	360
80	210
100	90

43.

x	y
10	43.3
20	50
30	53
40	56.8
50	58.8
60	60.8

44.

x	y
5	29
10	40
15	45.6
20	50
25	53.3
30	56

45.

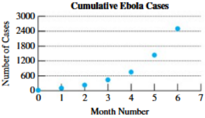
x	y
2	0.326
4	2.57
6	10.8
8	16.8
10	17.9
5	6
7	14.8

46.

x	y
0	0.05
2	0.45
4	2.94
5	5.8
6	8.8
7	10.6
8	11.5
10	11.9

47. During a recent outbreak of Ebola in western Africa, the cumulative number of cases y was reported t months after April 1. **(See Example 7, p. 425)**

Month Number (t)	Cumulative Cases (y)
0	18
1	105
2	230
3	438
4	752
5	1437
6	2502



a. Use a graphing utility to find a model of the form $y = ab^t$. Round a to 1 decimal place and b to 3 decimal places.

b. Write the function from part (a) as an exponential function of the form $y = ae^{bt}$.

c. Use either model to predict the number of Ebola cases 8 months after April 1 if this trend continues. Round to the nearest thousand.

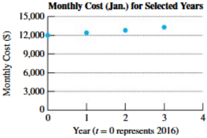
d. Would it seem reasonable for this trend to continue indefinitely?

e. Use a graphing utility to find a logistic model $y = \frac{c}{1 + ae^{-bt}}$. Round a and c to the nearest whole number and b to 2 decimal places.

f. Use the logistic model from part (e) to predict the number of Ebola cases 8 months after April 1. Round to the nearest thousand.

48. The monthly costs for a small company to do business have been increasing over time due in part to inflation. The table gives the monthly cost y (in \$) for the month of January for selected years. The variable t represents the number of years since 2016.

Year ($t = 0$ is 2016)	Monthly Costs (\$) y
0	12,000
1	12,400
2	12,800
3	13,300

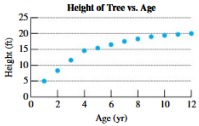


a. Use a graphing utility to find a model of the form $y = ab^t$. Round a to the nearest whole unit and b to 3 decimal places.

b. Write the function from part (a) as an exponential function with base e.

c. Use either model to predict the monthly cost for January in the year 2023 if this trend continues. Round to the nearest hundred dollars.

49. The age of a tree t (in yr) and its corresponding height $H(t)$ are given in the table. (**See Example 8, p. 428**)



Age of Tree (yr) t	Height (ft) $H(t)$
1	5
2	8.3
3	11.6
4	14.6
5	15.4
6	16.5
7	17.5
8	18.3
9	19
10	19.4
11	19.7
12	20

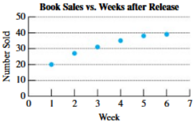
a. Write a model of the form $H(t) = a + b \ln t$. Round a and b to 2 decimal places.

b. Use the model to predict the age of a tree if it is 25 ft high. Round to the nearest year.

c. Is it reasonable to assume that this logarithmic trend will continue indefinitely? Why or why not?

50. The sales of a book tend to increase over the short term as word-of-mouth makes the book "catch on." The number of books sold $N(t)$ for a new novel t weeks after release at a certain bookstore is given in the table for the first 6 weeks.

Weeks t	Number Sold $N(t)$
1	20
2	27
3	31
4	35
5	38
6	39



a. Find a model of the form $N(t) = a + b \ln t$. Round a and b to 1 decimal place.

b. Use the model to predict the sales in week 7. Round to the nearest whole unit.

c. Is it reasonable to assume that this logarithmic trend will continue? Why or why not?

Mixed Exercises

For Exercises 51–54, solve for x.

51. $\ln(x + a) - b = c$

52. $3 \ln(ax) + b = 0, a \neq 0$

53. $\ln(2x) = 2c + 3 \ln b$ where $b > 0$

54. $2(\ln x - \ln c) = 3 \ln b$ where $b, c > 0$

55. Suppose that \$1000 is invested, and at the end of 6 years, the value of the account is \$1211.67. Use the model $A = Pe^{rt}$ to determine the average rate of return r under continuous compounding.

56. The half-life of cobalt-60 is 5.3 yr. Use the model $Q(t) = Q_{0e}^{-kt}$ to determine the time required for 20% of a 250-g sample of cobalt-60 to decay. Round to the nearest tenth of a year.

57. Suppose that a scientist wants to culture HeLa cells but uses a cool environment so that the growth rate is slowed. Initially, there are 1.2×10^6 cells, and after 24 hr, the number of cells has increased to 2.16×10^6. Use the model $P(t) = P_0e^{kt}$ to find the doubling time for HeLa cells in this cooler environment. Round to the nearest hour.

58. A lab technician starts a culture of HeLa cells. The cell bank from which the cells were obtained indicates that under optimal conditions, the doubling time of the cells is 24 hr. So, the number of cells $N(t)$ can be approximated by $N(t) = N_0 \cdot 2^{t/24}$ where t is the number of hours since the culture started and N_0 is the initial number of cells. If the technician finds that there are 45 million cells after 60 hr, what was the initial number of cells in the culture? Round to the nearest million.

59. A van is purchased new for $29,200.

- **a.** Write a linear function of the form $y = mt + b$ to represent the value y of the vehicle t years after purchase. Assume that the vehicle is depreciated by $2920 per year.
- **b.** Suppose that the vehicle is depreciated so that it holds only 80% of its value from the previous year. Write an exponential function of the form $y = V_0b^t$, where V_0 is the initial value and t is the number of years after purchase.
- **c.** To the nearest dollar, determine the value of the vehicle after 5 yr and after 10 yr using the linear model.
- **d.** To the nearest dollar, determine the value of the vehicle after 5 yr and after 10 yr using the exponential model.

60. A delivery truck is purchased new for $54,000.

- **a.** Write a linear function of the form $y = mt + b$ to represent the value y of the vehicle t years after purchase. Assume that the vehicle is depreciated by $6750 per year.
- **b.** Suppose that the vehicle is depreciated so that it holds 70% of its value from the previous year. Write an exponential function of the form $y = V_0b^t$, where V_0 is the initial value and t is the number of years after purchase.
- **c.** To the nearest dollar, determine the value of the vehicle after 4 yr and after 8 yr using the linear model.
- **d.** To the nearest dollar, determine the value of the vehicle after 4 yr and after 8 yr using the exponential model.

61. The number of new cryptocurrencies y changes day-to-day, but the approximate number of such currencies in circulation for selected years is given in the table. Year 0 represents 2015.

Year x	Number of Cryptocurrencies y
0	491
1	551
2	617
3	1334
4	2995
5	5392

- **a.** Graph the data on a graphing device.
- **b.** Based on the shape of the graph, which function type would provide the best model: linear, exponential, or logarithmic?
- **c.** Use a graphing utility to find an appropriate model, $y = f(x)$.
- **d.** Use a graphing utility to graph the function and the data points.
- **e.** Use the model to approximate the number of cryptocurrencies in year 6. Round to the nearest whole unit.

62. The approximate number of Twitter users y (in millions) is given in the table for selected years. Year 1 represents 2013.

Year, x	Number of Twitter Users (millions) y
1	204
2	255
3	302
4	310
5	327
6	336
7	330

a. Graph the data on a graphing device.

b. Based on the shape of the graph, which function type would provide the best model: linear, exponential, or logarithmic?

c. Use a graphing utility to find an appropriate model, $y = f(x)$. Round the coefficients to one decimal place.

d. Use a graphing utility to graph the function and the data points.

e. Use the model to approximate the number of Twitter users in year 8. Round to the nearest million.

63. The population of a country $P(t)$ (in millions) can be approximated by $P(t) = \frac{740}{1 + 8.265e^{-0.0175t}}$ where t is the number of years since January 1, 1900.

a. What was the population of the country on January 1, 1900? Round to the nearest million.

b. Use the function to predict the population on January 1, 2030. Round to the nearest million.

c. From the model, during which year would the population reach 460 million?

Write About It

64. Why is it important to graph a set of data before trying to find an equation or function to model the data?

65. How does the average rate of change differ for a linear function versus an increasing exponential function?

66. Explain the difference between an exponential growth model and a logistic growth model.

67. Explain how to convert an exponential expression b^t to an exponential expression base e.

Expanding Your Skills

68. The monthly payment P (in $) to pay off a loan of amount A (in $) at an interest rate r in t years is given by

$$P = \frac{\frac{Ar}{12}}{1 - \left(1 + \frac{r}{12}\right)^{-12t}}.$$

a. Solve for t (note that there are numerous equivalent algebraic forms for the result).

b. Interpret the meaning of the resulting relationship.

69. Suppose that a population follows a logistic growth pattern, with a limiting population N. If the initial population is denoted by P_0, and t is the amount of time elapsed, then the population P can be represented by

$$P = \frac{P_0 N}{P_0 + (N - P_0)e^{-kt}}.$$

where k is a constant related to the growth rate.

a. Solve for t (note that there are numerous equivalent algebraic forms for the result).

b. Interpret the meaning of the resulting relationship.

Chapter 3 Review

Chapter 3 Key Concepts

Lesson 3.1 Inverse Function

Key Concepts	Reference
A function f is **one-to-one** if for a and b in the domain of f, if $a \neq b$, then $f(a) \neq f(b)$, or equivalently, if $f(a) = f(b)$, then $a = b$.	p. 330
Horizontal line test: A function defined by $y = f(x)$ is one-to-one if no horizontal line intersects the graph in more than one point.	p. 330
Function **g is the inverse of f** if $(f \circ g)(x) = x$ for all x in the domain of g and $(g \circ f)(x) = x$ for all x in the domain of f.	p. 333
Procedure to find $f^{-1}(x)$: **1.** Replace $f(x)$ with y. **2.** Interchange x and y. **3.** Solve for y. **4.** Replace y with $f^{-1}(x)$.	p. 334

Lesson 3.2 Exponential Functions

Key Concepts	Reference
Let b be a real number with $b > 0$ and $b \neq 1$. Then for any real number x, a function of the form $f(x) = b^x$ is an **exponential function of base b.**	p. 346
For the graph of an exponential function $f(x) = b^x$, • If $b > 1$, f is an increasing exponential function or an **exponential growth function**. • If $0 < b < 1$, f is a decreasing exponential function or an **exponential decay function**. • The domain is $(-\infty, \infty)$. • The range is $(0, \infty)$. • The line $y = 0$ is a horizontal asymptote. • The function passes through $(0, 1)$.	p. 348
The irrational number e is the limiting value of the expression $\left(1 + \frac{1}{x}\right)^x$ as x approaches ∞. $e \approx 2.71828$	p. 349
If P dollars in principal is invested or borrowed at an annual interest rate r for t years, then $I = Prt$ Simple interest $A = P\left(1 + \frac{r}{n}\right)^{nt}$ Future value A with interest compounded n times per year. $A = Pe^{rt}$ Future value A with interest compounded continuously.	p. 352

Lesson 3.3 Logarithmic Functions

Key Concepts	Reference
If x and b are positive real numbers such that $b \neq 1$, then $y = log_b x$ is called the logarithmic function base b, where $\underbrace{y = \log_b x}_{\text{logarithmic form}}$ is equivalent to $\underbrace{b^y = x}_{\text{exponential form}}$.	p. 364
The functions $f(x) = \log_b x$ and $g(x) = b^x$ are inverses.	p. 375
Basic properties of logarithms: **1.** $\log_b 1 = 0$ **2.** $\log_b b = 1$ **3.** $\log_b b^x = x$ **4.** $b^{\log_b x} = x$	p. 370
$y = \log_{10} x$ is written as $y = \log x$ and is called the **common logarithmic function.** $y = \log_e x$ is written as $y = \ln x$ and is called the **natural logarithmic function.**	p. 367
Given $f(x) = \log_b x$, • If $b > 1$, f is an increasing function. • f $0 < b < 1$, f is a decreasing function. • The domain is $(0, \infty)$. • The range is $(-\infty, \infty)$. • The line $x = 0$ is a vertical asymptote. • The function passes through $(1, 0)$.	p. 373
The domain of $f(x) = \log_b x$ is $\{x \mid x > 0\}$.	p. 374

Lesson 3.4 Properties of Logarithms

Key Concepts	Reference
Let b, x, and y be positive real numbers with $b \neq 1$. Then, $\log_b(xy) = \log_b x + \log_b y$ (Product property) $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$ (Quotient property) $\log_b x^p = p \log_b x$ (Power property)	p. 386
Change-of-base formula: For positive real numbers a and b, where $a \neq 1$ and $b \neq 1$, $\log_b x = \frac{\log_a x}{\log_a b}$.	p. 393

Lesson 3.5 Exponential and Logarithmic Equations and Applications

Key Concepts	Reference
Equivalence property of exponential expressions: If b, x, and y are real numbers with $b > 0$ and $b \neq 1$, then $b^x = b^y$ implies that $x = y$.	p. 398

Steps to solve exponential equations by using logarithms: **1.** Isolate the exponential expression on one side of the equation. **2.** Take a logarithm of the same base on both sides of the equation. **3.** Use the power property of logarithms to “bring down” the exponent. **4.** Solve the resulting equation.	p. 400
Equivalence property of logarithmic expressions: If b, x, and y are positive real numbers and $b \neq 1$, then $\log_b x = \log_b y$ implies that $x = y$.	p. 403
Guidelines to solve a logarithmic equation: **1.** Isolate the logarithms on one side of the equation. **2.** Use the properties of logarithms to write the equation in the form $\log_b x = k$, where k is a constant. **3.** Write the equation in exponential form. **4.** Solve the equation from step 3. **5.** Check the potential solution(s) in the original equation.	p. 405

Lesson 3.6 Modeling with Exponential and Logarithmic Functions

Key Concepts	Reference
The function defined by $y = y_0e^{kt}$ represents exponential growth if $k > 0$ and exponential decay if $k < 0$.	p. 417
An exponential expression can be rewritten as an expression of a different base. In particular, to convert to base e, we have b^t is equivalent to $e^{(\ln b)t}$.	p. 420
A **logistic growth function** is a function of the form $y = \frac{c}{1 + ae^{-bt}}$. A logistic growth function imposes a limiting value on the dependent variable.	p. 422

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Chapter 3 Test

1. Given $f(x) = 4x^3 - 1$,

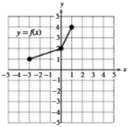
  a. Write an equation for $f^{-1}(x)$.

  b. Verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

2. The graph of f is given.

  a. Is f a one-to-one function?

  b. If f is a one-to-one function, graph f^{-1} on the same coordinate system as f.



3. Given $f(x) = \frac{x+3}{x-4}$, write an equation for the inverse function.

For Exercises 4–7,

  a. Write the domain and range of f in interval notation.

  b. Write an equation of the inverse function.

  c. Write the domain and range of f^{-1} in interval notation.

4. $f(x) = -x^2 + 1, x \leq 0$

5. $f(x) = \log x$

6. $f(x) = 3^x + 1$

7. $f(x) = \sqrt{x+5}$

For Exercises 8–11,

  a. Graph the function.

  b. Write the domain in interval notation.

  c. Write the range in interval notation.

  d. Write an equation of the asymptote.

8. $f(x) = \left(\frac{1}{3}\right)^x + 2$

9. $g(x) = 2^{x-4}$

10. $h(x) = -\ln x$

11. $k(x) = \log_2(x+1) - 3$

12. Write the statement in exponential form. $\ln(x + y) = a$

For Exercises 13–18, evaluate the logarithmic expression without using a calculator.

13. $\log_9 \frac{1}{81}$

14. $\log_6 216$

15. $\ln e^8$

16. $\log 10^{-4}$

17. $10^{\log(a^2 + b^2)}$

18. $\log_{1/2} 1$

For Exercises 19–20, write the domain of the function in interval notation.

19. $f(x) = \log(7 - 2x)$

20. $g(x) = \log_4(x^2 - 25)$

For Exercises 21–22, write the logarithm as a sum or difference of logarithms. Simplify each term as much as possible.

21. $\ln\left(\frac{x^5 y^2}{w\sqrt[3]{z}}\right)$

22. $\log\left(\frac{\sqrt{a^2 + b^2}}{10^4}\right)$

For Exercises 23–24, write the logarithmic expression as a single logarithm with coefficient 1, and simplify as much as possible.

23. $6\log_2 a - 4\log_2 b + \frac{2}{3}\log_2 c$

24. $\frac{1}{2}\ln(x^2 - x - 12) - \frac{1}{2}\ln(x - 1)$

For Exercises 25–26, use $\log_b 2 \approx 0.289$, $\log_b 3 \approx 0.458$, and $\log_b 5 \approx 0.671$ to approximate the value of the given logarithms.

25. $\log_b 72$

26. $\log_b\left(\frac{1}{125}\right)$

For Exercises 27–36, solve the equation. Write the solution set with exact values and give approximate solutions to 4 decimal places.

27. $2^{5y+1} = 4y^{-3}$

28. $5^{x+3} + 3 = 56$

29. $2^{c+7} = 3^{2c+3}$

30. $7e^{4x} - 2 = 12$

31. $e^{2x} + 7e^x - 8 = 0$

32. $\log_5(3 - x) = \log_5(x + 1)$

33. $5 \ln(x + 2) + 1 = 16$

34. $\log x + \log(x - 1) = \log 12$

35. $-3 + \log_4 x = -\log_4(x + 30)$

36. $\log 3 + \log(x + 3) = \log(4x + 5)$

For Exercises 37–38, solve for the indicated variable.

37. $S = 92 - k \ln(t + 1)$ for t

38. $A = P\left(1 + \frac{r}{n}\right)^{nt}$ for t

39. Suppose that $10,000 is invested and the account grows to $13,566.25 in 5 yr.

- **a.** Use the model $A = Pe^{rt}$ to determine the average rate of return under continuous compounding. Round to the nearest tenth of a percent.
- **b.** Using the interest rate from part (a), how long will it take the investment to reach $50,000? Round to the nearest tenth of a year.

40. The number of bacteria in a culture begins with approximately 10,000 organisms at the start of an experiment. If the bacteria double every 5 hr, the model $P(t) = 10{,}000(2)^{t/5}$ represents the population $P(t)$ after t hours.

- **a.** Write a function of the form $P(t) = P_0e^{kt}$ to model the population.
- **b.** Determine the amount of time required for the population to grow to 5 million. Round to the nearest hour.

41. The population $P(t)$ of a herd of deer on an island can be modeled by $P(t) = \frac{1200}{1 + 2e^{-0.12t}}$, where t represents the number of years since the park service has been tracking the herd.

- **a.** Evaluate $P(0)$ and interpret its meaning in the context of this problem.
- **b.** Use the function to predict the deer population after 4 yr. Round to the nearest whole unit.
- **c.** Use the function to predict the deer population after 8 yr.
- **d.** Determine the number of years required for the deer population to reach 900. Round to the nearest year.
- **e.** What value will the term $\frac{2}{e^{0.12t}}$ approach as $t \to \infty$?
- **f.** Determine the limiting value of $P(t)$.

42. The number N of visitors to a new website is given in the table t weeks after the website was launched.

t	0	1	2	3	4
N	24	50	121	270	640

- **a.** Use a graphing utility to find an equation of the form $N = ab^t$ to model the data. Round a to 1 decimal place and b to 3 decimal places.
- **b.** Use a graphing utility to graph the data and the model from part (a).
- **c.** Use the model to predict the number of visitors to the website 10 weeks after launch. Round to the nearest thousand.

GO ONLINE Go online for more practice problems.

Chapter 3 Cumulative Review Exercises

For Exercises 1–2, simplify the expression.

1. $\dfrac{3x^{-1}-6x^{-2}}{2x^{-2}-x^{-1}}$

2. $\dfrac{5}{\sqrt[3]{2x^2}}$

3. Factor. a^3-b^3-a+b

4. Perform the operations and write the answer in scientific notation. $\dfrac{(3.0\times10^{7})(8.2\times10^{-3})}{1.23\times10^{-5}}$

For Exercises 5–13, solve the equations and inequalities. Write the solution sets to the inequalities in interval notation.

5. $5\le3+|2x-7|$

6. $3x(x-1)=x+6$

7. $\sqrt{t+3}+4=t+1$

8. $9^{2m-3}=27^{m+1}$

9. $-x^3-5x^2+4x+20<0$

10. $|5x-1|=|3-4x|$

11. $(x^2-9)^2-2(x^2-9)-35=0$

12. $\log_2(3x-1)=\log_2(x+1)+3$

13. $\dfrac{x-4}{x+2}\le0$

14. Find all the zeros of $f(x)=x^4+10x^3+10x^2+10x+9$

15. Given $f(x)=x^2-16x+55$,

- **a.** Does the graph of the parabola open upward or downward?
- **b.** Find the vertex of the parabola.
- **c.** Identify the maximum or minimum point.
- **d.** Identify the maximum or minimum value of the function.
- **e.** Identify the x-intercept(s).
- **f.** Identify the y-intercept.
- **g.** Write an equation for the axis of symmetry.
- **h.** Write the domain in interval notation.
- **i.** Write the range in interval notation.

16. Graph. $f(x)=-1.5x^2(x-2)^3(x+1)$

17. Given $f(x)=\dfrac{3x+6}{x-2}$,

- **a.** Write an equation of the vertical asymptote(s).
- **b.** Write an equation of the horizontal or slant asymptote.
- **c.** Graph the function.

18. Given $f(x)=2^{x+2}-3$,

- **a.** Write an equation of the asymptote.
- **b.** Write the domain in interval notation.
- **c.** Write the range in interval notation.

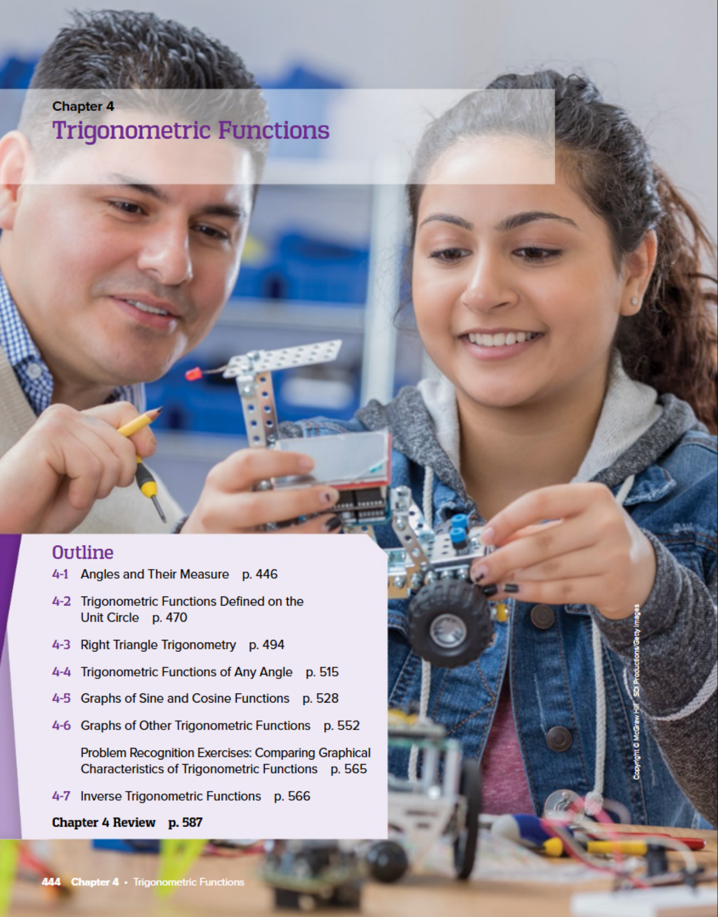
19. Write the expression as a single logarithm and simplify.
$\log 40+\log 50-\log 2$

20. Given the one-to-one function defined by $f(x)=\sqrt[3]{x-4}+1$, write an equation for $f^{-1}(x)$.

Go online for more practice problems.

Chapter 4

Trigonometric Functions



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Launch Activity: Make Use of Structure

Revolutions of the Wheel

Your robotics team is preparing to compete in a local competition. Once the course for the competition is complete your team can view the layout and measure distances of the path and angles of the turns prior to the event. All coding for this course must be completed before the competition begins. For this competition, the standard radius of each wheel is 3.5 cm.

One teammate from your robotics team measured the first straight segment of the competition path and found the length to be 4 feet 3 inches long. Coding for the robot's movement does not use distance as input but rather revolutions of the wheel.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer these questions: How many revolutions will the wheel of the robot make in the first straight segment of the course? How many degrees will the wheel travel for this section of the course?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

GO ONLINE to answer these questions

Lesson 4-1

Angles and Their Measure

What Will You Learn?

After completing this lesson, you should be able to:

- Find Degree Measure
- Find Radian Measure
- Determine Coterminal Angles
- Compute Arc Length of a Sector of a Circle
- Compute Linear and Angular Speed
- Compute the Area of a Sector of a Circle

Learn: Find Degree Measure

The Tour de France is the most famous bicycle race in the world, spanning 3500 km (2175 mi) in 21 days through France and two mountain ranges. The speed of a racer depends on a number of variables, including the gear ratio and the cadence. The gear ratio determines the number of times the rear wheel turns with each pedal stroke, and the cadence is the number of revolutions of the pedals per minute. In this lesson, we study angles and their measure, and the relationship between angular and linear speeds to study such applications as the speed of a bicycle.

Photo Link/Photodisc/Getty Images

A **ray** is a part of a line that consists of an endpoint and all points on the line to one side of the endpoint. In Figure 4-1, ray $\overrightarrow{PQ}$ is named by using the endpoint P and another point Q on the ray. Notice that the rays $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ are different because the initial points are different and the rays extend in opposite directions.

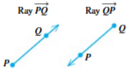


Figure 4-1

Good Practices

A ray has only one endpoint, which is always written first when naming the ray.

An **angle** is formed by rotating a ray about its endpoint. The starting position of the ray is called the **initial side** of the angle, and the final position of the ray is called the **terminal side.** The common endpoint is called the **vertex** of the angle, and the vertex is often named by a capital letter such as *A* (Figure 4-2).

Angle *A* in Figure 4-2 can be denoted by $\angle A$ (read as "angle *A*") or by $\angle BAC$, where *B* is a point on the initial side of the angle, *A* represents the vertex, and *C* is a point on the terminal side. Alternatively, Greek letters such as θ (theta), α (alpha), β (beta), and γ (gamma) are often used to denote angles.

Insights When denoting an angle such as $\angle BAC$, the vertex is the middle letter.

An angle is in **standard position** if its vertex is at the origin in the *xy*-plane and its initial side is the positive *x*-axis. In Figure 4-3, angle α is drawn in standard position.

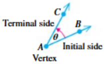


Figure 4-2

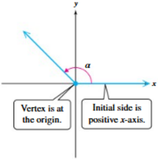


Figure 4-3

The **measure** of an angle quantifies the direction and amount of rotation from the initial side to the terminal side. The measure of an angle is *positive* if the rotation is counterclockwise, and the measure is *negative* if the rotation is clockwise. One unit with which to measure an angle is the **degree.** One full rotation of a ray about its endpoint is 360 degrees, denoted 360°. Therefore, 1° is $\frac{1}{360}$ of a full rotation. Figure 4-4 shows a variety of angles and their measures.

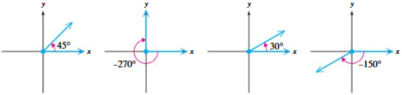
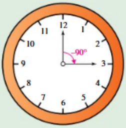


Figure 4-4

Insights

The hands of a clock each resemble a ray. When the minute hand moves from 12:00 to 12:15, the rotation is $-90°$.


12
1
2
3
4
5
6
7
8
9
10
11
-90°

If the measure of an angle θ is 30°, we denote the measure of θ as $m(\theta) = 30°$ or simply $\theta = 30°$. We may also refer to θ as a 30° angle, rather than using the more formal, but cumbersome language "an angle whose measure is 30°." We also introduce several key terms associated with the measure of an angle (Figure 4-5).

A *right* angle is a 90° angle and a *straight* angle is a 180° angle. An *acute* angle is one with measure strictly between 0° and 90°, while an *obtuse* angle measures strictly bewteen 90° and 180°.

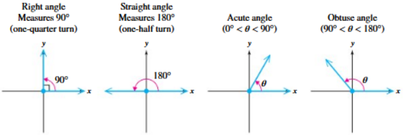

Right angle
Measures 90°
(one-quarter turn)
Straight angle
Measures 180°
(one-half turn)
Acute angle
$(0° < \theta < 90°)$
Obtuse angle
$(90° < \theta < 180°)$
y
x
90°
180°
θ

Figure 4-5

If the sum of the measures of two angles is 90°, we say that the angles are **complementary** (for example, the *complement* of a 20° angle is a 70° angle and vice versa). If the sum of the measures of two angles is 180°, we say that the angles are **supplementary** (for example, the supplement of a 20° angle is a 160° angle and vice versa).

A degree can be divided into 60 equal parts called **minutes** (min or ′), and each minute is divided into 60 equal parts called **seconds** (sec or ″). Note that the terms *minutes* and *seconds* when used in the context of angular measure are sometimes called **arcminutes** and **arcseconds** to avoid confusion with measurements of time.

- 1 min $= \left(\frac{1}{60}\right)^{\circ}$ or $1' = \left(\frac{1}{60}\right)^{\circ}$
- 1 sec $= \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^{\circ}$ or $1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^{\circ}$

For example, 74 degrees, 42 minutes, 15 seconds is denoted $74°42'15''$.

Example 1: Converting from Degrees, Minutes, Seconds (DMS) to Degree Decimal Form

Convert 74°42′15″ to decimal degrees. Round to 4 decimal places.

Solution:

Convert the minute and second portions of the angle to degrees. Choose the conversion factors so that the original units (minutes and seconds) "cancel," leaving the measurement in degrees.

$$74°42'15'' = 74° + (42\ \cancel{\text{min}}) \cdot \left(\frac{1°}{60\ \cancel{\text{min}}}\right) + (15\ \cancel{\text{sec}}) \cdot \left(\frac{1°}{3600\ \cancel{\text{sec}}}\right)$$

$$= 74° + 0.7° + 0.00416°$$

$$\approx 74.7042°$$

Apply the Skills

1. Convert 131°12′33″ to decimal degrees. Round to 4 decimal places.

Example 2: Converting from Decimal Degrees to Degrees, Minutes, Seconds (DMS) Form

Convert 159.26° to degree, minute, second form.

Solution:

$159.26° = 159° + 0.26°$	Write the decimal as a whole number part plus a fractional part. The fractional part of 1° needs to be converted from degrees to minutes and seconds.
$= 159° + 0.26° \cdot \left(\frac{60'}{1°}\right)$	Use the conversion factor 60′ = 1° to convert to minutes.
$= 159° + 15.6'$	
$= 159° + 15' + 0.6'$	Write 15.6′ as a whole number part plus a fractional part.
$= 159° + 15' + 0.6' \cdot \left(\frac{60''}{1'}\right)$	Now convert the fractional part of 1 minute to seconds. Use the conversion factor 60″ = 1′ to convert to seconds.
$= 159° + 15' + 36''$	
$= 159°15'36''$	

Apply the Skills

2. Convert 26.48° to degree, minute, second form.

Learn: Find Radian Measure

Degree measure is used extensively in many applications of engineering, surveying, and navigation. Another type of angular measure that is better suited for applications in trigonometry and calculus is **radian measure.** To begin, we define a **central angle** as an angle with the vertex at the center of a circle.

Definition of One Radian

A central angle that intercepts an arc on the circle with length equal to the radius of the circle has a measure of **1 radian** (Figure 4-6).

Note: One radian may be denoted as 1 radian, 1 rad, or simply 1. That is, radian measure carries no units.

When two lines or rays cross a circle, the part of the circle between the intersection points is called the *intercepted arc* and is often denoted by s.

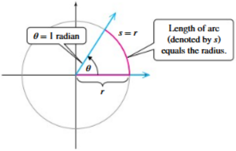


Figure 4-6

Any central angle can be measured in radians by dividing the length s of the intercepted arc by the radius r. For example, in Figure 4-7, the length of the red arc is $2r$ (twice the radius).

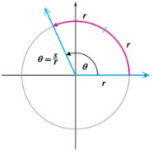


Figure 4-7

Therefore, the measure of angle θ is given by

$$\theta = \frac{s}{r} = \frac{2r}{r} = 2 \text{ (2 radians)}$$

Definition of Radian Measure of an Angle

The radian measure of a central angle θ subtended by an arc of length s on a circle of radius r is given by $\theta = \frac{s}{r}$.

You may have an intuitive feel for angles measured in degrees (for example, 90° is one-quarter of a full rotation). However, radian measure is unfamiliar. From Figure 4-6, notice that an angle of 1 radian (1 rad) is approximately 57.3°. If we divide the circumference of a circle into arcs of length r (Figure 4-8), we see that there are just over 6 rad in one full rotation. In fact, we can show that there are exactly 2π rad in one full rotation.

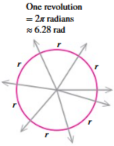


Figure 4-8

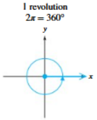
Insights

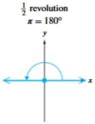
Radian measure carries no units because it is measured as a ratio of two lengths with the same units (the units associated with s/r "cancel"). So, 2π rad is simply written as 2π. It is universally understood that the measure is in radians. Sometimes the notation "rad" is included for emphasis, but is not necessary.

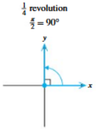
Recall that π is defined as the ratio of the circumference of a circle to its diameter d. Therefore, the circumference C is given by $C = \pi d$ or equivalently $C = 2\pi r$, where r is the radius of the circle. The circumference is the arc length of a full circle, and dividing this by the radius gives the number of radians in one revolution.

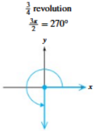
$$\overset{\text{Number of radians in one revolution}}{\theta} = \frac{s}{r} = \frac{\overset{\text{Arc length of one revolution}}{\text{circumference}}}{\text{radius}} = \frac{2\pi r}{r} = 2\pi \approx 6.28$$

The angular measure of one full rotation is 2π (2π rad). Therefore, we have the following relationships.









The statement $\pi = 180°$ gives us a conversion factor to convert between degree measure and angular measure.

Converting Between Degree and Radian Measure

- To convert from degrees to radians, multiply the degree measure by $\frac{\pi}{180°}$.
- To convert from radians to degrees, multiply the radian measure by $\frac{180°}{\pi}$.

Good Practices

Some angles are used frequently in the study of trigonometry. Their degree measures and equivalent radian measures are worth memorizing.

$$30° = \frac{\pi}{6} \qquad 45° = \frac{\pi}{4} \qquad 60° = \frac{\pi}{3} \qquad 90° = \frac{\pi}{2}$$

Example 3: Converting from Degrees to Radians

Convert from degrees to radians.

a. 210° **b.** −135°

Solution:

a. $210° \cdot \left(\frac{\pi\text{ rad}}{180°}\right) = \frac{\overset{7}{\cancel{210}}\pi}{\underset{6}{\cancel{180}}}\text{rad} = \frac{7\pi}{6}\text{rad} = \frac{7\pi}{6}$

The units of degrees "cancel" in the numerator and denominator in the first step, leaving units of radians.

b. $-135° \cdot \left(\frac{\pi}{180°}\right) = -\frac{\overset{3}{\cancel{135}}\pi}{\underset{4}{\cancel{180}}} = -\frac{3\pi}{4}$

The units of "rad" are implied in the numerator of the conversion factor $\frac{\pi}{180°}$.

Apply the Skills

3. Convert from degrees to radians.

a. 300° **b.** −70°

Example 4: Converting from Radians to Degrees

Convert from radians to degrees.

a. $\frac{\pi}{12}$ **b.** $-\frac{4\pi}{3}$

Solution:

a. $\frac{\pi}{12}\text{rad}\cdot\left(\frac{180^\circ}{\pi\,\text{rad}}\right)=\left(\frac{\overset{15}{\cancel{180}}\,\cancel{\pi}}{\underset{1}{\cancel{12}}\,\cancel{\pi}}\right)^\circ=15^\circ$

The units of rad "cancel" in the numerator and denominator in the first step, leaving units of degrees.

b. $-\frac{4\pi}{3}\cdot\left(\frac{180^\circ}{\pi}\right)=\left(-\frac{\overset{240}{\cancel{720}}\,\cancel{\pi}}{\underset{1}{\cancel{3}}\,\cancel{\pi}}\right)^\circ=-240^\circ$

The units of "rad" are implied in the denominator of the conversion factor $\frac{180^\circ}{\pi}$.

Insights

To help determine which conversion factor to use, remember that you want the original units to "cancel," leaving the new unit of measurement in the numerator.

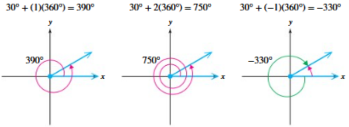
Apply the Skills

4. Convert from radians to degrees.

a. $\frac{\pi}{18}$ **b.** $-\frac{7\pi}{4}$

Learn: Determine Coterminal Angles

Two angles in standard position with the same initial side and same terminal side are called **coterminal angles.** Figure 4-9 illustrates three angles in standard position that are coterminal to 30°. Notice that each angle is 30° plus or minus some number of full revolutions clockwise or counterclockwise.



$30^\circ + (1)(360^\circ) = 390^\circ$
$30^\circ + 2(360^\circ) = 750^\circ$
$30^\circ + (-1)(360^\circ) = -330^\circ$
y
390°
x
y
750°
x
y
−330°
x

Figure 4-9 Coterminal Angles

Point of Interest

Aerial snowboarding is a winter sport in which competitors perform aerial tricks after launching from the sides of a halfpipe. The degree of difficulty of a move is measured in part by the number of full rotations that a competitor completes through the air. Ulrik Badertscher of Norway was the first to rotate through the the air an amazing 4.5 full rotations, an angle of 1620°.

Example 5: Finding Coterminal Angles between 0° and 360°

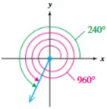
Find an angle coterminal to θ between 0° and 360°.

a. $\theta = 960°$ **b.** $\theta = -225°$

Solution:

a. $\theta = 960°$ is more than 360°. Therefore, we will subtract some multiple of 360° to get an angle coterminal to θ between 0° and 360°.

- One revolution measures 360°.
- Two revolutions measure 720°.
- Three revolutions measure 1080°.



360°(2) is subtracted from 960° because θ is between 2 and 3 revolutions.

Therefore,

$960° - (360°)(2) = 960° - 720° = 240°$

Insights

The number of revolutions contained in 960° can be found by dividing 960° by 360°.

2 revolutions

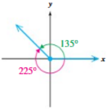
$$\begin{array}{r} 2 \\ 360\overline{)960} \\ \underline{720} \\ 240 \end{array}$$

remainder

$\theta = 960°$ is two full revolutions plus 240°.

b. An angle of −225° is less than one revolution. Therefore, we will add 360°(1) to obtain a positive angle between 0° and 360° and coterminal to −225°.

$-225° + 360° = 135°$



Apply the Skills

5. Find an angle coterminal to θ between 0° and 360°.

a. $\theta = 1230°$ **b.** $\theta = -315°$

Two angles in standard position are coterminal if their measures differ by a multiple of 360° or 2π rad. The angles shown in Figure 4-10 are coterminal to an angle of $\frac{\pi}{6}$ rad (or 30°).

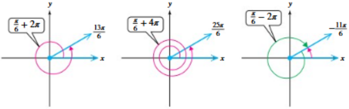


Figure 4-10 Coterminal Angles

Example 6: Finding Coterminal Angles between 0 and 2π

Find an angle coterminal to θ on the interval $[0, 2\pi)$.

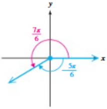
a. $\theta = -\frac{5\pi}{6}$ **b.** $\theta = \frac{13\pi}{2}$

Solution:

a. One revolution is 2π rad or equivalently $\frac{12\pi}{6}$.

Therefore, adding any multiple of $\frac{12\pi}{6}$ to $-\frac{5\pi}{6}$ results in an angle coterminal to $-\frac{5\pi}{6}$.

$$-\frac{5\pi}{6} + \frac{12\pi}{6} = \frac{7\pi}{6}$$



b. One revolution is 2π rad, or equivalently $\frac{4\pi}{2}$.

We can divide $\frac{13\pi}{2} \div \frac{4\pi}{2} = \frac{13\pi}{2} \cdot \frac{2}{4\pi} = \frac{13}{4} = 3\frac{1}{4}$.

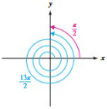
Therefore, $\frac{13\pi}{2}$ is three full revolutions plus $\frac{1}{4}$ of a revolution.

Subtracting three revolutions results in an angle coterminal to $\frac{13\pi}{2}$.

Subtract 3 revolutions.

Fractional part of a revolution.

$$\frac{13\pi}{2} - 3\left(\frac{4\pi}{2}\right) = \frac{13\pi}{2} - \frac{12\pi}{2} = \frac{\pi}{2}$$



Apply the Skills

6. Find an angle coterminal to θ on the interval $[0, 2\pi)$.

a. $\theta = -\frac{\pi}{8}$ **b.** $\theta = \frac{19\pi}{4}$

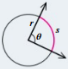
Learn: Compute Arc Length of a Sector of a Circle

By definition, the measure of a central angle θ in radians equals the length s of the intercepted arc divided by the radius r, that is, $\theta = \frac{s}{r}$. Solving this relationship for s gives $s = r\theta$, which enables us to compute arc length if the measure of the central angle and radius are known.

Arc Length

Given a circle of radius r, the length s of an arc intercepted by a central angle θ (in radians) is given by

$$s = r\theta$$



Example 7: Determining Arc Length

Find the length of the arc made by an angle of 105° on a circle of radius 15 cm. Give the exact arc length and round to the nearest tenth of a centimeter.

Solution:

$\theta = 105° \cdot \frac{\pi}{180°} = \frac{7\pi}{12}$ Convert θ to radians.

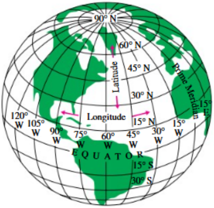
$s = r\theta = (15 \text{ cm}) \cdot \frac{7\pi}{12} = 8.75\pi \text{ cm}$ Apply the arc length formula.

The arc is $8.75\pi \approx 27.5$ cm.

Apply the Skills

7. Find the length of the arc made by an angle of 220° on a circle of radius 9 in.

The Earth is approximately spherical, and the most common way to locate points on the surface is by using *latitude* and *longitude*. These coordinates are measured in degrees and represent central angles measured from the center of the Earth.


90° N
60° N
45° N
30° N
Latitude
Prime Meridian
15° E
120° W
105° W
90° W
75° W
60° W
45° W
30° W
15° W
Longitude
15° N
EQUATOR
15° S
30° S

The **equator** is an imaginary circle around the Earth equidistant between the north and south poles. **Latitude** is the angular measure of a central angle measuring north (N) or south (S) from the equator. There are 90° of latitude measured north from the equator and 90° of latitude measured south from the equator. The equator has a latitude of 0°, the north pole has a latitude of 90°N, and the south pole has a latitude of 90°S (Figure 4-11).

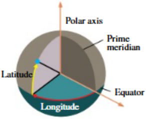

Polar axis
Prime meridian
Latitude
Equator
Longitude

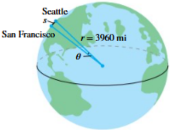
Figure 4-11

Lines of **longitude,** called meridians, are circles that pass through both poles and run perpendicular to the equator. By international agreement, 0° longitude is taken to be the meridian line through Greenwich, England. This is called the **prime meridian.** So, longitude is the angular measure of a central angle east (E) or west (W) of the prime meridian. The Earth is divided into 360° of longitude. There are 180° east (E) of the prime meridian and 180° west (W) of the prime meridian.

For example, New York City is located at 40.7°N, 74.0°W. This means that New York City is located 40.7° north of the equator and 74.0° west of the prime meridian.

Example 8: Determining the Distance Between Cities

Seattle, Washington, is located at 47.6°N, 122.4°W, and San Francisco, California, is located at 37.8°N, 122.3°W. Since the longitudes are nearly the same, the cities are roughly due north-south of each other. Using the difference in latitude, approximate the distance between the cities assuming that the radius of the earth is 3960 mi. Round to the nearest mile.


Seattle
s
San Francisco
r = 3960 mi
θ

Solution:

$47.6° - 37.8° = 9.8°$	Calculate the difference in latitude.
$\theta = 9.8° \cdot \frac{\pi}{180°} \approx 0.17104$	Convert θ to radians.
$s = r\theta$	Apply the arc length formula.
$s = (3960)(0.17104) \approx 677$ mi	

Check Your Work

When applying the formula $s = r\theta$, always use radian measure for θ.

Apply the Skills

8. Lincoln, Nebraska, is located at 40.8°N, 96.7°W and Dallas, Texas, is located at 32.8°N, 96.7°W. Since the longitudes are the same, the cities are north-south of each other. Using the difference in latitudes, approximate the distance between the cities assuming that the radius of the Earth is 3960 mi. Round to the nearest mile.

Learn: Compute Linear and Angular Speed

Consider a ceiling fan rotating at a constant rate. For a small increment of time, suppose that point A on the tip of a blade travels a distance s_1. In the same amount of time, point B will travel a shorter distance s_2 (Figure 4-12). Therefore, point A on the tip has a greater linear speed v than point B. However, each point has the same *angular* speed ω (omega) because they move through the same angle for a given unit of time.

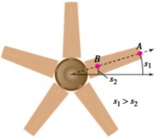


Figure 4-12

Angular and Linear Speed

If a point on a circle of radius r moves through an angle of θ radians in time t, the angular and linear speeds of the point are

$$\text{angular speed: } \omega = \frac{\theta}{t}$$
$$\text{linear speed: } v = \frac{s}{t} \quad \text{or} \quad v = \frac{r\theta}{t} \quad \text{or} \quad v = r\omega$$

Insights

In an application, if a rate is given in degrees, radians, or revolutions per unit time, it is an angular speed. If a rate is given in linear units (such as feet) per unit time, it is a linear speed.

Example 9: Finding Linear and Angular Speed

A ceiling fan rotates at 90 rpm (revolutions per minute). For a point at the tip of a 2-ft blade,

a. Find the angular speed.

b. Find the linear speed. Round to the nearest whole unit.

Solution:

a. For each revolution of the blade, the point moves through an angle of 2π radians.

$$\omega = \frac{\theta}{t} = \frac{90\ \cancel{\text{rev}}}{\text{min}} \cdot \frac{2\pi\ \text{rad}}{\cancel{\text{rev}}} = 180\pi\ \text{rad/min} = 180\pi/\text{min}$$

b. $v = r\omega$

$$v = (2\ \text{ft})\left(\frac{180\pi}{\text{min}}\right) = 360\pi\ \text{ft/min} \approx 1131\ \text{ft/min}$$

Insights

Converting the result to miles per hour, we have

$$\frac{1131\ \cancel{\text{ft}}}{\cancel{\text{min}}} \cdot \frac{1\ \text{mi}}{5280\ \cancel{\text{ft}}} \cdot \frac{60\ \cancel{\text{min}}}{1\ \text{hr}} \approx 12.9\ \text{mph.}$$

Apply the Skills

9. A bicycle wheel rotates at 2 revolutions per second.

a. Find the angular speed.

b. How fast does the bicycle travel (in ft/sec) if the wheel is 2.2 ft in diameter? Round to the nearest tenth.

Point of Interest

The Earth rotates at a constant angular speed of 360° in 24 hr or equivalently 15° in 1 hr. Therefore, each time zone is approximately 15° of longitude in width (with local variations) and is 1 hr earlier than the zone immediately to the east.

Learn: Compute the Area of a Sector of a Circle

A sector of a circle is a "pie-shaped" wedge of a circle bounded by the sides of a central angle and the intercepted arc (Figure 4-13).

The area of a sector of a circle is proportional to the measure of the central angle. The expression $\frac{\theta}{2\pi}$ is the fractional amount of a full rotation represented by angle θ (in radians). So the area of a sector formed by θ is

$$\underbrace{\frac{\theta}{2\pi}}_{\text{Fractional amount of full circle}} \cdot \underbrace{(\pi r^2)}_{\text{Area of full circle}} = \underbrace{\frac{1}{2}r^2\theta}_{\text{Area of sector}}$$

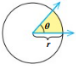


Figure 4-13

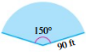
Area of a Sector

The area A of a sector of a circle of radius r with central angle θ (in radians) is given by

$$A = \frac{1}{2}r^2\theta$$

Example 10: Determining the Area of a Sector

A crop sprinkler rotates through an angle of 150° and sprays water a distance of 90 ft. Find the amount of area watered. Round to the nearest whole unit.



Solution:

To use the formula $A = \frac{1}{2}r^2\theta$, we need to convert θ to radians.

$$150° \cdot \frac{\pi}{180°} = \frac{5\pi}{6}$$

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(90\text{ ft})^2\left(\frac{5\pi}{6}\right) = 3375\pi\text{ ft}^2 \approx 10{,}603\text{ ft}^2$$

Apply the Skills

10. A sprinkler rotates through an angle of 120° and sprays water a distance of 30 ft. Find the amount of area watered. Round to the nearest whole unit.

Practice 4-1

Practice Exercises

Prerequisite Review

R.1. Solve for the specified variable.
$p = hz$ for z

R.2. If a plane travels 280 mph for 3.5 hr, find the distance traveled.

For Exercises R.3–R.5, convert the unit of time.

R.3. 210 min = __________ hr

R.4. 120 sec = __________ min

R.5. 7200 sec = __________ hr

R.6. Find the circumference of a circle with a radius of $2\frac{1}{2}$ m.

- **a.** Give the exact answer in terms of π.
- **b.** Approximate the answer by using 3.14 for π. Round to 1 decimal place.

R.7. Determine the area of a circle with a diameter of 40 ft. Use 3.14 for π. Round to the nearest whole unit.

Concept Connections

1. The measure of an angle is (positive/negative) __________ if its rotation from the initial side to the terminal side is clockwise. If the rotation is counterclockwise, then the measure is __________.

2. An angle with its vertex at the origin of an xy-coordinate plane and with initial side on the positive x-axis is in __________ position.

3. Two common units used to measure angles are __________ and __________.

4. One degree is what fractional amount of a full rotation?

5. An angle that measures 360° has a measure of __________ radians.

6. Two angles are called __________ if the sum of their measures is 90°. Two angles are called __________ if the sum of their measures is 180°.

7. An angle with measure ________ or ________ radians is a right angle. A straight angle has a measure of ________ or ________ radians.

8. A(n) __________ angle has a measure between 0° and 90°, whereas a(n) ________ angle has a measure between 90° and 180°.

9. One degree is equally divided into 60 parts called ______________.

10. One minute is equally divided into 60 parts called ________________.

11. 1° = ________′ = ________″

12. An angle with its vertex at the center of a circle is called a(n) ___________ angle.

13. A central angle of a circle that intercepts an arc equal in length to the radius of the circle has a measure of ___________.

14. Which angle has a greater measure, 2° or 2 radians?

15. To convert from radians to degrees, multiply by__________. To convert from degrees to radians, multiply by __________.

16. Two angles are __________ if they have the same initial side and same terminal side.

17. The measure of all angles coterminal to $\frac{7\pi}{4}$ differ from $\frac{7\pi}{4}$ by a multiple of __________.

18. The measure of all angles coterminal to 112° differ from 112° by a multiple of __________.

19. The length s of an arc made by an angle θ on a circle of radius r is given by the formula _________ , where θ is measured in ___________.

20. To locate points on the surface of the Earth that are north or south of the equator, we measure the ___________ of the point. To locate points that are east or west of the prime meridian, we measure the ___________ of the point.

21. The relationship $v = \frac{\text{arc length}}{\text{time}}$ represents the ___________ speed of a point traveling in a circular path.

22. The symbol ω is typically used to denote __________ speed and represents the number of radians per unit time that an object rotates.

23. A wedge of a circle, similar in shape to a slice of pie, is called a(*n*) __________ of the circle.

24. The area A of a sector of a circle of radius r with central angle θ is given by the formula __________, where θ is measured in __________.

Learn: Find Degree Measure

For Exercises 25–26, sketch the angles in standard position.

25. a. $60°$ **b.** $225°$ **c.** $-210°$ **d.** $-86°$

26. a. $30°$ **b.** $120°$ **c.** $-135°$ **d.** $-73°$

For Exercises 27–30, convert the given angle to decimal degrees. Round to 4 decimal places. (See Example 1, p. 449)

27. $17°34'$ **28.** $215°47'$

29. $54°36'55''$ **30.** $23°42'48''$

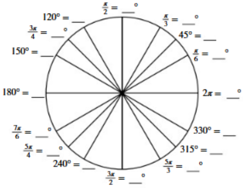
For Exercises 31–34, convert the given angle to DMS (degree-minute-second) form. Round to the nearest second if necessary. (See Example 2, p. 449)

31. $46.418°$ **32.** $82.074°$

33. $-84.64°$ **34.** $-61.46°$

Learn: Find Radian Measure

35. Use the conversion factor $\pi = 180°$ along with the symmetry of the circle to complete the degree or radian measure of the missing angles.


$\frac{\pi}{2} = \_\_\_°$
$120° = \_\_\_$
$\frac{\pi}{3} = \_\_\_°$
$\frac{3\pi}{4} = \_\_\_°$
$45° = \_\_\_$
$150° = \_\_\_$
$\frac{\pi}{6} = \_\_\_°$
$180° = \_\_\_$
$2\pi = \_\_\_°$
$\frac{7\pi}{6} = \_\_\_°$
$330° = \_\_\_$
$\frac{5\pi}{4} = \_\_\_°$
$315° = \_\_\_$
$240° = \_\_\_$
$\frac{5\pi}{3} = \_\_\_°$
$\frac{3\pi}{2} = \_\_\_°$

36. Insert the appropriate symbol <, >, or = in the blank.

a. $\frac{5\pi}{6} \square 120°$

b. $-\frac{4\pi}{3} \square -270°$

For Exercises 37–40, convert from degrees to radians. Give the answers in exact form in terms of π. (See Example 3, p. 452)

37. $75°$ **38.** $240°$

39. $-210°$ **40.** $-195°$

For Exercises 41–44, convert from degrees to radians. Round to 4 decimal places.

41. $-64.6°$ **42.** $-312.4°$

43. $12°6'36''$ **44.** $108°42'9''$

For Exercises 45–56, convert from radians to decimal degrees. Round to 1 decimal place if necessary. (See Example 4, p. 453)

45. $\frac{\pi}{4}$ **46.** $\frac{11\pi}{6}$

47. $-\frac{5\pi}{3}$ **48.** $-\frac{7\pi}{6}$

49. $\frac{5\pi}{18}$ **50.** $\frac{7\pi}{9}$

51. $-\frac{2\pi}{5}$ **52.** $-\frac{3\pi}{8}$

53. 2.7 **54.** 5.3

55. $\frac{9\pi}{2}$ **56.** 7π

Learn: Determine Coterminal Angles

For Exercises 57–64, find a positive angle and a negative angle that is coterminal to the given angle. (See Examples 5–6, pp. 454–455)

57. 57° **58.** 313°

59. −105° **60.** −12°

61. $\frac{5\pi}{6}$ **62.** $\frac{3\pi}{4}$

63. $-\frac{3\pi}{2}$ **64.** $-\pi$

For Exercises 65–70, find an angle between 0° and 360° or between 0 and 2π that is coterminal to the given angle. (See Examples 5–6, pp. 454–455)

65. 1521° **66.** −603°

67. $\frac{17\pi}{4}$ **68.** $\frac{11\pi}{3}$

69. $-\frac{7\pi}{18}$ **70.** $-\frac{5\pi}{9}$

Learn: Compute Arc Length of a Sector of a Circle

For Exercises 71–74, find the exact length of the arc intercepted by a central angle θ on a circle of radius r. Then round to the nearest tenth of a unit. (See Example 7, p. 456)

71. $\theta = \frac{\pi}{3}$, $r = 12$ cm

72. $\theta = \frac{5\pi}{6}$, $r = 4$ m

73. $\theta = 135°$, $r = 10$ in.

74. $\theta = 315°$, $r = 2$ yd

75. A 6-ft pendulum swings through an angle of 40°36′. What is the length of the arc that the tip of the pendulum travels? Round to the nearest hundredth of a foot.

76. A gear with a 1.2-cm radius moves through an angle of 220°15′. What distance does a point on the edge of the gear move? Round to the nearest tenth of a centimeter.

77. Which location would best fit the coordinates 12°S, 77°W?

a. Paris, France **b.** Lima, Peru

c. Miami, Florida **d.** Moscow, Russia

78. a. What is the geographical relationship between two points that have the same latitude?

b. What is the geographical relationship between two points that have the same longitude?

For Exercises 79–82, assume that the Earth is approximately spherical with radius 3960 mi. Approximate the distances to the nearest mile. (See Example 8, p. 457)

79. Barrow, Alaska (71.3°N, 156.8°W), and Kailua, Hawaii (19.7°N, 156.1°W), have approximately the same longitude, which means that they are roughly due north-south of each other. Use the difference in latitude to approximate the distance between the cities.

80. Rochester, New York (43.2°N, 77.6°W), and Richmond, Virginia (37.5°N, 77.5°W), have approximately the same longitude, which means that they are roughly due north-south of each other. Use the difference in latitude to approximate the distance between the cities.

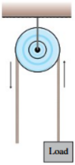
81. Raleigh, North Carolina (35.8°N, 78.6°W), is located north of the equator, and Quito, Ecuador (0.3°S, 78.6°W), is located south of the equator. The longitudes are the same, indicating that the cities are due north-south of each other. Use the difference in latitude to approximate the distance between the cities.

82. Trenton, New Jersey (40.2°N, 74.8°W), is located north of the equator, and Ayacucho, Peru (13.2°S, 74.2°W), is located south of the equator. The longitudes are nearly the same, indicating that the cities are roughly due north-south of each other. Use the difference in latitude to approximate the distance between the cities.

83. A pulley is 16 cm in diameter.

a. Find the distance the load will rise if the pulley is rotated 1350°. Find the exact distance in terms of π and then round to the nearest centimeter.

b. Through how many degrees should the pulley rotate to lift the load 100 cm? Round to the nearest degree.

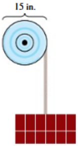


84. A pulley is 1.2 ft. in diameter.

a. Find the distance the load will rise if the pulley is rotated 630°. Find the exact distance in terms of π and then round to the nearest tenth of a foot.

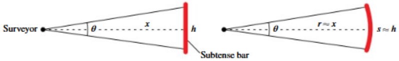
b. Through how many degrees should the pulley rotate to lift the load 24 ft? Round to the nearest degree.

85. A hoist is used to lift a pallet of bricks. The drum on the hoist is 15 in. in diameter. How many degrees should the drum be rotated to lift the pallet a distance of 6 ft? Round to the nearest degree.



86. A winch on a sailboat is 8 in. in diameter and is used to pull in the "sheets" (ropes used to control the corners of a sail). To the nearest degree, how far should the winch be turned to pull in 2 ft of rope?

Before the widespread introduction of electronic devices to measure distances, surveyors used a subtense bar to measure a distance *x* that is not directly measurable. A subtense bar is a bar of known length *h* with marks or "targets" at either end. The surveyor measures the angle *θ* formed by the location of the surveyor's scope and the top and bottom of the bar (this is the angle *subtended* by the bar). Since the angle and height of the bar are known, right triangle trigonometry can be used to find the horizontal distance. Alternatively, if the distance from the surveyor to the bar is large, then the distance can be approximated by the radius *r* of the arc *s* intercepted by the bar. Use this information for Exercises 87–88.



87. A surveyor uses a subtense bar to find the distance across a river. If the angle of sight between the bottom and top marks on a 2-m bar is 57′18″, approximate the distance across the river between the surveyor and the bar. Round to the nearest meter.

88. A surveyor uses a subtense bar to find the distance across a canyon. If the angle of sight between the bottom and top marks on a 2-m bar is 24′33″, approximate the distance across the river between the surveyor and the bar. Round to the nearest meter.

Learn: Compute Linear and Angular Speed

89. A circular paddle wheel of radius 3 ft is lowered into a flowing river. The current causes the wheel to rotate at a speed of 12 rpm. To 1 decimal place,

a. What is the angular speed? (**See Example 9, p. 459**)

b. Find the speed of the current in ft/min.

c. Find the speed of the current in mph.

90. An energy-efficient hard drive has a 2.5-in. diameter and spins at 4200 rpm.

a. What is the angular speed?

b. How fast in in./min does a point on the edge of the hard drive spin? Give the exact speed and the speed rounded to the nearest in./min.

91. A $7\frac{1}{4}$-in.-diameter circular saw has 24 teeth and spins at 5800 rpm.

a. What is the angular speed?

b. What is the linear speed of one of the "teeth" on the outer edge of the blade? Round to the nearest inch per minute.

92. On a weed-cutting device, a thick nylon line rotates on a spindle at 3000 rpm.

a. Determine the angular speed.

b. Determine the linear speed (to the nearest inch per minute) of a point on the tip of the line if the line is 5 in.

93. A truck has 2.5-ft tires (in diameter).

a. What distance will the truck travel with one rotation of the wheels? Give the exact distance and an approximation to the nearest tenth of a foot.

b. How far will the truck travel with 10,000 rotations of the wheels? Give the exact distance and an approximation to the nearest foot.

c. If the wheels turn at 672 rpm, what is the angular speed?

d. If the wheels turn at 672 rpm, what is the linear speed in feet per minute? Give the exact distance and an approximation to the nearest whole unit.

e. If the wheels turn at 672 rpm, what is the linear speed in miles per hour? Round to the nearest mile per hour. (*Hint*: 1 mi = 5280 ft and 1 hr = 60 min.)

94. A bicycle has 25-in. wheels (in diameter).

a. What distance will the bicycle travel with one rotation of the wheels? Give the exact distance and an approximation to the nearest tenth of an inch.

b. How far will the bicycle travel with 200 rotations of the wheels? Give the exact distance and approximations to the nearest inch and nearest foot.

c. If the wheels turn at 80 rpm, what is the angular speed?

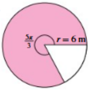
d. If the wheels turn at 80 rpm, what is the linear speed in inches per minute? Give the exact speed and an approximation to the nearest inch per minute.

e. If the wheels turn at 80 rpm, what is the linear speed in miles per hour? Round to the nearest mile per hour. (*Hint*: 1 ft = 12 in., 1 mi = 5280 ft, and 1 hr = 60 min.)

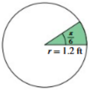
Learn: Compute the Area of a Sector of a Circle

For Exercises 95–98, find the exact area of the sector. Then round the result to the nearest tenth of a unit. (See Example 10, p. 460)

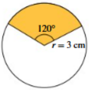
95.



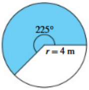
96.



97.



98.



99. A slice of a circular pizza 12 in. in diameter is cut into a wedge with a 45° angle. Find the area and round to the nearest tenth of a square inch.

100. A circular cheesecake 9 in. in diameter is cut into a slice with a 20° angle. Find the area and round to the nearest tenth of a square inch.

101. The back wiper blade on an SUV extends 3 in. from the pivot point to a distance of 17 in. from the pivot point. If the blade rotates through an angle of 175°, how much area does it cover? Round to the nearest square inch.

102. A robotic arm rotates through an angle of 160°. It sprays paint between a distance of 0.5 ft and 3 ft from the pivot point. Determine the amount of area that the arm makes. Round to the nearest square foot.

Mixed Exercises

For Exercises 103–108, find the (a) complement and (b) supplement of the given angle.

103. 16.21°

104. 49.87°

105. 18°13′37″

106. 22°9′54″

107. 9°42′7″

108. 82°15′3″

109. The second hand of a clock moves from 12:10 to 12:30.

- **a.** How many degrees does it move during this time?
- **b.** How many radians does it move during this time?
- **c.** If the second hand is 10 in. in length, determine the exact distance that the tip of the second hand travels during this time.
- **d.** Determine the exact angular speed of the second hand in radians per second.
- **e.** What is the exact linear speed (in inches per second) of the tip of the second hand?
- **f.** What is the amount of area that the second hand sweeps out during this time? Give the exact area in terms of π and then approximate to the nearest square inch.

110. The minute hand of a clock moves from 12:10 to 12:15.

- **a.** How many degrees does it move during this time?
- **b.** How many radians does it move during this time?
- **c.** If the minute hand is 9 in. in length, determine the exact distance that the tip of the minute hand travels during this time.
- **d.** Determine the exact angular speed of the minute hand in radians per minute.
- **e.** What is the exact linear speed (in inches per minute) of the tip of the minute hand?
- **f.** What is the amount of area that the minute hand sweeps out during this time? Give the exact area in terms of π and then approximate to the nearest square inch.

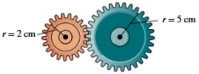
111. The Earth's orbit around the Sun is elliptical (oval shaped). However, the elongation is small, and for our discussion here, we take the orbit to be circular with a radius of approximately 93,000,000 mi.

- **a.** Find the linear speed (in mph) of the Earth through its orbit around the Sun. Round to the nearest hundred miles per hour.
- **b.** How far does the Earth travel in its orbit in one day? Round to the nearest thousand miles.

112. The Earth completes one full rotation around its axis (poles) each day.

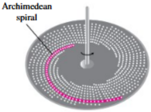
- **a.** Determine the angular speed (in radians per hour) of the Earth during its rotation around its axis.
- **b.** The Earth is nearly spherical with a radius of approximately 3960 mi. Find the linear speed of a point on the surface of the Earth rounded to the nearest mile per hour.

113. Two gears are calibrated so that the smaller gear drives the larger gear. For each rotation of the smaller gear, how many degrees will the larger gear rotate?



114. Two gears are calibrated so that the larger gear drives the smaller gear. The larger gear has a 6-in. radius, and the smaller gear has a 1.5-in. radius. For each rotation of the larger gear, by how many degrees will the smaller gear rotate?

115. A spinning-disc confocal microscope contains a rotating disk with multiple small holes arranged in a series of nested Archimedean spirals. An intense beam of light is projected through the holes, enabling biomedical researchers to obtain detailed video images of live cells. The spinning disk has a diameter of 55 mm and rotates at a rate of 1800 rpm. At the edge of the disk,

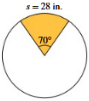


a. Find the angular speed.

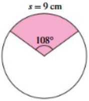
b. Find the linear speed. Round to the nearest whole unit.

For Exercises 116–119, approximate the area of the shaded region to 1 decimal place. In the figure, *s* represents arc length, and *r* represents the radius of the circle.

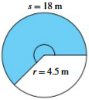
116. $s = 28$ in.



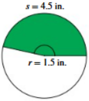
117. $s = 9$ cm



118. $s = 18$ m



119. $s = 4.5$ in.

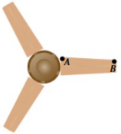


Write About It

120. Explain what is meant by 1 radian. Explain what is meant by 1°.

121. For an angle drawn in standard position, explain how to determine in which quadrant the terminal side lies.

122. As the fan rotates (see figure), which point *A* or *B* has a greater angular speed? Which point has a greater linear speed? Why?



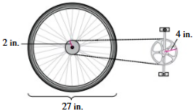
123. If an angle of a sector is held constant, but the radius is doubled, how will the arc length of the sector and area of the sector be affected?

124. If an angle of a sector is doubled, but the radius is held constant, how will the arc length of the sector and the area of the sector be affected?

Expanding Your Skills

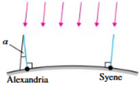
125. When a person pedals a bicycle, the front sprocket moves a chain that drives the back wheel and propels the bicycle forward. For each rotation of the front sprocket, the chain moves a distance equal to the circumference of the front sprocket. The back sprocket is smaller, so it will simultaneously move through a greater rotation. Furthermore, since the back sprocket is rigidly connected to the back wheel, each rotation of the back sprocket generates a rotation of the wheel.

Suppose that the front sprocket of a bicycle has a 4-in. radius and the back sprocket has a 2-in. radius.



- **a.** How much chain will move with one rotation of the pedals (front sprocket)?
- **b.** How many times will the back sprocket turn with one rotation of the pedals?
- **c.** How many times will the wheels turn with one rotation of the pedals?
- **d.** If the wheels are 27 in. in diameter, how far will the bicycle travel with one rotation of the pedals?
- **e.** If the bicyclist pedals 80 rpm, what is the linear speed (in ft/min) of the bicycle?
- **f.** If the bicyclist pedals 80 rpm, what is the linear speed (in mph) of the bicycle? (*Hint*: 1 mi = 5280 ft, and 1 hr = 60 min)

126. In the third century B.C., the Greek astronomer Eratosthenes approximated the Earth's circumference. On the summer solstice at noon in Alexandria, Egypt, Eratosthenes measured the angle α of the Sun relative to a line perpendicular to the ground. At the same time in Syene (now Aswan), located on the Tropic of Cancer, the Sun was directly overhead.



- **a.** If $\alpha = \frac{1}{50}$ of a circle, find the measure of α in degrees. (In Eratosthenes' time, the degree measure had not yet been defined.)
- **b.** If the distance between Alexandria and Syene is 5000 stadia, find the circumference of the Earth measured in stadia.
- **c.** If 10 stadia $\approx$ 1 mi, find Eratosthenes' approximation of the circumference of the Earth in miles (the modern-day approximation at the equator is 24,900 mi).

127. The space shuttle program involved 135 crewed space flights in 30 yr. In addition to supplying and transporting astronauts to the International Space Station, space shuttle missions serviced the Hubble Space Telescope and deployed satellites. For a particular mission, a space shuttle orbited the Earth in 1.5 hr at an altitude of 200 mi.

- **a.** Determine the angular speed (in radians per hour) of the shuttle.
- **b.** Determine the linear speed of the shuttle in miles per hour. Assume that the Earth's radius is 3960 mi. Round to the nearest hundred miles per hour.

128. What is the first time (to the nearest second) after 12:00 midnight for which the minute hand and hour hand of a clock make a 120° angle?

Technology Connections

For Exercises 129–130, use the functions in the ANGLE menu on your calculator to

a. Convert from decimal degrees to DMS (degree-minute-second) form and

b. Convert from DMS form to decimal degrees.

On some calculators, the ″ symbol is accessed by hitting ALPHA followed by +.

129. a. –216.479° **b.** 42°13′5.9″

130. a. –14.908° **b.** 71°19′4.7″

For Exercises 131–132, use a calculator to convert from degrees to radians.

131. a. 147°26′9″ **b.** –228.459°

132. a. 36°4′47″ **b.** –25.716°

For Exercises 133–134, use a calculator to convert from radians to degrees.

133. a. $\frac{4\pi}{9}$ **b.** –5.718

134. a. $\frac{11\pi}{18}$ **b.** –1.356

Point of Interest

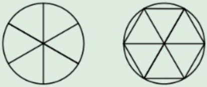
Why 60?

One hour may be subdivided into 60 minutes, and each minute may be further divided into 60 seconds. The 360 degrees in a circle may be similarly subdivided by dividing each degree into 60 minutes and each minute into 60 seconds. (These smaller measurements of angle are sometimes referred to as *arcminutes* and *arcseconds* to avoid confusion with time measurements.) Both the measurements for time and for angles are based on the number 60. A number system based on 60 is called *sexagesimal*.

The base 60 system used for both time and degree measure is usually attributed to the ancient Sumerians and Babylonians or possibly Egyptians, although it is difficult to be sure of a specific origin. One advantage of a base 60 system is that there are many divisors of 60: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60.

135. Perhaps old habits die hard, but why not use 100 minutes in an hour? Why not consider 100 degrees for a circle? Interestingly, in 1793, during the French Revolution, a new system of time was declared called *French Revolutionary Time*. Timepieces during that period had a 10-hour day with 100 minutes in an hour and 100 seconds in a minute. Even today, there are still manufacturers that make watches and clocks that keep decimal time. Write a short report on *French Revolutionary Time* and the timepieces created to keep time in this fashion.

136. When a circle is divided into 6 parts, what is the measure (in degrees) of each central angle? If we inscribe a regular hexagon in a circle of radius r, what is the length of each side of the hexagon?



137. For a circle of radius r, the length s of an arc intercepted by a central angle θ (in radians) is given by $s = r\theta$. What would be an equivalent formula if θ is measured in degrees?

Go online for more practice problems.

Lesson 4-2

Trigonometric Functions Defined on the Unit Circle

What Will You Learn?

After completing this lesson, you should be able to:

- Evaluate Trigonometric Functions Using the Unit Circle
- Identify the Domains of the Trigonometric Functions
- Use Fundamental Trigonometric Identities
- Apply the Periodic and Even and Odd Function Properties of Trigonometric Functions
- Approximate Trigonometric Functions on a Calculator

Learn: Evaluate Trigonometric Functions Using the Unit Circle

The functions we have studied so far have been used to model phenomena such as exponential growth and decay, projectile motion, and profit and cost, to name a few. However, none of the functions in our repertoire represent cyclical behavior such as the orbits of the Moon and planets, the variation in air pressure that produces sound, the back-and-forth oscillations of a stretched spring, and so on. To model these behaviors, we introduce six new functions called *trigonometric functions*.

Historically, the study of trigonometry arose from the need to study relationships among the angles and sides of a triangle. An alternative approach is to define trigonometric functions as *circular functions*. To begin, we define the **unit circle** as the circle of radius 1 unit, centered at the origin. The unit circle consists of all points (x, y) that satisfy the equation $x^2 + y^2 = 1$. For example, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is a point on the unit circle because $\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$ (Figure 4-14).

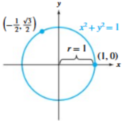


Figure 4-14

Insights

The circumference of the unit circle is $2\pi r = 2\pi(1) = 2\pi$. So, the value $t = \pi$ represents one-half of a revolution and falls on the point (–1, 0). Likewise the value $t = \frac{\pi}{2}$ represents one-quarter of a revolution and falls on the point (0, 1).

Now suppose we wrap the real number line around the unit circle by placing the point $t = 0$ from the number line on the point (1, 0) on the circle. Positive real numbers from the number line ($t > 0$) wrap onto the unit circle in the counterclockwise direction. Negative numbers from the number line ($t < 0$) wrap onto the unit circle in the clockwise direction (Figure 4-15). For a value of t greater than 2π (or less than -2π) more than one revolution around the unit circle is required.

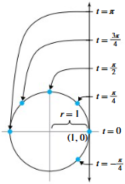


Figure 4-15

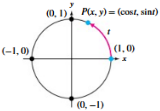
The purpose of wrapping the real number line around the unit circle is to associate each real number t with a unique point (x, y) on the unit circle.

For a real number t corresponding to a point $P(x, y)$ on the unit circle, we can use the coordinates of P to define six trigonometric functions of t (Table 4-1). Notice that rather than using letters such as f, g, h, and so on, trigonometric functions are given the word names sine, cosine, tangent, cosecant, secant, and cotangent. These functions are abbreviated as "sin," "cos," "tan," "csc," "sec," and "cot," respectively. The value t is the input value or **argument** of each function.

Let $P(x, y)$ be the point associated with a real number t measured along the circumference of the unit circle from the point (1, 0).

TABLE 4-1 Unit Circle Definitions of the Trigonometric Functions

Function Name	Definition
sine	$\sin t = y$
cosine	$\cos t = x$
tangent	$\tan t = \frac{y}{x}$ $(x \neq 0)$
cosecant	$\csc t = \frac{1}{y}$ $(y \neq 0)$
secant	$\sec t = \frac{1}{x}$ $(x \neq 0)$
cotangent	$\cot t = \frac{x}{y}$ $(y \neq 0)$



Note:

- If P is on the y-axis [either (0, 1) or (0, −1)], then $x = 0$ and the tangent and secant functions are undefined.
- If P is on the x-axis [either (1, 0) or (−1, 0)], then $y = 0$ and the cotangent and cosecant functions are undefined.

Example 1: Evaluating Trigonometric Functions

Suppose that the real number t corresponds to the point $P\left(-\frac{2}{3}, -\frac{\sqrt{5}}{3}\right)$ on the unit circle. Evaluate the six trigonometric functions of t.

Solution:

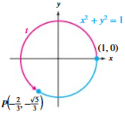
$$\sin t = y = -\frac{\sqrt{5}}{3} \qquad \cos t = x = -\frac{2}{3}$$

$$\tan t = \frac{y}{x} = \frac{-\frac{\sqrt{5}}{3}}{-\frac{2}{3}} = -\frac{\sqrt{5}}{3} \cdot \left(-\frac{3}{2}\right) = \frac{\sqrt{5}}{2}$$

$$\csc t = \frac{1}{y} = \frac{1}{-\frac{\sqrt{5}}{3}} = -\frac{3}{\sqrt{5}} = -\frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

$$\sec t = \frac{1}{x} = \frac{1}{-\frac{2}{3}} = -\frac{3}{2}$$

$$\cot t = \frac{x}{y} = \frac{-\frac{2}{3}}{-\frac{\sqrt{5}}{3}} = -\frac{2}{3} \cdot \left(-\frac{3}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$



Insights

In the figure in Example 1, t was arbitrarily taken to be positive as noted by the red arc wrapping counterclockwise. However, we can just as easily use a negative value of t, terminating at the same point in Quadrant III.

Apply the Skills

1. Suppose that the real number t corresponds to the point $P\left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$ on the unit circle. Evaluate the six trigonometric functions of t.

Let $P(x, y)$ be the point on the unit circle associated with a real number $t \geq 0$. Let $\theta \geq 0$ be the central angle in standard position measured in radians with terminal side through P and arc length t (Figure 4-16). Because the radius of the unit circle is 1, the arc length formula $s = r\theta$ becomes $t = 1 \cdot \theta$ or simply $t = \theta$.

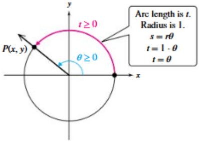


Figure 4-16

A similar argument can be made for $t < 0$ and $\theta < 0$. The arc length formula is $s = r(-\theta)$, or equivalently $-t = r(-\theta)$, which also implies that $t = \theta$ (Figure 4-17).

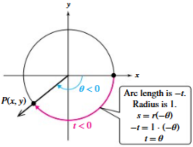


Figure 4-17

Check Your Work

If θ is negative, then the opposite of θ is used in the arc length formula to ensure that the length of the arc is positive.

From this discussion, we have the following important result. The real number t taken along the circumference of the unit circle gives the radian measure of the corresponding central angle. That is, $\theta = t$ radians. Furthermore, this one-to-one correspondence between the real number t and the radian measure of the central angle θ means that the trigonometric functions can also be defined as functions of θ.

Trigonometric Functions of Real Numbers and Angles

If $\theta = t$ radians, then

$\sin t = \sin\theta$	$\cos t = \cos\theta$	$\tan t = \tan\theta$
$\csc t = \csc\theta$	$\sec t = \sec\theta$	$\cot t = \cot\theta$

We now want to determine the values of the trigonometric functions for several "special" values of t corresponding to central angles θ that are integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Consider the point $P(x, y)$ on the unit circle corresponding to $t = \frac{\pi}{4}$ (Figure 4-18). Point P lies on the line $y = x$, and the coordinates of P can be written as (x, x).

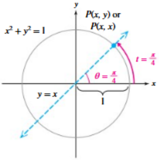


Figure 4-18

Substituting the coordinates of (x, x) into the equation $x^2 + y^2 = 1$, we have

$$\begin{aligned} x^2 + x^2 &= 1 \\ 2x^2 &= 1 \\ x^2 &= \frac{1}{2} \\ x &= \sqrt{\frac{1}{2}} \\ x &= \frac{1}{\sqrt{2}} \end{aligned}$$

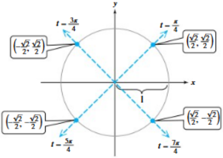
Choose x positive for a first quadrant point.

Rationalizing the denominator, we have $x = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

Since $y = x$, point P has coordinates

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ or } \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

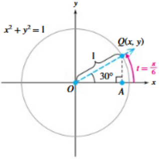
From the symmetry of the circle, the points corresponding to $t = \frac{3\pi}{4}$, $t = \frac{5\pi}{4}$, and $t = \frac{7\pi}{4}$ have coordinates $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, respectively (Figure 4-19).



y
$t = \frac{3\pi}{4}$
$t = \frac{\pi}{4}$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
x
1
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$t = \frac{5\pi}{4}$
$t = \frac{7\pi}{4}$

Figure 4-19

Now consider the point $Q(x, y)$ on the unit circle corresponding to $t = \frac{\pi}{6}$ (Figure 4-20). Dropping a line segment from Q perpendicular to the x-axis at point A, we construct a right triangle ($\triangle OAQ$). The acute angles in $\triangle OAQ$ are 30° and 60°, and the hypotenuse of the triangle is 1 unit.



y
$x^2 + y^2 = 1$
$Q(x, y)$
1
$t = \frac{\pi}{6}$
30°
O
A
x

Figure 4-20

Placing two such triangles adjacent to one another on opposite sides of the x-axis, we have an equilateral triangle ($\triangle OBQ$) with sides of 1 unit and angles of 60° (Figure 4-21). From $\triangle OBQ$ we have $2y = 1$. So, $y = \frac{1}{2}$.

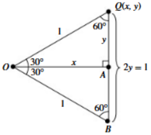

$Q(x, y)$
1
60°
y
O
30°
30°
x
A
$2y = 1$
1
60°
B

Figure 4-21

Since $Q(x, y)$ is a point on the unit circle, we can substitute $y = \frac{1}{2}$ into the equation $x^2 + y^2 = 1$ and solve for x.

$$x^2 + y^2 = 1$$
$$x^2 + \left(\frac{1}{2}\right)^2 = 1$$
$$x^2 + \frac{1}{4} = 1$$
$$x^2 = \frac{3}{4}$$
$$x = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

So, Q has coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Using symmetry, we can also find coordinates of the points on the unit circle corresponding to $t = \frac{5\pi}{6}$, $t = \frac{7\pi}{6}$, and $t = \frac{11\pi}{6}$ (Figure 4-22).

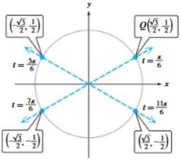
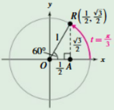

y
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$Q\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$t = \frac{5\pi}{6}$
$t = \frac{\pi}{6}$
x
$t = \frac{7\pi}{6}$
$t = \frac{11\pi}{6}$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

Figure 4-22

Using similar reasoning, we can show that the point $R(x, y)$ corresponding to $t = \frac{\pi}{3}$ has coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Insights

The value $t = \frac{\pi}{3}$ corresponds to the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle. $\triangle OAR$ is congruent to the triangle shown in Figure 4-20, but is oriented with the 60° angle in standard position rather than the 30° angle in standard position. As a result, notice that the x- and y-coordinates of points Q and R are reversed.


y
$R\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
1
$\frac{\sqrt{3}}{2}$
$t = \frac{\pi}{3}$
60°
O
$\frac{1}{2}$
A
x

We summarize our findings in Figure 4-23 for selected values of *t* and the corresponding central angle θ.

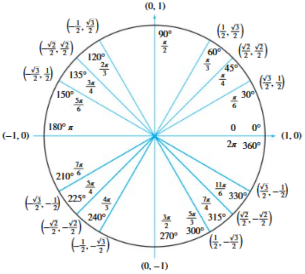

(0, 1)
90°
$\frac{\pi}{2}$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
60°
$\frac{\pi}{3}$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
45°
$\frac{\pi}{4}$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
30°
$\frac{\pi}{6}$
0 0°
2π 360°
(1, 0)
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
120°
$\frac{2\pi}{3}$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
135°
$\frac{3\pi}{4}$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
150°
$\frac{5\pi}{6}$
180° π
(−1, 0)
210° $\frac{7\pi}{6}$
$\frac{5\pi}{4}$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
225°
$\frac{4\pi}{3}$
240°
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\frac{3\pi}{2}$
270°
$\frac{5\pi}{3}$
300°
$\frac{7\pi}{4}$
315°
$\frac{11\pi}{6}$
330°
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
(0, −1)

Figure 4-23

Figure 4-23 looks daunting, but using the symmetry of the circle and recognizing several patterns can make the coordinates of these special points easy to determine. First we recommend memorizing the coordinates of the three special points in Quadrant I.

- Note that for $t = \frac{\pi}{4}$, the *x*- and *y*-coordinates are both $\frac{\sqrt{2}}{2}$, or equivalently $\frac{1}{\sqrt{2}}$.
- For $t = \frac{\pi}{6}$ and $t = \frac{\pi}{3}$, the *x*- and *y*-coordinates are reversed.
- The value $\frac{\sqrt{3}}{2} \approx 0.866$ is greater than the value $\frac{1}{2} = 0.5$. For $t = \frac{\pi}{6}$, the *x*-coordinate is greater than the *y*-coordinate. Therefore, *x* must be $\frac{\sqrt{3}}{2}$. For $t = \frac{\pi}{3}$, the *x*-coordinate is less than the *y*-coordinate. Therefore, *x* must be $\frac{1}{2}$.
- For the special points in Quadrants II, III, and IV, use the values $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, and $\frac{\sqrt{2}}{2}$ with the appropriate signs attached.

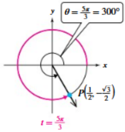
Example 2: Evaluating Trigonometric Functions

Evaluate the six trigonometric functions of the real number t.

a. $t = \frac{5\pi}{3}$ **b.** $t = -\frac{5\pi}{4}$

Solution:

a. $t = \frac{5\pi}{3}$ corresponds to the point $(x, y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ on the unit circle.



y
$\theta = \frac{5\pi}{3} = 300°$
x
$P\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$t = \frac{5\pi}{3}$

$$\sin\frac{5\pi}{3} = y = -\frac{\sqrt{3}}{2}$$

$$\cos\frac{5\pi}{3} = x = \frac{1}{2}$$

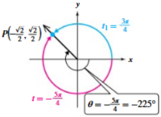
$$\tan\frac{5\pi}{3} = \frac{y}{x} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2} \cdot \frac{2}{1} = -\sqrt{3}$$

$$\csc\frac{5\pi}{3} = \frac{1}{y} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\sec\frac{5\pi}{3} = \frac{1}{x} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot\frac{5\pi}{3} = \frac{x}{y} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2} \cdot \left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

b. Since the circumference of the unit circle is 2π, the values $t = -\frac{5\pi}{4}$ and $t_1 = -\frac{5\pi}{4} + 2\pi = \frac{3\pi}{4}$ have the same location in Quadrant II on the unit circle. Both correspond to the point $(x, y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, or equivalently $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.



y
$t_1 = \frac{3\pi}{4}$
$P\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
x
$t = -\frac{5\pi}{4}$
$\theta = -\frac{5\pi}{4} = -225°$

$$\sin\left(-\frac{5\pi}{4}\right) = y = \frac{\sqrt{2}}{2}$$

$$\cos\left(-\frac{5\pi}{4}\right) = x = -\frac{\sqrt{2}}{2}$$

$$\tan\left(-\frac{5\pi}{4}\right) = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2} \cdot \left(-\frac{2}{\sqrt{2}}\right) = -1$$

$$\csc\left(-\frac{5\pi}{4}\right) = \frac{1}{y} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

$$\sec\left(-\frac{5\pi}{4}\right) = \frac{1}{x} = \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$$

$$\cot\left(-\frac{5\pi}{4}\right) = \frac{x}{y} = \frac{\frac{-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -\frac{\sqrt{2}}{2} \cdot \frac{2}{\sqrt{2}} = -1$$

Apply the Skills

Evaluate the six trigonometric functions of the real number t.

a. $t = \frac{5\pi}{6}$ **b.** $t = -\frac{3\pi}{4}$

Learn: Identify the Domains of the Trigonometric Functions

From Table 4-1, $\sin t = y$ and $\cos t = x$ have no restrictions on their domain. However, $\tan t = \frac{y}{x}$ and $\sec t = \frac{1}{x}$ are undefined for all values of t corresponding to the points (0, 1) and (0, −1) on the unit circle. These are $t = \frac{\pi}{2}$, $t = \frac{3\pi}{2}$, and all other odd multiples of $\frac{\pi}{2}$ (Figure 4-24).

- $\tan t$ and $\sec t$ are undefined for $\ldots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{(2n+1)\pi}{2}$, for all integers n.
- Likewise, $\cot t = \frac{x}{y}$ and $\csc t = \frac{1}{y}$ are undefined for all values of t corresponding to the points (1, 0) and (−1, 0) on the unit circle. These are $t = 0$, $t = \pi$, $t = 2\pi$, and all other multiples of π (Figure 4-24).
- $\cot t$ and $\csc t$ are undefined for $\ldots -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots, n\pi$ for all integers n.

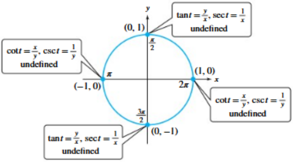

y
x
(0, 1)
π/2
tan t = y/x, sec t = 1/x undefined
cot t = x/y, csc t = 1/y undefined
π
(−1, 0)
(1, 0)
2π
cot t = x/y, csc t = 1/y undefined
3π/2
(0, −1)
tan t = y/x, sec t = 1/x undefined

Figure 4-24

Insights

For integer values of n, the formula $\frac{(2n+1)\pi}{2}$ generates odd multiples of $\frac{\pi}{2}$.

For example, if $n = 6$:

$$\frac{[2(6)+1]\pi}{2} = \frac{13\pi}{2}.$$

The formula $n\pi$ generates multiples of π.

For example, if $n = 6$:

$$nx = (6)\pi = 6\pi.$$

Domains of the Trigonometric Functions

Function	**Domain***	**Notes**
$f(t) = \sin t$	All real numbers	No restrictions
$f(t) = \cos t$	All real numbers	No restrictions
$f(t) = \tan t$	$\left\{t \middle\vert t \neq \frac{(2n+1\pi)}{2} \text{ for all integers } n\right\}$	Exclude odd multiples of $\frac{\pi}{2}$.
$f(t) = \cot t$	$\{t \mid t \neq n\pi \text{ for all integers } n\}$	Exclude multiples of π.
$f(t) = \sec t$	$\left\{t \middle\vert t \neq \frac{2n+1\pi}{2} \text{ for all integers } n\right\}$	Exclude odd multiples of $\frac{\pi}{2}$.
$f(t) = \csc t$	$\{t \mid t \neq n\pi \text{ for all integers } n\}$	Exclude multiples of π.

* The range of each trigonometric function will be discussed in Lessons 4.5 and 4.6 when we cover the graphs of the functions.

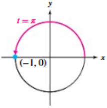
Example 3: Evaluate the Trigonometric Functions for Given Values of *t*

Evaluate the six trigonometric functions of the real number *t*.

a. $t = \pi$ **b.** $t = \frac{5\pi}{2}$

Solution:

a. The value $t = \pi$ corresponds to $(-1, 0)$ on the unit circle.

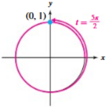


$\sin \pi = y = 0$ $\csc \pi = \frac{1}{y}$ is undefined because $\frac{1}{0}$ is undefined.

$\cos \pi = x = -1$ $\sec \pi = \frac{1}{x} = \frac{1}{-1} = -1$

$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0$ $\cot \pi = \frac{x}{y}$ is undefined because $\frac{-1}{0}$ is undefined.

b. The circumference of the unit circle is 2π. Therefore, since $t = \frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$, the value $t = \frac{5\pi}{2}$ corresponds to the point $(0, 1)$ on the unit circle.



$\sin \frac{5\pi}{2} = y = 1$ $\csc \frac{5\pi}{2} = \frac{1}{y} = \frac{1}{1} = 1$

$\cos \frac{5\pi}{2} = x = 0$ $\sec \frac{5\pi}{2} = \frac{1}{x}$ is undefined because $\frac{1}{0}$ is undefined.

$\tan \frac{5\pi}{2} = \frac{x}{y}$ is undefined because $\frac{1}{0}$ is undefined. $\cot \frac{5\pi}{2} = \frac{x}{y} = \frac{0}{1} = 0$

Apply the Skills

3. Evaluate the six trigonometric functions of the real number *t*.

a. $t = -2\pi$ **b.** $\frac{3\pi}{2}$

Insights

Determining the location of a real number *t* wrapped onto the unit circle is equivalent to finding the point where the terminal side of the central angle θ intercepts the unit circle. This is true for θ measured in radians and drawn in standard position. In Example 3(b), the angle $\theta = \frac{5\pi}{2}$ is coterminal to $\frac{\pi}{2}$, which intercepts the unit circle at (0, 1).

Learn: Use Fundamental Trigonometric Identities

You may have already noticed several relationships among the six trigonometric functions that follow directly from their definitions. For example, for a real number t corresponding to a point (x, y) on the unit circle, $\sin t = y$ and $\csc t = \frac{1}{y}$. So, the sine and cosecant functions are reciprocals for all t in their common domain. Table 4-2 summarizes the reciprocal relationships among the trigonometric functions along with two important quotient properties. These relationships should be committed to memory.

Check Your Work

It is important to note that the argument *must be* included when denoting the value of a trigonometric function. For example, we write "$\tan t = \frac{\sin t}{\cos t}$" not "$\tan = \frac{\sin}{\cos}$".

TABLE 4-2 Reciprocal and Quotient Identities

Trigonometric Functions	Relationship
$\csc t = \frac{1}{\sin t}$ or $\sin t = \frac{1}{\csc t}$	$\sin t$ and $\csc t$ are reciprocals.
$\sec t = \frac{1}{\cos t}$ or $\cos t = \frac{1}{\sec t}$	$\cos t$ and $\sec t$ are reciprocals.
$\cot t = \frac{1}{\tan t}$ or $\tan t = \frac{1}{\cot t}$	$\tan t$ and $\cot t$ are reciprocals.
$\tan t = \frac{\sin t}{\cos t}$	$\tan t$ is the ratio of $\sin t$ and $\cos t$.
$\cot t = \frac{\cos t}{\sin t}$	$\cot t$ is the ratio of $\cos t$ and $\sin t$.

Example 4: Using the Reciprocal and Quotient Identities

Given that $\sin t = \frac{5}{8}$ and $\cos t = \frac{\sqrt{39}}{8}$, use the reciprocal and quotient identities to find the values of the other trigonometric functions of t.

Solution:

Given the values of $\sin t$ and $\cos t$, we can use the quotient identities to find $\tan t$ and $\cot t$.

$$\tan t = \frac{\sin t}{\cos t} = \frac{\frac{5}{8}}{\frac{\sqrt{39}}{8}} = \frac{5}{8} \cdot \frac{8}{\sqrt{39}} = \frac{5}{\sqrt{39}} = \frac{5}{\sqrt{39}} \cdot \frac{\sqrt{39}}{\sqrt{39}} = \frac{5\sqrt{39}}{39}$$

$$\cot t = \frac{\cos t}{\sin t} = \frac{\frac{\sqrt{39}}{8}}{\frac{5}{8}} = \frac{\sqrt{39}}{8} \cdot \frac{8}{5} = \frac{\sqrt{39}}{5}$$

The remaining two functions can be found by using the reciprocal identities.

$$\csc t = \frac{1}{\sin t} = \frac{1}{\frac{5}{8}} = \frac{8}{5}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{\sqrt{39}}{8}} = \frac{8}{\sqrt{39}} = \frac{8}{\sqrt{39}} \cdot \frac{\sqrt{39}}{\sqrt{39}} = \frac{8\sqrt{39}}{39}$$

Apply the Skills

4. Given $\sin t = \frac{3}{7}$ and $\cos t = \frac{2\sqrt{10}}{7}$ use the reciprocal and quotient identities to find the values of the other trigonometric functions of t.

Consider a real number t corresponding to the point $P(x, y)$ on the unit circle. Since $\cos t = x$ and $\sin t = y$, point P can be labeled $P(\cos t, \sin t)$. See Figure 4-25. Furthermore, since P is on the unit circle, it satisfies the equation $x^2 + y^2 = 1$. So,

$(\cos t)^2 + (\sin t)^2 = 1$ or $(\sin t)^2 + (\cos t)^2 = 1$.

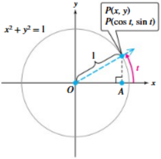


Figure 4-25

It is universally preferred to write the exponent associated with a trigonometric function between the name of the function and its argument. For example, $(\sin t)^2$ is most often written as $\sin^2 t$. So, the relationship $(\sin t)^2 + (\cos t)^2 = 1$ is written as $\sin^2 t + \cos^2 t = 1$. This relationship is called a "Pythagorean identity" because $\sin^2 t + \cos^2 t = 1$ is a statement of the Pythagorean theorem for $\triangle OAP$.

Table 4-3 summarizes three Pythagorean identities. The second and third relationships can be derived by dividing both sides of the equation $\sin^2 t + \cos^2 t = 1$ by $\cos^2 t$ and $\sin^2 t$, respectively (see Exercises 47 and 48).

TABLE 4-3 Pythagorean Identities

Trigonometric Functions	Pythagorean Identity
$\sin t$ and $\cos t$	$\sin^2 t + \cos^2 t = 1$
$\tan t$ and $\sec t$	$\tan^2 t + 1 = \sec^2 t$
$\cot t$ and $\csc t$	$1 + \cot^2 t = \csc^2 t$

Example 5: Using the Pythagorean Identities

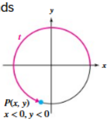
Given that $\tan t = \frac{12}{5}$ for $\pi < t < \frac{3\pi}{2}$, use an appropriate identity to find the value of $\sec t$.

Solution:

$\tan^2 t + 1 = \sec^2 t$	The real number t corresponds to a point $P(x, y)$ on the unit circle in Quadrant III. Since the value of $\tan t$ is known, we can use the relationship $\tan^2 t + 1 = \sec^2 t$ to find the value of $\sec t$.

$\sec t = \pm\sqrt{\tan^2 t + 1}$

$\sec t = -\sqrt{\tan^2 t + 1}$ Since t is between π and $\frac{3\pi}{2}$, it corresponds to a point $P(x, y)$ on the unit circle in Quadrant III. Since P is in Quadrant III, the values of both x and y are negative. So, $\sec t = \frac{1}{x}$ is negative.



$$\sec t = -\sqrt{\left(\frac{12}{5}\right)^2 + 1}$$
$$= -\sqrt{\frac{144}{25} + \frac{25}{25}} = -\sqrt{\frac{169}{25}} = -\frac{13}{5}$$

Apply the Skills

5. Given that $\csc t = \frac{5}{4}$ for $\frac{\pi}{2} < t < \pi$, use an appropriate identity to find the value of $\cot t$.

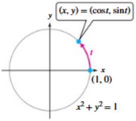
Sometimes it is beneficial to express one trigonometric function in terms of another. This is demonstrated in Example 6.

Example 6: Expressing a Trigonometric Function in Terms of Another Trigonometric Function

For a given real number t, express $\sin t$ in terms of $\cos t$.

Solution:

Let $P(x, y) = (\cos t, \sin t)$ be the point on the unit circle determined by t.



$x^2 + y^2 = 1$ Equation of unit circle

$\cos^2 t + \sin^2 t = 1$ Substitute $x = \cos t$ and $y = \sin t$.

$\sin^2 t = 1 - \cos^2 t$

$\sin t = \pm\sqrt{1 - \cos^2 t}$ The sign is determined by the quadrant in which P lies.

Apply the Skills

6. For a given real number t, express $\tan t$ in terms of $\sec t$.

Learn: Apply the Periodic and Even and Odd Function Properties of Trigonometric Functions

Many cyclical patterns occur in nature such as the changes of the seasons, the rise and fall of the tides, and the phases of the Moon. In many cases, we can predict these behaviors by determining the period of one complete cycle. For example, an observer on the Earth would note that the Moon changes from a new moon to a full moon and back again to a new moon in approximately 29.5 days. If $m(t)$ represents the percentage of the Moon seen on day t, then

$$m(t + 29.5) = m(t) \text{ for all } t \text{ in the domain of } m.$$

We say that function m is periodic because it repeats at regular intervals. In this example, the period is 29.5 days because this is the shortest time required to complete one full cycle.

Definition of a Periodic Function

A function f is **periodic** if $f(t + p) = f(t)$ for some constant p.

The smallest positive value p for which f is periodic is called the **period** of f.

The values of the six trigonometric functions of t are determined by the corresponding point $P(x, y)$ on the unit circle. Since the circumference of the unit circle is 2π, adding (or subtracting) 2π to t results in the same terminal point (x, y). Consequently, the values of the trigonometric functions are the same for t and $t + 2n\pi$.

In Exercises 121 and 122, we show that the sine and cosine functions are periodic with period 2π. Likewise, their reciprocal functions, cosecant and secant, are periodic with period 2π. However, the period of the tangent and cotangent functions is π, which we show in Exercises 119 and 120.

Periodic Properties of Trigonometric Functions

Function	Period	Property
Sine	2π	$\sin(t + 2\pi) = \sin t$
Cosine	2π	$\cos(t + 2\pi) = \cos t$
Cosecant	2π	$\csc(t + 2\pi) = \csc t$
Secant	2π	$\sec(t + 2\pi) = \sec t$
Tangent	π	$\tan(t + \pi) = \tan t$
Cotangent	π	$\cot(t + \pi) = \cot t$

Given a periodic function f, if the period is p, then $f(t + p) = f(t)$. It is also true that $f(t + np) = f(t)$ for any integer n. That is, adding any integer multiple of the period to a domain element of a periodic function results in the same function value. This is demonstrated in Example 7.

Example 7: Applying the Periodic Properties of the Trigonometric Functions

Given $\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$, determine the value of $\sin\frac{49\pi}{12}$.

Solution:

The period of the sine function is 2π or equivalently $\frac{24\pi}{12}$. Therefore, adding or subtracting any multiple of $\frac{24\pi}{12}$ to the argument $\frac{\pi}{12}$ results in the same value of the sine function.

$$\sin\frac{49\pi}{12} = \sin\left(\frac{\pi}{12} + \frac{48\pi}{12}\right) = \sin\left[\frac{\pi}{12} + 2\left(\frac{24\pi}{12}\right)\right] = \sin\left(\frac{\pi}{12} + 2(2\pi)\right)$$
$$= \sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Apply the Skills

7. Given $\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$, determine the value of $\sin\left(-\frac{15\pi}{8}\right)$.

Recall that a function *f* is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. Suppose that t is a real number associated with a point $P(x, y)$ on the unit circle. Then $-t$ is associated with the point $Q(x, -y)$. See Figure 4-26.

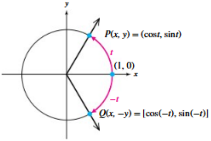


Figure 4-26

Notice that $\cos(-t) = \cos t = x$, so the cosine function is an even function. The sine function, however, is an odd function because $\sin t = y$, but $\sin(-t) = -y$. So, $\sin(-t) = -\sin t$. The tangent function is also an odd function because $\tan(-t) = \frac{-y}{x}$ and $\tan t = \frac{y}{x}$. Therefore, $\tan(-t) = -\tan t$. The functions secant, cosecant, and cotangent carry the same even and odd properties as their reciprocals: cosine, sine, and tangent, respectively.

Even and Odd Properties of Trigonometric Functions

Function	Evaluate at t and $-t$	Property
Sine	$\sin t = y$ and $\sin(-t) = -y$	$\sin(-t) = -\sin t$ (odd function)
Cosine	$\cos t = x$ and $\cos(-t) = x$	$\cos(-t) = \cos t$ (even function)
Cosecant	$\csc t = \frac{1}{y}$ and $\csc(-t) = \frac{1}{-y}$	$\csc(-t) = -\csc t$ (odd function)
Secant	$\sec t = \frac{1}{x}$ and $\sec(-t) = \frac{1}{x}$	$\sec(-t) = \sec t$ (even function)
Tangent	$\tan t = \frac{y}{x}$ and $\tan(-t) = \frac{-y}{x}$	$\tan(-t) = -\tan t$ (odd function)
Cotangent	$\cot t = \frac{x}{y}$ and $\cot(-t) = \frac{x}{-y}$	$\cot(-t) = -\cot t$ (odd function)

Example 8: Simplifying Expressions Using the Even-Odd Properties of Trigonometric Functions

Use the properties of the trigonometric functions to simplify.

a. $4\cos t + \cos(-t)$

b. $\tan(-3t) - \tan(3t + \pi)$

Solution:

a. $4\cos t + \cos(-t)$ The cosine function is even. $\cos(-t) = \cos t$

$= 4\cos t + \cos t$

$= 5\cos t$

b. $\tan(-3t) - \tan(3t + \pi)$ The tangent function is odd. $\tan(-3t) = -\tan 3t$

$= -\tan 3t - \tan 3t$ The period of the tangent function is π. Therefore, $\tan(3t + \pi) = \tan 3t$.

$= -2\tan 3t$

Check Your Work

A factor from the argument of a function cannot be factored out in front of the function. For example,

$$f(2x) \neq 2f(x)$$

In step 2 of Example 8(b), the value –1 that appears in front of the first term was not factored out, but rather is a result of the odd function property of the tangent function.

Apply the Skills

8. Use the properties of the trigonometric functions to simplify.

a. $\sec t - 2\sec(-t)$

b. $\sin(2t + 2\pi) - \sin(-2t)$

Learn: Approximate Trigonometric Functions on a Calculator

From Figure 4-23 (on page 34), we display the coordinates of the points on the unit circle that correspond to *t* values in multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. From this information we can find the values of the six trigonometric functions for these real numbers. However, it is often the case that the exact value of a trigonometric function for a real number *t* cannot be found analytically. In such a case, we can use a calculator to approximate the function values.

Example 9: Approximating Trigonometric Functions Using a Calculator

Use a calculator to approximate the function values. Round to 4 decimal places.

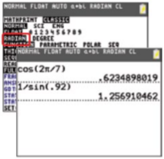
a. $\cos\frac{2\pi}{7}$

b. $\csc 0.92$

Solution:

The real number *t* corresponds to a point $P(x, y)$ on the unit circle. The value of *t* is the same as the radian measure of the central angle θ formed by the positive *x*-axis and the ray from the origin to point *P*. Therefore, we must first be sure that the calculator is in radian mode.

Graphing Calculator


NORMAL FLOAT AUTO a+bi RADIAN CL
MATHPRINT CLASSIC
NORMAL SCI ENG
FLOAT 0123456789
RADIAN DEGREE
FUNCTION PARAMETRIC POLAR SEQ

```
NORMAL FLOAT AUTO a+bi RADIAN CL
cos(2π/7)
                    .6234898019
1/sin(.92)
                    1.256910462
```

Scientific Calculator (radian mode)	Rounded
a. $2 \times \pi \div 7 =$ COS	0.6235
b. 0.92 SIN x^{-1}	1.2569

This takes the reciprocal of the sine function (cosecant).

Apply the Skills

9. Use a calculator to approximate the function values. Round to 4 decimal places.

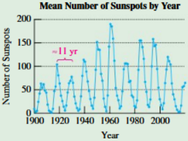
a. $\tan 1.4$

b. $\cot\frac{\pi}{8}$

Point of Interest (Cyclical Data)

In September 1859, a gigantic *coronal mass ejection* erupted from the Sun, sending a huge amount of electromagnetic radiation toward the Earth. This geomagnetic storm was the largest on record to have struck the Earth, causing telegraph lines to short out and compasses to go haywire. The aurora borealis or "northern lights" following the event could be seen as far south as Cuba.

Solar storms are natural phenomena that occur as a result of the rise and fall of the Sun's magnetic activity. Although the cause of solar storms is not completely understood, they seem to be cyclical. Scientists have established a correlation between solar storms and the number of sunspots that cycle over a period of approximately 11 yr. The effects of a massive solar storm today could bring potentially devastating disruption to power grids, satellite communication, and air travel. As a result, solar activity and "space weather" are studied intensely by NASA and NOAA.



(*Source*: SILSO data/image, Royal Observatory of Belgium, Brussels)

Practice 4-2

Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, determine if the function is even, odd, or neither.

R.1. $k(x) = 11x^3 + 12x$

R.2. $r(x) = \sqrt{25 - (x + 2)^2}$

R.3. $q(x) = \sqrt{23 + x^2}$

For Exercises R.4–R.5, write the domain of the function in set-builder notation.

R.4. $p(v) = \frac{v + 3}{v + 4}$

R.5. $k(q) = \frac{q^2}{2q^2 + 3q - 27}$

Concept Connections

1. The graph of $x^2 + y^2 = 1$ is known as the ________ circle. It has a radius of length _____ and center at the ______________.

2. The circumference of the unit circle is ______ . So, the value $t = \frac{\pi}{2}$ represents ______ of a revolution.

3. When $\tan t = 0$, the value of $\cot t$ is _____________.

4. The value of $\sec t = \frac{1}{\cos t}$. Therefore, when $\cos t = 0$, the value of $\sec t$ is _____________.

5. On the interval $[0, 2\pi)$, $\sin t = 0$ for $t =$ _____ and $t =$ _____. Since $\csc t = \frac{1}{\sin t}$, the value of $\csc t$ is ___________ for these values of t.

6. The domain of the trigonometric functions ________ and __________ is all real numbers.

7. Which trigonometric functions have domain $\left\{t \,\middle|\, t \neq \frac{(2n + 1)\pi}{2} \text{ for all integers } n\right\}$?

8. A function f is _____________ if $f(t + p) = f(t)$ for some constant p.

9. The period of the tangent and cotangent functions is ______. The period of the sine, cosine, cosecant, and secant functions is ______.

10. The cosine function is an ___________ function because $\cos(-t) = \cos t$. The sine function is an __________ function because $\sin(-t) = -\sin t$.

Learn: Evaluate Trigonometric Functions Using the Unit Circle

For Exercises 11–14, determine if the point lies on the unit circle.

11. $\left(\frac{\sqrt{10}}{5}, \frac{\sqrt{15}}{5}\right)$

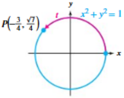
12. $\left(\frac{\sqrt{61}}{8}, -\frac{\sqrt{2}}{8}\right)$

13. $\left(-\frac{\sqrt{7}}{10}, -\frac{2\sqrt{23}}{10}\right)$

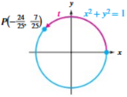
14. $\left(-\frac{2\sqrt{34}}{17}, \frac{3\sqrt{17}}{17}\right)$

For Exercises 15–18, the real number *t* corresponds to the point *P* on the unit circle. Evaluate the six trigonometric functions of *t*. (See Example 1, p. 472)

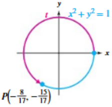
15.



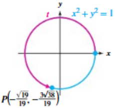
16.



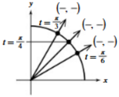
17.



18.

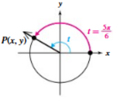

y
x
t
$x^2+y^2=1$
$P\left(-\frac{\sqrt{19}}{19}, -\frac{3\sqrt{38}}{19}\right)$

19. Fill in the ordered pairs on the unit circle corresponding to each real number t.

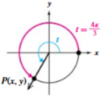

y
x
(–, –)
(–, –)
(–, –)
$t=\frac{\pi}{3}$
$t=\frac{\pi}{4}$
$t=\frac{\pi}{6}$

For Exercises 20–23, identify the coordinates of point *P*. Then evaluate the six trigonometric functions of *t*. (See Example 2, p. 477)

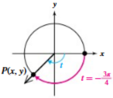
20.


y
x
P(x, y)
t
$t=\frac{5\pi}{6}$

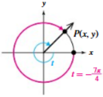
21.


y
x
P(x, y)
t
$t=\frac{4\pi}{3}$

22.


y
x
P(x, y)
t
$t=-\frac{3\pi}{4}$

23.


y
x
P(x, y)
t
$t=-\frac{7\pi}{4}$

For Exercises 24–26, identify the ordered pairs on the unit circle corresponding to each real number *t*.

24. **a.** $t=\frac{2\pi}{3}$
  b. $t=-\frac{5\pi}{4}$
  c. $t=\frac{5\pi}{6}$

25. **a.** $t=\frac{7\pi}{6}$
  b. $t=-\frac{3\pi}{4}$
  c. $t=\frac{4\pi}{3}$

26. **a.** $t=\frac{5\pi}{3}$
  b. $t=\frac{7\pi}{4}$
  c. $t=-\frac{\pi}{6}$

For Exercises 27–32, evaluate the trigonometric function at the given real number.

27. $f(t)=\cos t;\ t=\frac{2\pi}{3}$

28. $g(t)=\sin t;\ t=\frac{5\pi}{4}$

29. $h(t)=\cot t;\ t=\frac{11\pi}{6}$

30. $s(t)=\sec t;\ t=\frac{5\pi}{3}$

31. $z(t)=\csc t;\ t=\frac{7\pi}{4}$

32. $r(t)=\tan t;\ t=\frac{5\pi}{6}$

Learn: Identify the Domains of the Trigonometric Functions

For Exercises 33–38, select the domain of the trigonometric function.

a. All real numbers

b. $\left\{t \middle| t \neq \frac{(2n+1)\pi}{2} \text{ for all integers } n\right\}$

c. $\{t \mid t \neq n\pi \text{ for all integers } n\}$

33. $f(t)=\sin t$

34. $f(t)=\tan t$

35. $f(t)=\cot t$

36. $f(t)=\cos t$

37. $f(t)=\sec t$

38. $f(t)=\csc t$

For Exercises 39–42, evaluate the function if possible. (See Example 3, p. 479)

39. **a.** $\sin 0$ **b.** $\cot \pi$
c. $\tan 3\pi$ **d.** $\sec \pi$
e. $\csc 0$ **f.** $\cos \pi$

40. **a.** $\cos\left(-\frac{\pi}{2}\right)$ **b.** $\csc\frac{\pi}{2}$
c. $\cot\frac{5\pi}{2}$ **d.** $\tan\frac{\pi}{2}$
e. $\sec\frac{3\pi}{2}$ **f.** $\sin\frac{\pi}{2}$

41. **a.** $\sin\frac{3\pi}{2}$ **b.** $\cos\frac{7\pi}{2}$
c. $\tan\frac{3\pi}{2}$ **d.** $\csc\left(-\frac{\pi}{2}\right)$
e. $\sec 1.5\pi$ **f.** $\cot\frac{\pi}{2}$

42. **a.** $\cot 0$ **b.** $\cos 2\pi$
c. $\csc \pi$ **d.** $\tan 0$
e. $\sin(-3\pi)$ **f.** $\sec 0$

Learn: Use Fundamental Trigonometric Identities

For Exercises 43–46, given the values for $\sin t$ and $\cos t$, use the reciprocal and quotient identities to find the values of the other trigonometric functions of t. (See Example 4, p. 480)

43. $\sin t = \frac{\sqrt{5}}{3}$ and $\cos t = \frac{2}{3}$

44. $\sin t = \frac{3}{4}$ and $\cos t = \frac{\sqrt{7}}{4}$

45. $\sin t = -\frac{\sqrt{39}}{8}$ and $\cos t = -\frac{5}{8}$

46. $\sin t = \frac{28}{53}$ and $\cos t = -\frac{45}{53}$

For Exercises 47–48, derive the given identity from the Pythagorean identity, $\sin^2 t + \cos^2 t = 1$.

47. $\tan^2 t + 1 = \sec^2 t$

48. $1 + \cot^2 t = \csc^2 t$

For Exercises 49–54, use an appropriate Pythagorean identity to find the indicated value. (See Example 5, p. 481)

49. Given $\cos t = -\frac{7}{25}$ for $\frac{\pi}{2} < t < \pi$, find the value of $\sin t$.

50. Given $\sin t = -\frac{8}{17}$ for $\pi < t < \frac{3\pi}{2}$, find the value of $\cos t$.

51. Given $\cot t = \frac{45}{28}$ for $\pi < t < \frac{3\pi}{2}$, find the value of $\csc t$.

52. Given $\csc t = -\frac{41}{40}$ for $\frac{3\pi}{2} < t < 2\pi$, find the value of $\cot t$.

53. Given $\tan t = -\frac{11}{60}$ for $\frac{3\pi}{2} < t < 2\pi$, find the value of $\sec t$.

54. Given $\sec t = -\frac{37}{35}$ for $\frac{\pi}{2} < t < \pi$, find the value of $\tan t$.

55. Write $\sin t$ in terms of $\cos t$ for **(See Example 6, p. 482)**
- **a.** t in Quadrant I.
- **b.** t in Quadrant III.

56. Write $\tan t$ in terms of $\sec t$ for
- **a.** t in Quadrant II.
- **b.** t in Quadrant IV.

57. Write $\cot t$ in terms of $\csc t$ for
- **a.** t in Quadrant I.
- **b.** t in Quadrant III.

58. Write $\cos t$ in terms of $\sin t$ for
- **a.** t in Quadrant II.
- **b.** t in Quadrant IV.

Learn: Apply the Periodic and Even and Odd Function Properties of Trigonometric Functions

59. Given that $\cos\frac{29\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$, determine the value of $\cos\frac{5\pi}{12}$. **(See Example 7, p. 484)**

60. Given that $\sec\frac{11\pi}{12} = \sqrt{2} - \sqrt{6}$, determine the value of $\sec\left(-\frac{13\pi}{12}\right)$.

61. Given $\tan\left(-\frac{\pi}{8}\right) = 1 - \sqrt{2}$, determine the value of $\cot\frac{7\pi}{8}$.

62. Given that $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$, determine the value of $\csc\frac{21\pi}{10}$.

For exercises 63–66, use the periodic properties of the trigonometric functions to simplify each expression to a single function of *t*.

63. $\sin(t + 2\pi) \cdot \cot(t + \pi)$

64. $\sin(t + 2\pi) \cdot \sec(t + 2\pi)$

65. $\csc(t + 2\pi) \cdot \cos(t + 2\pi)$

66. $\tan(t + \pi) \cdot \csc(t + 2\pi)$

For Exercises 67–74, use the even-odd and periodic properties of the trigonometric functions to simplify. (See Example 8, p. 485)

67. $\csc t - 4\csc(-t)$

68. $\tan(-t) - 3\tan t$

69. $-\cot(-t + \pi) - \cot t$

70. $\sec(t + 2\pi) - \sec(-t)$

71. $-2\sin(3t + 2\pi) - 3\sin(-3t)$

72. $\cot(-3t) - 3\cot(3t + \pi)$

73. $\cos(-2t) - \cos 2t$

74. $\sec(-2t) + 3\sec 2t$

Learn: Approximate Trigonometric Functions on a Calculator

Use a calculator to approximate the function values. Round to 4 decimal places. (See Example 9, p. 486)

75. a. $\sin(-0.15)$

b. $\cos\frac{2\pi}{5}$

76. a. $\cos\left(-\frac{7\pi}{11}\right)$

b. $\sin 0.96$

77. a. $\cot\frac{12\pi}{7}$

b. $\sec 5.43$

78. a. $\csc 7.58$

b. $\tan\frac{3\pi}{8}$

Mixed Exercises

For Exercises 79–80, evaluate sin *t*, cos *t*, and tan *t* for the real number *t*.

79. a. $t = \frac{2\pi}{3}$

b. $t = -\frac{4\pi}{3}$

80. a. $t = -\frac{11\pi}{6}$

b. $t = \frac{\pi}{6}$

For Exercises 81–86, identify the values of *t* on the interval $[0, 2\pi]$ that make the function undefined (if any).

81. $y = \sin t$

82. $y = \cot t$

83. $y = \tan t$

84. $y = \cos t$

85. $y = \csc t$

86. $y = \sec t$

For Exercises 87–92, select all properties that apply to the trigonometric function.

a. The function is even.

b. The function is odd.

c. The period is 2π.

d. The period is π.

e. The domain is all real numbers.

f. The domain is all real numbers excluding odd multiples of $\frac{\pi}{2}$.

g. The domain is all real numbers excluding multiples of π.

87. $f(t) = \sin(t)$

88. $f(t) = \tan(t)$

89. $f(t) = \sec(t)$

90. $f(t) = \cot(t)$

91. $f(t) = \csc(t)$

92. $f(t) = \cos(t)$

For Exercises 93–98, simplify using properties of trigonometric functions.

93. $\sin^2(t + 2\pi) + \cos^2 t + \tan^2(t + \pi)$

94. $\cot(-t) \cdot \sin(t + 2\pi) + \cos^2 t \cdot \sec(-t + 2\pi)$

95. $\sec^2\left(-\frac{\pi}{3}\right) + \tan^2\left(-\frac{\pi}{3}\right)$

96. $\sin^2\left(\frac{\pi}{6}\right) - \cos^2\left(-\frac{\pi}{6}\right)$

97. $\sec^2\left(-\frac{\pi}{4}\right) - \csc^2\left(\frac{7\pi}{6}\right)$

98. $\tan^2\left(\frac{2\pi}{3}\right) + \csc^2\left(-\frac{5\pi}{4}\right)$

99. In a small coastal town, the monthly revenue received from tourists rises and falls throughout the year. The tourist revenue peaks during the months when the town holds seafood cooking competitions. The function $f(t) = 5.6\cos\left(\frac{\pi}{3}t\right) + 11.2$ represents the monthly revenue $f(t)$, in tens of thousands of dollars, for a month t, where $t = 0$ represents April.

a. Complete the table and give the period of the function. Round to 1 decimal place.

t	0	1	2	3	4	5
$f(t)$						

t	6	7	8	9	10	11
$f(t)$						

b. What are the months of peak revenue, and what is the revenue for those months?

100. The fluctuating brightness of a distant star is given by the function
$f(d) = 3.8 + 0.25\sin\left(\frac{2\pi}{3}d\right)$
where d is the number of days and $f(d)$ is the apparent brightness.
Complete the table and give the period of the function. Round to 2 decimal places.

d	0	1	2	3	4	5
$f(d)$						

For Exercises 101–104, identify each function as even, odd, or neither.

101. $f(t) = t^2 \sin t$

102. $g(t) = t\cos t$

103. $z(t) = t^3 \tan t$

104. $h(t) = t^3 + \sec t$

Write About It

105. Describe the changes in $\sin t$ and $\cos t$ as t increases from 0 to $\frac{\pi}{2}$.

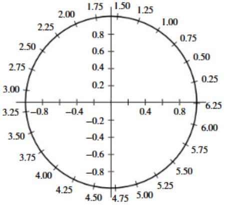
106. Describe the changes in $\cos t$ and $\sec t$ as t increases from 0 to $\frac{\pi}{2}$.

107. Explain why $-1 \le \cos t \le 1$ and $-1 \le \sin t \le 1$ for all real numbers t.

108. Do the trigonometric functions satisfy the definition of a function learned in Chapter 1? Explain your answer.

Expanding Your Skills

For Exercises 109–114, use the figure to estimate the value of (a) $\sin t$ and (b) $\cos t$ for the given value of t.



109. $t = 0.5$

110. $t = 1.25$

111. $t = 3.75$

112. $t = 2.75$

113. $t = 5$

114. $t = 3$

For Exercises 115–120, use the figure from Exercises 109–114 to approximate the solutions to the equation over the interval $[0, 2\pi)$.

115. $\sin t = 0.2$

116. $\sin t = -0.4$

117. $\cos t = -0.4$

118. $\cos t = 0.6$

119. Show that $\tan(t + \pi) = \tan t$.

120. Show that $\cot(t + \pi) = \cot t$

121. Prove that the period of $f(t) = \sin t$ is 2π.

122. Prove that the period of $f(t) = \cos t$ is 2π.

123. Given that $\tan t = a$ for t in Quadrant II, write expressions for $\sin t$ and $\cos t$.

124. Given that $\cos t = a$ and that t is in Quadrant III, find the values of $\sin t$ and $\tan t$.

125. For an arbitrary value of t, write an expression for $\sin t$ and $\cos t$ in terms of $\tan t$.

126. For an arbitrary value of t, write an expression for $\sin t$ and $\tan t$ in terms of $\cos t$.

Technology Connections

127. **a.** Complete the table to show that $\sin t \approx t$ for values of t close to zero. Round to 5 decimal places.

t	$\sin t$	$\frac{\sin t}{t}$
0.2		
0.1		
0.01		
0.001		

b. What value does the ratio $\frac{\sin t}{t}$ seem to approach as $t \longrightarrow 0$?

128. Consider the expression $\frac{1-\cos t}{t}$.

a. What is the value of the numerator for $t = 0$?

b. What is the value of the denominator for $t = 0$?

c. The expression $\frac{1-\cos t}{t}$ is undefined at $t = 0$. Complete the table to investigate the value of the expression *close* to $t = 0$. Round to 5 decimal places.

t	$\frac{1-\cos t}{t}$
0.2	
0.1	
0.01	
0.001	

Go online for more practice problems.

Lesson 4-3

Right Triangle Trigonometry

What Will You Learn?

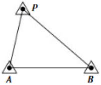
After completing this lesson, you should be able to:

- Evaluate Trigonometric Functions of Acute Angles
- Use Fundamental Trigonometric Identities
- Use Trigonometric Functions in Applications

Learn: Evaluate Trigonometric Functions of Acute Angles

The science of land surveying encompasses the measurement and mapping of land using mathematics and specialized equipment and technology. Surveyors provide data relevant to the shape and contour of the Earth's surface for engineering, mapmaking, and construction projects.

One technique used by surveyors to map a landscape is called *triangulation*. The surveyor first measures the distance between two fixed points, *A* and *B*. Then the surveyor measures the bearing from *A* and the bearing from *B* to a third point *P*. From this information, the surveyor can compute the measures of the angles of the triangle formed by the points, and can use trigonometry to find the lengths of the unknown sides.



To use trigonometry in such applications, we present alternative definitions of the trigonometric functions using right triangles. Consider a right triangle with an acute angle θ. The longest side in the triangle is the hypotenuse ("hyp") and is opposite the right angle (Figure 4-27). The two legs of the triangle will be distinguished by their relative positions to θ. The leg that lies on one ray of angle θ is called the adjacent leg ("adj") and the leg that lies across the triangle from θ is called the opposite leg ("opp").

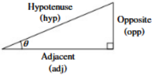


Figure 4-27

We now define the six trigonometric functions as functions whose input (or **argument**) is an acute angle θ. The output value for each function will be one of the six possible ratios of the lengths of the sides of the triangle (Table 4-4).

The definitions of the six trigonometric functions of acute angles are given in Table 4-4 along with examples based on Figure 4-28.

TABLE 4-4 Definition of Trigonometric Functions of Acute Angles

Function Name	Definition	Example
sine	$\sin\theta = \frac{\text{opp}}{\text{hyp}}$	$\sin\theta = \frac{3\text{ ft}}{5\text{ ft}} = \frac{3}{5}$
cosine	$\cos\theta = \frac{\text{adj}}{\text{hyp}}$	$\cos\theta = \frac{4\text{ ft}}{5\text{ ft}} = \frac{4}{5}$
tangent	$\tan\theta = \frac{\text{opp}}{\text{adj}}$	$\tan\theta = \frac{3\text{ ft}}{4\text{ ft}} = \frac{3}{4}$
cosecant	$\csc\theta = \frac{\text{hyp}}{\text{opp}}$	$\csc\theta = \frac{5\text{ ft}}{3\text{ ft}} = \frac{5}{3}$
secant	$\sec\theta = \frac{\text{hyp}}{\text{adj}}$	$\sec\theta = \frac{5\text{ ft}}{4\text{ ft}} = \frac{5}{4}$
cotangent	$\cot\theta = \frac{\text{adj}}{\text{opp}}$	$\cot\theta = \frac{4\text{ ft}}{3\text{ ft}} = \frac{4}{3}$

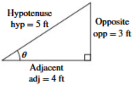


Figure 4-28

Good Practices

The mnemonic device "SOH-CAH-TOA" may help you remember the ratios for $\sin\theta$, $\cos\theta$, and $\tan\theta$, respectively.

Sine: Opp over Hyp

Cosine: Adj over Hyp

Tangent: Opp over Adj

The notation $\sin\theta$ is read as "sine of theta" (likewise, $\cos\theta$ is read as "cosine of theta" and so on). Also notice that the output value of a trigonometric function is unitless because the common units of length "cancel" within each ratio.

The definitions of the trigonometric functions given in Table 4-4 are consistent with the definitions based on the unit circle from Lesson 4.2. For example, consider a real number $0 < t < \frac{\pi}{2}$ that corresponds to the point $P(x, y)$ on the unit circle (Figure 4-29). Suppose that we drop a perpendicular line segment from P to the x-axis at point A. Then $\triangle OAP$ is a right triangle. The lengths of the legs are x and y, and the hypotenuse is 1 unit. Furthermore, the radian measure of $\angle AOP$ is equal to t. If we let θ represent the degree measure of $\angle AOP$, we have

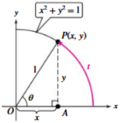


Figure 4-29

Unit circle definition

$\sin t = y$

$\cos t = x$

$\tan t = \frac{y}{x}$

Right triangle definition

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$

It is also very important to note that the values of the trigonometric functions depend only on the measure of the angle, not the size of the triangle. For any given acute angle θ, a series of right triangles can be formed by drawing a line segment perpendicular to the initial side of θ with endpoints on the initial and terminal sides of θ (Figure 4-30). Triangles $\triangle ABC$ and $\triangle ADE$ are similar triangles with common angle θ. Therefore, the corresponding sides are proportional, and the value of each trigonometric function of θ will be the same regardless of which triangle is used. For example, $\sin\theta = \frac{x}{b} = \frac{y}{d}$.

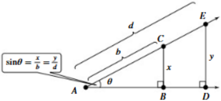


Figure 4-30

In Example 1, we find the values of the six trigonometric functions of an acute angle within a right triangle where the lengths of two legs are given.

Example 1: Evaluating Trigonometric Functions by First Applying the Pythagorean Theorem

Suppose that a right triangle has legs of length 4 cm and 7 cm. Evaluate the six trigonometric functions for the smaller acute angle.

Solution:

A right triangle $\triangle ABC$ meeting the conditions of this example is drawn in Figure 4-31. Since $\angle B$ is opposite the shorter leg, $\angle B$ is the smaller acute angle.

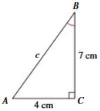
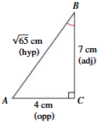


Figure 4-31

To find the values of all six trigonometric functions, we also need to know the length of the hypotenuse. The hypotenuse is always opposite the right angle.

$c^2 = 4^2 + 7^2$	Apply the Pythagorean theorem.
$c^2 = 16 + 49$	Simplify terms with exponents.
$c^2 = 65$ $c = \sqrt{65}$	Since c represents the length of a side of a triangle, take the positive square root of 65.

Relative to angle *B*, the 4-cm side is the opposite side. The 7-cm side is the adjacent leg, and the hypotenuse is $c = \sqrt{65}$.



$$\sin B = \frac{\text{opp}}{\text{hyp}} = \frac{4}{\sqrt{65}} = \frac{4}{\sqrt{65}} \cdot \frac{\sqrt{65}}{\sqrt{65}} = \frac{4\sqrt{65}}{65}$$
$$\cos B = \frac{\text{adj}}{\text{hyp}} = \frac{7}{\sqrt{65}} = \frac{7}{\sqrt{65}} \cdot \frac{\sqrt{65}}{\sqrt{65}} = \frac{7\sqrt{65}}{65}$$
$$\tan B = \frac{\text{opp}}{\text{adj}} = \frac{4}{7}$$
$$\csc B = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{65}}{4}$$
$$\sec B = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{65}}{7}$$
$$\cot B = \frac{\text{adj}}{\text{opp}} = \frac{7}{4}$$

Check Your Work

In Example 1, the opposite (opp) and adjacent (adj) legs are labeled in the triangle relative to angle *B*.

Apply the Skills

1. Suppose that a right triangle $\triangle ABC$ has legs of length 6 cm and 3 cm. Evaluate the six trigonometric functions for angle *B*, where angle *B* is the larger acute angle.

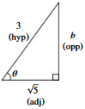
In Example 1, we were given the lengths of the sides of a right triangle and asked to find the values of the six trigonometric functions. This process can be reversed. Given the value of one trigonometric function of an acute angle, we can find the lengths of the sides of a representative triangle, and therefore the values of the other trigonometric functions.

Example 2: Using a Known Value of a Trigonometric Function to Determine Other Function Values

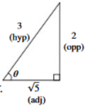
Suppose that $\cos\theta = \frac{\sqrt{5}}{3}$ for the acute angle θ. Evaluate $\tan\theta$.

Solution:

Since $\cos\theta$ is the ratio of the length of the leg adjacent to θ and the length of the hypotenuse, it is convenient to construct a triangle with adjacent leg $\sqrt{5}$ units and hypotenuse 3 units. Using the Pythagorean theorem, we can find the length of the opposite leg.



$(\sqrt{5})^2 + b^2 = (3)^2$	Apply the Pythagorean theorem.
$5 + b^2 = 9$	Simplify terms with exponents.
$b^2 = 4$	
$b = 2$	Take the positive square root of 4.
$\tan\theta = \frac{opp}{adj} = \frac{2}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$	Once the proper ratio is identified for $\tan\theta$, rationalize the denominator.



Apply the Skills

2. Suppose that $\sin\theta = \frac{3}{4}$ for the acute angle θ. Evaluate $\sec\theta$.

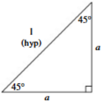
In Example 3, we find the sine, cosine, and tangent of 45° by using a right triangle approach.

Example 3: Determine the Trigonometric Function Values of a 45° Angle

Evaluate $\sin 45°$, $\cos 45°$, and $\tan 45°$.

Solution:

A right triangle with an acute angle of 45° must have a second acute angle of 45° because the sum of the angles must equal 180°. This is called an isoceles right triangle or a 45°–45°–90° triangle. Since all such triangles are similar, we can choose one of the sides to be of arbitrary length. We have chosen the hypotenuse to be 1 unit. The sides opposite the 45° angles are equal in length and have been labeled a.



$$a^2 + a^2 = 1^2$$

Apply the Pythagorean theorem.

$$2a^2 = 1$$

$$a^2 = \frac{1}{2}$$

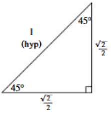
$$a = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Take the positive square root of $\frac{1}{2}$.
Rationalize the denominator.

$$\sin 45° = \frac{\text{opp}}{\text{hyp}} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{\text{adj}}{\text{hyp}} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{\text{opp}}{\text{adj}} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$



Apply the Skills

3. Evaluate csc 45°, sec 45°, and cot 45°.

Insights

The values of sin 45°, cos 45°, and tan 45° found in Example 3 are consistent with the values of $\sin\frac{\pi}{4}$, $\cos\frac{\pi}{4}$, and $\tan\frac{\pi}{4}$ found by using the unit circle.

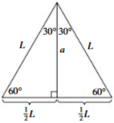
A right triangle with a 30° angle will also have a 60° angle so that the sum of the angular measures is 180°. Such a triangle is called a 30°–60°–90° triangle. Interestingly, as we will show in Example 4, the length of the shorter leg of a 30°–60°–90° triangle is always one-half the length of the hypotenuse.

Example 4: Determine the Trigonometric Function Values of a 60° Angle

Evaluate sin 60°, cos 60°, and tan 60°.

Solution:

Suppose that we start with an equilateral triangle with sides of length L. All angles in the triangle are equal to 60°. Any altitude a of this triangle bisects a 60° angle as well as the opposite side. Therefore, we can create two congruent right triangles with legs of length a and $\frac{1}{2}L$ and hypotenuse of length L.



Use the Pythagorean theorem to find the altitude a in terms of L.

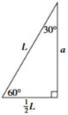
$\left(\frac{1}{2}L\right)^2 + a^2 = L^2$ Apply the Pythagorean theorem.

$\frac{1}{4}L^2 + a^2 = L^2$ Simplify terms with exponents.

$a^2 = \frac{3}{4}L^2$ Combine like terms.

$a = \sqrt{\frac{3}{4}}L$ Take the positive square root.

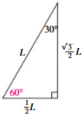
$a = \frac{\sqrt{3}}{2}L$ Simplify the radical.



$$\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\frac{\sqrt{3}}{2}L}{L} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{\frac{1}{2}L}{L} = \frac{1}{2}$$

$$\tan 60° = \frac{\text{opp}}{\text{adj}} = \frac{\frac{\sqrt{3}}{2}L}{\frac{1}{2}L} = \left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{2}{1}\right) = \sqrt{3}$$



Apply the Skills

4. Evaluate $\sin 30°$, $\cos 30°$, and $\tan 30°$.

From Examples 3 and 4, we can summarize the trigonometric function values of the "special" angles 30°, 45°, and 60°, or equivalently $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ radians (Table 4-5).

TABLE 4-5 Trigonometric Function Values of Special Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$30° = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$45° = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60° = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Good Practices

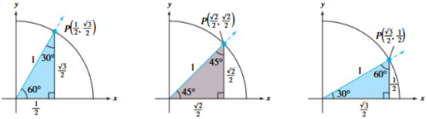
As a memory device, note the 1-2-3 pattern in the numerator for the sine function for the special angles.

$$\sin 30° = \frac{\sqrt{1}}{2}$$
$$\sin 45° = \frac{\sqrt{2}}{2}$$
$$\sin 60° = \frac{\sqrt{3}}{2}$$

For cosine, the order is reversed, 3-2-1.

$$\cos 30° = \frac{\sqrt{3}}{2}$$
$$\cos 45° = \frac{\sqrt{2}}{2}$$
$$\cos 60° = \frac{\sqrt{1}}{2}$$

To help you remember the values of the trigonometric functions for 30°, 45°, and 60°, you can refer to the first quadrant of the unit circle. The values $t=\frac{\pi}{6}$, $t=\frac{\pi}{4}$, and $t=\frac{\pi}{3}$ correspond to the central angles $\theta=30°$, $\theta=45°$, and $\theta=60°$, respectively (Figure 4-32).



y
x
$P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
1
30°
$\frac{\sqrt{3}}{2}$
60°
$\frac{1}{2}$
y
x
$P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
1
45°
$\frac{\sqrt{2}}{2}$
45°
$\frac{\sqrt{2}}{2}$
y
x
$P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
1
60°
$\frac{1}{2}$
30°
$\frac{\sqrt{3}}{2}$

Figure 4-32

Example 5: Simplifying Expressions Involving Trigonometric Functions of Special Angles

Simplify the expressions.

a. $\tan 60° - \tan 30°$ **b.** $2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$

Solution:

a. $\tan 60° - \tan 30° = \sqrt{3} - \frac{\sqrt{3}}{3}$ Evaluate tan 60° and tan 30°.

$= \frac{3}{3}\cdot\frac{\sqrt{3}}{1} - \frac{\sqrt{3}}{3}$ Write the first term with a common denominator of 3.

$= \frac{2\sqrt{3}}{3}$ Combine like terms.

b. $2\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$ Evaluate $\sin\frac{\pi}{6}$ and $\cos\frac{\pi}{6}$.

$= \frac{\sqrt{3}}{2}$ Multiply.

Apply the Skills

5. Simplify the expressions.

a. $\cot 60° - \cot 30°$ **b.** $\sin\frac{\pi}{3}\cos\frac{\pi}{6} + \cos\frac{\pi}{3}\sin\frac{\pi}{6}$

Learn: Use Fundamental Trigonometric Identities

In Lesson 4.2, we observed several relationships among the trigonometric functions that follow directly from the definitions of the functions. The reciprocal and quotient identities also follow from the right triangle defintions of trigonometric functions of acute angles (Table 4-6). For example, since $\sin\theta = \frac{\text{opp}}{\text{hyp}}$ and $\csc\theta = \frac{\text{hyp}}{\text{opp}}$ for an acute angle θ, we know that $\sin\theta$ and $\csc\theta$ are reciprocals.

TABLE 4-6 Reciprocal and Quotient Identities

Trigonometric Functions	Relationship
$\csc\theta = \frac{1}{\sin\theta}$ or $\sin\theta = \frac{1}{\csc\theta}$	$\sin\theta$ and $\csc\theta$ are reciprocals.
$\sec\theta = \frac{1}{\cos\theta}$ or $\cos\theta = \frac{1}{\sec\theta}$	$\cos\theta$ and $\sec\theta$ are reciprocals.
$\cot\theta = \frac{1}{\tan\theta}$ or $\tan\theta = \frac{1}{\cot\theta}$	$\tan\theta$ and $\cot\theta$ are reciprocals.
$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\tan\theta$ is the ratio of $\sin\theta$ and $\cos\theta$.
$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$\cot\theta$ is the ratio of $\cos\theta$ and $\sin\theta$.

We can also show that the Pythagorean identities follow from the right triangle definitions of the trigonometric functions of an acute angle θ. Consider a right triangle with legs of length a and b and hypotenuse c (Figure 4-33).

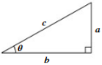


Figure 4-33

From the Pythagorean theorem, we know that $a^2 + b^2 = c^2$. Dividing both sides by the positive real number c^2 we have

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$ Divide by c^2.

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \left(\frac{c}{c}\right)^2$$ Rewrite each term as the power of a quotient.

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$ From Figure 4-33, $\sin\theta = \frac{a}{c}$ and $\cos\theta = \frac{b}{c}$.

$$\sin^2\theta + \cos^2\theta = 1$$

Table 4-7 summarizes this and two other Pythagorean identities. The second and third relationships can be derived by dividing both sides of the equation $a^2 + b^2 = c^2$ by b^2 and a^2, respectively.

TABLE 4-7 Pythagorean Identities

Trigonometric Functions	Pythagorean Identity
$\sin t$ and $\cos t$	$\sin^2 t + \cos^2 t = 1$
$\tan t$ and $\sec t$	$\tan^2 t + 1 = \sec^2 t$
$\cot t$ and $\csc t$	$1 + \cot^2 t = \csc^2 t$

Visualizing the ratios of the lengths of the sides of a right triangle can also help us understand the cofunction identities.

The two acute angles in a right triangle are complementary because the sum of their measures is 90°. Symbolically, the complement of angle θ is $(90° - \theta)$. In Figure 4-34, notice that side a is opposite angle θ but is the adjacent leg to angle $(90° - \theta)$. Likewise, side b is adjacent to angle θ but opposite angle $(90° - \theta)$.

From these relationships, we have

$$\sin\theta = \cos(90^\circ - \theta) = \frac{a}{c}$$

$$\tan\theta = \cot(90^\circ - \theta) = \frac{a}{b}$$

$$\sec\theta = \csc(90^\circ - \theta) = \frac{c}{b}$$

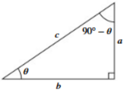


Figure 4-34

Notice that the sine of angle θ equals the cosine of the complement $(90^\circ - \theta)$. For this reason, the sine and cosine functions are called *cofunctions*. More generally, for an acute angle θ, two trigonometric functions f and g are **cofunctions** if

$$f(\theta) = g(90^\circ - \theta) \text{ and } g(\theta) = f(90^\circ - \theta)$$

Cofunction Identities

Cofunctions of complementary angles are equal.

$\sin\theta = \cos(90^\circ - \theta)$	$\cos\theta = \sin(90^\circ - \theta)$	Sine and cosine are cofunctions.
$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$	
$\tan\theta = \cot(90^\circ - \theta)$	$\cot\theta = \tan(90^\circ - \theta)$	Tangent and cotangent are cofunctions.
$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$	
$\sec\theta = \csc(90^\circ - \theta)$	$\csc\theta = \sec(90^\circ - \theta)$	Secant and cosecant are cofunctions.
$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$	

Insights

Cosine means "*complement's sine.*"

Cotangent means "*complement's tangent.*"

Cosecant means "*complement's secant.*"

Example 6: Using the Cofunction Identities

For each function value, find a cofunction with the same value.

a. $\cot 15^\circ = 2 + \sqrt{3}$ **b.** $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Solution:

a. The complement of 15° is $(90^\circ - 15^\circ) = 75^\circ$.

Cotangent and tangent are cofunctions.

Therefore, $\cot 15^\circ = \tan 75^\circ = 2 + \sqrt{3}$.

b. The complement of $\frac{\pi}{6}$ is $\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\left(\frac{3\pi}{6}-\frac{\pi}{6}\right)=\frac{2\pi}{6}=\frac{\pi}{3}$.

Cosine and sine are cofunctions.

Therefore, $\cos\frac{\pi}{6}=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$.

Apply the Skills

6. For each function value, find a cofunction with the same value.

a. $\tan 22.5° = \sqrt{2}-1$ **b.** $\sec\frac{\pi}{3}=2$

Learn: Use Trigonometric Functions in Applications

In many applications of right triangle trigonometry, angles are measured relative to an imaginary horizontal line of reference. An angle of elevation is an angle measured upward from a horizontal line of reference. An angle of depression is an angle measured downward from a horizontal line of reference (Figure 4-35).

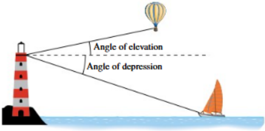
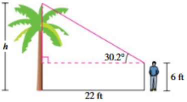


Figure 4-35

Example 7: Using Trigonometry to Find the Height of a Tree

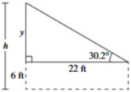
Palm trees are easy to transplant relative to similarly sized broad-leaved trees. At one tree farm, palm trees are harvested once they reach a height of 20 ft. Suppose that the distance along the ground from the base of the tree to a farm worker is 22 ft. Then, an instrument called a clinometer is held at an eye level of 6 ft to measure the angle of elevation to the top of the tree. If the angle of elevation is 30.2°, is the tree tall enough to harvest?



Solution:

To find the height of the tree, we will use the right triangle to find the vertical leg *y* and then add 6 ft.

Relative to the 30.2° angle, we know the adjacent side is 22 ft and we want to know the length of the opposite side y.



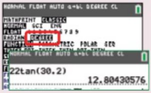
The two trigonometric functions that are defined by the opposite and adjacent legs are tangent and cotangent. To solve for y, we can use the relationship

$\tan 30.2° = \frac{y}{22}$ or $\cot 30.2° = \frac{22}{y}$. Using the tangent function we have

$\tan 30.2° = \frac{y}{22}$	We use the tangent function because it is easily approximated on a calculator.
$y = 22 \tan 30.2°$	Multiply both sides by 22.
$y \approx 12.8$ ft	Use a calculator to approximate $22 \tan 30.2°$.
$h \approx 12.8 \text{ ft} + 6 \text{ ft}$	To find the height of the tree, add 6 ft.
≈ 18.8 ft	The tree is approximately 18.8 ft and is not ready to harvest.

Check Your Work

To approximate $\tan 30.2°$ on a calculator, be sure that the calculator is in degree mode.



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NORMAL FLOAT AUTO a+bi DEGREE CL
MATHPRINT CLASSIC
NORMAL SCI ENG
FLOAT      6 7 8 9
RADIAN DEGREE
FUNCTION     TRIC POLAR SEQ
```

```
NORMAL FLOAT AUTO a+bi DEGREE CL
22tan(30.2)
                12.80430576
```

Apply the Skills

7. Suppose that the distance along the ground from a farm worker to the base of a palm tree is 16 ft. If the angle of elevation from an eye level of 5 ft to the top of the tree is 45.9°, is the tree tall enough to harvest (at least 20 ft tall)?

Point of Interest

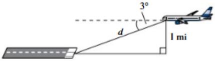
A variety of instruments are used to measure angles. Historically, quadrants and sextants were used for early celestial navigation. Theodolites (or transits) are used for surveying, and clinometers are used in construction to measure slope.

Geoff Miller

Nineteenth-century sextant, courtesy of Geoff Miller

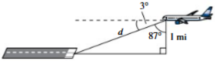
Example 8: Using Trigonometry in Aeronautical Science

A pilot flying an airplane at an altitude of 1 mi sights a point at the end of a runway. The angle of depression is 3°. What is the distance *d* from the plane to the point on the runway? Round to the nearest tenth of a mile.



Solution:

The complement of the 3° angle of depression is the acute angle 87° within the right triangle.



Relative to the 87° angle, we know that the adjacent side is 1 mi, and we want to find the length of the hypotenuse. The two trigonometric functions that are defined by the adjacent leg and hypotenuse are the cosine and secant functions.

Using the relationship $\cos 87° = \frac{1}{d}$, we have $d = \frac{1}{\cos 87°} \approx 19.1$.

Therefore, the plane is approximately 19.1 mi from the end of the runway.

Apply the Skills

8. If a 15-ft ladder is leaning against a wall at an angle of 62° with the ground, how high up the wall will the ladder reach? Round to the nearest tenth of a foot.

Practice 4-3

Practice Exercises

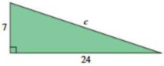
Prerequisite Review

R.1. Simplify the radical. $\sqrt{45}$

R.2. Rationalize the denominator. $\frac{14}{\sqrt{7}}$

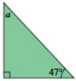
R.3. Rationalize the denominator. $\frac{9}{\sqrt{18}}$

R.4. Use the Pythagorean theorem to find the length of the missing side *c*.



R.5. In a right triangle, one leg measures 24 ft and the hypotenuse measures 26 ft. Find the length of the missing side.

R.6. Find the measure of angle *a*.



Concept Connections

1. In a right triangle with an acute angle θ, the longest side in the triangle is the ____________ and is opposite the ____________ angle.

2. The leg of a right triangle that lies on one ray of angle θ is called the ____________ leg, and the leg that lies across the triangle from θ is called the ____________ leg.

3. The mnemonic device "SOH-CAH-TOA" stands for the ratios $\sin\theta = \frac{\square}{\square}$, $\cos\theta = \frac{\square}{\square}$, and $\tan\theta = \frac{\square}{\square}$.

4. Complete the reciprocal and quotient identities. $\csc\theta = \frac{1}{\square}$, $\sec\theta = \frac{1}{\square}$, $\cot\theta = \frac{1}{\square}$ or $\frac{\square}{\square}$

5. Given the lengths of two sides of a right triangle, we can find the length of the third side by using the ____________ theorem.

6. An ____________ right triangle is a right triangle in which the two legs are of equal length. The two acute angles in this triangle each measure ____________°.

7. The length of the shorter leg of a 30°–60°–90° triangle is always ____________ the length of the hypotenuse.

8. For the six trigonometric functions $\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$, and $\cot\theta$, identify the three reciprocal pairs.

9. $\tan\theta$ is the ____________ of $\sin\theta$ and $\cos\theta$.

10. Complete the Pythagorean identities. $\sin^2\theta +$ ____________ $= 1$, ____________ $+ 1 = \sec^2\theta$, $1 + \cot^2\theta =$

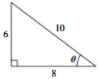
11. The two acute angles in a right triangle are complementary because the sum of their measures is ____________°.

12. The sine of angle θ equals the cosine of ____________. For this reason, the sine and cosine functions are called ____________.

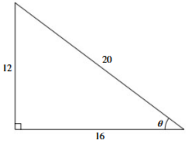
Learn: Evaluate Trigonometric Functions of Acute Angles

For Exercises 13–14, find the exact values of the six trigonometric functions for angle θ.

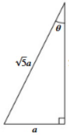
13. a.



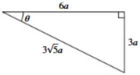
b.



14. a.

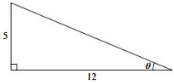


b.

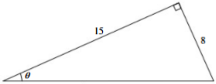


For Exercises 15–18, first use the Pythagorean theorem to find the length of the missing side. Then find the exact values of the six trigonometric functions for angle θ. (See Example 1, p. 496)

15.



16.



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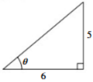
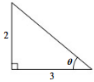


18.



For Exercises 19–22, first use the Pythagorean theorem to find the length of the missing side of the right triangle. Then find the exact values of the six trigonometric functions for the angle θ opposite the shortest side. (See Example 1, p. 496)

19. Leg = 2 cm, leg = 6 cm

20. Leg = 3 cm, leg = 15 cm

21. Leg = $2\sqrt{7}$ m, hypotenuse = $2\sqrt{11}$ m

22. Leg = $5\sqrt{3}$ in., hypotenuse = $2\sqrt{21}$ in.

In Exercises 23–24, given the value of one trigonometric function of an acute angle θ, find the values of the remaining five trigonometric functions of θ. (See Example 2, p. 498)

23. $\tan\theta = \frac{4}{7}$

24. $\cos\theta = \frac{\sqrt{10}}{10}$

For Exercises 25–30, assume that θ is an acute angle. (See Example 2, p. 498)

25. If $\cos\theta = \frac{\sqrt{21}}{7}$, find $\csc\theta$.

26. If $\sin\theta = \frac{\sqrt{17}}{17}$, find $\cot\theta$.

27. If $\sec\theta = \frac{3}{2}$, find $\sin\theta$.

28. If $\csc\theta = 3$, find $\cos\theta$.

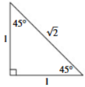
29. If $\tan\theta = \frac{\sqrt{15}}{9}$, find $\cos\theta$.

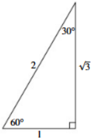
30. If $\cot\theta = \frac{\sqrt{3}}{2}$, find $\cos\theta$.

For Exercise 31, use the isosceles right triangle and the 30°–60°–90° triangle to complete the table. (See Examples 3–4, pp. 498–499)

31.

θ	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
$30° = \frac{\pi}{6}$						
$45° = \frac{\pi}{4}$						
$60° = \frac{\pi}{3}$						





32. a. Evaluate $\sin 60^\circ$.

b. Evaluate $\sin 30^\circ + \sin 30^\circ$.

c. Are the values in parts (a) and (b) the same?

For Exercises 33–38, find the exact value of each expression without the use of a calculator. (See Example 5, p. 501)

33. $\sin\frac{\pi}{4} \cdot \cot\frac{\pi}{3}$

34. $\tan\frac{\pi}{6} \cdot \csc\frac{\pi}{4}$

35. $3\cos\frac{\pi}{3} + 4\sin\frac{\pi}{6}$

36. $2\cos\frac{\pi}{6} - 5\sin\frac{\pi}{3}$

37. $\csc^2 60^\circ - \sin^2 45^\circ$

38. $\cos^2 45^\circ + \tan^2 60^\circ$

Learn: Use Fundamental Trigonometric Identities

For Exercises 39–44, determine whether the statement is true or false for an acute angle θ by using the fundamental identities. If the statement is false, provide a counterexample by using a special angle: $\frac{\pi}{3}$, $\frac{\pi}{4}$, or $\frac{\pi}{6}$.

39. $\sin\theta \cdot \tan\theta = 1$

40. $\cos^2\theta \cdot \tan^2\theta = \sin^2\theta$

41. $\sin^2\theta + \tan^2\theta + \cos^2\theta = \sec^2\theta$

42. $\csc\theta \cdot \cot\theta = \sec\theta$

43. $\frac{1}{\tan\theta} \cdot \cot\theta + 1 = \csc^2\theta$

44. $\sin\theta \cdot \cos\theta \cdot \tan\theta + 1 = \cos^2\theta$

For Exercises 45–50, given the function value, find a cofunction of another angle with the same value. (See Example 6, p. 503)

45. $\tan 75^\circ = 2 + \sqrt{3}$

46. $\sec\frac{\pi}{12} = \sqrt{6} - \sqrt{2}$

47. $\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3}$

48. $\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$

49. $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

50. $\cot\frac{\pi}{6} = \sqrt{3}$

For Exercises 51–54, use a calculator to approximate the function values to 4 decimal places. Be sure that your calculator is in the correct mode.

51. a. $\cos 48.2^\circ$

b. $\sin 2^\circ 55' 42''$

c. $\tan\frac{3\pi}{8}$

52. a. $\sin 12.6^\circ$

b. $\tan 19^\circ 36' 18''$

c. $\cos\left(\frac{5\pi}{22}\right)$

53. a. $\csc 39.84^\circ$

b. $\sec\frac{\pi}{18}$

c. $\cot 0.8$

54. a. $\cot 18.46^\circ$

b. $\csc\frac{2\pi}{9}$

c. $\sec 1.25$

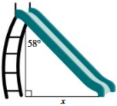
Learn: Use Trigonometric Functions in Applications

55. An observer at the top of a 462-ft cliff measures the angle of depression from the top of a cliff to a point on the ground to be 5°. What is the distance from the base of the cliff to the point on the ground? Round to the nearest foot. **(See Example 7, p. 504)**

56. A lamppost casts a shadow of 18 ft when the angle of elevation of the Sun is 33.7°. How high is the lamppost? Round to the nearest foot.

57. A 30-ft boat ramp makes a 7° angle with the water. What is the height of the ramp above the water at the ramp's highest point? Round to the nearest tenth of a foot. **(See Example 8, p. 506)**

58. A backyard slide is designed for a child's playground. If the top of the 10-ft slide makes an angle of 58° from the vertical, how far out from the base of the steps will the slide extend? Round to the nearest tenth of a foot.

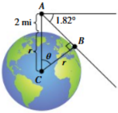


59. The Lookout Mountain Incline Railway, located in Chattanooga, Tennessee, is 4972 ft long and runs up the side of the mountain at an average incline of 17°. What is the gain in altitude? Round to the nearest foot.

60. A 12-ft ladder leaning against a house makes a 64° angle with the ground. Will the ladder reach a window sill that is 10.5 ft up from the base of the house?

61. According to National Football League (NFL) rules, all crossbars on goalposts must be 10 ft from the ground. However, teams are allowed some freedom on how high the vertical posts on each end may extend, as long as they measure at least 30 ft. A measurement on an NFL field taken 100 yd from the goalposts yields an angle of 7.8° from the ground to the top of the posts. If the crossbar is 10 ft from the ground, do the goalposts satisfy the NFL rules?

62. A scientist standing at the top of a mountain 2 mi above sea level measures the angle of depression to the ocean horizon to be 1.82°. Use this information to approximate the radius of the Earth to the nearest mile. (*Hint*: The line of sight $\overleftrightarrow{AB}$ is tangent to the Earth and forms a right angle with the radius at the point of tangency.)



63. A scenic overlook along the Pacific Coast Highway in Big Sur, California, is 280 ft above sea level. A 6-ft-tall hiker standing at the overlook sees a sailboat and estimates the angle of depression to be 30°. Approximately how far off the coast is the sailboat? Round to the nearest foot.

64. An airplane traveling 400 mph at a cruising altitude of 6.6 mi begins its descent. If the angle of descent is 2° from the horizontal, determine the new altitude after 15 min. Round to the nearest tenth of a mile.

Mixed Exercises

For Exercises 65–70, given that $\sin A = \frac{4}{5}$, $\tan B = \frac{5}{12}$, $\cos C = \frac{8}{17}$, and $\cot D = \frac{24}{7}$, evaluate each expression. Assume that A, B, C, and D are all on the interval $\left(0, \frac{\pi}{2}\right)$ or (0°, 90°).

65. $\cos A \cdot \sec D \cdot \cot C$

66. $\sin B \cdot \csc C \cdot \cos D$

67. $\sin A \cdot \csc B \cdot \sec D \cdot \cos B$

68. $\csc^2 A \cdot \sin D \cdot \cot B$

69. $\sin(90° - B) \cdot \tan D \cdot \cot C \cdot \csc A$

70. $\cos\left(\frac{\pi}{2} - A\right) \cdot \tan C \cdot \cos D \cdot \sin B$

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For Exercises 71–75, use the fundamental trigonometric identities as needed.

71. Given that $\sin x° \approx 0.3746$, approximate the given function values. Round to 4 decimal places.

- **a.** $\cos(90 - x)°$
- **b.** $\cos x°$
- **c.** $\tan x°$
- **d.** $\sin(90 - x)°$
- **e.** $\cot(90 - x)°$
- **f.** $\csc x°$

72. Given that $\cos x \approx 0.6691$, approximate the given function values. Round to 4 decimal places.

- **a.** $\sin x$
- **b.** $\sin\left(\frac{\pi}{2} - x\right)$
- **c.** $\tan x$
- **d.** $\cos\left(\frac{\pi}{2} - x\right)$
- **e.** $\sec x$
- **f.** $\cot\left(\frac{\pi}{2} - x\right)$

73. Given that $\cos\frac{\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4}$, give the exact function values.

- **a.** $\sin\frac{5\pi}{12}$
- **b.** $\sin\frac{\pi}{12}$
- **c.** $\sec\frac{\pi}{12}$

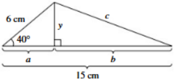
74. Given that $\tan 36° = \sqrt{5 - 2\sqrt{5}}$, give the exact function values.

- **a.** $\sec 36°$
- **b.** $\csc 54°$
- **c.** $\cot 54°$

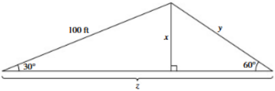
75. Simplify each expression to a single trigonometric function.

- **a.** $\sec\theta \cdot \tan\left(\frac{\pi}{2} - \theta\right)$
- **b.** $\cot^2\theta \cdot \csc(90° - \theta) \cdot \sin\theta$

76. Find the lengths a, b, y, and c. Round to the nearest tenth of a centimeter.

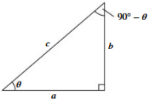


77. Find the exact lengths x, y, and z.



Write About It

78. Use the figure to explain why $\tan\theta = \cot(90° - \theta)$.



79. Explain the difference between an angle of elevation and an angle of depression.

Expanding Your Skills

For Exercises 80–85, simplify each expression to obtain one of the following expressions.

a. 1 **b.** $\cos\theta$ **c.** $\tan\theta$

80. $\dfrac{\cos\theta\csc\theta}{\cot\theta}$

81. $\dfrac{1}{\csc\theta\tan\theta}$

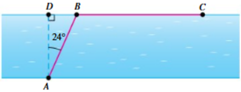
82. $\dfrac{\sin\theta\cos\theta}{1 - \sin^2\theta}$

83. $\sec\theta - \tan\theta\sin\theta$

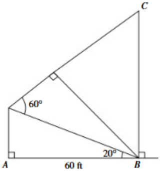
84. $\dfrac{\tan^2\theta + 1}{\tan\theta + \cot\theta}$

85. $\dfrac{\sin^2\theta\sec\theta}{1 + \sec\theta} + \cos\theta$

86. An athlete is in a boat at point A, $\frac{1}{4}$ mi from the nearest point D on a straight shoreline. She can row at a speed of 3 mph and run at a speed of 6 mph. Her planned workout is to row to point D and then run to point C farther down the shoreline. However, the current pushes her at an angle of 24° from her original path so that she comes ashore at point B, 2 mi from her final destination at point C. How many minutes will her trip take? Round to the nearest minute.



87. Given that the distance from A to B is 60 ft, find the distance from B to C. Round to the nearest foot.



88. In the figure, $\overline{CD} = 15$, $\overline{DE} = 8$, $\tan\alpha = \frac{4}{3}$, and $\sin\beta = \frac{3}{5}$. Find the lengths of

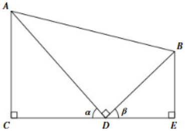
a. $\overline{AC}$

b. $\overline{AD}$

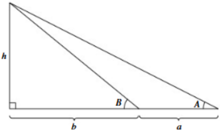
c. $\overline{DB}$

d. $\overline{BE}$

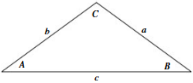
e. $\overline{AB}$



89. For the given triangle, show that $a = h\cot A - h\cot B$.



90. Show that the area A of the triangle is given by $A = \frac{1}{2}bc\sin A$.



91. Use a cofunction relationship to show that the product $(\tan 1°)(\tan 2°)(\tan 3°) \cdot \cdots \cdot (\tan 87°)(\tan 88°)(\tan 89°)$ is equal to 1.

Technology Connections

For Exercises 92–95, use a calculator to approximate the values of the left- and right-hand sides of each statement for $A = 30°$ and $B = 45°$. Based on the approximations from your calculator, determine if the statement appears to be true or false.

92. a. $\sin(A + B) = \sin A + \sin B$

b. $\sin(A + B) = \sin A \cos B + \cos A \sin B$

93. a. $\tan(A - B) = \tan A - \tan B$

b. $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

94. a. $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{1 + \cos A}{2}}$

b. $\cos\left(\frac{A}{2}\right) = \frac{1}{2}\cos A$

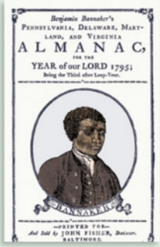
95. a. $\tan\left(\frac{B}{2}\right) = \frac{1 - \cos B}{\sin B}$

b. $\tan\left(\frac{B}{2}\right) = \frac{\sin B}{1 + \cos B}$

Point of Interest

Washington, D.C., is a unique American city in that it was established by the United States Constitution to serve as the nation's capital. After land was ceded by Maryland and Virginia, the seat of the national government was built along the Potomac River in what is now called the District of Columbia. Interestingly, one of the surveyors tasked with mapping the land and creating the boundaries for the city was a free African American man named Benjamin Banneker.

Benjamin Banneker was born in 1731 in the then slave state of Maryland. His mother, a free Black woman, and his father, a freed man, ran a family farm near Baltimore. Benjamin's life was somewhat of an historical anomaly. He learned to read from his maternal grandmother and for a short time attended a Quaker school, but like many people of color during that time, he was largely self-educated. He was an intellectually curious man with a knack for mathematics and invention. Among his accomplishments in early life, he is known for constructing an irrigation system for the family farm and hand-carving a wooden clock that kept near perfect time.



Benjamin Bannaker's
PENNSYLVANIA, DELAWARE, MARY-
LAND, AND VIRGINIA
ALMANAC,
FOR THE
YEAR of our LORD 1795;
Being the Third after Leap-Year.

BANNAKER

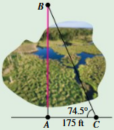
—PRINTED FOR—
And Sold by JOHN FISHER, *Stationer*,
BALTIMORE.

After meeting the prominent land surveyor George Ellicott, Banneker developed a keen interest in astronomy and surveying and quickly excelled in both fields. In 1791, Ellicott hired Banneker to assist in surveying the territory for the nation's capital.

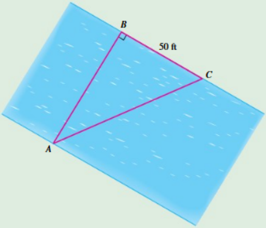
Banneker also used his knowledge of mathematics and astronomy to publish his own widely distributed almanac that included astronomical information (information on the tides, phases of the Moon, eclipses, and solar table calculations), along with medical treatments of the time and his own personal essays.

Library of Congress Geography and Map Division [G3850 1792 .E42 Vault]; Science History Images/Alamy Stock Photo

96. A zip line is to be built between two towers labeled *A* and *B* across a wetland area. To approximate the distance of the zip line, a surveyor marks a third point *C*, a distance of 175 ft from one end of the zip line and perpendicular to the zip line. The measure of $\angle ACB$ is 74.5°. How long is the zip line? Round to the nearest foot.



97. To determine the width of a river from point *A* to point *B*, a surveyor walks downriver 50 ft along a line perpendicular to $\overline{AB}$ to a new position at point *C*. The surveyor determines that the measure of $\angle ACB$ is 60°. Find the exact width of the river from point *A* to point *B*.



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Lesson 4-4

Trigonometric Functions of Any Angle

What Will You Learn?

After completing this lesson, you should be able to:

- Evaluate Trigonometric Functions of Any Angle
- Determine Reference Angles
- Evaluate Trigonometric Functions Using Reference Angles

Learn: Evaluate Trigonometric Functions of Any Angle

In many applications we encounter angles that are not acute. For example, a robotic arm may have a range of motion of 360°, or an object may rotate in a repeated circular pattern through all possible angles. For these situations, we need to extend the definition of trigonometric functions to any angle.

In Figure 4-36, the real number t corresponds to a point $Q(a, b)$ on the unit circle. Angle θ is a second quadrant angle passing through point $P(x, y)$ with terminal side passing through Q. Suppose that we drop perpendicular line segments from points P and Q to the x-axis to form two similar right triangles. For $\triangle ORP$, the hypotenuse r is given by $r = \sqrt{x^2 + y^2}$ and the lengths of the two legs are $|x|$ and y. For $\triangle OSQ$, the hypotenuse is 1 unit and the lengths of the legs are $|a|$ and b.

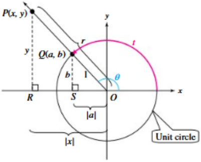


Figure 4-36

Since $\triangle ORP$ and $\triangle OSQ$ are similar triangles, the ratios of their corresponding sides are proportional, which implies that $\frac{b}{1} = \frac{y}{r}$. Furthermore, since $\sin t = b$, we have $\sin t = \frac{y}{r}$. We also know that there is a one-to-one correspondence between the real number t and the radian measure of the central angle θ in standard position with terminal side through P. From this correspondence, we define $\sin\theta = \sin t = \frac{y}{r}$. Similar logic can be used to define the other five trigonometric functions for a point P in any quadrant or on the x- or y-axes. The results are summarized in Table 4-8 on page 516.

Let θ be an angle in standard position with point $P(x, y)$ on the terminal side, and let $r = \sqrt{x^2 + y^2} \neq 0$ represent the distance from $P(x, y)$ to $(0, 0)$.

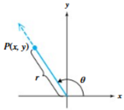
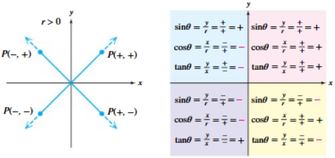


TABLE 4-8 Trigonometric Functions of Any Angle

Trigonometric Function	Restriction
$\sin\theta = \frac{y}{r}$	none
$\cos\theta = \frac{x}{r}$	none
$\tan\theta = \frac{y}{x}$	$x \neq 0$
$\csc\theta = \frac{r}{y}$	$y \neq 0$
$\sec\theta = \frac{r}{x}$	$x \neq 0$
$\cot\theta = \frac{x}{y}$	$y \neq 0$

Point P is on the terminal side of angle θ and not at the origin, which means that $r = \sqrt{x^2 + y^2} \neq 0$. Therefore, the signs of the values of the trigonometric functions depend solely on the signs of x and y. For an acute angle, the values of the trigonometric functions are positive because x and y are both positive in the first quadrant. For angles in the other quadrants, we determine the signs of the trigonometric functions by analyzing the signs of x and y (Figure 4-37). Note that the reciprocal functions cosecant, secant, and cotangent will have the same signs as sine, cosine, and tangent, respectively, for a given value of θ.



r > 0
y
P(−, +)
P(+, +)
x
P(−, −)
P(+, −)
y
$\sin\theta = \frac{y}{r} = \frac{+}{+} = +$
$\cos\theta = \frac{x}{r} = \frac{-}{+} = -$
$\tan\theta = \frac{y}{x} = \frac{+}{-} = -$
$\sin\theta = \frac{y}{r} = \frac{+}{+} = +$
$\cos\theta = \frac{x}{r} = \frac{+}{+} = +$
$\tan\theta = \frac{y}{x} = \frac{+}{+} = +$
x
$\sin\theta = \frac{y}{r} = \frac{-}{+} = -$
$\cos\theta = \frac{x}{r} = \frac{-}{+} = -$
$\tan\theta = \frac{y}{x} = \frac{-}{-} = +$
$\sin\theta = \frac{y}{r} = \frac{-}{+} = -$
$\cos\theta = \frac{x}{r} = \frac{+}{+} = +$
$\tan\theta = \frac{y}{x} = \frac{-}{+} = -$

Figure 4-37

Example 1: Evaluating Trigonometric Functions

Let $P(-2, -5)$ be a point on the terminal side of angle θ drawn in standard position. Find the values of the six trigonometric functions of θ.

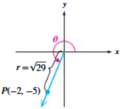
Solution:

We first draw θ in standard position and label point P.

The distance between $P(-2, -5)$ and $(0, 0)$ is

$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-5)^2} = \sqrt{4 + 25} = \sqrt{29}$.

We have $x = -2$, $y = -5$, and $r = \sqrt{29}$.



y
θ
x
$r = \sqrt{29}$
P(−2, −5)

$$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{29}} = \frac{-5}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = -\frac{5\sqrt{29}}{29} \qquad \csc\theta = \frac{r}{y} = \frac{\sqrt{29}}{-5} = -\frac{\sqrt{29}}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{29}} = \frac{-2}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = -\frac{2\sqrt{29}}{29} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{29}}{-2} = -\frac{\sqrt{29}}{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-5}{-2} = \frac{5}{2} \qquad \cot\theta = \frac{x}{y} = \frac{-2}{-5} = \frac{2}{5}$$

Apply the Skills

1. Let $P(-5, -7)$ be a point on the terminal side of angle θ drawn in standard position. Find the values of the six trigonometric functions of θ.

Learn: Determine Reference Angles

For nonacute angles θ, we often find values of the six trigonometric functions by using the related *reference angle*.

Definition of a Reference Angle

Let θ be an angle in standard position. The **reference angle** for θ is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Figure 4-38 shows the reference angles θ' for angles on the interval $[0, 2\pi)$ drawn in standard position for each of the four quadrants.

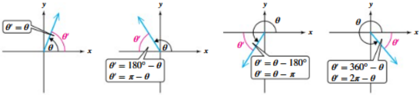


Figure 4-38

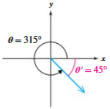
Example 2: Finding Reference Angles

Find the reference angle θ'.

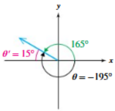
a. $\theta = 315°$ **b.** $\theta = -195°$ **c.** $\theta = 3.5$ **d.** $\theta = \frac{25\pi}{4}$

Solution:

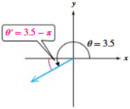
a. $\theta = 315°$ is a fourth quadrant angle. The acute angle it makes with the x-axis is $\theta' = 360° - 315° = 45°$.



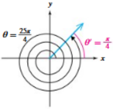
b. $\theta = -195^\circ$ is a second quadrant angle coterminal to 165°. The acute angle it makes with the *x*-axis is $\theta' = 180^\circ - 165^\circ = 15^\circ$.



c. $\theta = 3.5$ is measured in radians. Since $\pi \approx 3.14$ and $\frac{3\pi}{2} \approx 4.71$, we know that $\pi < 3.5 < \frac{3\pi}{2}$ implying that 3.5 is in the third quadrant. The reference angle is $\theta' = 3.5 - \pi \approx 0.3584$.



d. $\theta = \frac{25\pi}{4}$ is a first quadrant angle coterminal to $\frac{\pi}{4}$. The angle $\frac{\pi}{4}$ is an acute angle and is its own reference angle. Therefore, $\theta' = \frac{\pi}{4}$.



Apply the Skills

2. Find the reference angle θ'.

a. $\theta = 150^\circ$ **b.** $\theta = -157.5^\circ$ **c.** $\theta = 5$ **d.** $\theta = \frac{13\pi}{3}$

Learn: Evaluate Trigonometric Functions Using Reference Angles

In Figure 4-39, an angle θ is drawn in standard position along with its accompanying reference angle θ'. A right triangle can be formed by dropping a perpendicular line segment from a point $P(x, y)$ on the terminal side of θ to the *x*-axis. The length of the vertical leg of the triangle is $|y|$, the length of the horizontal leg is $|x|$, and the hypotenuse is r.

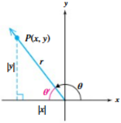


Figure 4-39

Now suppose we compare the values of the cosine function of θ and θ'.

$$\cos\theta = \frac{x}{r} \text{ and } \cos\theta' = \frac{\text{adj}}{\text{hyp}} = \frac{|x|}{r}$$

Notice that the ratio for $\cos\theta$ has x in the numerator and the ratio for $\cos\theta'$ has $|x|$. It follows that $\cos\theta$ and $\cos\theta'$ are equal except possibly for their signs. A similar argument holds for the other five trigonometric functions. This leads to the following procedure to evaluate a trigonometric function based on reference angles.

Evaluating Trigonometric Functions Using Reference Angles

To find the value of a trigonometric function of a given angle θ,

1. Determine the function value of the reference angle θ'.
2. Affix the appropriate sign based on the quadrant in which θ lies.

Before we present examples of this process, take a minute to review the values of the trigonometric functions of 30°, 45°, and 60° (Table 4-9).

TABLE 4-9 Trigonometric Function Values of Special Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$30^\circ = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$45^\circ = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60^\circ = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Example 3: Using Reference Angles to Evaluate Functions

Evaluate the functions. **a.** $\sin 240^\circ$ **b.** $\tan(-225^\circ)$ **c.** $\sec\frac{11\pi}{6}$

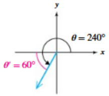
Solution:

a. $\theta = 240^\circ$ is in Quadrant III.

The reference angle is $240^\circ - 180^\circ = 60^\circ$

Since $\sin\theta$ is negative in Quadrant III,

$\sin 240^\circ = -\sin 60^\circ = -\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$

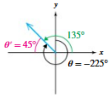


b. $\theta = -225^\circ$ is an angle in Quadrant II, coterminal to 135°.

The reference angle is $180^\circ - 135^\circ = 45^\circ$.

Since $\tan\theta$ is negative in Quadrant II,

$\tan(-225^\circ) = -\tan 45^\circ = -(1) = -1$

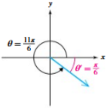


c. $\theta = \frac{11\pi}{6}$ is in Quadrant IV.

The reference angle is $2\pi - \frac{11\pi}{6} = \frac{12\pi}{6} - \frac{11\pi}{6} = \frac{\pi}{6}$

Since $\cos\theta$ and its reciprocal $\sec\theta$ are positive in Quadrant IV,

$$\sec\frac{11\pi}{6} = \sec\frac{\pi}{6} = \frac{2\sqrt{3}}{3}$$



Apply the Skills

3. Evaluate the functions.

a. $\cos\frac{5\pi}{6}$ **b.** $\cot(-120^\circ)$ **c.** $\csc\frac{7\pi}{4}$

Example 4: Evaluate Trigonometric Functions

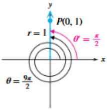
Evaluate the functions. **a.** $\sec\frac{9\pi}{2}$ **b.** $\sin(-510^\circ)$

Solution:

a. $\frac{9\pi}{2}$ is coterminal to $\frac{\pi}{2}$.

The terminal side of $\frac{\pi}{2}$ is on the positive *y*-axis, where we have selected the arbitrary point (0, 1).

$\sec\frac{9\pi}{2} = \sec\frac{\pi}{2} = \frac{r}{x} \rightarrow \frac{1}{0}$ is undefined.

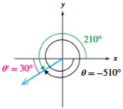


b. -510° is coterminal to 210°, which is a third-quadrant angle.

The reference angle is $210^\circ - 180^\circ = 30^\circ$.

Since $\sin\theta$ is negative in Quadrant III,

$\sin(-510^\circ) = -\sin 30^\circ = -\left(\frac{1}{2}\right) = -\frac{1}{2}$



Apply the Skills

4. Evaluate the functions.

a. $\csc(11\pi)$ **b.** $\cos(-600^\circ)$

In Example 5, we use a reference angle to determine the values of trigonometric functions based on given information about θ.

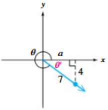
Example 5: Evaluating Trigonometric Functions

Given $\sin\theta = -\frac{4}{7}$ and $\cos\theta > 0$, find $\cos\theta$ and $\tan\theta$.

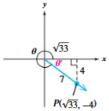
Solution:

First note that $\sin\theta < 0$ and $\cos\theta > 0$ in Quadrant IV.

We can label the reference angle θ' and then draw a representative triangle with opposite leg of length 4 units and hypotenuse of length 7 units.



Using the Pythagorean theorem, we can determine the length of the adjacent leg.



$$\begin{aligned} a^2 + (4)^2 &= (7)^2 \\ a^2 + 16 &= 49 \\ a^2 &= 33 \\ a &= \sqrt{33} \end{aligned}$$

Choose the positive square root for the length of a side of a triangle.

$$\cos\theta = \cos\theta' = \frac{\sqrt{33}}{7}$$

$\cos\theta$ is positive in Quadrant IV.

$$\tan\theta = -\tan\theta' = -\frac{4}{\sqrt{33}} = -\frac{4}{\sqrt{33}} \cdot \frac{\sqrt{33}}{\sqrt{33}} = -\frac{4\sqrt{33}}{33}$$

$\tan\theta$ is negative in Quadrant IV.

Insights

With the reference angle and representative triangle drawn in Quadrant IV, we can find point $P\left(\sqrt{33}, -4\right)$ on the terminal side of θ.

$$\cos\theta = \frac{x}{r} = \frac{\sqrt{33}}{7}$$
$$\tan\theta = \frac{y}{x} = \frac{-4}{\sqrt{33}}$$

Apply the Skills

5. Given $\cos\theta = -\frac{3}{8}$ and $\sin\theta < 0$, find $\sin\theta$ and $\tan\theta$.

The identities involving trigonometric functions of acute angles presented in Lesson 4.3 are also true for trigonometric functions of non-acute angles provided that the functions are well defined at θ.

Pythagorean Identities	Reciprocal Identities	Quotient Identities
$\sin^2\theta + \cos^2\theta = 1$	$\csc\theta = \frac{1}{\sin\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\tan^2\theta + 1 = \sec^2\theta$	$\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$
$1 + \cot^2\theta = \csc^2\theta$	$\cot\theta = \frac{1}{\tan\theta}$	

In Example 6, we are given information about an angle θ and will find the values of the trigonometric functions by applying the fundamental identities and by using reference angles.

Example 6: Using Fundamental Identities

Given $\cos\theta = -\frac{3}{5}$ for θ in Quadrant II, find $\sin\theta$ and $\tan\theta$.

Solution:

Using Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + \left(-\frac{3}{5}\right)^2 = 1$$

$$\sin^2\theta + \frac{9}{25} = 1$$

$$\sin^2\theta = 1 - \frac{9}{25}$$

$$\sin^2\theta = \frac{25}{25} - \frac{9}{25}$$

$$\sin^2\theta = \frac{16}{25}$$

$$\sin\theta = \pm\frac{4}{5}$$

In Quadrant II, $\sin\theta > 0$. Therefore, choose $\sin\theta = \frac{4}{5}$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

Alternative Approach

Label the reference angle θ' and then draw a representative triangle with adjacent leg of length 3 units and hypotenuse of length 5 units.

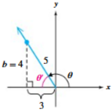
Using the Pythagorean theorem, we can determine the length of the opposite leg.

$$(3)^2 + b^2 = (5)^2$$

$$9 + b^2 = 25$$

$$b^2 = 16$$

$$b = 4$$



In Quadrant II, $\sin\theta > 0$.

Therefore, $\sin\theta = \sin\theta' = \frac{4}{5}$.

In Quadrant II, $\tan\theta < 0$.

Therefore, $\tan\theta = -\tan\theta' = -\frac{4}{3}$.

Insights

With the reference angle and representative triangle drawn in Quadrant II, we can find point $P(-3, 4)$ on the terminal side of θ. Therefore, $x = -3$, $y = 4$, and $r = 5$.

$$\sin\theta = \frac{y}{r} = \frac{4}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{4}{-3}$$

Apply the Skills

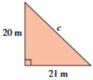
6. Given $\sin\theta = -\frac{5}{13}$ for θ in Quadrant IV, find $\cos\theta$ and $\tan\theta$.

Practice 4-4

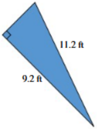
Practice Exercises

Prerequisite Review

R.1. Find the length of the missing side of the right triangle.



R.2. Find the length of the third side using the Pythagorean theorem. Round the answer to the nearest tenth of a foot.



R.3. Use the distance formula to find the distance from (2, 5) to (–3, 10).

Concept Connections

1. The distance from the origin to a point $P(x, y)$ is given by __________.

2. Angles with terminal sides on the coordinate axes are referred to as __________ angles.

3. If θ is an angle in standard position, the __________ angle for θ is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

4. The values $\tan\theta$ and $\sec\theta$ are undefined for odd multiples of __________.

5. The values $\cot\theta$ and $\csc\theta$ are undefined for multiples of __________.

6. For what values of θ is $\sin\theta$ greater than 1?

Learn: Evaluate Trigonometric Functions of Any Angle

7. Let $P(x, y)$ be a point on the terminal side of an angle θ drawn in standard position and let r be the distance from P to the origin. Fill in the boxes to form the ratios defining the six trigonometric functions.

a. $\sin\theta = \frac{\square}{\square}$ **b.** $\cos\theta = \frac{\square}{\square}$ **c.** $\tan\theta = \frac{\square}{\square}$

d. $\csc\theta = \frac{\square}{\square}$ **e.** $\sec\theta = \frac{\square}{\square}$ **f.** $\cot\theta = \frac{\square}{\square}$

8. Fill in the cells in the table with the appropriate sign for each trigonometric function for θ in Quadrants I, II, III, and IV. The signs for the sine function are done for you.

	Quadrant			
	I	**II**	**III**	**IV**
$\sin\theta$	+	+	−	−
$\cos\theta$				
$\tan\theta$				
$\csc\theta$				
$\sec\theta$				
$\cot\theta$				

For Exercises 9–14, given the stated conditions, identify the quadrant in which θ lies.

9. $\sin\theta < 0$ and $\tan\theta > 0$

10. $\csc\theta > 0$ and $\cot\theta < 0$

11. $\sec\theta < 0$ and $\tan\theta < 0$

12. $\cot\theta < 0$ and $\sin\theta < 0$

13. $\cos\theta > 0$ and $\cot\theta < 0$

14. $\cot\theta < 0$ and $\sec\theta < 0$

For Exercises 15–20, a point is given on the terminal side of an angle θ drawn in standard position. Find the values of the six trigonometric functions of θ. (See Example 1, p. 516)

15. (5, –12) **16.** (–8, 15) **17.** (–3, –5)

18. $(2, -3)$ **19.** $\left(-\frac{3}{2}, 2\right)$ **20.** $\left(5, -\frac{8}{3}\right)$

21. Complete the table for the given angles.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$0^\circ = 0$						
$90^\circ = \frac{\pi}{2}$						
$180^\circ = \pi$						
$270^\circ = \frac{3\pi}{2}$						
$360^\circ = 2\pi$						

22. Evaluate the expression for $A = 90^\circ$, $B = 180^\circ$, and $C = 270^\circ$.

$$\frac{\sin A + 2\cos B}{\sec B - 3\csc C}$$

Learn: Determine Reference Angles

For Exercises 23–30, find the reference angle for the given angle. (See Example 2, p. 517)

23. a. 135° **b.** -330°

c. 660° **d.** -690°

24. a. -120° **b.** 225°

c. -1035° **d.** 510°

25. a. $\frac{2\pi}{3}$ **b.** $-\frac{5\pi}{6}$

c. $\frac{13\pi}{4}$ **d.** $-\frac{10\pi}{3}$

26. a. $-\frac{5\pi}{4}$ **b.** $\frac{11\pi}{6}$

c. $\frac{17\pi}{3}$ **d.** $\frac{19\pi}{6}$

27. a. $\frac{20\pi}{17}$ **b.** $-\frac{99\pi}{20}$

c. 110° **d.** -422°

28. a. $\frac{25\pi}{11}$ **b.** $\frac{-27\pi}{14}$

c. -512° **d.** 1280°

29. a. 1.8 **b.** 1.8π

c. 5.1 **d.** 5.1°

30. a. 0.6 **b.** 0.6π

c. 100 **d.** 100°

Learn: Evaluate Trigonometric Functions Using Reference Angles

For Exercises 31–54, use reference angles to find the exact value. (See Examples 3–4, pp. 519–520)

31. $\sin 120^\circ$

32. $\cos 225^\circ$

33. $\cos\frac{4\pi}{3}$

34. $\sin\frac{5\pi}{6}$

35. $\sec(-330^\circ)$

36. $\csc(-225^\circ)$

37. $\sec\frac{13\pi}{3}$

38. $\csc\frac{5\pi}{3}$

39. $\cot 240^\circ$

40. $\tan(-150^\circ)$

41. $\tan\frac{5\pi}{4}$

42. $\cot\left(-\frac{3\pi}{4}\right)$

43. $\cos(-630^\circ)$

44. $\sin 630^\circ$

45. $\sin\frac{17\pi}{6}$

46. $\cos\left(-\frac{11\pi}{4}\right)$

47. $\sec 1170^\circ$

48. $\csc 750^\circ$

49. $\csc(-5\pi)$

50. $\sec 5\pi$

51. $\tan(-2400^\circ)$

52. $\cot 900^\circ$

53. $\cot\frac{42\pi}{8}$

54. $\tan\frac{18\pi}{4}$

For Exercises 55–58, find two angles between 0° and 360° for the given condition.

55. $\sin\theta = \frac{1}{2}$

56. $\cos\theta = -\frac{\sqrt{2}}{2}$

57. $\cot\theta = -\sqrt{3}$

58. $\tan\theta = \sqrt{3}$

For Exercises 59–62, find two angles between 0 and 2π for the given condition.

59. $\sec\theta = \sqrt{2}$

60. $\csc\theta = -\frac{2\sqrt{3}}{3}$

61. $\tan\theta = -\frac{\sqrt{3}}{3}$

62. $\cot\theta = 1$

In Exercises 63–68, find the values of the trigonometric functions from the given information. (See Example 5, p. 521)

63. Given $\tan\theta = -\frac{20}{21}$ and $\cos\theta < 0$, find $\sin\theta$ and $\cos\theta$.

64. Given $\cot\theta = \frac{11}{60}$ and $\sin\theta < 0$, find $\cos\theta$ and $\sin\theta$.

65. Given $\sin\theta = -\frac{3}{10}$ and $\tan\theta > 0$, find $\cos\theta$ and $\cot\theta$.

66. Given $\cos\theta = -\frac{5}{8}$ and $\csc\theta > 0$, find $\sin\theta$ and $\tan\theta$.

67. Given $\sec\theta = \frac{\sqrt{58}}{7}$ and $\cot\theta < 0$, find $\csc\theta$ and $\cos\theta$.

68. Given $\csc\theta = -\frac{\sqrt{26}}{5}$ and $\cos\theta < 0$, find $\sin\theta$ and $\cot\theta$.

For Exercises 69–74, use fundamental trigonometric identities to find the values of the functions. (See Example 6, p. 522)

69. Given $\cos\theta = \frac{20}{29}$ for θ in Quadrant IV, find $\sin\theta$ and $\tan\theta$.

70. Given $\sin\theta = -\frac{8}{17}$ for θ in Quadrant III, find $\cos\theta$ and $\cot\theta$.

71. Given $\tan\theta = -4$ for θ in Quadrant II, find $\sec\theta$ and $\cot\theta$.

72. Given $\sec\theta = 5$ for θ in Quadrant IV, find $\csc\theta$ and $\cos\theta$.

73. Given $\cot\theta = \frac{5}{4}$ for θ in Quadrant III, find $\csc\theta$ and $\sin\theta$.

74. Given $\csc\theta = \frac{7}{3}$ for θ in Quadrant II, find $\cot\theta$ and $\cos\theta$.

Mixed Exercises

For Exercises 75–76, find the sign of the expression for θ in each quadrant.

75.

		I	II	III	IV
a.	$\sin\theta\cos\theta$				
b.	$\frac{\tan\theta}{\cos\theta}$				

76.

		I	II	III	IV
a.	$\frac{\cot\theta}{\sin\theta}$				
b.	$\tan\theta\sec\theta$				

For Exercises 77–84, suppose that θ is an acute angle. Identify each statement as true or false. If the statement is false, rewrite the statement to give the correct answer for the right side.

77. $\cos(180° - \theta) = -\cos\theta$

78. $\tan(180° - \theta) = \tan\theta$

79. $\tan(180° + \theta) = \tan\theta$

80. $\sin(180° + \theta) = -\sin\theta$

81. $\csc(\pi - \theta) = -\csc\theta$

82. $\sec(\pi + \theta) = \sec\theta$

83. $\cos(\pi + \theta) = -\cos\theta$

84. $\sin(\pi + \theta) = -\sin\theta$

For Exercises 85–90, find the value of each expression.

85. $\sin 30° \cdot \cos 150° \cdot \sec 60° \cdot \csc 120°$

86. $\cos 45° \cdot \sin 240° \cdot \tan 135° \cdot \cot 60°$

87. $\cos^2\frac{5\pi}{4} - \sin^2\frac{2\pi}{3}$

88. $\sin^2\frac{11\pi}{6} + \cos^2\frac{4\pi}{3}$

89. $\dfrac{2\tan\frac{11\pi}{6}}{1 - \tan^2\frac{11\pi}{6}}$

90. $\dfrac{\cot^2\frac{4\pi}{3} - 1}{2\cot\frac{4\pi}{3}}$

For Exercises 91–94, verify the statement for the given values.

91. $\sin(A - B) = \sin A\cos B - \cos A\sin B$; $A = 240°, B = 120°$

92. $\cos(B - A) = \cos B\cos A + \sin B\sin A$; $A = 330°, B = 120°$

93. $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\,\tan B}$; $A = 210°$, $B = 120°$

94. $\cot(A + B) = \frac{\cot A\,\cot B - 1}{\cot A + \cot B}$; $A = 300°$, $B = 150°$

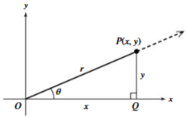
For Exercises 95–96, give the exact values if possible. Otherwise, use a calculator and approximate the result to 4 decimal places.

95. a. $\sin 30°$ **b.** $\sin(30\pi)$ **c.** $\sin 30$

96. a. $\cos(0.25\pi)$ **b.** $\frac{\cos\pi}{0.25}$ **c.** $\cos(25°)$

Write About It

97. Explain why neither $\sin\theta$ nor $\cos\theta$ can be greater than 1. Refer to the figure for your explanation.



98. Explain why $\tan\theta$ is undefined at $\theta = \frac{\pi}{2}$ but $\cot\theta$ is defined at $\theta = \frac{\pi}{2}$.

Expanding Your Skills

For Exercises 99–100, for part (a), find four angles between 0° and 720° that satisfy the given condition. For part (b), find four angles between 0° and 360° that satisfy the given condition.

99. **a.** $\cos\theta = -\frac{\sqrt{2}}{2}$

b. $\cos 2\theta = -\frac{\sqrt{2}}{2}$

100. **a.** $\tan\theta = \sqrt{3}$

b. $\tan 2\theta = \sqrt{3}$

For Exercises 101–102, for part (a), find four angles between 0 and 3π that satisfy the given condition. For part (b), find four angles between 0 and π that satisfy the given condition.

101. **a.** $\csc\theta = \sqrt{2}$

b. $\csc 3\theta = \sqrt{2}$

102. **a.** $\sin\theta = \frac{\sqrt{3}}{2}$

b. $\sin 3\theta = \frac{\sqrt{3}}{2}$

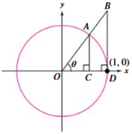
103. **a.** Graph the point (3, 4) on a rectangular coordinate system and draw a line segment connecting the point to the origin. Find the slope of the line segment.

b. Draw another line segment from the point (3, 4) to meet the *x*-axis at a right angle, forming a right triangle with the *x*-axis as one side. Find the tangent of the acute angle that has the *x*-axis as its initial side.

c. Compare the results in part (a) and part (b).

104. The circle shown is centered at the origin with a radius of 1. The segment $\overline{BD}$ is tangent to the circle at D. Match the length of each segment with the appropriate trigonometric function.

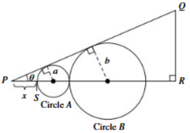
a. $\overline{AC}$ **i.** $\tan\theta$
b. $\overline{BD}$ **ii.** $\cos\theta$
c. $\overline{OB}$ **iii.** $\sin\theta$
d. $\overline{OC}$ **iv.** $\sec\theta$



105. Circle *A*, with radius *a*, and circle *B*, with radius *b*, are tangent to each other and to $\overline{PQ}$ (see figure). $\overline{PR}$ passes through the center of each circle. Let *x* be the distance from point *P* to a point *S* where $\overline{PR}$ intersects circle *A* on the left. Let θ denote $\angle RPQ$.

a. Show that $\sin\theta = \frac{a}{x+a}$ and $\sin\theta = \frac{b}{x+2a+b}$.

b. Use the results from part (a) to show that $\sin\theta = \frac{b-a}{b+a}$.



Go online for more practice problems.

Lesson 4-5

Graphs of Sine and Cosine Functions

What Will You Learn?

After completing this lesson, you should be able to:

- Graph $y = \sin x$ and $y = \cos x$
- Graph $y = A\sin x$ and $y = A\cos x$
- Graph $y = A\sin Bx$ and $y = A\cos Bx$
- Graph $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$
- Model Sinusoidal Behavior

Learn: Graph $y = \sin x$ and $y = \cos x$

Throughout this text, we have been progressively introducing categories of functions and analyzing their properties and graphs. In each case, we begin with a fundamental "parent" function such as $y = x^2$ and then develop interesting variations on the graph by applying transformations. For example, $y = a(x - h)^2 + k$ is a family of quadratic functions whose properties help us model physical phenomena such as projectile motion.

In a similar manner we will graph the basic functions $y = \sin x$ and $y = \cos x$ and then analyze their variations. To begin, note that we will use variable x (rather than θ or t) to represent the independent variable.

To develop the graph of $y = \sin x$ we return to the notion of a number line wrapped onto the unit circle. The points on the number line correspond to the real numbers x in the domain of the sine function. The value of $\sin x$ is the y-coordinate of the corresponding point on the unit circle (Figure 4-40).

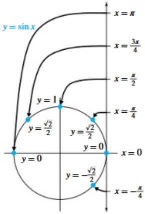


Figure 4-40

Table 4-10 gives several values of x and the corresponding values of $\sin x$. Plotting these as ordered pairs gives the graph of $y = \sin x$ in Figure 4-41.

TABLE 4-10 Values of Sin x for Common Values of x

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0

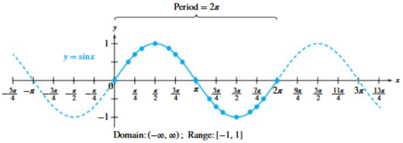


Figure 4-41

Insights

The decimal approximations of the coordinates involving irrational numbers can be used to sketch the graph. For example,

$$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) \approx (0.79, 0.71).$$

Recall that the sine function is periodic with period 2π. This means that the pattern shown between 0 and 2π repeats infinitely far to the left and right as denoted by the dashed portion of the curve (Figure 4-41). In Figure 4-42, one complete cycle of the graph of $y = \sin x$ is shown over the interval $[0, 2\pi]$. Notice the relationship between the y-coordinates on the unit circle and the graph of $y = \sin x$.

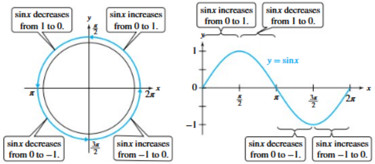

sin x decreases from 1 to 0.
sin x increases from 0 to 1.
y
$\frac{\pi}{2}$
π
2π
x
sin x decreases from 0 to −1.
$\frac{3\pi}{2}$
sin x increases from −1 to 0.
sin x increases from 0 to 1.
sin x decreases from 1 to 0.
y
1
0
−1
y = sin x
x
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
sin x decreases from 0 to −1.
sin x increases from −1 to 0.

Figure 4-42

In a similar fashion, we can graph $y = \cos x$. Table 4-11 (see page 530) gives several values of x with the corresponding values of $\cos x$. Plotting these as ordered pairs gives the graph of $y = \cos x$ in Figure 4-43.

TABLE 4-11 Values of Cos x for Common Values of x

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

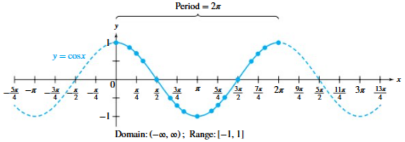


Figure 4-43

Insights

When we define $\sin t = y$ and $\cos t = x$, the values of x and y refer to the coordinates of a point on the unit circle $x^2 + y^2 = 1$.

When we define $y = \sin x$ and $y = \cos x$, the variable x has taken the role of t (a real number on a number line wrapped onto the unit circle) and y is the corresponding sine or cosine value.

In Figure 4-44, the graphs of $y = \sin x$ and $y = \cos x$ are shown on the interval $[-2\pi, 2\pi]$ for comparison. From the graphs, we make the following observations.

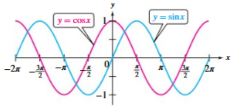


Figure 4-44

Characteristics of the Graphs of $y = \sin x$ and $y = \cos x$

- The domain is $(-\infty, \infty)$.
- The range is $[-1, 1]$.
- The period is 2π.
- The graph of $y = \sin x$ is symmetric with respect to the origin. $y = \sin x$ is an odd function.
- The graph of $y = \cos x$ is symmetric with respect to the y-axis. $y = \cos x$ is an even function.
- The graphs of $y = \sin x$ and $y = \cos x$ differ by a horizontal shift of $\frac{\pi}{2}$.

Learn: Graph $y = A\sin x$ and $y = A\cos x$

Figures 4-45 and 4-46 show one complete cycle of the graphs of $y = \sin x$ and $y = \cos x$. When we graph variations of these functions we want to graph the key points. These are the relative minima, relative maxima, and x-intercepts. Notice that the graphs of both the sine and cosine function alternate between a relative minimum or maximum and an x-intercept for each quarter of a period.

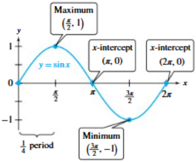


Figure 4-45

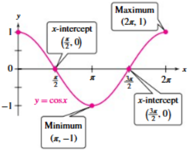


Figure 4-46

We begin analyzing variations of the sine and cosine graphs by graphing functions of the form $y = A\sin x$ and $y = A\cos x$.

Example 1: Graphing $y = A\sin x$ and $y = A\cos x$

Graph the function and identify the key points on one full period.

a. $y = 3\sin x$ **b.** $y = -\frac{1}{2}\cos x$

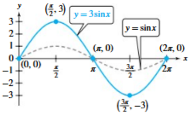
Solution:

Recall from Lesson 1.6 that the graph of $y = A \cdot f(x)$ is the graph of $y = f(x)$ with

- A vertical stretch if $|A| > 1$ or
- A vertical shrink if $0 < |A| < 1$.
- A reflection across the x-axis if $A < 0$.

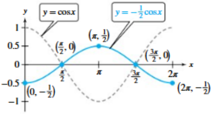
a. Given $y = 3\sin x$, the value of A is 3, so the graph of $y = 3\sin x$ is the graph of $y = \sin x$ with a vertical stretch by a factor of 3.

The dashed curve $y = \sin x$ is shown for comparison.



b. Given $y = -\frac{1}{2}\cos x$, the value of A is $-\frac{1}{2}$, so the graph of $y = -\frac{1}{2}\cos x$ is the graph of $y = \cos x$ with a vertical shrink and a reflection across the x-axis.

The dashed curve, $y = \cos x$, is shown for comparison.



Apply the Skills

1. Graph the function and identify the key points on one full period.

a. $y = 2\cos x$ b. $y = -\frac{1}{3}\sin x$

Notice that the graph of $y = 3\sin x$ from Example 1(a) deviates 3 units above and below the *x*-axis. For the graphs of $y = |A|\sin x$ and $y = |A|\cos x$, the value of the vertical scaling factor $|A|$ is called the **amplitude** of the function. The amplitude represents the amount of deviation from the central position of the sine wave. The amplitude is half the distance between the maximum value of the function and the minimum value of the function. In Example 1(a), the amplitude is $|3| = 3$, and in Example 1(b), the amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$.

Learn: Graph $y = A\sin Bx$ and $y = A\cos Bx$

Recall from Lesson 1.6 that the graph of $y = f(Bx)$ is the graph of $y = f(x)$ with a horizontal shrink or stretch. This means that for $y = \sin x$ and $y = \cos x$, the factor *B* will affect the period of the graph. For example, Figures 4-47 and 4-48 show the graphs of $y = \sin 2x$ and $y = \sin\frac{1}{2}x$, respectively, as compared to the parent function $y = \sin x$.

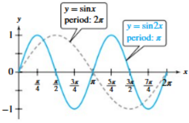


Figure 4-47

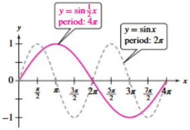


Figure 4-48

The period of the functions $y = \sin x$ and $y = \cos x$ is 2π. That is, the graphs of $y = \sin x$ and $y = \cos x$ show one complete cycle on the interval $0 \le x \le 2\pi$. To determine a comparable interval for one complete cycle of the graphs of $y = \sin Bx$ and $y = \cos Bx$ for $B > 0$, we can solve the following inequality.

$$\begin{aligned} 0 \le\ & Bx \le 2\pi \\ \frac{0}{B} \le\ & \frac{Bx}{B} \le \frac{2\pi}{B} \\ 0 \le\ & x \le \frac{2\pi}{B} \end{aligned}$$

"Starting" value for the cycle

"Ending" value for the cycle

This tells us that for $B > 0$, the period of $y = \sin Bx$ and $y = \cos Bx$ is $\frac{2\pi}{B}$.

Amplitude and Period of the Sine and Cosine Functions

For $y = A\sin Bx$ and $y = A\cos Bx$ and $B > 0$, the amplitude and period are Amplitude $= |A|$ and Period $= \frac{2\pi}{B}$

To analyze the period of a sine or cosine function with a negative coefficient on x, we would first rewrite $y = A\sin(-Bx)$ and $y = A\cos(-Bx)$ using the odd and even properties. That is, for $B > 0$,

Rewrite $y = \sin(-Bx)$ as $y = -\sin Bx$ because $y = \sin x$ is an odd function.

Rewrite $y = \cos(-Bx)$ as $y = \cos Bx$ because $y = \cos x$ is an even function.

Example 2: Graphing $y = A\sin Bx$

Given $f(x) = 4\sin 3x$,

a. Identify the amplitude and period.

b. Graph the function and identify the key points on one full period.

Solution:

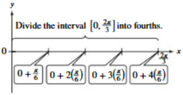
a. $f(x) = 4\sin 3x$ is in the form $f(x) = A\sin Bx$ with $A = 4$ and $B = 3$.

The amplitude is $|A| = |4| = 4$.

The period is $\frac{2\pi}{B} = \frac{2\pi}{3}$.

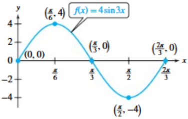
b. The period $\frac{2\pi}{3}$ is shorter than 2π, which tells us that the graph is compressed horizontally. For $y = \sin x$, one complete cycle can be graphed for x on the interval $0 \le x \le 2\pi$. For $y = 4\sin 3x$, one complete cycle can be graphed on the interval defined by $0 \le 3x \le 2\pi$.

$$0 \le 3x \le 2\pi$$
$$\frac{0}{3} \le \frac{3x}{3} \le \frac{2\pi}{3}$$
$$0 \le x \le \frac{2\pi}{3}$$



Dividing the period into fourths, we have increments of $\frac{1}{4}\left(\frac{2\pi}{3}\right) = \frac{\pi}{6}$.

Since the amplitude is 4, the maximum and minimum points have y-coordinates of 4 and -4.



Apply the Skills

2. Given $f(x) = 2\sin 4x$,

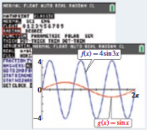
a. Identify the amplitude and period.

b. Graph the function and identify the key points on one full period.

TECHNOLOGY CONNECTIONS

Graphing a Trigonometric Function

To graph a trigonometric function, first be sure that the calculator is in radian mode. The graph of $f(x) = 4\sin 3x$ is shown along with the graph of $g(x) = \sin x$ for comparison. Notice that the graph of f deviates more from the x-axis than g because the amplitude is greater. The period of f is smaller and as a result, shows the sine wave "compressed" horizontally



Learn: Graph $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$

Recall that the graph of $y = f(x - h)$ is the graph of $y = f(x)$ with a horizontal shift of $|h|$ units. If $h > 0$, the shift is to the right, and if $h < 0$, the shift is to the left. The graphs of $y = A\sin(Bx - C)$ and $y = A\cos(Bx - C)$ may have both a change in period and a horizontal shift. In Example 3, we illustrate how to graph such a function.

Example 3: Graphing $y = A\cos(Bx - C)$

Given $y = \cos\left(2x + \frac{\pi}{2}\right)$,

a. Identify the amplitude and period.

b. Graph the function and identify the key points on one full period.

Solution:

The function is of the form $y = A\cos(Bx - C)$, where $A = 1$, $B = 2$, and $C = -\frac{\pi}{2}$.

a. $y = \cos\left(2x + \frac{\pi}{2}\right)$ can be written as $y = 1 \cdot \cos\left(2x + \frac{\pi}{2}\right)$.

The amplitude is $|A| = |1| = 1$.

The period is $\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$.

b. To find an interval over which this function completes one cycle, solve the inequality.

$$0 \le 2x + \frac{\pi}{2} \le 2\pi$$

$$-\frac{\pi}{2} \le 2x \le 2\pi - \frac{\pi}{2} \qquad \text{Subtract } \frac{\pi}{2}.$$

$$-\frac{\pi}{2} \le 2x \le \frac{4\pi}{2} - \frac{\pi}{2} \qquad \text{Write the right side with a common denominator. Then divide by 2.}$$

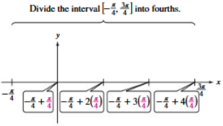
$$-\frac{\pi}{2} \le 2x \le \frac{3\pi}{2}$$

$$-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$$

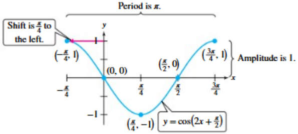
"Starting" value for the cycle

"Ending" value for the cycle

Dividing the period into fourths, we have increments of $\frac{1}{4}(\pi) = \frac{\pi}{4}$.



Divide the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ into fourths.
y
x
$-\frac{\pi}{4}$
$-\frac{\pi}{4} + \frac{\pi}{4}$
$-\frac{\pi}{4} + 2\left(\frac{\pi}{4}\right)$
$-\frac{\pi}{4} + 3\left(\frac{\pi}{4}\right)$
$-\frac{\pi}{4} + 4\left(\frac{\pi}{4}\right)$
$\frac{3\pi}{4}$



Insights

Note that the distance between the "ending" point and "starting" point of the cycle shown is

$$\frac{3\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{4\pi}{4}$$
$$= \pi$$

which is the period of the function.

Apply the Skills

3. Given $y = \cos\left(3x - \frac{\pi}{2}\right)$,

a. Identify the amplitude and period.

b. Graph the function and identify the key points on one full period.

In Example 3, the graph of $y = \cos\left(2x + \frac{\pi}{2}\right)$ is the graph of $y = \cos x$ compressed horizontally by a factor of 2 and shifted to the left $\frac{\pi}{4}$ units. The shift to the left is called the *phase shift*. In general, given $y = A\sin(Bx - C)$ and $y = A\cos(Bx - C)$ for $B > 0$, the horizontal transformations are controlled by the variables B and C. The value of B controls the horizontal shrink or stretch, and therefore the period of the function. For $B > 0$, the phase shift can be determined by solving the following inequality.

The phase shift is the left endpoint.

$$\begin{aligned} 0 &\le Bx - C \le 2\pi \\ C &\le Bx \le 2\pi + C && \text{Add } C \text{ to each part.} \\ \frac{C}{B} &\le \frac{Bx}{B} \le \frac{2\pi + C}{B} && \text{Divide by } B\ (B > 0). \\ \frac{C}{B} &\le x \le \frac{2\pi + C}{B} && \text{The graph is shifted horizontally } \tfrac{C}{B} \text{ units.} \\ & && \text{The phase shift is } \tfrac{C}{B}. \end{aligned}$$

The phase shift is a horizontal shift of a trigonometric function. To find the vertical shift, recall that the graph of $y = f(x) + D$ is the graph of $y = f(x)$ shifted $|D|$ units upward for $D > 0$ and $|D|$ units downward if $D < 0$. We are now ready to summarize the properties of the graphs of $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$.

Properties of the General Sine and Cosine Functions

Consider the graphs of $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$ with $B > 0$.

1. The amplitude is $|A|$.
2. The period is $\frac{2\pi}{B}$.
3. The phase shift is $\frac{C}{B}$.
4. The vertical shift is D.
5. One full cycle is given on the interval $0 \le Bx - C \le 2\pi$.
6. The domain is $-\infty < x < \infty$.
7. The range is $-|A| + D \le y \le |A| + D$.

Example 4: Graphing $y = A\cos(Bx - C) + D$

Given $y = 2\cos(4x - 3\pi) + 5$,

- **a.** Identify the amplitude, period, phase shift, and vertical shift.
- **b.** Graph the function and identify the key points on one full period.

Solution:

a. $y = 2\cos(4x - 3\pi) + 5$ has the form $y = A\cos(Bx - C) + D$, where $A = 2$, $B = 4$, $C = 3\pi$, and $D = 5$.

The amplitude is $|A| = |2| = 2$.

The period is $\frac{2\pi}{B} = \frac{2\pi}{4} = \frac{\pi}{2}$.

The phase shift is $\frac{C}{B} = \frac{3\pi}{4}$.

The vertical shift is D. Since $D = 5 > 0$, the shift is upward 5 units.

b. To find an interval over which this function completes one cycle, solve the inequality.

$$0 \le 4x - 3\pi \le 2\pi$$

$$3\pi \le 4x \le 2\pi + 3\pi \qquad \text{Add } 3\pi.$$

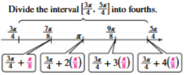
$$\frac{3\pi}{4} \le \frac{4x}{4} \le \frac{5\pi}{4} \qquad \text{Divide by 4.}$$

$$\frac{3\pi}{4} \le x \le \frac{5}{4}$$

"Starting" value for the cycle

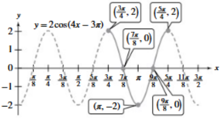
"Ending" value for the cycle

Dividing the period into fourths, we have increments of $\frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{8}$.

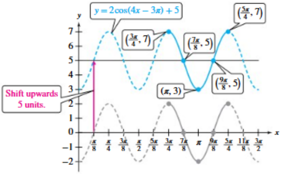

Divide the interval $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ into fourths.
$\frac{3\pi}{4}$
$\frac{7\pi}{8}$
π
$\frac{9\pi}{8}$
$\frac{5\pi}{4}$
$\frac{3\pi}{4} + \frac{\pi}{8}$
$\frac{3\pi}{4} + 2\left(\frac{\pi}{8}\right)$
$\frac{3\pi}{4} + 3\left(\frac{\pi}{8}\right)$
$\frac{3\pi}{4} + 4\left(\frac{\pi}{8}\right)$

First sketch the function on the interval $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ without the vertical shift (solid gray curve).

The dashed curve is a continuation of this pattern.



Now apply the vertical shift upwards 5 units



Apply the Skills

4. Given $y = 2\cos(3x - \pi) + 3$,

a. Identify the amplitude, period, phase shift, and vertical shift.

b. Graph the function and identify the key points on one full period.

In Example 4, notice that the *x*-intercepts of the graph before the vertical shift become the points where the graph intersects the line $y = 5$ after the vertical shift. The curve oscillates above and below the line $y = 5$ rather than the *x*-axis. In a sense, $y = 5$ is the "central" value or "equilibrium" value of the function. This line represents the midpoint of the range, and the amplitude tells us by how much the function deviates from this line.

Example 5: Graphing $y = A\sin(Bx - C) + D$

Given $y = 3\sin\left(-\frac{\pi}{4}x - \frac{\pi}{2}\right) - 4$,

a. Identify the amplitude, period, phase shift, and vertical shift.

b. Graph the function and identify the key points on one full period.

Solution:

a. First note that the coefficient on x in the argument is not positive.

We want to write the function in the form $y = A\sin(Bx - C) + D$, where $B > 0$.

$$y = 3\sin\left[-1\left(\frac{\pi}{4}x + \frac{\pi}{2}\right)\right] - 4$$

Factor out −1 from the argument.

$$y = -3\sin\left(\frac{\pi}{4}x + \frac{\pi}{2}\right) - 4$$

The sine function is an odd function. $\sin(-x) = -\sin(x)$

$$y = -3\sin\left[\frac{\pi}{4}x - \left(-\frac{\pi}{2}\right)\right] + (-4)$$

$$A = -3, B = \frac{\pi}{4}, C = -\frac{\pi}{2}, \text{ and } D = -4$$

Write the equation in the form $y = A\sin(Bx - C) + D$, where $B > 0$.

The amplitude is $|A| = |-3| = 3$.

The period is $\frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{4}} = 2\pi \cdot \frac{4}{\pi} = 8$.

The phase shift is $\frac{C}{B} = \frac{-\frac{\pi}{2}}{\frac{\pi}{4}} = -\frac{\pi}{2} \cdot \frac{4}{\pi} = -2$.

The vertical shift is D. Since $D = -4 < 0$, the shift is downward 4 units.

Insights

The negative sign on the phase shift indicates that the shift is to the left.

b. To find an interval over which this function completes one cycle, solve the inequality.

$$0 \le \frac{\pi}{4}x - \left(-\frac{\pi}{2}\right) \le 2\pi$$

Add $-\frac{\pi}{2}$.

$$-\frac{\pi}{2} \le \frac{\pi}{4}x \le \frac{3\pi}{2}$$

$$\frac{4}{\pi}\cdot\left(-\frac{\pi}{2}\right) \le \frac{4}{\pi}\cdot\left(\frac{\pi}{4}x\right) \le \frac{4}{\pi}\cdot\left(\frac{3\pi}{2}\right)$$

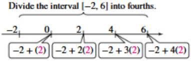
Multiply by $\frac{4}{\pi}$.

$$-2 \le x \le 6$$

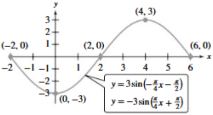
"Starting" value for the cycle

"Ending" value for the cycle

Dividing the period into fourths, we have increments of $\frac{1}{4}(8) = 2$.

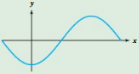


First sketch the function without the vertical shift.

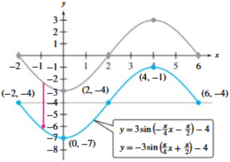


Insights

For the equation $y = -3\sin\left(\frac{\pi}{4}x + \frac{\pi}{2}\right)$, since $A = -3$, the general shape of the curve "starting" at the phase shift is an inverted sine curve.



Now apply the vertical shift downward 4 units.



Apply the Skills

5. Given $y = -2\sin\left(-\frac{\pi}{6}x - \frac{\pi}{2}\right) + 1$,

- **a.** Identify the amplitude, period, phase shift, and vertical shift.
- **b.** Graph the function and identify the key points on one full period.

Check Your Work

A factor from the argument of a function cannot be factored out in front of the function. For example,

$$f(2x) \neq 2f(x).$$

In step 2 of Example 5(a), the value –1 that appears in front of the function was not factored out, but rather is a result of the odd function property of the sine function.

$$\sin(-x) = -\sin(x)$$

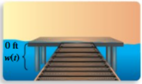
Learn: Model Sinusoidal Behavior

To this point, we have taken an equation of a sine or cosine function and sketched its graph. Now we reverse the process. In Example 6, we take observed data that follow a "wavelike" pattern similar to a sine or cosine graph and build a model. When the graph of a data set is shaped like a sine or cosine graph, we say the graph is **sinusoidal.**

Example 6: Modeling the Level of the Tide

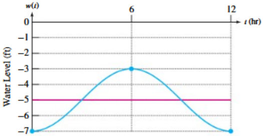
The water level relative to the top of a boat dock varies with the tides. One particular day, low tide occurs at midnight and the water level is 7 ft below the dock. The first high tide of the day occurs at approximately 6:00 A.M., and the water level is 3 ft below the dock. The next low tide occurs at noon and the water level is again 7 ft below the dock.

Assuming that this pattern continues indefinitely and behaves like a cosine wave, write a function of the form $w(t) = A\cos(Bt - C) + D$. The value $w(t)$ is the water level (in ft) relative to the top of the dock, t hours after midnight.



Solution:

Plotting the water level at midnight, 6:00 A.M., and noon helps us visualize the curve. By inspection, the curve behaves like a cosine reflected across the t-axis and shifted down 5 units.



$|A| = \frac{1}{2}[-3 - (-7)] = \frac{1}{2}(4) = 2$

The amplitude of the curve is half the distance between the highest value and lowest value.

Vertical shift: $D = \frac{-7 + (-3)}{2} = -5$

The midpoint of the range gives us the vertical shift.

$P = \frac{2\pi}{B}$, which implies that $12 = \frac{2\pi}{B}$.

One complete cycle takes place between $t = 0$ and $t = 12$ (from low tide to low tide). Therefore, the period P is $12 - 0 = 12$.

Therefore, $B = \frac{2\pi}{12} = \frac{\pi}{6}$.

Since a minimum value of the curve occurs at $t = 0$, there is no phase shift for this cosine function, implying that $C = 0$.

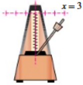
Finally, the amplitude is 2, however, we take A to be negative because the graph was reflected across the t-axis before being shifted downward.

$w(t) = -2\cos\left(\frac{\pi}{6}t\right) - 5$

Substitute $A = -2$, $B = \frac{\pi}{6}$, $C = 0$, and $D = -5$, into $w(t) = A\cos(Bt - C) + D$.

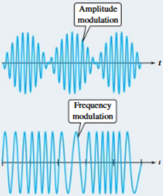
Apply the Skills

6. *A* mechanical metronome uses an inverted pendulum that makes a regular, rhythmic click as it swings to the left and right. With each swing, the pendulum moves 3 in. to the left and right of the center position. The pendulum is initially pulled to the right 3 in. and then released. It returns to its starting position in 0.8 sec. Assuming that this pattern continues indefinitely and behaves like a cosine wave, write a function of the form $x(t) = A\cos(Bt - C) + D$. The value $x(t)$ is the horizontal position (in inches) relative to the center line of the pendulum.



Point of Interest

AM (amplitude modulation) and FM (frequency modulation) radio are ways of broadcasting radio signals by sending electromagnetic waves through space from a transmitter to a receiver. Each method transmits information in the form of electromagnetic waves. AM works by modulating (or varying) the amplitude of the signal while the frequency remains constant. FM works by varying the frequency and keeping the amplitude constant. In 1873, James Maxwell showed mathematically that electromagnetic waves could propagate through free space. Today the far-reaching applications of wireless radio technology include use in televisions, computers, cell phones, and even deep-space radio communications.



Practice 4-5

Practice Exercises

Prerequisite Review

For Exercises R.1–R.5, solve the equation.

R.1. $h(x) = |x| - 1$ **R.2.** $n(x) = |x + 1|$

R.3. $b(x) = 2|x|$ **R.4.** $h(x) = \frac{1}{3}|x|$

R.5. $p(x) = -|x|$

Concept Connections

1. The value of sin$x$ (increases/decreases) __________ on $\left(0, \frac{\pi}{2}\right)$ and (increases/decreases) __________ on $\left(\frac{\pi}{2}, \pi\right)$.

2. The value of cos$x$ (increases/decreases) __________ on $\left(0, \frac{\pi}{2}\right)$ and (increases/decreases) __________ on $\left(\frac{\pi}{2}, \pi\right)$.

3. The graph of $y = \sin x$ and $y = \cos x$ differ by a horizontal shift of __________ units.

4. Given $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$, for $B > 0$ the amplitude is __________, the period is __________, the phase shift is __________, and the vertical shift is __________.

5. The sine function is an (even/odd) ________ function because $\sin(-x) =$ __________. The cosine function is an (even/odd) __________ function because $\cos(-x) =$ __________.

6. Given $B > 0$, how would the equation $y = A\sin(-Bx - C) + D$ be rewritten to obtain a positive coefficient on x?

7. Given $B > 0$, how would the equation $y = A\cos(-Bx - C) + D$ be rewritten to obtain a positive coefficient on x?

8. Given $y = \sin(Bx)$ and $y = \cos(Bx)$, for $B > 1$, is the period less than or greater than 2π? If $0 < B < 1$, is the period less than or greater than 2π?

Learn: Graph $y = \sin x$ and $y = \cos x$

9. From memory, sketch $y = \sin x$ on the interval $[0, 2\pi]$.

10. From memory, sketch $y = \cos x$ on the interval $[0, 2\pi]$.

11. For $y = \cos x$,

a. The domain is __________.

b. The range is _________.

c. The amplitude is _____.

d. The period is _____.

e. The cosine function is symmetric to the _____ -axis.

f. On the interval $[0, 2\pi]$, the x-intercepts are ________________ .

g. On the interval $[0, 2\pi]$, the maximum points are ________ and ________ , and the minimum point is _________.

12. For $y = \sin x$,

a. The domain is _________.

b. The range is _________.

c. The amplitude is _____.

d. The period is _____.

e. The sine function is symmetric to the __________.

f. On the interval $[0, 2\pi]$, the x-intercepts are ______________ .

g. On the interval $[0, 2\pi]$, the maximum point is __________ and the minimum point is __________.

13. a. Over what interval(s) taken between 0 and 2π is the graph of $y = \sin x$ increasing?

b. Over what interval(s) taken between 0 and 2π is the graph of $y = \sin x$ decreasing?

14. a. Over what interval(s) taken between 0 and 2π is the graph of $y = \cos x$ increasing?

b. Over what interval(s) taken between 0 and 2π is the graph of $y = \cos x$ decreasing?

Learn: Graph $y = A\sin x$ and $y = A\cos x$

For Exercises 15–16, identify the amplitude of the function.

15. a. $y = 7\sin x$

b. $y = \frac{1}{7}\sin x$

c. $y = -7\sin x$

16. a. $y = 2\cos x$

b. $y = \frac{1}{2}\cos x$

c. $y = -2\cos x$

17. By how many units does the graph of $y = \frac{1}{4}\cos x$ deviate from the x-axis?

18. By how many units does the graph of $y = -5\sin x$ deviate from the x-axis?

For Exercises 19–24, graph the function and identify the key points on one full period. (See Example 1, p. 531)

19. $y = 5\cos x$ **20.** $y = 4\sin x$

21. $y = \frac{1}{2}\sin x$ **22.** $y = \frac{1}{4}\cos x$

23. $y = -2\cos x$ **24.** $y = -3\sin x$

Learn: Graph $y = A\sin Bx$ and $y = A\cos Bx$

For Exercises 25–26, identify the period.

25. a. $\sin 2x$ **b.** $\sin 2\pi x$

c. $\sin\left(-\frac{2}{3}x\right)$

26. a. $\cos\frac{1}{3}x$ **b.** $\cos(-3\pi x)$

c. $\cos\frac{1}{3}\pi x$

For Exercises 27–32

a. Identify the amplitude and period.

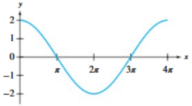
b. Graph the function and identify the key points on one full period. **(See Example 2, p. 533)**

27. $y = 2\cos 3x$ **28.** $y = 6\sin 4x$

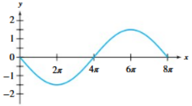
29. $y = 4\sin\frac{\pi}{3}x$ **30.** $y = 5\cos\frac{\pi}{6}x$

31. $y = \sin\left(-\frac{1}{3}x\right)$ **32.** $y = \cos\left(-\frac{1}{2}x\right)$

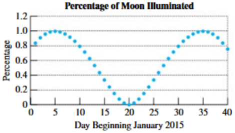
33. Write a function of the form $f(x) = A\cos Bx$ for the given graph.



34. Write a function of the form $g(x) = A\sin Bx$ for the given graph.



35. The graph shows the percentage in decimal form of the Moon illuminated for the first 40 days of a recent year. (*Source*: Astronomical Applications Department, U.S. Naval Observatory: http://aa.usno.navy.mil)



a. On approximately which days during this time period did a full moon occur? (*A* full moon corresponds to 100% or 1.0.)

b. On which day was there a new moon (no illumination)?

c. From the graph, approximate the period of a synodic month. *A* synodic month is the period of one lunar cycle (full moon to full moon).

36. A respiratory cycle is defined as the beginning of one breath to the beginning of the next breath. The rate of air intake r (in L/sec) during a respiratory cycle for a physically fit male can be approximated by $r(t) = 0.9\sin\frac{\pi}{3.5}t$, where t is the number of seconds into the cycle. A positive value for r represents inhalation and a negative value represents exhalation.

- **a.** How long is the respiratory cycle?
- **b.** What is the maximum rate of air intake?
- **c.** Graph one cycle of the function. On what interval does inhalation occur? On what interval does exhalation occur?

Learn: Graph $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$

For Exercises 37–38, identify the phase shift and indicate whether the shift is to the left or to the right.

37. a. $\cos\left(x - \frac{\pi}{3}\right)$
  b. $\cos\left(2x - \frac{\pi}{3}\right)$
  c. $\cos\left(3\pi x + \frac{\pi}{3}\right)$

38. a. $\sin\left(x + \frac{\pi}{8}\right)$
  b. $\sin\left(2\pi x - \frac{\pi}{8}\right)$
  c. $\sin\left(4x - \frac{\pi}{8}\right)$

For Exercises 39–44,

- **a.** Identify the amplitude, period, and phase shift.
- **b.** Graph the function and identify the key points on one full period. (**See Example 3, p. 534**)

39. $y = 2\cos(x + \pi)$

40. $y = 4\sin\left(x + \frac{\pi}{2}\right)$

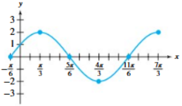
41. $y = \sin\left(2x - \frac{\pi}{3}\right)$

42. $y = \cos\left(3x - \frac{\pi}{4}\right)$

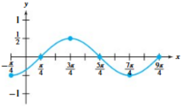
43. $y = -6\cos\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

44. $y = -5\sin\left(\frac{1}{3}x + \frac{\pi}{6}\right)$

45. Write a function of the form $f(x) = A\cos(Bx - C)$ for the given graph.



46. Write a function of the form $f(x) = A\sin(Bx - C)$ for the given graph.



47. Given $y = -2\sin\left(2x - \frac{\pi}{6}\right) - 7$,

- **a.** Is the period less than or greater than 2π?
- **b.** Is the phase shift to the left or right?
- **c.** Is the vertical shift upward or downward?

48. Given $y = \cos\left(\frac{1}{2}x + \pi\right) + 4$,

- **a.** Is the period less than or greater than 2π?
- **b.** Is the phase shift to the left or right?
- **c.** Is the vertical shift upward or downward?

For Exercises 49–52, rewrite the equation so that the coefficient on *x* is positive.

49. $y = \cos\left(-2x + \frac{\pi}{6}\right) - 4$

50. $y = 4\cos(-3x - \pi) + 5$

51. $y = \sin\left(-2x + \frac{\pi}{6}\right) - 4$

52. $y = 4\sin(-3x - \pi) + 5$

53. Given $y = 2\sin\left(-\frac{\pi}{6}x + \frac{\pi}{2}\right) - 3$, is the phase shift to the right or left?

54. Given $y = -4\cos\left(-\frac{\pi}{6}x - \frac{\pi}{2}\right) + 1$, is the phase shift to the right or left?

For Exercises 55–68,

a. Identify the amplitude, period, phase shift, and vertical shift.

b. Graph the function and identify the key points on one full period.
(See Examples 4–5, pp. 536–538)

55. $h(x) = 3\sin(4x - \pi) + 5$

56. $g(x) = 2\sin(3x - \pi) - 4$

57. $f(x) = 4\cos\left(3x - \frac{\pi}{2}\right) - 1$

58. $k(x) = 5\cos\left(2x - \frac{\pi}{2}\right) + 1$

59. $y = \frac{1}{2}\sin\left(-\frac{1}{3}x\right)$

60. $y = \frac{2}{3}\sin\left(-\frac{1}{2}x\right)$

61. $v(x) = 1.6\cos(-\pi x)$

62. $m(x) = 2.4\cos(-4\pi x)$

63. $y = 2\sin(-2x - \pi) + 5$

64. $y = 3\sin(-4x - \pi) - 7$

65. $p(x) = -\cos\left(-\frac{\pi}{2}x - \pi\right) + 2$

66. $q(x) = -\cos\left(-\frac{\pi}{3}x - \pi\right) - 2$

67. $y = \sin\left(-\frac{\pi}{4}x - \frac{\pi}{2}\right) - 3$

68. $y = \sin\left(-\frac{\pi}{3}x - \frac{\pi}{2}\right) + 4$

69. The temperature T (in °F) for a midwestern city over a several day period in April can be approximated by $T(t) = -5.9\cos(0.262t - 1.245) + 48.2$, where t is the number of hours since midnight on day 1.

a. What is the period of the function? Round to the nearest hour.

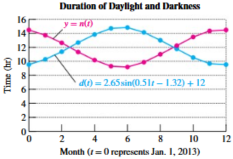
b. What is the significance of the term 48.2 in this model?

c. What is the significance of the factor 5.9 in this model?

d. What was the minimum temperature for the day? When did it occur?

e. What was the maximum temperature for the day? When did it occur?

70. The duration of daylight and darkness varies during the year due to the angle of the Sun in the sky. The model $d(t) = 2.65\sin(0.51t - 1.32) + 12$ approximates the amount of daylight $d(t)$ (in hours) for a northern city as a function of the time t (in months) after January 1 for a recent year; that is, $t = 0$ is January 1, $t = 1$ is February 1, and so on. The model $y = n(t)$ represents the amount of darkness as a function of t.



a. Describe the relationship between the graphs of the functions and the line $y = 12$.

b. Use the result of part (a) and a transformation of $y = d(t)$ to write an equation representing n as a function of t.

c. What do the points of intersection of the two graphs represent?

d. What do the relative minima and relative maxima of the graphs represent?

e. What does $T(t) = d(t) + n(t)$ represent?

Learn: Model Sinusoidal Behavior

71. The probability of precipitation in Modesto, California, varies from a peak of 0.34 (34%) in January to a low of 0.04 (4%) in July. Assume that the percentage of precipitation varies monthly and behaves like a cosine curve.

a. Write a function of the form $P(t) = A\cos(Bt - C) + D$ to model the precipitation probability. The value $P(t)$ is the probability of precipitation (as a decimal), for month t, with January as $t = 1$.

b. Graph the function from part (a) on the interval [0, 13] and plot the points (1, 0.34), (7, 0.04), and (13, 0.34) to check the accuracy of your model.

72. The monthly high temperature for Atlantic City, New Jersey, peaks at an average high of 86° in July and goes down to an average high of 64° in January. Assume that this pattern for monthly high temperatures continues indefinitely and behaves like a cosine wave.

a. Write a function of the form $H(t) = A\cos(Bt - C) + D$ to model the average high temperature. The value $H(t)$ is the average high temperature for month t, with January as $t = 0$.

b. Graph the function from part (a) on the interval [0, 13] and plot the points (0, 64), (6, 86), and (12, 64) to check the accuracy of your model.

73. An adult human at rest inhales and exhales approximately 500 mL of air (called the tidal volume) in approximately 5 sec. However, at the end of each exhalation, the lungs still contain a volume of air, called the functional residual capacity (FRC), which is approximately 2000 mL.
(See Example 6, p. 540)

a. What volume of air is in the lungs after inhalation?

b. What volume of air is in the lungs after exhalation?

c. What is the period of a complete respiratory cycle?

d. Write a function $V(t) = A\cos Bt + D$ to represent the volume of air in the lungs t seconds after the end of an inhalation.

e. What is the average amount of air in the lungs during one breathing cycle?

f. During hyperventilation, breathing is more rapid with deep inhalations and exhalations. What parts of the equation from part (d) change?

74. The times for high and low tides are given in the table for a recent day in Jacksonville Beach, Florida. The times are rounded to the nearest hour, and the tide levels are measured relative to mean sea level (MSL).

a. Write a model $h(t) = A\cos(Bt - C)$ to represent the tide level $h(t)$ (in feet) in terms of the amount of time t elapsed since midnight.

b. Use the model from part (a) to estimate the tide level at 3:00 P.M.

	Time (hr after midnight)	Height Relative to MSL (ft)
High tide	0	3.4
Low tide	6	–3.4
High tide	12	3.4
Low tide	18	–3.4
High tide	24	3.4

75. The data in the table represent the monthly power bills (in dollars) for a homeowner.

a. Enter the data in a graphing utility and use the sinusoidal regression tool (SinReg) to find a model of the form $A(t) = a\sin(bt + c) + d$, where $A(t)$ represents the amount of the bill for month t ($t = 1$ represents January, $t = 2$ represents February, and so on).

b. Graph the data and the resulting function.

Month, *t*	1	2	3	4	5	6
Amount ($)	104.73	66.13	48.99	56.04	85.51	98.57

Month, *t*	7	8	9	10	11	12
Amount ($)	125.08	124.48	113.93	81.06	63.30	71.85

76. The data in the table represent the duration of daylight $d(t)$ (in hours) for Houston, Texas, for the first day of the month, t months after January 1 for a recent year. (*Source*: Astronomical Applications Department, U.S. Naval Observatory: http://aa.usno.navy.mil)

a. Enter the data in a graphing utility and use the sinusoidal regression tool (SinReg) to find a model of the form $d(t) = a\sin(bt + c) + d$.

b. Graph the data and the resulting function.

Month, *t*	1	2	3	4	5
Amount (hr)	10.28	10.80	11.57	12.48	13.65

Month, *t*	6	7	8	9	10	11
Amount (hr)	14.02	13.55	12.73	11.87	11.00	10.38

Mixed Exercises

For Exercises 77–78, write the range of the function in interval notation.

77. a. $y = 8\cos(2x - \pi) + 4$

b. $y = -3\cos\left(x + \frac{\pi}{3}\right) - 5$

78. a. $y = -6\sin\left(3x - \frac{\pi}{2}\right) - 2$

b. $y = 2\sin(3x + 2\pi) + 12$

79. Given $f(x) = \cos x$ and $h(x) = 3x + 2$, find $(h \circ f)(x)$ and for $(h \circ f)(x)$,

a. Find the amplitude.

b. Find the period.

c. Write the domain in interval notation.

d. Write the range in interval notation.

80. Given $g(x) = \sin x$ and $k(x) = 6x$, find $(g \circ k)(x)$ and for $(g \circ k)(x)$,

a. Find the amplitude.

b. Find the period.

c. Write the domain in interval notation.

d. Write the range in interval notation.

81. Write a function of the form $y = A\sin(Bx - C) + D$ that has period $\frac{\pi}{3}$, amplitude 4, phase shift $\frac{\pi}{2}$, and vertical shift 5.

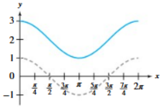
82. Write a function of the form $y = A\cos(Bx - C) + D$ that has period $\frac{\pi}{4}$, amplitude 2, phase shift $-\frac{\pi}{3}$, and vertical shift 7.

83. Write a function of the form $y = A\cos(Bx - C) + D$ that has period 16, phase shift -4, and range $3 \leq y \leq 7$.

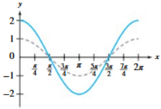
84. Write a function of the form $y = A\sin(Bx - C) + D$ that has period 8, phase shift -2, and range $-14 \le y \le -6$.

85. Describe the transformation applied to the dashed curve to make the solid curve. Choose from: change in amplitude, change in period, phase shift, or vertical shift. Then write an equation for the solid curve in terms of the cosine function.

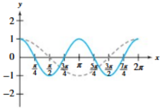
a.



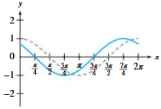
b.



c.

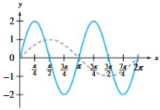


d.

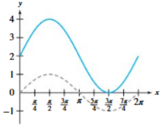


86. Describe the transformations (there may be more than one) applied to the dashed curve to make the solid curve. Choose from: change in amplitude, change in period, phase shift, or vertical shift. Then write an equation for the solid curve in terms of the sine function.

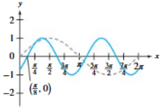
a.



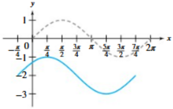
b.



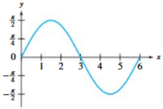
c.



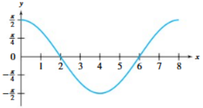
d.



87. Write an equation of the form $y = A\sin Bx$ to model the graph.



88. Write an equation of the form $y = A\cos Bx$ to model the graph.

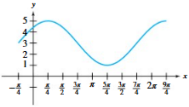


For Exercises 89–90,

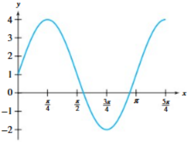
a. Write an equation of the form $y = A\cos(Bx - C) + D$ with $A > 0$ to model the graph.

b. Write an equation of the form $y = A\sin(Bx - C) + D$ with $A > 0$ to model the graph.

89.



90.



For Exercises 91–94, explain how to graph the given function by performing transformations on the "parent" graphs $y = \sin x$ and $y = \cos x$.

91. a. $y = \sin 2x$

b. $y = 2\sin x$

92. a. $y = \frac{1}{3}\cos x$

b. $y = \cos\frac{1}{3}x$

93. a. $y = \sin(x + 2)$

b. $y = \sin x + 2$

94. a. $y = \cos x - 4$

b. $y = \cos(x - 4)$

Write About It

95. Is $f(x) = \sin x$ one-to-one? Explain why or why not.

96. Is $f(x) = \cos x$ one-to-one? Explain why or why not.

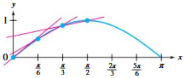
97. If f and g are both periodic functions with period P, is $(f + g)(x)$ also periodic? Explain why or why not.

Expanding Your Skills

98. Explain why a function that is increasing on its entire domain cannot be periodic.

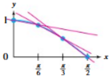
For Exercises 99–100, find the average rate of change on the given interval. Give the exact value and an approximation to 4 decimal places. Verify that your results are reasonable by comparing the results to the slopes of the lines given in the graph.

99. $f(x) = \sin x$



a. $\left[0, \frac{\pi}{6}\right]$ **b.** $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ **c.** $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

100. $f(x) = \cos x$



a. $\left[0, \frac{\pi}{6}\right]$ **b.** $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ **c.** $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

For Exercises 101–104, use your knowledge of the graphs of the sine function and linear functions to determine the number of solutions to the equation.

101. $\sin x = x - 2$

102. $\cos x = -x$

103. $\sin 2x = -2$

104. $2\sin 2x = -2$

For Exercises 105–106, graph the piecewise-defined function.

105. $g(x) = \begin{cases} \sin x \text{ for } 0 \le x \le \pi \\ -\sin x \text{ for } \pi < x \le 2\pi \end{cases}$

106. $f(x) = \begin{cases} \cos x \text{ for } 0 \le x \le \frac{\pi}{4} \\ \sin x \text{ for } \frac{\pi}{4} < x \le \frac{\pi}{2} \end{cases}$

Technology Connections

107. Functions a and m approximate the duration of daylight, respectively, for Albany, New York, and Miami, Florida, for a given year for day t. The value $t = 1$ represents January 1, $t = 2$ represents February 1, and so on.

$$a(t) = 12 + 3.1\sin\left[\frac{2\pi}{365}(t - 80)\right]$$

$$m(t) = 12 + 1.6\sin\left[\frac{2\pi}{365}(t - 80)\right]$$

a. Graph the two functions with a graphing utility and comment on the difference between the two graphs.

b. Both functions have a constant term of 12. What does this represent graphically and in the context of this problem?

c. What do the factors 3.1 and 1.6 represent in the two functions?

d. What is the period of each function?

e. What does the horizontal shift of 80 units represent in the context of this problem?

f. Use the Intersect feature to approximate the points of intersection.

g. Interpret the meaning of the points of intersection.

For Exercises 108–109, we demonstrate that trigonometric functions can be approximated by polynomial functions over a given interval in the domain.

Graph functions f, g, h, and k on the viewing window $-4 \le x \le 4$, $-4 \le y \le 4$. Then use a Table feature on a graphing utility to evaluate each function for the given values of x. How do functions g, h, and k compare to function f for x values farther from 0? [*Hint*: For a given natural number n, the value $n!$, read as "n factorial," is defined as $n! = n(n-1)(n-2)\cdots 1$. For example, $3! = 3 \cdot 2 \cdot 1 = 6$.]

108.

Function	$x =$ 0.1	$x =$ 0.5	$x =$ 1.0	$x =$ 1.5
$f(x) = \cos x$				
$g(x) = 1 - \frac{x^2}{2!}$				
$h(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$				
$k(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$				

109.

Function	$x =$ 0.1	$x =$ 0.5	$x =$ 1.0	$x =$ 1.5
$f(x) = \sin x$				
$g(x) = x - \frac{x^3}{3!}$				
$h(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$				
$k(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$				

For Exercises 110–111, use a graph to solve the equation on the given interval.

110. $\cos\left(2x - \frac{\pi}{3}\right) = 0.5$ on $[0, \pi]$

Viewing window: $\left[0, \pi, \frac{\pi}{3}\right]$ by $\left[-1, 1, \frac{1}{2}\right]$

111. $\sin\left(2x + \frac{\pi}{4}\right) = 1$ on $[0, 2\pi]$

Viewing window: $\left[0, 2\pi, \frac{\pi}{8}\right]$ by $[-2, 2, 1]$

For Exercises 112–113, use a graph to solve the equation on the given interval. Round the answer to 2 decimal places.

112. $\sin\left(x - \frac{\pi}{4}\right) = -e^x$ on $[-\pi, \pi]$

Viewing window: $\left[-\pi, \pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$

113. $6\cos\left(x + \frac{\pi}{6}\right) = \ln x$ on $[0, 2\pi]$

Viewing window: $\left[0, 2\pi, \frac{\pi}{2}\right]$ by $[-7, 7, 1]$

Variable Amplitude: The graphs of functions such as $f(x) = a\cos x$ and $g(x) = a\sin x$ are curves with a *constant* amplitude, a. If we have a nonconstant coefficient $a(x)$, we will have a *variable* amplitude with the graph of sine or cosine bounded by the graphs of $y = a(x)$ and $y = -a(x)$. Use this information for Exercises 114–115.

114. Graph $f(x) = 0.5x\sin x$ on the viewing window $[-5\pi, 5\pi, \pi]$ by $[-8, 8, 1]$ and give the equations of the bounding functions.

115. Graph $g(x) = \cos x \sin 12\,x$ on the viewing window $[0, 2.5\pi, 0.25\pi]$ by $[-1, 1, 0.5]$ and give the equations of the bounding functions.

Go online for more practice problems.

Lesson 4-6

Graphs of Other Trigonometric Functions

What Will You Learn?

After completing this lesson, you should be able to:

- Graph the Secant and Cosecant Functions
- Graph the Tangent and Cotangent Functions

Learn: Graph the Secant and Cosecant Functions

In this lesson, we turn our attention to the graphs of the secant, cosecant, tangent, and cotangent functions. Each of these functions has vertical asymptotes and, for this reason, we often use them to model patterns involving unbounded behavior.

For example, the distance $d(x)$ between an observer 30 ft from a straight highway and a police car traveling down the highway is given by $d(x) = 30\sec x$. The independent variable x is the angle formed from the perpendicular line from the road to the observer and from the observer to the police car (Figure 4-49). Notice that for x values close to $90°\left(\frac{\pi}{2}\text{radians}\right)$, the distance $d(x)$ approaches infinity. This means that the graph of $d(x) = 30\sec x$ will have a vertical asymptote at $x = \frac{\pi}{2}$.

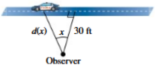


Figure 4-49

To graph the cosecant and secant functions, we first review their reciprocal relationships.

$$y = \csc x = \frac{1}{\sin x} \quad \text{and} \quad y = \sec x = \frac{1}{\cos x}$$

- If $\sin x = 0$, then $\frac{1}{\sin x}$ is undefined, and the graph of $y = \csc x$ has a vertical asymptote (the numerator is nonzero and the denominator is zero). This occurs when x is a multiple of $\pi : x = n\pi$ for all integers n.
- If $\cos x = 0$, then $\frac{1}{\cos x}$ is undefined, and the graph of $y = \sec x$ has a vertical asymptote (the numerator is nonzero and the denominator is zero). This occurs when x is an odd multiple of $\frac{\pi}{2} : x = \frac{(2n+1)\pi}{2}$ for all integers n.

To sketch the graph of $y = \csc x$, we construct a table with several values of x and the corresponding values of $\sin x$ and $\csc x$ (Table 4-12). The cosecant function is an odd function, so the points plotted in Quadrants I and II each have a mirror image in Quadrants III and IV. Furthermore, both the sine and cosecant functions are periodic, and their graphs show a repeated pattern in intervals of 2π (Figure 4-50).

TABLE 4-12 Values of Sin *x* and Csc *x* for Common Values of *x*

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2} \approx 0.71$	$\frac{\sqrt{3}}{2} \approx 0.87$	1	$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{\sqrt{2}}{2} \approx 0.71$	$\frac{1}{2}$	0
$\csc x$	Undefined	2	$\sqrt{2} \approx 1.4$	$\frac{2\sqrt{3}}{3} \approx 1.2$	1	$\frac{2\sqrt{3}}{3} \approx 1.2$	$\sqrt{2} \approx 1.4$	2	Undefined

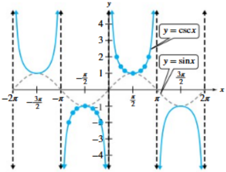


Figure 4-50

Notice that the relative maxima and minima from the sine function correspond to the relative minima and maxima for the cosecant function. The *x*-intercepts of the sine graph give the location of the vertical asymptotes for the graph of $y = \csc x$.

The graphs of the secant function and cosine function are similarly related. The secant function is an even function, and therefore its graph is symmetric with respect to the *y*-axis. The asymptotes of the graph of $y = \sec x$ occur at odd multiples of $\frac{\pi}{2}$ where $\cos x = 0$ (Figure 4-51).

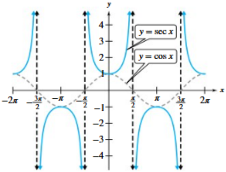


Figure 4-51

Table 4-13 summarizes key properties of the graphs of $y = \csc x$ and $y = \sec x$.

TABLE 4-13 Graphs of the Cosecant and Secant Functions

Function	$y = \csc x$	$y = \sec x$
Domain	$\{x \mid x \neq n\pi$ for all integers $n\}$	$\left\{x \middle\vert x \neq \frac{(2n+1)\pi}{2} \text{ for all integers } n\right\}$
Range	$\{y \mid y \leq -1$ or $y \geq 1\}$	$\{y \mid y \leq -1$ or $y \geq 1\}$
Amplitude	None ($y = \csc x$ increases and decreases without bound)	None ($y = \sec x$ increases and decreases without bound)
Period	2π	2π
Vertical Asymptotes	$x = n\pi$ (multiples of π)	$x = \frac{(2n+1)\pi}{2}$ $\left(\text{odd multiples of } \frac{\pi}{2}\right)$
Symmetry	Origin (The cosecant function is an odd function.)	y-axis (The secant function is an even function.)

To graph variations of $y = \csc x$ or $y = \sec x$, we use the graph of the related reciprocal function for reference.

Example 1: Graphing a Variation of $y = \sec x$

Graph $y = 3\sec 2x$.

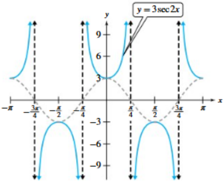
Solution:

We first graph the reciprocal function $y = 3\cos 2x$ (shown in gray) using the techniques from Lesson 4.5.

The amplitude is $|3| = 3$.

The period is $\frac{2\pi}{2} = \pi$.

There is no phase shift or vertical shift.



The graph of $y = 3\sec 2x$ (shown in blue) has vertical asymptotes at the odd multiples of $\frac{\pi}{4}$ where the graph of $y = 3\cos 2x$ has x-intercepts.

Apply the Skills

1. Graph $y = 2\sec 4x$.

In Example 1 we built the graph of $y = 3\sec 2x$ by using reference points (x-intercepts and relative extrema) from the graph of $y = 3\cos 2x$. To graph a cosecant or secant function involving a vertical shift, we recommend first graphing the function *without* the vertical shift. This is to make use of the x-intercepts of the reciprocal function to find the vertical asymptotes of the cosecant or secant function. This is demonstrated in Example 2.

Example 2: Graphing a Cosecant Function with a Vertical Shift

Graph $y = \csc\left(x - \frac{\pi}{4}\right) + 3$.

Solution:

First graph the function without the vertical shift, $y = \csc\left(x - \frac{\pi}{4}\right)$.

For the related reciprocal function $y = \sin\left(x - \frac{\pi}{4}\right)$, we have:

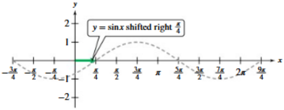
The amplitude is 1.

The period is 2π.

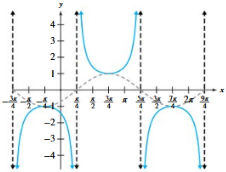
The phase shift is $\frac{\pi}{4}$.

The x-intercepts occur at

$x = \ldots, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$

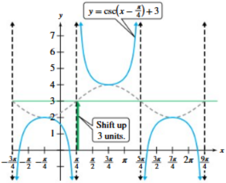


The vertical asymptotes of $y = \csc\left(x - \frac{\pi}{4}\right)$ pass through the x-intercepts of the related sine function.



Next, apply the vertical shift 3 units upward to graph $y = \csc\left(x - \frac{\pi}{4}\right) + 3$.

Notice that the range of $y = \csc\left(x - \frac{\pi}{4}\right) + 3$ is $(-\infty, 2] \cup [4, \infty)$.



Apply the Skills

2. Graph $y = \csc(x + \pi) - 2$.

Learn: Graph the Tangent and Cotangent Functions

We now investigate the graph $y = \tan x$ and begin with several observations. The graph of $y = \tan x = \frac{\sin x}{\cos x}$ will have

- An x-intercept where the numerator equals zero. The value $\sin x = 0$ when x is a multiple of π; that is, $x = n\pi$ for all integers n (Figure 4-52).
- A vertical asymptote where the denominator equals zero. The value $\cos x = 0$ when x is an odd multiple of $\frac{\pi}{2}$; that is, at $x = \frac{(2n+1)\pi}{2}$ for all integers n (Figure 4-52).

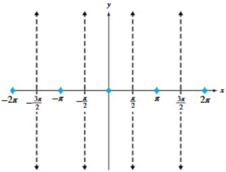


Figure 4-52

The tangent function is an odd function and is symmetric with respect to the origin. Therefore, each Quadrant I point given in Table 4-14 has a mirror image in Quadrant III (Figure 4-53). The period of the tangent function is π, so one complete period occurs between two consecutive vertical asymptotes. We can then sketch more cycles to the left and right as desired.

TABLE 4-14 Values of Tan x for Common Values of x

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	0	$\frac{\sqrt{3}}{3} \approx 0.58$	1	$\sqrt{3} \approx 1.73$	Undefined

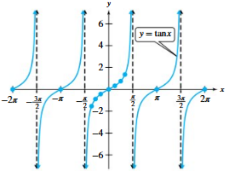


Figure 4-53

We need to use particular care graphing the function near the asymptotes; for example, we can find a value of $\tan x$ arbitrarily large if x is taken sufficiently close to $\frac{\pi}{2}$ from the left. Therefore, $\tan x$ approaches ∞ as x approaches $\frac{\pi}{2}$ from the left (Table 4-15). Likewise, $\tan x$ approaches $-\infty$ as x approaches $\frac{\pi}{2}$ from the right.

TABLE 4-15 Values of Tan$x$ for x Near $\frac{\pi}{2}$

As $x \to \left(\frac{\pi}{2}\right)^-$, $\tan x \to \infty$

As $x \to \left(\frac{\pi}{2}\right)^+$, $\tan x \to -\infty$

x	1.52	1.54	1.56	$\frac{\pi}{2}$	1.58	1.60	1.62
$\tan x$	≈19.7	≈32.5	≈92.6	Undefined	≈−108.6	≈−34.2	≈−20.3

To graph the cotangent function, we make the following observations. The graph of $y = \cot x = \frac{\cos x}{\sin x}$ will have

- An x-intercept where the numerator equals zero. The value $\cos x = 0$ when x is an odd multiple of $\frac{\pi}{2}$; that is, at $x = \frac{(2n+1)\pi}{2}$ or all integers n.
- A vertical asymptote where the denominator equals zero. The value $\sin x = 0$ when x is a multiple of π; that is, at $x = n\pi$ for all integers n.

The cotangent function is an odd function and is symmetric with respect to the origin. Therefore, each Quadrant I point given in Table 4-16 has a mirror image in Quadrant III (Figure 4-54). The period of the cotangent function is π, so one complete period occurs between two consecutive vertical asymptotes. We can then sketch more cycles to the left and right as desired.

TABLE 4-16 Values of Cot$x$ for Common Values of x

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cot x$	Undefined	$\sqrt{3} \approx 1.73$	1	$\frac{\sqrt{3}}{3} \approx 0.58$	0

Take a few minutes to review the key characteristics of the tangent and cotangent functions and their graphs.

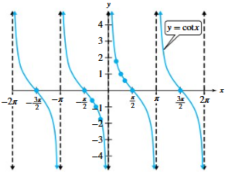


Figure 4-54

Graphs of the Tangent and Cotangent Functions

Function	$y = \tan x$	$y = \cot x$
Domain	$\left\{x \mid x \neq \frac{(2n+1)\pi}{2} \text{ for all integers } n\right\}$	$\{x \mid x \neq n\pi \text{ for all integers } n\}$
Range	All real numbers	All real numbers
Amplitude	None ($y = \tan x$ is unbounded.)	None ($y = \cot x$ is unbounded.)
Period	π	π
Vertical Asymptotes	$x = \frac{(2n+1)\pi}{2}$ (odd multiples of $\frac{\pi}{2}$)	$x = n\pi$ (multiples of π)
Symmetry	Origin (The tangent function is an odd function.)	Origin (The cotangent function is an odd function.)

Now consider variations on the tangent and cotangent functions $y = A\tan(Bx - C) + D$ and $y = A\cot(Bx - C) + D$ with $B > 0$. (If $B < 0$, we can rewrite the equations using the odd function property of the tangent and cotangent functions.)

- $|A|$ is the vertical scaling factor.
- If $A < 0$, the graph is reflected across the x-axis.
- The period is $\frac{\pi}{B}$.
- The phase shift is $\frac{C}{B}$.
- The vertical shift is D.

To graph variations of the tangent or cotangent functions, we graph the functions first without the vertical shift using the guidelines in the box below. The vertical shift is applied once the other transformations are accounted for.

Guidelines to Graph $y = A\tan(Bx - C)$ and $y = A\cot(Bx - C)$

1. Find an interval between two consecutive vertical asymptotes and graph the asymptotes as dashed lines.
   - For $y = A\tan(Bx - C)$ and $B > 0$, solve the inequality $-\frac{\pi}{2} < Bx - C < \frac{\pi}{2}$.
   - For $y = A\cot(Bx - C)$ and $B > 0$, solve the inequality $0 < Bx - C < \pi$.
2. Plot an x-intercept halfway between the asymptotes from step 1.
3. Sketch the general shape of the "parent" function between the asymptotes (if $A < 0$, the parent function is reflected across the x-axis). If more accuracy is desired, consider evaluating the function for x values midway between the x-intercept and the asymptotes found in step 1.
   - For $y = A\tan x$, the y-coordinates for x values midway between the x-intercept and the asymptotes will be $-A$ and A.
   - For $y = A\cot x$, the y-coordinates for x values midway between the x-intercept and the asymptotes will be A and $-A$.
4. Sketch additional cycles to the right or left as desired.

Example 3: Graphing a Variation of $y = \tan x$

Graph $f(x) = 3\tan\frac{\pi}{4}x$.

Solution:

One complete period of the "parent" function $y = \tan x$ occurs between consecutive asymptotes on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the related interval for $f(x) = 3\tan\frac{\pi}{4}x$, solve the following inequality.

$$-\frac{\pi}{2} < \frac{\pi}{4}x < \frac{\pi}{2}$$

Multiply by $\frac{4}{\pi}$.

$$\frac{4}{\pi}\cdot\left(-\frac{\pi}{2}\right) < \frac{4}{\pi}\cdot\left(\frac{\pi}{4}x\right) < \frac{4}{\pi}\cdot\left(\frac{\pi}{2}\right)$$

$$-2 < x < 2$$

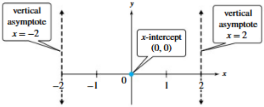
vertical asymptote $x = -2$

vertical asymptote $x = 2$

One complete cycle of $f(x) = 3\tan\frac{\pi}{4}x$ occurs between the asymptotes $x = -2$ and $x = 2$.

The midpoint of the interval $(-2, 2)$ is $\frac{-2+2}{2} = 0$.

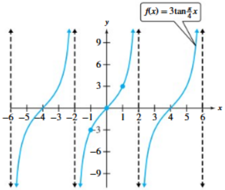
The function has an x-intercept at $(0, 0)$.



$$f(-1) = 3\tan\left[\frac{\pi}{4}(-1)\right] = 3\tan\left(-\frac{\pi}{4}\right) = 3(-1)$$
$$= -3$$

$x = -1$ and $x = 1$ are midway between the x-intercept and the vertical asymptotes.

$$f(1) = 3\tan\left[\frac{\pi}{4}(1)\right] = 3\tan\left(\frac{\pi}{4}\right) = 3(1) = 3$$



Apply the Skills

3. Graph $y = 4\tan\frac{\pi}{2}x$.

If the graph of a tangent or cotangent function involves a vertical shift, we recommend graphing the function without the vertical shift first.

Example 4: Graphing a Variation of $y = \cot x$

Graph $y = \cot\left(x + \frac{\pi}{4}\right) + 2$.

Solution:

We first graph $g(x) = \cot\left(x + \frac{\pi}{4}\right)$.

One complete period of the "parent" function $y = \cot x$ occurs between consecutive asymptotes on the interval $(0, \pi)$. To find the related interval for $g(x) = \cot\left(x + \frac{\pi}{4}\right)$, solve the following inequality.

$$0 < x + \frac{\pi}{4} < \pi$$

$$-\frac{\pi}{4} < x < \pi - \frac{\pi}{4} \qquad \text{Subtract } \frac{\pi}{4}.$$

$$-\frac{\pi}{4} < x < \frac{3\pi}{4}$$

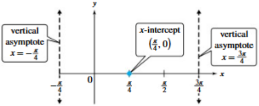
vertical asymptote $x = -\frac{\pi}{4}$

vertical asymptote $x = \frac{3\pi}{4}$

One complete cycle of $g(x) = \cot\left(x + \frac{\pi}{4}\right)$ occurs between the asymptotes $x = -\frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

The midpoint of the interval $\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$ is $\frac{1}{2}\left(-\frac{\pi}{4} + \frac{3\pi}{4}\right) = \frac{1}{2}\left(\frac{2\pi}{4}\right) = \frac{\pi}{4}$.

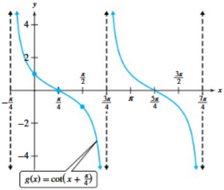
$g(x) = \cot\left(x + \frac{\pi}{4}\right)$ has an x-intercept at $\left(\frac{\pi}{4}, 0\right)$.

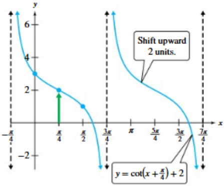


$$g(0) = \cot\left[0 + \frac{\pi}{4}\right] = \cot\left(\frac{\pi}{4}\right) = 1$$

$$g\left(\frac{\pi}{2}\right) = \cot\left[\frac{\pi}{2} + \frac{\pi}{4}\right] = \cot\left(\frac{3\pi}{4}\right) = -1$$

If more accuracy in the graph is desired, plot a few more points. The values $x = 0$ and $x = \frac{\pi}{2}$ are midway between the x-intercept and the vertical asymptotes.





Apply the Skills

4. Graph $y = \cot\left(x + \frac{\pi}{2}\right) - 3$.

Practice 4-6

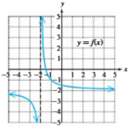
Practice Exercises

Prerequisite Review

R.1. Write the domain of the function in interval notation.

$$f(x) = \frac{x^2 - 16}{x - 4}$$

R.2. Refer to the graph of the function to complete the statement.



a. As $x \longrightarrow -2^-$, $f(x) \longrightarrow$ __________.

b. As $x \longrightarrow -2^+$, $f(x) \longrightarrow$ __________.

R.3. Determine the vertical asymptotes of the graph of the function.

$$f(x) = \frac{6}{x - 5}$$

Concept Connections

1. At values of x for which $\sin x = 0$, the graph of $y = \csc x$ will have a __________ __________. This occurs for $x =$ __________ for all integers n.

2. At values of x for which $\cos x = 0$, the graph of $y = \sec x$ will have a __________ __________. This occurs for x at odd multiples of __________.

3. The relative maxima on the graph of $y = \sin x$ correspond to the __________ __________ on the graph of $y = \csc x$.

4. The graph of $y = \csc x$ is symmetric with respect to the __________. The graph of $y = \sec x$ is symmetric with respect to the __________-axis.

5. If a function is an odd function, then each point (x, y) in Quadrant I will have a corresponding point (__________, __________) in Quadrant __________.

6. The range of $y = \tan x$ and $y = \cot x$ is __________.

7. The graphs of both $y = \tan x$ and $y = \cot x$ are symmetric with respect to the __________.

8. For the functions $y = A\tan(Bx - C)$ and $y = A\cot(Bx - C)$ with $B > 0$, the vertical scaling factor is __________, the period is __________, and the phase shift is __________.

Learn: Graph the Secant and Cosecant Functions

9. Sketch the graph of $y = \csc x$ from memory. Use the graph of $y = \sin x$ for reference.

10. Sketch the graph of $y = \sec x$ from memory. Use the graph of $y = \cos x$ for reference.

For Exercises 11–16, identify the statements among a–h that follow directly from the given condition about *x*.

a. $\csc x$ is undefined.

b. $\sec x$ is undefined.

c. The graph of $y = \sec x$ has a relative maximum at x.

d. The graph of $y = \csc x$ has a relative minimum at x.

e. The graph of $y = \sec x$ has a vertical asymptote.

f. The graph of $y = \csc x$ has a vertical asymptote.

g. The graph of $y = \csc x$ has a relative maximum at x.

h. The graph of $y = \sec x$ has a relative minimum at x.

11. $\sin x = 0$

12. $\cos x = 0$

13. The graph of $y = \cos x$ has a relative maximum at x.

14. The graph of $y = \sin x$ has a relative minimum at x.

15. The graph of $y = \cos x$ has a relative minimum at x.

16. The graph of $y = \sin x$ has a relative maximum at x.

For Exercises 17–32, graph one period of the function. (See Examples 1–2, pp. 554–555)

17. $y = 2\csc x$

18. $y = \frac{1}{4}\sec x$

19. $y = -5\sec x$

20. $y = -\frac{1}{3}\csc x$

21. $y = 3\csc\frac{x}{3}$

22. $y = -4\sec\frac{x}{2}$

23. $y = \sec 2\pi x$

24. $y = \csc 3\pi x$

25. $y = \csc\left(x - \frac{\pi}{4}\right)$

26. $y = \sec\left(x + \frac{\pi}{3}\right)$

27. $y = -2\sec(2\pi x + \pi)$

28. $y = -\csc\left(\frac{\pi}{3}x + \frac{\pi}{2}\right)$

29. $y = 2\csc\left(2x + \frac{\pi}{4}\right) + 1$

30. $y = 3\sec\left(x - \frac{\pi}{3}\right) - 3$

31. $y = \sec\left(-x - \frac{\pi}{4}\right) - 2$

32. $y = -\csc(-x + \pi) + 4$

For Exercises 33–34, write the range of the function in interval notation.

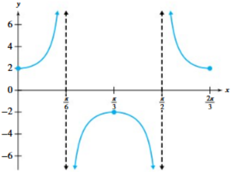
33. a. $y = -4\csc(2x - \pi) + 7$

b. $y = 2\csc\left(\frac{\pi}{3}x\right) - 10$

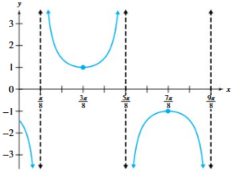
34. a. $y = -3\sec(5\pi x) - 1$

b. $y = 5\sec\left(3x - \frac{\pi}{2}\right) - 2$

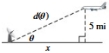
35. Write a function of the form $y = A\sec Bx$ for the given graph.



36. Write a function of the form $y = \csc(Bx - C)$ for the given graph.



37. A plane flying at an altitude of 5 mi travels on a path directly over a radar tower.

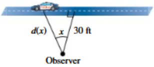


a. Express the distance $d(\theta)$ (in miles) between the plane and the tower as a function of the angle θ in standard position from the tower to the plane.

b. If $d(\theta) = 5$, what is the measure of the angle and where is the plane located relative to the tower?

c. Can the value of θ be π? Explain your answer in terms of the function d.

38. The distance $d(x)$ (in feet) between an observer 30 ft from a straight highway and a police car traveling down the highway is given by $d(x) = 30\sec x$, where x is the angle (in degrees) between the observer and the police car.



a. Use a calculator to evaluate $d(x)$ for the given values of x. Round to the nearest foot.

x	45	60	70	80	89
$d(x)$					

b. Try experimenting with values of x closer to 90°. What happens as $x \longrightarrow 90°$?

Learn: Graph the Tangent and Cotangent Functions

39. a. Graph $y = \tan x$ on the interval $[-\pi, \pi]$.

b. How many periods of the tangent function are shown on the interval $[-\pi, \pi]$?

40. a. Graph $y = \cot x$ on the interval $[-\pi, \pi]$.

b. How many periods of the cotangent function are shown on the interval $[-\pi, \pi]$?

For Exercises 41–42, graph one complete period of the function. Identify the *x*-intercept and evaluate the function for *x* values midway between the *x*-intercept and the asymptotes. (See Example 3, p. 559)

41. a. $y = \frac{1}{2}\tan \pi x$

b. $y = -3\tan \pi x$

42. a. $y = 2\cot \pi x$

b. $y = -\frac{1}{4}\cot \pi x$

For Exercises 43–58, graph the function. (See Examples 3–4, pp. 559–560)

43. $y = \tan 2x$

44. $y = \cot 3x$

45. $y = \cot(-2x)$

46. $y = \tan(-x)$

47. $y = \tan\left(\frac{1}{2}x\right)$

48. $y = \cot\left(\frac{1}{3}x\right)$

49. $y = 4\cot 2x$

50. $y = -3\tan 4x$

51. $y = -2\tan \frac{\pi}{3}x$

52. $y = 5\cot \frac{\pi}{4}x$

53. $y = \cot\left(2x + \frac{\pi}{3}\right)$

54. $y = \tan\left(3x - \frac{\pi}{4}\right)$

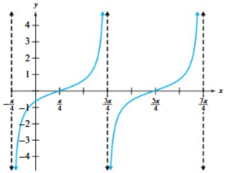
55. $y = -\tan 3x + 4$

56. $y = -\cot 2x - 3$

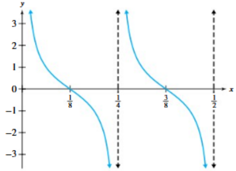
57. $y = 3\tan(\pi x + \pi) - 2$

58. $y = 2\cot\left(2\pi x - \frac{\pi}{2}\right) + 1$

59. Write a function of the form $y = \tan(Bx - C)$ for the given graph.



60. Write a function of the form $y = \cot(Bx)$ for the given graph.



Mixed Exercises

For Exercises 61–64, given $y = f(x)$ and $y = g(x)$,

a. Find $(f \circ g)(x)$ and graph the resulting function.

b. Find $(g \circ f)(x)$ and graph the resulting function.

61. $f(x) = \tan x$ and $g(x) = \frac{x}{4}$

62. $f(x) = -2x$ and $g(x) = \cot x$

63. $f(x) = -3x$ and $g(x) = \csc x$

64. $f(x) = \frac{x}{3}$ and $g(x) = \sec x$

For Exercises 65–68, complete the statements for the function provided.

65. $f(x) = \tan x$

a. As $x \to -\frac{\pi}{2}^-$, $f(x) \to$ ____________

b. As $x \to -\frac{\pi}{2}^+$, $f(x) \to$ ____________

66. $f(x) = \cot x$

a. As $x \to 0^-$, $f(x) \to$ ____________

b. As $x \to 0^+$, $f(x) \to$ ____________

67. $f(x) = \csc x$

a. As $x \to 0^-$, $f(x) \to$ ____________

b. As $x \to 0^+$, $f(x) \to$ ____________

68. $f(x) = \sec x$

a. As $x \to \frac{\pi}{2}^-$, $f(x) \to$ ____________

b. As $x \to \frac{\pi}{2}^+$, $f(x) \to$ ____________

Write About It

69. Explain how to find two consecutive vertical asymptotes of $y = A\tan(Bx - C)$ for $B > 0$.

70. Explain how to find two consecutive vertical asymptotes of $y = A\cot(Bx - C)$ for $B > 0$.

71. Explain how to graph $y = A\sec(Bx - C) + D$.

72. Explain how to graph $y = A\csc(Bx - C) + D$.

Expanding Your Skills

For Exercises 73–76, solve each equation for x on the interval $[0, 2\pi)$.

73. $\tan x = 1$

74. $\sec x = -1$

75. $\csc x = 1$

76. $\cot x = -1$

For Exercises 77–78, graph the piecewise-defined function.

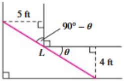
77. $$f(x) = \begin{cases} -\frac{3\sqrt{3}}{2\pi}x - \frac{3\sqrt{3}}{2} & \text{for} \quad x \le -\frac{\pi}{3} \\ \tan x & \text{for} \quad -\frac{\pi}{3} < x < \frac{\pi}{3} \\ \sqrt{3} & \text{for} \quad x \ge \frac{\pi}{3} \end{cases}$$

Hint: Graph the function on the interval $[-\pi, \pi]$. Use increments of $\frac{\pi}{3}$ on the x-axis and increments of $\sqrt{3}$ on the y-axis.

78. $$g(x) = \begin{cases} |x| + 1 & \text{for} \quad x \le -\frac{\pi}{3} \\ \sec x & \text{for} \quad -\frac{\pi}{3} < x < \frac{\pi}{3} \\ |x| + 1 & \text{for} \quad x \ge \frac{\pi}{3} \end{cases}$$

Hint: Graph the function on the interval $[-\pi, \pi]$.

79. Show that the maximum length L (in feet) of a beam that can fit around the corner shown in the figure is $L = 5\sec\theta + 4\csc\theta$.



Technology Connections

80. Graph the functions $y = \tan x$ and $y = x + \frac{x^3}{3} + \frac{2x^5}{15}$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. How do the functions compare for values of x taken close to 0?

81. Graph the functions $y = \sec x$ and $y = 1 + \frac{x^2}{2} + \frac{5x^4}{24}$ on the interval $[-\pi, \pi]$. How do the functions compare for values of x taken close to 0?

82. Given $f(x) = x^2$, $g(x) = \tan x$, and $h(x) = \sec x$,

a. Find $(f \circ h)(x)$.

b. Graph $g(x)$ and $(f \circ h)(x)$ together using the ZTRIG window. The relationship between the two graphs will be studied in calculus. For a given value of x in the domain of $g(x) = \tan x$, $y = \sec^2 x$ gives the slope of a line tangent to g at x.

83. Given $r(x) = -x^2$, $s(x) = \cot x$, and $t(x) = \csc x$,

a. Find $(r \circ t)(x)$

b. Graph $s(x)$ and $(r \circ t)(x)$ together on the ZTRIG window. The relationship between the two graphs will be studied in calculus. For a given value of x in the domain of $s(x) = \cot x$, $y = -\csc^2 x$ gives the slope of a line tangent to s at x.

Problem Recognition Exercises

Comparing Graphical Characteristics of Trigonometric Functions

For Exercises 1–16, identify which functions shown here (*f*, *g*, *h*, and so on) have the given characteristics.

$f(x) = \sin\left(\frac{\pi}{2}x\right) + 3$ $\quad g(x) = -3\cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$ $\quad h(x) = 3\sin\left(-\frac{1}{2}x - \frac{\pi}{4}\right)$

$k(x) = -3\sec(2x + \pi)$ $\quad m(x) = 2\csc\left(2x - \frac{\pi}{2}\right) - 3$ $\quad n(x) = 3\tan\left(x - \frac{\pi}{2}\right)$

$p(x) = -2\cot\left(\frac{1}{2}x + \pi\right)$ $\quad t(x) = -3 + 2\cos x$

1. Has an amplitude of 3

2. Has an amplitude of 2

3. Has no amplitude

4. Has a period of 2π

5. Has a period of 4π

6. Has a period of π

7. Has a vertical shift upward from the parent graph

8. Has a vertical shift downward from the parent graph

9. Has no asymptotes

10. Has no *x*-intercepts

11. Has no *y*-intercept

12. Has a range of all real numbers

13. Has domain of all real numbers

14. Has a phase shift of $-\frac{\pi}{2}$

15. Has a phase shift of $\frac{\pi}{2}$

16. Has no phase shift

GO ONLINE **Go online for more practice problems.**

Lesson 4-7

Inverse Trigonometric Functions

What Will You Learn?

After completing this lesson, you should be able to:

- Evaluate the Inverse Sine Function
- Evaluate the Inverse Cosine and Tangent Functions
- Approximate Inverse Trigonometric Functions on a Calculator
- Compose Trigonometric Functions and Inverse Trigonometric Functions
- Apply Inverse Trigonometric Functions
- Evaluate the Inverse Secant, Cosecant, and Cotangent Functions

Learn: Evaluate the Inverse Sine Function

Suppose that a yardstick casts a 4-ft shadow when the Sun is at an angle of elevation θ (Figure 4-55). It seems reasonable that we should be able to determine the angle of elevation from the relationship $\tan\theta = \frac{3}{4}$.

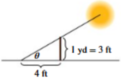


Figure 4-55

Until now, we have always been given an angle and then asked to find the sine, cosine, or tangent of the angle. However, finding the angle of elevation of the Sun from $\tan\theta = \frac{3}{4}$ requires that we reverse this process. Given the value of the tangent, we must find an angle that produced it. Therefore, we need to use the inverse of the tangent function.

We begin our study of the inverse trigonometric functions with the inverse of the sine function. First recall that a function must be one-to-one to have an inverse function. From the graph of $y = \sin x$ (Figure 4-56), we see that any horizontal line taken between $-1 \le y \le 1$ intersects the graph infinitely many times. Therefore, $y = \sin x$ is not a one-to-one function.

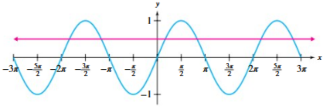


Figure 4-56

However, suppose we restrict the domain of $y = \sin x$ to the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (shown in blue in Figure 4-57).

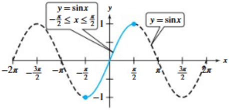


Figure 4-57

The graph of the restricted sine function is one-to-one and contains the entire range of y values $-1 \le y \le 1$.

Insights

The horizontal line test indicates that if no horizontal line intersects the graph of a function more than once, the function is one-to-one and has an inverse function. The restricted sine function is one-to-one (shown in blue in Figure 4-57).

The inverse of this restricted sine function is called the inverse sine function and is denoted by $\sin^{-1}$ or arcsin (shown in red in Figure 4-58).

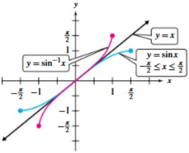


Figure 4-58

Recall that a point (a, b) on the graph of a function f corresponds to the point (b, a) on its inverse. So, points on the graph of $y = \sin x$ such as $\left(-\frac{\pi}{2}, -1\right)$ and $\left(\frac{\pi}{2}, 1\right)$ have their coordinates reversed on the graph of $y = \sin^{-1} x$. Also notice that the graphs of $y = \sin x$ and $y = \sin^{-1} x$ are symmetric with respect to the line $y = x$ as expected.

The Inverse Sine Function

The **inverse sine function** (or arc sine) denoted by $\sin^{-1}$ or arcsin is the inverse of the restricted sine function $y = \sin x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Therefore,

$$y = \sin^{-1} x \quad \Leftrightarrow \quad \sin y = x$$

$$y = \arcsin x \quad \Leftrightarrow \quad \sin y = x$$

where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Insights

The double arrow symbol ⇔ means that the statements to the left and right of the arrow are logically equivalent. That is, one statement follows from the other and vice versa.

- $y = \sin^{-1}x$ is read as "y equals the inverse sine of x" and $y = \arcsin x$ is read as "y equals the arc sine of x."
- To evaluate $y = \sin^{-1}x$ or $y = \arcsin x$ means to find an angle y between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive, whose sine value is x.

Check Your Work

The notation $\sin^{-1}x$ represents the inverse of the sine function, not the reciprocal. That is, $\sin^{-1}x \neq \frac{1}{\sin x}$.

Example 1: Evaluating the Inverse Sine Function

Find the exact values or state that the expression is undefined.

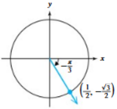
a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **b.** $\arcsin\frac{1}{2}$ **c.** $\sin^{-1}2$

Solution:

a. Let $y = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

Find an angle y on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = -\frac{\sqrt{3}}{2}$.

Then $\sin y = -\frac{\sqrt{3}}{2}$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

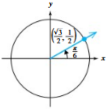


$y = -\frac{\pi}{3}$. Therefore $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.

b. Let $y = \arcsin\frac{1}{2}$.

Find an angle y on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \frac{1}{2}$.

Then $\sin y = \frac{1}{2}$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.



$y = \frac{\pi}{6}$. Therefore $\arcsin\frac{1}{2} = \frac{\pi}{6}$.

c. Let $y = \sin^{-1} 2$.

To evaluate $y = \sin^{-1} 2$ would mean that we find an angle y such that $\sin y = 2$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. However, recall that $-1 \le \sin y \le 1$ for any angle y. Therefore $\sin^{-1} 2$ is undefined.

Apply the Skills

1. Find the exact values or state that the expression is undefined.

a. $\sin^{-1}\frac{\sqrt{2}}{2}$ **b.** $\arcsin(-1)$ **c.** $\sin^{-1}(-3)$

Learn: Evaluate the Inverse Cosine and Tangent Functions

The inverse cosine function and the inverse tangent function are defined in a similar way. First, the domain of $y = \cos x$ and $y = \tan x$ must each be restricted to create a one-to-one function on an interval containing all values in the range.

The restricted cosine function is defined on $0 \le x \le \pi$ (Figure 4-59). The graph of the inverse cosine or "arccosine" (denoted by $\cos^{-1}$ or arccos) is shown in red in Figure 4-60.

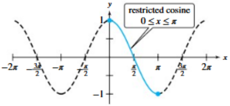


Figure 4-59

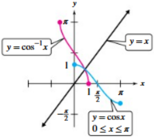


Figure 4-60

The restricted tangent function is defined on $-\frac{\pi}{2} < x < \frac{\pi}{2}$ (Figure 4-61). The graph of the inverse tangent or "arctangent" (denoted by $\tan^{-1}$ or arctan) is shown in red in Figure 4-62.

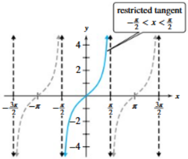


Figure 4-61

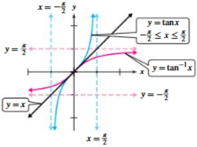


Figure 4-62

The restricted tangent function has vertical asymptotes at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. The inverse tangent function has horizontal asymptotes at $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$. We are now ready to summarize the definitions of the inverse functions for sine, cosine, and tangent.

Inverse Sine Function

We first consider the restricted function $y = \sin x$, where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $-1 \le y \le 1$.

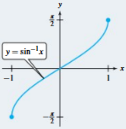
The **inverse sine function** (or arc sine), denoted by $\sin^{-1}$ or arcsin, is defined by

$$y = \sin^{-1} x \quad \Leftrightarrow \quad \sin y = x$$

$$y = \arcsin x \quad \Leftrightarrow \quad \sin y = x$$

$$-1 \le x \le 1 \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Graph



Inverse Cosine Function

We first consider the restricted function $y = \cos x$, where $0 \le x \le \pi$ and $-1 \le y \le 1$.

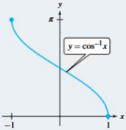
The **inverse function** (or arccosine), denoted by $\cos^{-1}$ or arccos, is defined by

$$y = \cos^{-1} x \quad \Leftrightarrow \quad \cos y = x$$

$$y = \arccos x \quad \Leftrightarrow \quad \cos y = x$$

$$-1 \le x \le 1 \quad \text{and} \quad 0 \le y \le \pi$$

Graph



Inverse Tangent Function

We first consider the restricted function $y = \tan x$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $y \in \mathbb{R}$. This restricted function has vertical asymptotes: $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$.

The **inverse tangent function** (or arctangent), denoted by $\tan^{-1}$ or arctan, is defined by

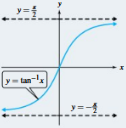
$$y = \tan^{-1} x \;\Leftrightarrow\; \tan y = x$$

$$y = \arctan x \;\Leftrightarrow\; \tan y = x$$

$$x \in \mathbb{R} \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Horizontal asymptotes: $y = -\frac{\pi}{2}$, $y = \frac{\pi}{2}$

Graph



Example 2: Evaluating Inverse Trigonometric Functions

Find the exact values.

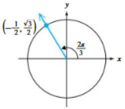
a. $\cos^{-1}\left(-\frac{1}{2}\right)$ **b.** $\tan^{-1}\sqrt{3}$ **c.** $\arctan(-1)$

Solution:

a. Let $y = \cos^{-1}\left(-\frac{1}{2}\right)$.

Then $\cos y = -\frac{1}{2}$ for $0 \le y \le \pi$.

Find an angle y on the interval $[0, \pi]$ such that $\cos y = -\frac{1}{2}$.



$y = \frac{2\pi}{3}$. Therefore, $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

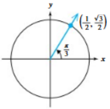
Check Your Work

Perhaps the most common error when evaluating the inverse trigonometric functions is to fail to recognize the restrictions on the range. For instance, in Example 2(a), the result of the inverse cosine function must be an angle between 0 and π.

b. Let $y = \tan^{-1}\sqrt{3}$.

Then $\tan y = \sqrt{3}$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Find an angle y on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan y = \sqrt{3}$.

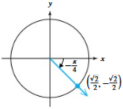


$y = \frac{\pi}{3}$. Therefore, $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

c. Let $y = \arctan(-1)$.

Then $\tan y = -1$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Find an angle y on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan y = -1$.



$y = -\frac{\pi}{4}$. Therefore, $\arctan(-1) = -\frac{\pi}{4}$.

Apply the Skills

2. Find the exact values.

a. $\cos^{-1}(-1)$ **b.** $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ **c.** $\arctan 0$

Learn: Approximate Inverse Trigonometric Functions on a Calculator

On a calculator we press the 2ND key followed by the SIN key, COS key, or TAN key to invoke $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$. By definition, the values of the inverse trigonometric functions are in radians. However, we often use the inverse functions in applications where the degree measure of an angle is desired. In Example 3, we approximate the values of several inverse trigonometric functions in both radians and degrees.

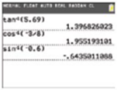
Example 3: Approximating Values of Inverse Functions

Use a calculator to approximate the function values in both radians and degrees.

a. $\tan^{-1}5.69$ **b.** $\cos^{-1}\left(-\frac{3}{8}\right)$ **c.** $\arcsin(-0.6)$

Solution:

Calculator in radian mode



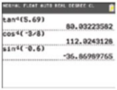
```
NORMAL FLOAT AUTO REAL RADIAN CL
tan⁻¹(5.69)
                 1.396826023
cos⁻¹(⁻3/8)
                 1.955193101
sin⁻¹(⁻0.6)
                -.6435011088
```

Press 2ND TAN

Press 2ND COS

Press 2ND SIN

Calculator in degree mode



```
NORMAL FLOAT AUTO REAL DEGREE CL
tan⁻¹(5.69)
                 80.03223582
cos⁻¹(⁻3/8)
                 112.0243128
sin⁻¹(⁻0.6)
                ⁻36.86989765
```

Apply the Skills

3. Use a calculator to approximate the function values in both radians and degrees.

a. $\tan^{-1}(-7.92)$ **b.** $\arccos\frac{2}{7}$ **c.** $\sin^{-1}(-0.81)$

We sometimes encounter applications in which we use the inverse trigonometric functions to find the value of an angle where the desired angle is not within the range of the inverse function. In such cases, we need to adjust the output value from the calculator to obtain an angle in the desired quadrant. This is demonstrated in Example 4.

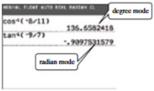
Example 4: Approximating Angles Based on Characteristics About the Angle

Use a calculator to approximate the degree measure (to 1 decimal place) or radian measure (to 4 decimal places) of the angle θ subject to the given conditions.

a. $\cos\theta = -\frac{8}{11}$ and $180° \le \theta \le 270°$ **b.** $\tan\theta = -\frac{9}{7}$ and $\frac{\pi}{2} < \theta < \pi$

Solution:

Entering each of these into the calculator, we receive the following output:



```
NORMAL FLOAT AUTO REAL RADIAN CL
cos⁻¹(⁻8/11)
                    136.6582418
tan⁻¹(⁻9/7)
                  ⁻.9097531579
```

degree mode

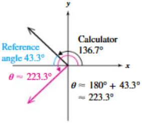
radian mode

a. $\cos^{-1}\left(-\frac{8}{11}\right) \approx 136.7°$

The calculator returns a second quadrant angle.

The reference angle is $180° - \cos^{-1}\left(-\frac{8}{11}\right) \approx 43.3°$.

The corresponding third quadrant angle is $\theta \approx 180° + 43.3° \approx 223.3°$.

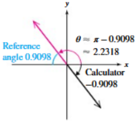


b. $\tan^{-1}\left(-\frac{9}{7}\right) \approx -0.9098$ (radians)

The calculator returns a negative fourth quadrant angle.

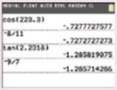
The reference angle is 0.9098.

The corresponding angle in Quadrant II is $\theta \approx \pi - 0.9098 \approx 2.2318$.



Check Your Work

The results of Example 4 can be checked on a calculator. Evaluate cos(223.3°) in degree mode and evaluate tan(2.2318) in radian mode. The results are approximately equal to –8/11 and –9/7, respectively.



```
NORMAL FLOAT AUTO REAL RADIAN CL
cos(223.3)
                       -.7277727577
-8/11
                       -.7272727273
tan(2.2318)
                       -1.285819075
-9/7
                       -1.285714286
```

Apply the Skills

4. Use a calculator to approximate the degree measure (to 1 decimal place) or radian measure (to 4 decimal places) of the angle θ subject to the given conditions.

a. $\sin\theta = -\frac{3}{7}$ and $180° \le \theta \le 270°$ **b.** $\tan\theta = -\frac{8}{3}$ and $\frac{\pi}{2} < \theta < \pi$

Learn: Compose Trigonometric Functions and Inverse Trigonometric Functions

Recall from Lesson 3.1 that inverse functions are defined such that

$$(f \circ f^{-1})(x) = f[f^{-1}(x)] = x \quad \text{and} \quad (f^{-1} \circ f)(x) = f^{-1}[f(x)] = x$$

When composing a trigonometric function with its inverse and vice versa, particular care must be taken regarding their domains.

Composing Trigonometric Functions and Their Inverses

$\sin(\sin^{-1}x) = x$ for $-1 \le x \le 1$

$\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\cos(\cos^{-1}x) = x$ for $-1 \le x \le 1$

$\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$

$\tan(\tan^{-1}x) = x$ for $x \in \mathbb{R}$

$\tan^{-1}(\tan x) = x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Example 5: Composing Inverse Trigonometric Functions

Find the exact values.

a. $\sin(\sin^{-1} 1)$ **b.** $\sin^{-1}\left(\sin \frac{7\pi}{6}\right)$

Solution:

a. $\sin(\sin^{-1} 1) = 1$	The value $x = 1$ lies in the domain of the inverse sine function. Therefore, we can apply the inverse property $\sin(\sin^{-1} x) = x$.
b. $\sin^{-1}\left(\sin \frac{7\pi}{6}\right)$	Since $\frac{7\pi}{6}$ is not on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it is not in the domain of the restricted sine function. Therefore, we cannot conclude that $\sin^{-1}(\sin x) = x$.
$= \sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$	Rewrite $\sin \frac{7\pi}{6}$ as an equivalent expression with an angle on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
$= -\frac{\pi}{6}$	Apply the property $\sin^{-1}(\sin x) = x$.

Insights

As an alternative method, we can simplify the expressions from Example 5 by applying the order of operations.

$$\sin[\sin^{-1}(1)] = \sin \frac{\pi}{2} = 1$$

and

$$\sin^{-1}\left(\sin \frac{7\pi}{6}\right) = \sin^{-1}\left(-\frac{1}{2}\right)$$
$$= -\frac{\pi}{6}$$

Apply the Skills

5. Find the exact values.

a. $\cos[\cos^{-1}(-1)]$ **b.** $\cos^{-1}\left(\cos \frac{4\pi}{3}\right)$

Example 6: Composing Trigonometric Functions and Inverse Trigonometric Functions

Find the exact value of $\cos\left[\tan^{-1}\left(-\frac{8}{15}\right)\right]$.

Solution:

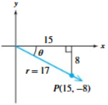
Let $\theta = \tan^{-1}\left(-\frac{8}{15}\right)$.

Since $-\frac{8}{15} < 0$, angle θ is on the interval $\left(-\frac{\pi}{2}, 0\right)$.

Let $P(15, -8)$ be a point on the terminal side of θ.

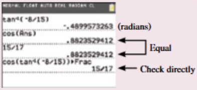
Then, $r = \sqrt{(15)^2 + (-8)^2} = \sqrt{289} = 17$.

$\cos\left[\tan^{-1}\left(-\frac{8}{15}\right)\right] = \cos\theta = \frac{15}{17}$


y
15
θ
x
8
r = 17
P(15, –8)

Check Your Work

You can confirm your answer from Example 6 on a calculator by applying the order of operations.


NORMAL FLOAT AUTO REAL RADIAN CL
(radians)
Equal
Check directly

```
tan⁻¹(⁻8/15)
                    ⁻.4899573263
cos(Ans)
                     .8823529412
15/17
                     .8823529412
cos(tan⁻¹(⁻8/15))▸Frac
                           15/17
```

Apply the Skills

6. Find the exact value of $\sin\left[\tan^{-1}\left(\frac{12}{5}\right)\right]$.

Example 7: Composing Trigonometric Functions and Inverse Trigonometric Functions

Find the exact value of $\sin\left[\cos^{-1}\left(-\frac{3}{7}\right)\right]$.

Solution:

Let $\theta = \cos^{-1}\left(-\frac{3}{7}\right)$.

Since $-\frac{3}{7} < 0$, angle θ is on the interval $\left(\frac{\pi}{2}, \pi\right)$.

Let $P(-3, y)$ be a point on the terminal side of θ.

From Figure 4-63:

$$
\begin{aligned}
(-3)^2 + y^2 &= 7^2 \\
9 + y^2 &= 49 \\
y^2 &= 40 \\
y &= \sqrt{40} = 2\sqrt{10}
\end{aligned}
$$

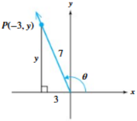
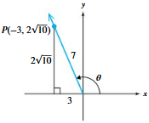


Figure 4-63


P(–3, 2√10)
y
2√10
7
θ
3
x

Therefore, $\sin\left[\cos^{-1}\left(-\frac{3}{7}\right)\right] = \sin\theta = \frac{2\sqrt{10}}{7}$.

Apply the Skills

7. Find the exact value of $\cos\left[\sin^{-1}\left(-\frac{2}{11}\right)\right]$.

In Example 8, we use a right triangle to visualize the composition of a trigonometric function and an inverse function, and then write an equivalent algebraic expression. This skill is used often in calculus.

Example 8: Writing a Trigonometric Expression as an Algebraic Expression

Write the expression $\tan\left(\sin^{-1}\frac{\sqrt{x^2-16}}{x}\right)$ as an algebraic expression for $x > 4$.

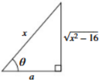
Solution:

Let $\theta = \sin^{-1}\frac{\sqrt{x^2-16}}{x}$.

Since $x > 4$, $\frac{\sqrt{x^2-16}}{x} > 0$, and we know that θ is an acute angle.

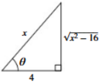
We can set up a representative right triangle using the relationship

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{x^2-16}}{x}$$



To find an expression for the adjacent side a, apply the Pythagorean theorem.

$$a^2 + \left(\sqrt{x^2-16}\right)^2 = x^2$$
$$a^2 + x^2 - 16 = x^2$$
$$a^2 = 16$$
$$a = 4$$



Therefore, $\tan\left(\sin^{-1}\frac{\sqrt{x^2-16}}{x}\right) = \tan\theta = \frac{\sqrt{x^2-16}}{4}$.

Apply the Skills

8. Write the expression $\tan\left(\sin^{-1}\frac{x}{\sqrt{x^2+9}}\right)$ as an algebraic expression for $x > 0$.

Learn: Apply Inverse Trigonometric Functions

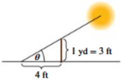
We now revisit the scenario that we used to introduce this lesson. Example 9 shows how inverse trigonometric functions can help us find the measure of an unknown angle in an application.

Example 9: Applying an Inverse Trigonometric Function

A yardstick casts a 4-ft shadow when the Sun is at an angle of elevation θ. Determine the angle of elevation of the Sun to the nearest tenth of a degree.

Solution:

The tip of the shadow and the points at the top and bottom of the yardstick form a right triangle with θ representing the angle of elevation of the Sun.



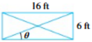
$\tan\theta = \frac{3}{4}$

$\theta = \tan^{-1}\frac{3}{4}$ Use the inverse tangent to find the acute angle θ whose tangent is $\frac{3}{4}$.

$\theta \approx 36.9°$ Be sure that your calculator is in degree mode.

Apply the Skills

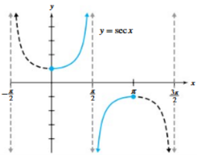
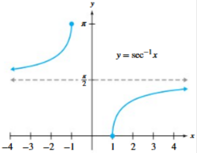
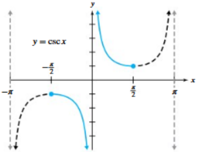
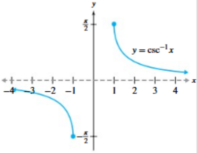
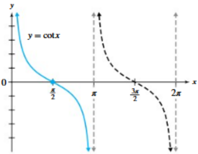
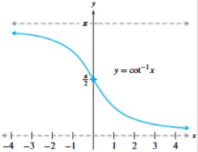
9. For the construction of a house, a 16-ft by 6-ft wooden frame is made. Find the angle that the diagonal beam makes with the base of the frame. Round to the nearest tenth of a degree.



Learn: Evaluate the Inverse Secant, Cosecant, and Cotangent Functions

We need to establish intervals over which the secant, cosecant, and cotangent functions are one-to-one before we can define their corresponding inverse functions. These intervals are chosen to accommodate the entire range of function values and to make the restricted functions one-to-one. The graphs of the restricted secant, cosecant, and cotangent functions and their inverses are shown in Table 4-17.

TABLE 4-17 The Inverse Secant, Cosecant, and Cotangent Functions

Inverse Function Relationship	Graphs
$y=\sec^{-1}x \Leftrightarrow \sec y=x$, where $\lvert x\rvert \ge 1$ and $0 \le y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y \le \pi$.	 
$y=\csc^{-1}x \Leftrightarrow \csc y=x$, where $\lvert x\rvert \ge 1$ and $-\frac{\pi}{2} \le y < 0$ or $0 < y \le \frac{\pi}{2}$.	 
$y=\cot^{-1}x \Leftrightarrow \cot y=x$, where x is any real number and $0 < y < \pi$.	 

Insights

It is important to note that there are infinitely many intervals over which the secant, cosecant, and cotangent functions can be restricted to define the inverse functions. These restrictions are not universally agreed upon.

A calculator does not have keys for inverse secant, cosecant, or cotangent. Therefore, to evaluate an inverse secant, cosecant, or cotangent on a calculator, we first rewrite the expression using inverse sine or cosine.

Example 10: Approximating the Value of an Inverse Secant, Cosecant, or Cotangent

Approximate each expression in radians, rounded to 4 decimal places.

a. $\csc^{-1}3$ **b.** $\cot^{-1}\left(-\frac{3}{4}\right)$

Solution:

a. Let $\theta = \csc^{-1}3$.

Then $\csc\theta = 3$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \ne 0$.

Since $y = \sin^{-1}x$ has the same range as $y = \csc^{-1}x$ for $x \ne 0$, we have:

$$\csc\theta = 3 \text{ implies that } \frac{1}{\sin\theta} = 3.$$

So, $\sin\theta = \frac{1}{3}$ and $\theta = \sin^{-1}\frac{1}{3} \approx 0.3398$.

b. Let $\theta = \cot^{-1}\left(-\frac{3}{4}\right)$.

Then $\cot\theta = -\frac{3}{4}$ for $0 < \theta < \pi$.

Since the argument $-\frac{3}{4}$ is negative, $\theta = \cot^{-1}\left(-\frac{3}{4}\right)$ must be a second quadrant angle.

The related expression using the inverse tangent function will *not* return a second quadrant angle but rather an angle on the interval $-\frac{\pi}{2} < \theta < 0$. So, we will rewrite the expression $\cot\theta = -\frac{3}{4}$ using the cosine function.

From Figure 4-64, $\cos\theta = -\frac{3}{5}$.

Therefore, $\theta = \cos^{-1}\left(-\frac{3}{5}\right) \approx 2.2143$.

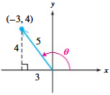


Figure 4-64

Apply the Skills

10. Approximate each expression in radians, rounded to 4 decimal places.

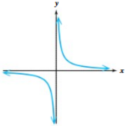
a. $\sec^{-1}(-4)$ **b.** $\cot^{-1}\left(-\frac{5}{12}\right)$

Practice 4-7

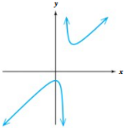
Practice Exercises

Prerequisite Review

R.1. Determine if the relation defines y as a one-to-one function of x.



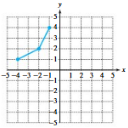
R.2. Determine if the relation defines y as a one-to-one function of x.



R.3. A one-to-one function is given. Write an equation for the inverse function.

$$g(x) = \frac{3-x}{4}$$

R.4. The graph of a function is given. Graph the inverse function.



Concept Connections

1. A function must be ______________ on its entire domain to have an inverse function.

2. If $\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$ is on the graph of $y = \cos x$, what is the related point on $y = \cos^{-1} x$?

3. The graph of $y = \tan^{-1} x$ has two ______________ (horizontal/vertical) asymptotes represented by the equations ______________ and ______________.

4. The domain of $y = \arctan x$ is ____________. The output is a real number (or angle in radians) between ____________ and ____________.

5. In interval notation, the domain of $y = \cos^{-1} x$ is ____________. The output is a real number (or angle in radians) between __________ and __________, inclusive.

6. In interval notation, the domain of $y = \sin^{-1} x$ is ____________. The output is a real number (or angle in radians) between __________ and __________, inclusive.

Learn: Evaluate the Inverse Sine Function

For Exercises 7–12, find the exact value or state that the expression is undefined. (See Example 1, p. 568)

7. $\arcsin \frac{\sqrt{2}}{2}$

8. $\sin^{-1}\left(-\frac{1}{2}\right)$

9. $\sin^{-1} \pi$

10. $\sin^{-1} \frac{3}{2}$

11. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

12. $\arcsin \frac{\sqrt{3}}{2}$

For Exercises 13–16, find the exact value.

13. $\sin^{-1} \frac{\sqrt{3}}{2} + \sin^{-1} \frac{1}{2}$

14. $\sin^{-1} \frac{\sqrt{2}}{2} - \sin^{-1}(-1)$

15. $2 \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) - \frac{\pi}{3}$

16. $\frac{\pi}{2} + 3 \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Learn: Evaluate the Inverse Cosine and Tangent Functions

For Exercises 17–28, find the exact value or state that the expression is undefined. (See Example 2, p. 571)

17. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$

18. $\tan^{-1}(-\sqrt{3})$

19. $\cos^{-1} 0$

20. $\tan^{-1} 0$

21. $\cos^{-1}(-2)$

22. $\arccos \frac{4}{3}$

23. $\arctan \frac{\sqrt{3}}{3}$

24. $\cos^{-1} \frac{\sqrt{2}}{2}$

25. $\tan^{-1} 1$

26. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

27. $\arccos \frac{\sqrt{3}}{2}$

28. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

For Exercises 29–32, find the exact value.

29. $\tan^{-1}(-1) + \tan^{-1}\sqrt{3}$

30. $\cos^{-1}\frac{\sqrt{2}}{2} + \cos^{-1}\frac{1}{2}$

31. $2\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \tan^{-1}\frac{\sqrt{3}}{3}$

32. $3\tan^{-1}1 + \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Learn: Approximate Inverse Trigonometric Functions on a Calculator

For Exercises 33–36, use a calculator to approximate the function values in both radians and degrees. (See Example 3, p. 572)

33. a. $\cos^{-1}\frac{3}{8}$

b. $\tan^{-1} 25$

c. $\arcsin 0.05$

34. a. $\sin^{-1} 0.93$

b. $\arccos 0.17$

c. $\arctan \frac{7}{4}$

35. a. $\tan^{-1}(-28)$

b. $\arccos \frac{\sqrt{3}}{5}$

c. $\sin^{-1}(-0.14)$

36. a. $\cos^{-1}(-0.75)$

b. $\tan^{-1}\frac{8}{3}$

c. $\arcsin \frac{\pi}{7}$

For Exercises 37–46, use a calculator to approximate the degree measure (to 1 decimal place) or radian measure (to 4 decimal places) of the angle θ subject to the given conditions. (See Example 4, p. 573)

37. $\cos\theta = -\frac{5}{6}$ and $180° < \theta < 270°$

38. $\sin\theta = -\frac{4}{5}$ and $180° < \theta < 270°$

39. $\tan\theta = -\frac{12}{5}$ and $90° < \theta < 180°$

40. $\cos\theta = -\frac{2}{13}$ and $180° < \theta < 270°$

41. $\sin\theta = \frac{12}{19}$ and $90° < \theta < 180°$

42. $\tan\theta = \frac{7}{15}$ and $180° < \theta < 270°$

43. $\cos\theta = -\frac{5}{8}$ and $\pi < \theta < \frac{3\pi}{2}$

44. $\tan\theta = -\frac{9}{5}$ and $\frac{\pi}{2} < \theta < \pi$

45. $\sin\theta = -\frac{17}{20}$ and $\pi < \theta < \frac{3\pi}{2}$

46. $\cos\theta = \frac{1}{17}$ and $\frac{3\pi}{2} < \theta < 2\pi$

Learn: Compose Trigonometric Functions and Inverse Trigonometric Functions

For Exercises 47–58, find the exact values. (See Example 5, p. 575)

47. $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

48. $\arcsin\left(\sin\frac{5\pi}{3}\right)$

49. $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$

50. $\sin\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

51. $\cos\left(\cos^{-1}\frac{2}{3}\right)$

52. $\arccos\left(\cos\frac{11\pi}{6}\right)$

53. $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

54. $\cos\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$

55. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

56. $\tan(\tan^{-1}2)$

57. $\tan[\arctan(-\pi)]$

58. $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$

For Exercises 59–70, find the exact values. (See Examples 6–7, pp. 575–576)

59. $\cos\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)$

60. $\sin\left(\cos^{-1}\frac{1}{2}\right)$

61. $\tan\left[\sin^{-1}\left(-\frac{2}{3}\right)\right]$

62. $\sin\left[\cos^{-1}\left(-\frac{2}{3}\right)\right]$

63. $\sin\left(\cos^{-1}\frac{3}{4}\right)$

64. $\sin\left(\tan^{-1}\frac{4}{3}\right)$

65. $\sin[\tan^{-1}(-1)]$

66. $\cos[\tan^{-1}(-1)]$

67. $\cos\left[\sin^{-1}\left(-\frac{2}{7}\right)\right]$

68. $\cos\left[\tan^{-1}\left(-\frac{5}{12}\right)\right]$

69. $\tan\left[\cos^{-1}\left(-\frac{5}{6}\right)\right]$

70. $\tan\left[\sin^{-1}\left(-\frac{\sqrt{5}}{3}\right)\right]$

For Exercises 71–76, write the given expression as an algebraic expression. It is not necessary to rationalize the denominator. (See Example 8, p. 577)

71. $\cos\left(\sin^{-1}\frac{x}{\sqrt{25+x^2}}\right)$ for $x > 0$.

72. $\cot\left(\cos^{-1}\frac{\sqrt{x^2-1}}{x}\right)$ for $x > 1$.

73. $\sin(\tan^{-1}x)$ for $x > 0$.

74. $\tan(\sin^{-1}x)$ for $|x| < 1$.

75. $\tan\left(\cos^{-1}\frac{3}{x}\right)$ for $x > 3$.

76. $\sin\left(\cos^{-1}\frac{\sqrt{x^2-25}}{x}\right)$ for $x > 5$.

Learn: Apply Inverse Trigonometric Functions

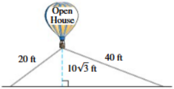
77. To meet the requirements of the Americans with Disabilities Act (ADA), a wheelchair ramp must have a slope of 1:12 or less. That is, for every 1 in. of "rise," there must be at least 12 in. of "run." (**See Example 9, p. 578**)

a. If a wheelchair ramp is constructed with the maximum slope, what angle does the ramp make with the ground? Round to the nearest tenth of a degree.

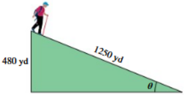
b. If the ramp is 22 ft long, how much elevation does the ramp provide? Round to the nearest tenth of a foot.

78. A student measures the length of the shadow of the Washington Monument to be 620 ft. If the Washington Monument is 555 ft tall, approximate the angle of elevation of the Sun to the nearest tenth of a degree.

79. A balloon advertising an open house is stabilized by two cables of lengths 20 ft and 40 ft tethered to the ground. If the perpendicular distance from the balloon to the ground is $10\sqrt{3}$ft, what is the degree measure of the angle each cable makes with the ground? Round to the nearest tenth of a degree if necessary.



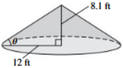
80. A group of campers hikes down a steep path. One member of the group has an altimeter to measure altitude. If the path is 1250 yd and the amount of altitude lost is 480 yd, what is the angle of incline? Round to the nearest tenth of a degree.



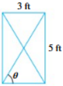
81. Navajo Tube Hill, a snow tubing hill in Utah, is 550 ft long and has a 75-ft vertical drop. Find the angle of incline of the hill. Round to the nearest tenth of a degree.

82. A ski run on Giant Steps Mountain in Utah is 1475 m long. The difference in altitude from the beginning to the end of the run is 350 m. Find the angle of the ski run. Round to the nearest tenth of a degree.

83. When granular material such as sand or gravel is poured onto a horizontal surface it forms a right circular cone. The angle that the surface of the cone makes with the horizontal is called the **angle of repose.** The angle of repose depends on a number of variables such as the shape of the particles and the amount of friction between them. “Stickier” particles have a greater angle of repose, and “slippery” particles have a smaller angle of repose. Find the angle of repose for the pile of dry sand. Round to the nearest degree.



84. For the construction of a bookcase, a 5-ft by 3-ft wooden frame is made with a cross-brace in the back for stability. Find the angle that the diagonal brace makes with the base of the frame. Round to the nearest tenth of a degree.



Learn: Evaluate the Inverse Secant, Cosecant, and Cotangent Functions

85. Show that $\sec^{-1}x = \cos^{-1}\frac{1}{x}$ for $x \ge 1$.

86. Show that $\csc^{-1}x = \sin^{-1}\frac{1}{x}$ for $x \ge 1$.

87. Show that $\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$ for $x \ge 1$.

88. Complete the table, giving the domain and range in interval notation for each inverse function.

Inverse Function	Domain	Range
$y = \sin^{-1}x$		
$y = \csc^{-1}x$		
$y = \cos^{-1}x$		
$y = \sec^{-1}x$		
$y = \tan^{-1}x$		
$y = \cot^{-1}x$		

For Exercises 89–94, find the exact values.

89. $\sec^{-1}\frac{2\sqrt{3}}{3}$

90. $\sec^{-1}(-\sqrt{2})$

91. $\csc^{-1}(-1)$

92. $\csc^{-1}(2)$

93. $\cot^{-1}\sqrt{3}$

94. $\cot^{-1}(1)$

For Exercises 95–100, use a calculator to approximate each expression in radians, rounded to 4 decimal places. (See Example 10, p. 580)

95. $\sec^{-1}\frac{7}{4}$

96. $\csc^{-1}\frac{6}{5}$

97. $\csc^{-1}(-8)$

98. $\sec^{-1}(-6)$

99. $\cot^{-1}\left(-\frac{8}{15}\right)$

100. $\cot^{-1}\left(-\frac{24}{7}\right)$

Mixed Exercises

For Exercises 101–104, find the exact value if possible. Otherwise find an approximation to 4 decimal places or state that the expression is undefined.

101. a. $\sin\frac{\pi}{4}$ **b.** $\sin^{-1}\frac{\pi}{4}$ **c.** $\sin^{-1}\frac{\sqrt{2}}{2}$

102. a. $\cos\frac{2\pi}{3}$ **b.** $\cos^{-1}\frac{2\pi}{3}$ **c.** $\cos^{-1}\left(-\frac{1}{2}\right)$

103. a. $\cos^{-1}\left(-\frac{\pi}{6}\right)$ **b.** $\cos\left(-\frac{\pi}{6}\right)$ **c.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

104. a. $\sin^{-1}\frac{7\pi}{6}$ **b.** $\sin\frac{7\pi}{6}$ **c.** $\sin^{-1}\left(-\frac{1}{2}\right)$

For Exercises 105–108, find the inverse function and its domain and range.

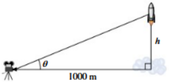
105. $f(x) = 3\sin x + 2$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

106. $g(x) = 6\cos x - 4$ for $0 \le x \le \pi$

107. $h(x) = \frac{\pi}{4} + \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

108. $k(x) = \pi + \sin x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

109. A video camera located at ground level follows the liftoff of an Atlas V rocket from the Kennedy Space Center. Suppose that the camera is 1000 m from the launch pad.

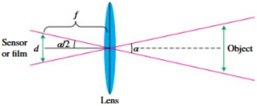


a. Write the angle of elevation θ from the camera to the rocket as a function of the rocket's height, h.

b. Without the use of a calculator, will the angle of elevation be less than 45° or greater than 45° when the rocket is 400 m high?

c. Use a calculator to find θ to the nearest tenth of a degree when the rocket's height is 400 m, 1500 m, and 3000 m.

110. The effective focal length f of a camera is the distance required for the lens to converge light to a single focal point. The angle of view α of a camera describes the angular range (either horizontally, vertically, or diagonally) that is imaged by a camera.

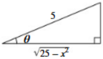


a. Show that $\alpha = 2\arctan\frac{d}{2f}$ where d is the dimension of the image sensor or film.

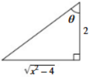
b. A typical 35-mm camera has image dimensions of 24 mm (vertically) by 36 mm (horizontally). If the focal length is 50 mm, find the vertical and horizontal viewing angles. Round to the nearest tenth of a degree.

For Exercises 111–114, use the relationship given in the right triangle and the inverse sine, cosine, and tangent functions to write θ as a function of x in three different ways. It is not necessary to rationalize the denominator.

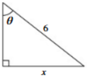
111.



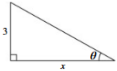
112.



113.



114.



For Exercises 115–120, find the exact solution to each equation.

115. $-2\sin^{-1}x - \pi = 0$

116. $3\cos^{-1}x - \pi = 0$

117. $6\cos^{-1}x - 3\pi = 0$

118. $4\sin^{-1}x + \pi = 0$

119. $4\tan^{-1}2x = \pi$

120. $6\tan^{-1}2x = 2\pi$

Write About It

121. Explain the difference between the reciprocal of a function and the inverse of a function.

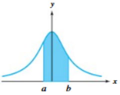
122. Explain the flaw in the logic:
$\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. Therefore, $\cos^{-1}\frac{\sqrt{2}}{2} = -\frac{\pi}{4}$.

123. In terms of angles, explain what is meant when we find $\sin^{-1}\left(-\frac{1}{2}\right)$.

124. Why must the domains of the sine, cosine, and tangent functions be restricted in order to define their inverse functions?

Expanding Your Skills

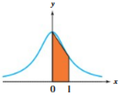
125. In calculus, we can show that the area below the graph of $f(x) = \frac{1}{1+x^2}$, above the x-axis, and between the lines $x = a$ and $x = b$ for $a < b$, is given by $\tan^{-1}b - \tan^{-1}a$.



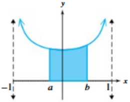
a. Find the area under the curve between $x = 0$ and $x = 1$.

b. Evaluate $f(0)$ and $f(1)$.

c. Find the area of the trapezoid defined by the points $(0, 0)$, $(1, 0)$, $[0, f(0)]$, and $[1, f(1)]$ to confirm that your answer from part (a) is reasonable.



126. In calculus, we can show that the area below the graph of $f(x) = \frac{1}{\sqrt{1-x^2}}$, above the x-axis, and between the lines $x = a$ and $x = b$ for $a < b$, is given by $\sin^{-1} b - \sin^{-1} a$.

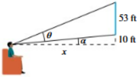


a. Find the area under the curve between $x = 0$ and $x = 0.5$.

b. Evaluate $f(0)$ and $f(0.5)$.

c. Find the area of the trapezoid defined by the points (0, 0), (1, 0), $[0, f(0)]$, and $[0.5, f(0.5)]$ to confirm that your answer from part (a) is reasonable.

127. The vertical viewing angle θ to a movie screen is the angle formed from the bottom of the screen to a viewer's eye to the top of the screen. Suppose that the viewer is sitting x horizontal feet from an IMAX screen 53 ft high and that the bottom of the screen is 10 vertical feet above the viewer's eye level. Let α be the angle of elevation to the bottom of the screen.



a. Write an expression for $\tan \alpha$.

b. Write an expression for $\tan(\alpha + \theta)$.

c. Using the relationships found in parts (a) and (b), show that $\theta = \tan^{-1}\frac{63}{x} - \tan^{-1}\frac{10}{x}$.

Technology Connections

128. Refer to the movie screen and observer in Exercise 127. The vertical viewing angle is given by $\theta = \tan^{-1}\frac{63}{x} - \tan^{-1}\frac{10}{x}$.

a. Find the vertical viewing angle (in radians) for an observer sitting 15, 25, and 35 ft away. Round to 2 decimal places.

b. Graph $y = \tan^{-1}\frac{63}{x} - \tan^{-1}\frac{10}{x}$ on a graphing utility on the window [0, 100, 10] by [0, 1, 0.1].

c. Use the Maximum feature on the graphing utility to estimate the distance that an observer should sit from the screen to produce the maximum viewing angle. Round to the nearest tenth of a foot.

129. a. Graph the functions $y = \tan^{-1} x$ and $y = x - \frac{x^3}{3} + \frac{x^5}{5}$ on the window $\left[-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}\right]$ by [−2, 2, 1].

b. Graph the functions $y = \tan^{-1} x$ and $y = \frac{x}{1 + \frac{x^2}{3 - x^2}}$ on the window $\left[-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}\right]$ by [−2, 2, 1].

c. How do the functions in parts (a) and (b) compare for values of x taken close to 0? The function values are close for x taken near 0.

Go online for more practice problems.

Chapter 4 Review

Chapter 4 Key Concepts

Lesson 4-1 Angles and Their Measure Reference

Key Concepts	Reference
The measure of an angle may be given in **degrees**, where 1° is $\frac{1}{360}$ of a full rotation. A measure of 1° can be further divided into 60 equal parts called **minutes**. Each minute can be divided into 60 equal parts called **seconds**.	pp. 447–448
The measure of an angle may be given in **radians**, where 1 radian is the measure of the central angle that intercepts an arc equal in length to the radius of the circle.	p. 450
$\theta = \frac{s}{r}$ gives the measure of a central angle θ (in radians), where s is the length of the arc intercepted by θ and r is the radius of the circle.	p. 451
The relationship $180° = \pi$ radians provides the conversion factors to convert from radians to degrees and vice versa.	p. 452
Two angles with the same initial and terminal sides are called **coterminal angles**. Coterminal angles differ in measure by an integer multiple of 360° or 2π radians.	p. 453
Given a circle of radius r, the length s of an arc intercepted by a central angle θ (in radians) is given by $s = r\theta$.	p. 456
Points on the Earth are identified by their latitude and longitude. **Latitude** is the angular measure of a central angle measuring north or south from the equator. Lines of **longitude**, also called meridians, are circles passing through both poles and running perpendicular to the equator.	p. 457
Suppose that a point on a circle of radius r moves through an angle of θ radians in time t. Then the angular and linear speeds of the point are given by angular speed: $\omega = \frac{\theta}{t}$ linear speed: $v = \frac{s}{t}$ or $v = \frac{r\theta}{t}$ or $v = r\omega$	p. 458
The area A of a sector of a circle of radius r with central angle θ (in radians) is given by $A = \frac{1}{2}r^2\theta$.	p. 460

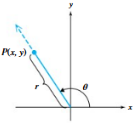
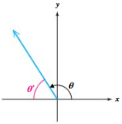
Lesson 4-2 Trigonometric Functions Defined on the Unit Circle

Key Concepts	Reference
The **unit circle** is a circle of radius r with center at the origin of a rectangular coordinate system: $x^2 + y^2 = 1$.	p. 470
The six trigonometric functions can be defined as functions of real numbers by using the unit circle. Let $P(x, y)$ be the point associated with a real number t measured along the circumference of the unit circle from the point (1, 0). $\sin t = y$ $\quad$ $\cos t = x$ $\quad$ $\tan t = \frac{y}{x}\ (x \neq 0)$ $\csc t = \frac{1}{y}\ (y \neq 0)$ $\quad$ $\sec t = \frac{1}{x}\ (x \neq 0)$ $\quad$ $\cot t = \frac{x}{y}\ (y \neq 0)$	p. 471
The real number t taken along the circumference of the unit circle gives the number of radians of the corresponding central angle θ; that is, $\theta = t$ radians.	p. 473
• The domain of both the sine function and cosine function is all real numbers. • The domain of both the tangent function and secant function excludes real numbers t that are odd multiples of $\frac{\pi}{2}$. • The domain of both the cotangent function and cosecant function excludes real numbers t that are multiples of π.	p. 478
A function f is **periodic** if $f(t + p) = f(t)$ for some constant p. The smallest positive value p for which f is periodic is called the **period** of f. • The period of the sine, cosine, secant, and cosecant functions is 2π. • The period of the tangent and cotangent functions is π.	p. 483
• The cosine and secant functions are even functions. $f(-x) = f(x)$. • The sine, cosecant, tangent, and cotangent functions are odd functions. $f(-x) = -f(x)$.	p. 485

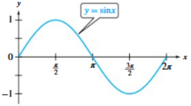
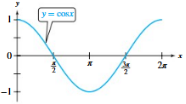
Lesson 4-3 Right Triangle Trigonometry

Key Concepts	Reference
Each trigonometric function of an acute angle θ is one of the six possible ratios of the sides of a right triangle containing θ. $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$, $\tan\theta = \frac{\text{opp}}{\text{adj}}$, $\csc\theta = \frac{\text{hyp}}{\text{opp}}$, $\sec\theta = \frac{\text{hyp}}{\text{adj}}$, $\cot\theta = \frac{\text{adj}}{\text{opp}}$	p. 495
Reciprocal Identities: $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta}$ **Quotient Identities**: $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$	p. 502
Pythagorean Identities: $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, $1 + \cot^2\theta = \csc^2\theta$	p. 502
The **cofunction identities** indicate that cofunctions of complementary angles are equal.	p. 503
An **angle of elevation** or **depression** from an observer to an object is used in many applications of trigonometric functions.	p. 504

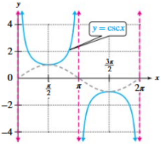
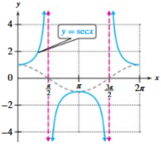
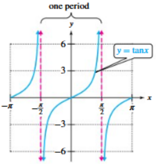
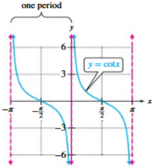
Lesson 4-4 Trigonometric Functions of Any Angle

Key Concepts	Reference
$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}\ (x \neq 0)$ $\csc\theta = \frac{r}{y}\ (y \neq 0)$ $\sec\theta = \frac{r}{x}\ (x \neq 0)$ $\cot\theta = \frac{x}{y}\ (y \neq 0)$ Let $P(x, y)$ be a point on the terminal side of angle θ drawn in standard position, and r be the distance from P to the origin. 	p. 516
Let θ be an nonquadrantal angle in standard position. The **reference angle** for θ is the acute angle θ' formed by the terminal side of θ and the horizontal axis. 	p. 517
The value of each trigonometric function of θ is the same as the corresponding function of the reference angle θ' except possibly for the sign.	p. 519

Lesson 4-5 Graphs of Sine and Cosine Functions

Key Concepts	Reference
The graph of $y = \sin x$ (for one period):  Period: 2π Amplitude: 1 Domain: $(-\infty, \infty)$ Range: $[-1, 1]$ Symmetric to the origin The graph of $y = \cos x$ (for one period):  Period: 2π Amplitude: 1 Domain: $(-\infty, \infty)$ Range: $[-1, 1]$ Symmetric to the y-axis	p. 530
Characteristics of $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$ with $B > 0$: 1. The amplitude is $\lvert A\rvert$. 2. The period is $\frac{2\pi}{B}$. 3. The phase shift is $\frac{C}{B}$. 4. The vertical shift is D. 5. One full cycle is given on the interval $0 \leq Bx - C \leq 2\pi$. 6. The domain is the set of real numbers. 7. The range is $-\lvert A\rvert + D \leq y \leq \lvert A\rvert + D$.	p. 536

Lesson 4-6 Graphs of Other Trigonometric Functions

Key Concepts		Reference
The graph of $y = \csc x = \frac{1}{\sin x}$ has vertical asymptotes where $\sin x = 0$.  Period: 2π Amplitude: None Domain: $\{x \mid x \neq n\pi$ for integers $n\}$ Range: $(-\infty, -1] \cup [1, \infty)$ Vertical asymptotes: $x = n\pi$ Symmetric to the origin (odd function)	The graph of $y = \sec x = \frac{1}{\cos x}$ has vertical asymptotes where $\cos x = 0$.  Period: 2π Amplitude: None Domain: $\left\{x \middle\vert x \neq \frac{(2n+1)\pi}{2} \text{ for integers } n\right\}$ Range: $(-\infty, -1] \cup [1, \infty)$ Vertical asymptotes: $x = \frac{(2n+1)\pi}{2}$ Symmetric to the *y*-axis (even function)	pp. 553–554
The graph of $y = \tan x = \frac{\sin x}{\cos x}$ has vertical asymptotes where $\cos x = 0$.  Period: π Amplitude: None Domain: $\left\{x \middle\vert x \neq \frac{(2n+1)\pi}{2} \text{ for integers } n\right\}$ Range: $(-\infty, \infty)$ Vertical asymptotes: $x = \frac{(2n+1)\pi}{2}$ Symmetric to the origin (odd function)	The graph of $y = \cot x = \frac{\cos x}{\sin x}$ has vertical asymptotes where $\sin x = 0$.  Period: π Amplitude: None Domain: $\{x \mid x \neq n\pi$ for integers $n\}$ Range: $(-\infty, \infty)$ Vertical asymptotes: $x = n\pi$ Symmetric to the origin (odd function)	pp. 556–557

To graph variations of $y = \csc x$ or $y = \sec x$, use the graph of the related reciprocal function for reference.	p. 554
To graph variations of $y = \tan x$ and $y = \cot x$, 1. First graph two consecutive asymptotes. 2. Plot an x-intercept halfway between the asymptotes. 3. Sketch the general shape of the "parent" function between the asymptotes. 4. Apply a vertical shift if applicable. 5. Sketch additional cycles to the right or left as desired.	p. 558

Lesson 4-7 Inverse Trigonometric Functions

Key Concepts	Reference
The **inverse sine function** (or arcsine), denoted by $\sin^{-1}$ or arcsin, is defined by $y = \sin^{-1}x \Leftrightarrow \sin y = x \quad \text{for} \quad -1 \le x \le 1 \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$ To compose the sine function and its inverse, $\sin(\sin^{-1}x) = x \quad \text{for} \quad -1 \le x \le 1 \quad \text{and} \quad \sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$	p. 567
The **inverse cosine function** (or arccosine), denoted by $\cos^{-1}$ or arccos, is defined by $y = \cos^{-1}x \Leftrightarrow \cos y = x \quad \text{for} \quad -1 \le x \le 1 \quad \text{and} \quad 0 \le y \le \pi.$ To compose the cosine function and its inverse, $\cos(\cos^{-1}x) = x \quad \text{for} \quad -1 \le x \le 1 \quad \text{and} \quad \cos^{-1}(\cos x) = x \quad \text{for} \quad 0 \le x \le \pi.$	p. 570
The **inverse tangent function** (or arctangent), denoted by $\tan^{-1}$ or arctan, is defined by $y = \tan^{-1}x \Leftrightarrow \tan y = x \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$ The graph of the inverse tangent function has horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}.$ To compose the tangent function and its inverse, $\tan(\tan^{-1}x) = x \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad \tan^{-1}(\tan x) = x \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$	p. 571

Go online for a detailed chapter summary.

Chapter 4 Test

1. Convert 15.36° to DMS (degree-minute-second) form. Round to the nearest second if necessary.

2. Convert 130.3° to radians. Round to 2 decimal places.

3. Find the exact length of the arc intercepted by a central angle of 27° on a circle of radius 5 ft.

4. A skateboard designed for rough surfaces has wheels with diameter 60 mm. If the wheels turn at 2200 rpm, what is the linear speed in mm per minute?

5. A pulley is 20 in. in diameter. Through how many degrees should the pulley rotate to lift a load 3 ft? Round to the nearest degree.

6. A circle has a radius of 9 yd. Find the area of a sector with a central angle of 120°.

7. For an acute angle θ, if $\sin\theta = \frac{5}{6}$, evaluate $\cos\theta$ and $\tan\theta$.

8. Evaluate $\tan\frac{\pi}{6} - \cot\frac{\pi}{6}$ without the use of a calculator.

9. Given $\sin\theta = \frac{5}{13}$, use a Pythagorean identity to find $\cos\theta$.

10. Given $\sec 75° = \sqrt{2} + \sqrt{6}$, find a cofunction of another angle with the same function value.

11. Use a calculator to approximate $\sin\frac{2\pi}{11}$ to 4 decimal places.

12. A flagpole casts a shadow of 20 ft when the angle of elevation of the Sun is 40°. How tall is the flagpole? Round to the nearest foot.

13. A newly planted tree is anchored by a covered wire running from the top of the tree to a post in the ground 5 ft from the base of the tree. If the angle between the wire and the top of the tree is 20°, what is the length of the wire? Round to the nearest foot.

For Exercises 14–21, evaluate the function or state that the function is undefined at the given value.

14. $\sin\left(-\frac{3\pi}{4}\right)$

15. $\tan 930°$

16. $\sec\frac{11\pi}{6}$

17. $\csc(-150°)$

18. $\cot 20\pi$

19. $\cos 690°$

20. $\tan\frac{7\pi}{3}$

21. $\sec(-630°)$

22. Given $\sin\theta = \frac{5}{8}$ and $\cos\theta < 0$, find $\tan\theta$.

23. Given $\sec\theta = \frac{4}{3}$ and $\sin\theta < 0$, find $\csc\theta$.

24. Given $\tan\theta = -\frac{4}{3}$ and θ is in Quadrant IV, find $\sec\theta$.

For Exercises 25–28, select the trigonometric function, $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = \tan t$ or $r(t) = \cot t$, with the given properties.

25. The function is odd, with period 2π, and domain of all real numbers.

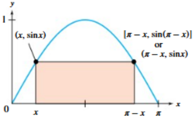
26. The function is odd, with period π, and domain of all real numbers excluding odd multiples of $\frac{\pi}{2}$.

27. The function is even, with period 2π, and domain of all real numbers.

28. The function is odd, with period π, and domain of all real numbers excluding multiples of π.

29. Use the even-odd and periodic properties of the trigonometric functions to simplify $\cos(-\theta - 2\pi) - \sin(\theta - 2\pi) \cdot \cot(\theta + \pi)$

30. Suppose that a rectangle is bounded by the x-axis and the graph of $y = \sin x$ on the interval $[0, \pi]$.



   a. Write a function that represents the area $A(x)$ of the rectangle for $0 < x < \frac{\pi}{2}$.

   b. Determine the area of the rectangle for $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

For Exercises 31–32, determine the amplitude, period, phase shift, and vertical shift for each function.

31. $y = \frac{3}{4}\sin\left(2\pi x - \frac{\pi}{6}\right)$

32. $y = -5\cos(5x + \pi) + 7$

For Exercises 33–36, graph one period of the function.

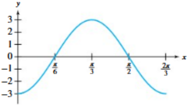
33. $y = -2\sin x$

34. $y = 3\cos 2x$

35. $y = \sin\left(2x - \frac{\pi}{4}\right)$

36. $y = 3 - 2\cos(2\pi x - \pi)$

37. Write a function of the form $f(x) = A\cos(Bx)$ for the given graph.



38. Identify each statement as true or false. If a statement is false, explain why.

a. The relative maxima of the graph of $y = \sin x$ correspond to the relative minima of the graph of $y = \csc x$.

b. The period of $y = \tan 2x$ is π.

c. The amplitude of $y = 2\cot x$ is 2.

d. The period of $y = \sec 2x$ is π.

e. The vertical asymptotes of the graph of $y = \cot x$ occur where the graph of $y = \cos x$ has x-intercepts.

For Exercises 39–40, graph one period of the function.

39. $y = -4\sec 2x$

40. $y = \csc\left(2x - \frac{\pi}{4}\right)$

For Exercises 41–42, graph two periods of the function.

41. $y = \tan 3x$

42. $y = -4\cot\left(x + \frac{\pi}{4}\right)$

43. Simplify each expression.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\frac{\sqrt{2}}{2}$

b. $\cos[\arctan(-\sqrt{3})]$

44. Use a calculator to approximate the degree measure (to 1 decimal place) of the angle θ subject to the given conditions.

$$\sin\theta = \frac{3}{8} \quad \text{and} \quad 90° \le \theta \le 180°$$

45. Find the exact value of $\sin^{-1}\left(\sin\frac{11\pi}{6}\right)$.

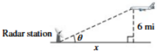
46. Find the exact value of $\tan\left(\cos^{-1}\frac{7}{8}\right)$.

47. Find the exact value of $\cos\left[\arcsin\left(-\frac{5}{6}\right)\right]$.

48. Write the expression $\tan\left(\cos^{-1}\frac{x}{\sqrt{x^2+25}}\right)$ for $x \ne 0$ as an algebraic expression.

49. A radar station tracks a plane flying at a constant altitude of 6 mi on a path directly over the station. Let θ be the angle of elevation from the radar station to the plane.

a. Write θ as a function of the plane's ground distance $x > 0$ from the station.



b. Without the use of a calculator, will the angle of elevation be less than 45° or greater than 45° when the plane's ground distance is 3.2 mi away?

c. Use a calculator to find θ to the nearest degree for $x = 3.2$, 1.6, and 0.5 mi.

Go online for more practice problems.

Chapter 4 Cumulative Review Exercises

For Exercises 1–9, solve the equations and inequalities. Write the solution set to inequalities in interval notation.

1. $|3x - 5| + 3 \geq 6$
2. $(x - 2)^2(x + 5) < 0$
3. $\frac{x+2}{x-3} \geq 2$
4. $3x^4 - x^2 - 10 = 0$
5. $\sqrt{x+7} = 4 - \sqrt{x-1}$
6. $2x^4 + 5x^3 - 29x^2 - 17x + 15 = 0$
7. $e^{3x} = 2$
8. $\log(3x + 2) - \log x = 3$
9. $\ln(x - 1) + \ln x = \ln 2$
10. Given $x^2 + y^2 - 4x + 6y + 9 = 0$
    - **a.** Write the equation of the circle in standard form.
    - **b.** Identify the center and radius.
    - **c.** Write the domain and range in interval notation.
11. Given $y = -x^2 + 2x - 4$
    - **a.** Identify the vertex.
    - **b.** Write the domain and range in interval notation.
12. Given $f(x) = x^4 - 2x^3 - 4x^2 + 8x$
    - **a.** Find the x-intercepts of the graph of f.
    - **b.** Determine the end behavior of the graph of f.
13. **a.** Graph $y = \frac{3x^2 + x - 5}{3x - 2}$.
    - **b.** Identify the asymptotes.
14. Given $y = -2\sin(4x - \pi) - 6$, identify the
    - **a.** Domain and range in interval notation.
    - **b.** Amplitude.
    - **c.** Period.
    - **d.** Phase shift.
    - **e.** Vertical shift.
15. Given $f(x) = \tan x$ and $g(x) = \frac{x}{2}$, evaluate
    - **a.** $(f \circ g)\left(\frac{\pi}{2}\right)$
    - **b.** $(g \circ f)(\pi)$
16. Evaluate $\cos(-450°)$.
17. Evaluate $\tan\left[\arcsin\left(-\frac{3}{8}\right)\right]$.
18. Write the logarithm as the sum or difference of logarithms. Simplify as much as possible.
    $\log\left(\frac{x^2y}{100z}\right)$
19. Divide. Write the answer in standard form, $a + bi$.
    $\frac{2 - 8i}{3 - 5i}$
20. Suppose that y varies inversely as x and directly as z. If y is 12 when x is 8 and z is 3, find the constant of variation k.

Go online for more practice problems.

Chapter 5

Analytic Trigonometry

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Launch Activity: Make Sense of the Problem

Conversions

The high school basketball team hosted a Walk-a-Thon fundraiser. Half of the proceeds from the fundraiser supported the team's trip to playoffs. The second half of the proceeds from the fundraiser went to Good Sports charity. During the Walk-a-Thon, 125 people walked with a step counter for a total of 304,800 steps. The basketball team calculated that it took 1950 steps to walk one mile. The fundraiser collected proceeds at a rate of $0.05 per inch walked by participants.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer this question: How much money was the basketball team able to raise for the Good Sports charity?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

GO ONLINE to answer these questions.

Lesson 5-1
Fundamental Trigonometric Identities

What Will You Learn?

After completing this lesson, you should be able to:

- Simplify Trigonometric Expressions
- Verify Trigonometric Identities
- Write an Algebraic Expression as a Trigonometric Expression

Learn: Simplify Trigonometric Expressions

In Chapter 5 we presented the six trigonometric functions by using two different approaches. We used the lengths of the sides of right triangles, and we used coordinates taken from points on the unit circle. The resulting definitions yield consistent results and give rise to thousands of relationships among the functions. In this lesson, we use algebra skills to simplify trigonometric expressions and verify identities.

You have already been introduced to the fundamental trigonometric identities given in in the following box.

Fundamental Identities

Reciprocal Identities

$\sin x = \frac{1}{\csc x}$

$\cos x = \frac{1}{\sec x}$

$\tan x = \frac{1}{\cot x}$

$\csc x = \frac{1}{\sin x}$

$\sec x = \frac{1}{\cos x}$

$\cot x = \frac{1}{\tan x}$

Quotient Identities

$\tan x = \frac{\sin x}{\cos x}$

$\cot x = \frac{\cos x}{\sin x}$

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Even and Odd Identities

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$

$\sec(-x) = \sec x$

$\cot(-x) = -\cot x$

Good Practices

It is often helpful to recognize alternative forms of the Pythagorean identities. For example, $\sin^2 x = 1 - \cos^2 x$ and $\cos^2 x = 1 - \sin^2 x$.

Each statement in the above box is true for every value of x for which the expressions within the statement are defined. For example,

$$\sin x = \frac{1}{\csc x} \text{ is true provided that } x \neq n\pi \text{ for integers } n.$$

The fundamental identities are tools to help us simplify trigonometric expressions and to verify other identities.

Example 1: Simplifying an Expression Using the Quotient and Reciprocal Identities

Simplify the expression. Write the final form with no fractions.

$$\sec^2 x \cot x \cos x$$

Solution:

$\sec^2 x \cot x \cos x$	
$= \frac{1}{\cos^2 x} \cdot \frac{\cos x}{\sin x} \cdot \frac{\cos x}{1}$	Write the trigonometric functions in terms of sine and cosine.
$= \frac{1}{\cancel{\cos^2 x}} \cdot \frac{\cancel{\cos x}}{\sin x} \cdot \frac{\cancel{\cos x}}{1}$	Simplify common factors from the numerator and denominator.
$= \frac{1}{\sin x}$	Simplify.
$= \csc x$	Apply the reciprocal identity relating the sine and cosecant functions.

Apply the Skills

1. Simplify. Write the final form with no fractions.

$$\tan x \cos^2 x \sec x$$

In Example 2, we simplify an expression by writing each term with a common denominator and adding the terms.

Insights

Recall that two fractions with different denominators b and d can be added as $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ provided that $b \neq 0$ and $d \neq 0$.

Example 2: Simplifying by Adding Fractional Expressions

Simplify the expression. Write the final form with no fractions. $\frac{\cos\theta}{1 + \sin\theta} + \tan\theta$

Solution:

$\frac{\cos\theta}{1 + \sin\theta} + \tan\theta$	
$= \frac{\cos\theta}{1 + \sin\theta} + \frac{\sin\theta}{\cos\theta}$	Write the trigonometric functions in terms of sine and cosine.
$= \frac{\cos\theta}{\cos\theta} \cdot \frac{\cos\theta}{(1 + \sin\theta)} + \frac{\sin\theta}{\cos\theta} \cdot \frac{(1 + \sin\theta)}{(1 + \sin\theta)}$	The least common denominator is $\cos\theta(1 + \sin\theta)$.
$= \frac{\cos^2\theta + \sin\theta + \sin^2\theta}{\cos\theta(1 + \sin\theta)}$	Add the fractions.
$= \frac{\sin^2\theta + \cos^2\theta + \sin\theta}{\cos\theta(1 + \sin\theta)}$	Group the squared terms because we recognize the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$.

$= \frac{1+\sin\theta}{\cos\theta(1+\sin\theta)}$	Substitute 1 for $\sin^2\theta + \cos^2\theta$.
$= \frac{\cancel{1+\sin\theta}}{\cos\theta\cancel{(1+\sin\theta)}}$	Simplify common factors from the numerator and denominator.
$= \frac{1}{\cos\theta}$	
$= \sec\theta$	Apply the reciprocal identity relating the cosine and secant functions.

Apply the Skills

2. Simplify. Write the final form with no fractions.

$$\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{1+\cos\theta}$$

In Example 3, we make use of factoring techniques to simplify a trigonometric expression.

Example 3: Simplifying a Trigonometric Expression by Factoring

Simplify. Write the final form with no fractions. $\frac{\tan^2 t - 1}{\tan t \sin t + \sin t}$

Solution:

$\frac{\tan^2 t - 1}{\tan t\,\sin t + \sin t}$	
$= \frac{(\tan t - 1)(\tan t + 1)}{\sin t(\tan t + 1)}$	The numerator factors as a difference of squares. The terms in the denominator share a common factor of $\sin t$.
$= \frac{(\tan t - 1)\cancel{(\tan t + 1)}}{\sin t\cancel{(\tan t + 1)}}$	Simplify common factors from the numerator and denominator.
$= \frac{\tan t - 1}{\sin t}$	Simplify.
$= \frac{\frac{\sin t}{\cos t} - 1}{\sin t}$	Write $\tan t$ as $\frac{\sin t}{\cos t}$.
$= \left(\frac{\sin t}{\cos t} - 1\right) \cdot \frac{1}{\sin t}$	Multiply the numerator of the complex fraction by the reciprocal of the denominator.
$= \frac{1}{\cos t} - \frac{1}{\sin t}$	Apply the distributive property.
$= \sec t - \csc t$	Apply the reciprocal identities.

Apply the Skills

3. Simplify. Write the final form with no fractions. $\frac{1-\sec^2 t}{\tan t - \tan t \sec t}$

Learn: Verify Trigonometric Identities

In Example 1, we showed that:

$$\sec^2 x \cot x \cos x = \csc x \tag{1}$$

Example 1 also illustrates that trigonometric expressions can be written in many different equivalent forms. Equation (1) is called an **identity** because it is true for all values of x for which the expressions on the left and right are defined.

Insights

Recall that a **conditional equation** is an equation that is true only for *some* values of the variable. For example, $\tan x = 1$ is true only for $x = \frac{\pi}{4} + n\pi$.

One method to verify that an equation is an identity is to manipulate one side of the equation (usually the more complicated side) until it takes the form of the other side of the equation. This process is the same as simplifying an expression, except that we know the final form.

While there is no explicit procedure to verify a trigonometric identity, we recommend some helpful guidelines.

Guidelines for Verifying a Trigonometric Identity

1. Work with one side of the equation (usually the more complicated side) and keep the other side in mind as your final goal.
2. Look for opportunities to apply the fundamental identities.
    - If the expression is a product or quotient of factors, consider the reciprocal and quotient identities.
    - If squared terms are present, look to see if the terms can be grouped in one of the forms of a Pythagorean identity.
    - If an expression involves a negative argument, consider using the even or odd function identities.
3. Apply basic algebraic techniques such as factoring, multiplying terms, combining like terms, and writing fractions with a common denominator.
4. Consider writing expressions explicitly in terms of sine and cosine.

Example 4: Verifying an Identity Containing Negative Arguments

Verify that the equation is an identity. $\frac{\cos(-x)\tan(-x)}{\sin x} = -1$

Solution:

$\frac{\cos(-x)\tan(-x)}{\sin x} = -1$

The left-hand side **(LHS)** is the more complicated side, so we will work with that side.

$$= \frac{\cos x(-\tan x)}{\sin x}$$

To simplify, we require the same arguments for each factor. The cosine function is an even function. So, $\cos(-x) = \cos x$. The tangent function is an odd function. So, $\tan(-x) = -\tan x$.

$$= \frac{\cos x\left(-\frac{\sin x}{\cos x}\right)}{\sin x}$$

Write tangent as the ratio of sine and cosine.

$$= \frac{-\sin x}{\sin x}$$

$$= -1 \quad \text{(RHS)}$$

This equals the right-hand side (**RHS**) of the original equation.

Therefore, we have verified the identity.

Apply the Skills

4. Verify that the equation is an identity. $\frac{\sin(-x)\cot(-x)}{\cos x} = 1$

When we verify an identity, we are trying to show that the expression on one side equals the expression on the other. Therefore, we cannot use the equality of the statement because we have not yet proved it. For this reason, we work with the left or right side of the equation, but not both. In Example 4, we worked only with the left side of the equation.

Example 5: Verifying an Identity by Combining Fractions

Verify that the equation is an identity. $\frac{1}{1-\cos x} - \frac{1}{1+\cos x} = 2\cot x \csc x$

Solution:

$$\frac{1}{1-\cos x} - \frac{1}{1+\cos x}$$

The LHS has two terms and the RHS has one term. Therefore, a good strategy is to combine the two terms on the left by adding the fractions.

$$= \frac{(1+\cos x)}{(1+\cos x)} \cdot \frac{1}{(1-\cos x)} - \frac{1}{(1+\cos x)} \cdot \frac{(1-\cos x)}{(1-\cos x)}$$

The LCD is $(1+\cos x)(1-\cos x)$.

$$= \frac{(1+\cos x)-(1-\cos x)}{(1+\cos x)(1-\cos x)}$$

Subtract the fractions.

$$= \frac{1+\cos x - 1 + \cos x}{1-\cos^2 x}$$

Apply the distributive property in the numerator. Multiply conjugates in the denominator.

$$= \frac{2\cos x}{\sin^2 x}$$

In the denominator, from the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, we know that $\sin^2 x = 1 - \cos^2 x$.

$$= 2 \cdot \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

The RHS of the original equation has factors of $\cot x$ and $\csc x$. Therefore, we group factors in the numerator and denominator conveniently to reach the desired result.

$$= 2\cot x \csc x \quad \text{(RHS)}$$

Apply the Skills

5. Verify that the equation is an identity. $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = 2\tan x\sec x$

Keep in mind that the Pythagorean identities have several alternative forms.

$\sin^2 x + \cos^2 x = 1$	$\tan^2 x + 1 = \sec^2 x$	$1 + \cot^2 x = \csc^2 x$
$1 - \cos^2 x = \sin^2 x$	$\sec^2 x - 1 = \tan^2 x$	$\csc^2 x - 1 = \cot^2 x$
$1 - \sin^2 x = \cos^2 x$	$\sec^2 x - \tan^2 x = 1$	$\csc^2 x - \cot^2 x = 1$

These alternative forms each involve a difference of squares that factors as a product of conjugates. This was illustrated in steps 2–5 of Example 5.

$$(1 + \cos x)(1 - \cos x) = 1 - \cos^2 x = \sin^2 x$$

Notice that upon simplification, the expression ultimately results in a single term. This pattern is sometimes helpful when we want to simplify a fraction with a denominator containing two terms as a fraction with a single-term denominator.

Example 6: Verifying an Identity by Using Conjugates

Verify that the equation is an identity. $\frac{1-\sin t}{1+\sin t} = (\sec t - \tan t)^2$

Solution:

$\frac{1-\sin t}{1+\sin t} = (\sec t - \tan t)^2$	Since the RHS has no fractions, one strategy might be to multiply the numerator and denominator of the LHS by the conjugate of the denominator. The purpose is to create the difference of squares $1 - \sin^2 t$ which then simplifies to the single term $\cos^2 t$.
$\frac{1-\sin t}{1+\sin t} = \frac{(1-\sin t)}{(1+\sin t)} \cdot \frac{(1-\sin t)}{(1+\sin t)}$	
$= \frac{(1-\sin t)^2}{1-\sin^2 t}$	The product of conjugates in the denominator results in a difference of squares.
$= \frac{(1-\sin t)^2}{\cos^2 t}$	Substitute $\cos^2 t$ for $1 - \sin^2 t$.
$= \left(\frac{1-\sin t}{\cos t}\right)^2$	Write the quotient of squares as the square of a quotient.
$= \left(\frac{1}{\cos t} - \frac{\sin t}{\cos t}\right)^2$	Use the property $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$.
$= (\sec t - \tan t)^2$ (RHS)	Apply reciprocal and quotient identities.

Apply the Skills

6. Verify that the equation is an identity. $1 - \frac{\sin^2 t}{1+\cos t} = \cos t$

It is important to understand that there are often many paths to simplifying a trigonometric identity. A different set of algebraic steps can be used, or we can manipulate the other side of the expression. For example, the expression in Example 6 can also be simplified by manipulating the RHS.

$(\sec t - \tan t)^2 = \left(\frac{1}{\cos t} - \frac{\sin t}{\cos t}\right)^2 = \left(\frac{1 - \sin t}{\cos t}\right)^2 = \frac{(1 - \sin t)^2}{\cos^2 t}$	Write the expression in terms of sin *t* and cos *t*. Combine terms on the left by subtracting fractions.
$= \frac{(1 - \sin t)^2}{1 - \sin^2 t} = \frac{(1 - \sin t)^2}{(1 - \sin t)(1 + \sin t)} = \frac{1 - \sin t}{1 + \sin t} = \text{LHS}$	Factor the denominator and simplify common factors in the numerator and denominator.

Example 7: Verifying an Identity Involving Logarithms

Verify that the equation is an identity. $\ln\lvert\sin x\rvert + \ln\lvert\sec x\rvert = \ln\lvert\tan x\rvert$

Solution:

$\ln\lvert\sin x\rvert + \ln\lvert\sec x\rvert$	The LHS is the more complicated side.
$= \ln\lvert\sin x\rvert + \ln\left\lvert\frac{1}{\cos x}\right\rvert$	Write sec *x* as $\frac{1}{\cos x}$.
$= \ln\left(\lvert\sin x\rvert \cdot \left\lvert\frac{1}{\cos x}\right\rvert\right)$	Apply the property $\ln x + \ln y = \ln(xy)$.

Good Practices

It is often helpful to use the rules and properties of logarithms. Recall that

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^p = p$
4. $b^{\log_b x} = x$
5. $\log_b(xy) = \log_b x + \log_b y$
6. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
7. $\log_b x^p = p\log_b x$

$= \ln\left\lvert\frac{\sin x}{\cos x}\right\rvert$	Apply the property of absolute value $\lvert a\rvert \cdot \lvert b\rvert = \lvert ab\rvert$.
$= \ln\lvert\tan x\rvert$ (RHS)	Write $\frac{\sin x}{\cos x}$ as tan *x*.

Apply the Skills

7. Verify that the equation is an identity. $\ln\lvert\cot x\rvert + \ln\lvert\sin x\rvert = \ln\lvert\cos x\rvert$

When verifying an identity, we generally work on one side of the equation only and manipulate the expression until it takes the form of the other side. It is logically equivalent, however, to verify an identity by transforming each side of the equation *independently* until we reach a common equivalent expression. This is demonstrated in Example 8.

Example 8: Verifying an Identity by Working Each Side Independently

Verify that the equation is an identity by manipulating each side of the equation independently.

$$\frac{\sin x}{1+\sin x} = \frac{\csc x - 1}{\cot^2 x}$$

Solution:

LHS: $\frac{\sin x}{1+\sin x}$

$$= \frac{(\sin x)}{(1+\sin x)} \cdot \frac{(1-\sin x)}{(1-\sin x)}$$

$$= \frac{\sin x(1-\sin x)}{(1-\sin^2 x)}$$

$$= \frac{\sin x - \sin^2 x}{\cos^2 x}$$

RHS: $\frac{\csc x - 1}{\cot^2 x}$

$$= \frac{\frac{1}{\sin x} - 1}{\frac{\cos^2 x}{\sin^2 x}} = \frac{\frac{\sin^2 x}{1} \cdot \left(\frac{1}{\sin x} - 1\right)}{\frac{\sin^2 x}{1} \cdot \left(\frac{\cos^2 x}{\sin^2 x}\right)}$$

$$= \frac{\sin x - \sin^2 x}{\cos^2 x}$$

Since the left-hand side equals the right-hand side, we have verified that the equation is an identity.

Insights

On the left side of the identity, we multiplied numerator and denominator by the conjugate of the denominator. Recall that the product of conjugates results in a difference of squares. The motivation is to use the Pythagorean relationship

$$1 - \sin^2 x = \cos^2 x.$$

Apply the Skills

8. Verify the equation is an identity by manipulating each side of the equation independently. $\frac{1-\cos x}{\cos x} = \frac{\tan x \sin x}{1+\cos x}$

The process used in Example 8 was to manipulate each side of the equation independently into a third, equivalent expression. In a rigorous setting, this method is frowned upon. In Example 8, we could have manipulated only one side (say the LHS) by first reaching the intermediate step $\frac{\sin x - \sin^2 x}{\cos^2 x}$. The remaining steps to verify the identity would be to reverse the steps used to manipulate the RHS.

Learn: Write an Algebraic Expression as a Trigonometric Expression

In some applications of calculus, it is advantageous to write an algebraic expression of the form $\sqrt{u^2+a^2}$, $\sqrt{a^2-u^2}$, or $\sqrt{u^2-a^2}$ as a trigonometric expression by way of an appropriate substitution. Notice that each algebraic expression looks similar to an expression that might be used to find the length of a side of a right triangle using the Pythagorean theorem. Therefore, it seems reasonable that these algebraic expressions may be linked to trigonometric expressions.

Example 9: Applying Trigonometric Substitution

Write $\sqrt{x^2+9}$ as a function of θ, where $0<\theta<\frac{\pi}{2}$, by making the substitution $x=3\tan\theta$.

Solution:

$\sqrt{x^2+9}=\sqrt{(3\tan\theta)^2+9}$ Substitute $3\tan\theta$ for x.

$=\sqrt{9\tan^2\theta+9}$ Square the expression $3\tan\theta$.

$=\sqrt{9(1+\tan^2\theta)}$ Factor.

$=\sqrt{9\sec^2\theta}$ Apply the Pythagorean identity $1+\tan^2\theta=\sec^2\theta$.

$=3\sec\theta$ Take the positive square root, since $0<\theta<\frac{\pi}{2}$.

Apply the Skills

9. Write $\sqrt{x^2-121}$ as a function of θ, where $0<\theta<\frac{\pi}{2}$, by making the substitution $x=11\sec\theta$.

From Example 9, we can illustrate the link between the algebraic form $\sqrt{x^2+9}$ and the trigonometric form $3\sec\theta$ by using a right triangle. The expression $\sqrt{x^2+9}$ represents the hypotenuse of a right triangle with legs of lengths x and 3 (Figure 5-1).

From the triangle, $\sec\theta=\frac{\sqrt{x^2+9}}{3}$, so $\sqrt{x^2+9}=3\sec\theta$.

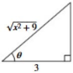


Figure 5-1

Practice 5-1

Practice Exercises

Prerequisite Review

R.1. Factor. $5r^2 + 33rs + 18s^2$

R.2. Find the least common denominator.

$\frac{3}{(c-3)(c+3)}; \frac{-6}{(c-3)(c-1)}$

R.3. Subtract. $\frac{3a}{a+9} - \frac{2}{a-5}$

R.4. Add. $\frac{y-4}{y-5} + \frac{5y^2 - 38y + 55}{y^2 - 25}$

R.5. Simplify. $\frac{\frac{8x^2}{19y^2}}{\frac{3x}{xy^2}}$

R.6. Simplify. $\frac{\frac{8}{n} + \frac{3}{n^2}}{\frac{6}{n^2} - \frac{7}{n}}$

R.7. Assume that y represents a positive real number. What is the conjugate of $\sqrt{y} - 24$?

R.8. Rationalize the denominator. $\frac{5}{6 - \sqrt{3}}$

Concept Connections

1. The value $\tan\theta$ is the quotient of ________ and ________, and $\cot\theta$ is the quotient of ________ and ________.

2. The value $\csc\theta$ is the reciprocal of ________, $\cot\theta$ is the reciprocal of ________, and $\sec\theta$ is the reciprocal of ________.

3. Given the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, write two alternative forms that involve the difference of squares.

4. Given the Pythagorean identity $\tan^2 x + 1 = \sec^2 x$, write two alternative forms that involve the difference of squares.

5. $\sin(-x) =$ ________, $\cos(-x) =$ ________, and $\tan(-x) =$ ________.

6. An equation that is true for all values of the variable except where the individual expressions are not defined is called an ________.

Learn: Simplify Trigonometric Expressions

For Exercises 7–10, find the least common denominator. Then rewrite each expression with the new denominator.

7. a. $\frac{1}{a}; \frac{1}{b}$

b. $\frac{1}{\cos x}; \frac{1}{\sin x}$

8. a. $\frac{a}{b}; a$

b. $\frac{\sin x}{\cos x}; \sin x$

9. a. $\frac{1}{1+a}; \frac{a}{b}$

b. $\frac{1}{1+\sin x}; \frac{\sin x}{\cos x}$

10. a. $\frac{1}{a}; \frac{a}{1-a}$

b. $\frac{1}{\cos x}; \frac{\cos x}{1-\cos x}$

For Exercises 11–12, multiply.

11. a. $(a+b)^2$

b. $(\cos x + \cot x)^2$

12. a. $(a-b)^2$

b. $(\sin x - \tan x)^2$

For Exercises 13–20, factor each expression.

13. a. $a^2 - b^2$

b. $\sin^2 x - \cos^2 x$

14. a. $a^2 + b^2$

b. $\tan^2 x + \sec^2 x$

15. a. $a^3 + b^3$

b. $\sin^3 x + \sec^3 x$

16. a. $a^3 - b^3$

b. $\cos^3 x - \cot^3 x$

17. a. $a^2 + 2ab + b^2$

b. $\cos^2 x + 2\cos x \csc x + \csc^2 x$

18. a. $a^2 - 2ab + b^2$

b. $\sin^2 x - 2\sin x \tan x + \tan^2 x$

19. a. $3a^2 - 4ab - 4b^2$

b. $3\sin^2 x - 4\sin x \cos x - 4\cos^2 x$

20. a. $4a^2 - 12ab + 5b^2$

b. $4\cos^2 x - 12\cos x \sin x + 5\sin^2 x$

For Exercises 21–36, simplify the expression. Write the final form with no fractions. (See Examples 1–3, pp. 599–600)

21. $\sin x \sec x \cot x$

22. $\cos x \tan x \csc x$

23. $\dfrac{\sin x}{\sec x \tan x}$

24. $\dfrac{\sin x}{\sec x \cot x}$

25. $\tan^2 x \cot x \csc x$

26. $\csc^2 x \tan^2 x \sec x$

27. $\dfrac{\tan^2 x}{\sin x \sec^2 x}$

28. $\dfrac{\tan x}{\sin x \sec^2 x}$

29. $\dfrac{\cos^2 x + 1}{\sin x} + \sin x$

30. $\dfrac{\sin^2 x + 1}{\cos^2 x} + 1$

31. $\dfrac{\csc\theta - \sin\theta}{\csc\theta}$

32. $\dfrac{\cos\theta - \sec\theta}{\sec\theta}$

33. $\dfrac{\sin^2 x + 2\sin x + 1}{\sin^2 x + \sin x}$

34. $\dfrac{\sin x \sec^2 x - \sin x}{\sin x \sec x + \sin x}$

35. $\dfrac{\cos^2 x - 2\cos x + 1}{\sin x \cos x - \sin x}$

36. $\dfrac{\tan^2 x + 2\tan x + 1}{\sin x \tan x - \sin x}$

Learn: Verify Trigonometric Identities

For Exercises 37–40, verify that the equation is an identity. (See Example 4, p. 601)

37. $\sin(-x) + \csc x = \cot x \cos x$

38. $\sin(-x) + \cot(-x)\cos(-x) = -\csc x$

39. $\cot x[\cot(-x) + \tan(-x)] = -\csc^2 x$

40. $\dfrac{\csc^2(-x)\tan x}{\cot(-x)} = -\sec^2 x$

For Exercises 41–46, verify that the equation is an identity. (See Example 5, p. 602)

41. $\dfrac{\sec x}{\tan x} - \dfrac{\tan x}{\sec x} = \cos x \cot x$

42. $\dfrac{\cot x}{\csc x} - \dfrac{\csc x}{\cot x} = -\sin x \tan x$

43. $\dfrac{1}{1 + \sin t} + \dfrac{1}{1 - \sin t} = 2\sec^2 t$

44. $\dfrac{1}{\sec t - 1} - \dfrac{1}{\sec t + 1} = 2\cot^2 t$

45. $\dfrac{1}{1 + \sec x} - \dfrac{1}{1 - \sec x} = 2\cot x \csc x$

46. $\dfrac{1}{1 - \csc x} - \dfrac{1}{1 + \csc x} = -2\tan x \sec x$

For Exercises 47–52, verify that the equation is an identity. (See Example 6, p. 603)

47. $\dfrac{\sin\theta}{\csc\theta - \cot\theta} = 1 + \cos\theta$

48. $\dfrac{\cos\theta}{\tan\theta + \sec\theta} = 1 - \sin\theta$

49. $\dfrac{1}{\cos x + \sin x \cos x} = \dfrac{1 - \sin x}{\cos^3 x}$

50. $\dfrac{1}{\sin x - \sin^2 x} = \dfrac{1 + \sin x}{\sin x \cos^2 x}$

51. $\dfrac{\sec x + 1}{\sec x - 1} = (\csc x + \cot x)^2$

52. $\dfrac{\sin x + \cos x}{\sin x - \cos x} = \dfrac{1 + 2\sin x \cos x}{1 - 2\cos^2 x}$

For Exercises 53–56, verify that the equation is an identity. (See Example 7, p. 604)

53. $\ln|\cos t| - \ln|\cot t| = \ln|\sin t|$

54. $\ln|\cot t| - \ln|\tan t| = 2\ln|\cot t|$

55. $\ln|\sec\theta + \tan\theta| = -\ln|\sec\theta - \tan\theta|$

56. $\ln|\csc\theta + \cot\theta| = -\ln|\csc\theta - \cot\theta|$

For Exercises 57–60, simplify each side of the equation independently to reach a common equivalent expression. (See Example 8, p. 605)

57. $\dfrac{\cos x}{1 - \sin x} = \sec x + \tan x$

58. $\dfrac{1 + \tan^2 x}{\tan x} = \dfrac{\cos x}{\sin x - \sin^3 x}$

59. $\cot^2 t - \cos^2 t = \csc^2 t \cos^2 t - \cos^2 t$

60. $\tan^2 t + \sin^2 t = \sec^2 t - \cos^2 t$

Learn: Write an Algebraic Expression as a Trigonometric Expression

For Exercises 61–68, write the given algebraic expression as a function of θ, where $0 < \theta < \frac{\pi}{2}$, by making the given substitution. (See Example 9, p. 606)

61. $\sqrt{x^2 + 25}$; Substitute $x = 5\tan\theta$.

62. $\sqrt{49 - x^2}$; Substitute $x = 7\sin\theta$.

63. $\sqrt{16 - (x - 1)^2}$; Substitute $x = 4\sin\theta + 1$.

64. $\sqrt{(x+2)^2-36}$; Substitute $x = 6\sec\theta - 2$.

65. $\frac{\sqrt{9-x^2}}{x}$; Substitute $x = 3\sin\theta$.

66. $\frac{\sqrt{x^2+16}}{x}$; Substitute $x = 4\tan\theta$.

67. $\frac{1}{(49x^2-1)^{3/2}}$; Substitute $x = \frac{1}{7}\sec\theta$.

68. $\frac{1}{(1-100x^2)^{3/2}}$; Substitute $x = \frac{1}{10}\sin\theta$.

Mixed Exercises

For Exercises 69–72, factor the expression and simplify factors using fundamental identities if possible.

69. $2\sin^2 x + 5\sin x - 3$

70. $2\cos^2 x - \cos x - 1$

71. $\sin^4 x - \cos^4 x$

72. $1 - \cot^4 x$

For Exercises 73–104, verify that the equation is an identity.

73. $(\cos^2\theta - 1)(\csc^2\theta - 1) = -\cos^2\theta$

74. $(1 - \sin^2\theta)(1 + \tan^2\theta) = 1$

75. $\sec x \cdot \frac{\tan x}{\sin x} = \sec^2 x$

76. $\sec x \cdot \frac{\cot x}{\sin x} = \csc^2 x$

77. $\frac{\sec x \sin x}{\csc x \cos x} = \tan^2 x$

78. $\frac{\sin x \cot x}{\cos x \tan x} = \cot x$

79. $\frac{2 + \sin^2 t - 3\sin^4 t}{\cos^2 t} = 2 + 3\sin^2 t$

80. $\frac{2\tan^4 x + 7\tan^2 x + 5}{\sec^2 x} = 2\tan^2 x + 5$

81. $\frac{\tan x \sin x}{\tan x - \sin x} = \frac{\sin x}{1 - \cos x}$

82. $\frac{\csc x \cos x}{\csc x - \sin x} = \sec x$

83. $(\sec\theta + \tan\theta)^2 = \frac{1 + \sin\theta}{1 - \sin\theta}$

84. $(\csc t + \cot t)^2 = \frac{1 + \cos t}{1 - \cos t}$

85. $(1 + \sin x)[1 + \sin(-x)] = \cos^2 x$

86. $[1 + \csc(-x)](1 + \csc x) = -\cot^2 x$

87. $\frac{\sin x}{\cos x + 1} = \frac{1 - \cos x}{\sin x}$

88. $\frac{\tan x}{\sec x - 1} = \frac{\sec x + 1}{\tan x}$

89. $\frac{1 + \sin\theta}{\tan\theta} = \frac{\csc\theta + 1}{\sec\theta}$

90. $\frac{\sin\theta + 1}{\csc\theta} = \frac{1 + \cos\theta}{\cot\theta}$

91. $\frac{\sin^3 t - \cos^3 t}{\cos t \sin t - \cos^2 t} = \sec t + \sin t$

92. $\frac{\cos^3 t + \sin^3 t}{\cos t \sin t - \sin^2 t} = \csc t - \cos t$

93. $\frac{\tan^4\theta - 4}{\sec^2\theta + 1} = \sec^2\theta - 3$

94. $\frac{\sin^4\theta - 9}{\cos^2\theta + 2} = \cos^2\theta - 4$

95. $\frac{\cot^3 z - 1}{\cot z - 1} - \cot z = \csc^2 z$

96. $\frac{\tan^3 z + 1}{\tan z + 1} + \tan z = \sec^2 z$

97. $\cos x \tan x - \sec x \cot x = -\cot x \cos x$

98. $\sin x \cot x + \cos x \tan^2 x = \sec x$

99. $\log|\cos x| - \log|\sin x| + \log|\sec x| = \log|\csc x|$

100. $\ln|\cot x| + \ln|\sec x| + \ln|\sin x| = 0$

101. $e^{\ln(\sin^2 x + \cos^2 x)} = 1$

102. $\log 100^{\sin x} = 2\sin x$

103. $\frac{1}{\sin x + \cos x} + \frac{1}{\sin x - \cos x} = \frac{2\sin x}{1 - 2\cos^2 x}$

104. $\frac{1}{\sin x + \cos x} - \frac{\sin x}{(\sin x + \cos x)^2} = \frac{\cos x}{1 + 2\sin x \cos x}$

105. Given $f(x) = \sqrt{1 - x^2}$ and $g(x) = \cos x$, show that $(f \circ g)(x) = |\sin x|$.

106. Given $f(x) = \frac{1}{\sqrt{x^2 + 1}}$ and $g(x) = \tan x$, show that $(f \circ g)(x) = |\cos x|$.

Write About It

107. Determine if the following logic is correct, and explain why or why not.

$\sin x \sec x = 1$ because

$$\sin\tfrac{\pi}{4}\sec\tfrac{\pi}{4} = \frac{\sqrt{2}}{2} \cdot \sqrt{2} = \frac{2}{2} = 1$$

108. Determine if the following procedure to verify an identity is correct, and explain why or why not.

$$\frac{\csc\theta - 1}{\cot\theta} = \frac{\cot\theta}{\csc\theta + 1}$$

Multiply both sides by the LCD: $\cot\theta(\csc\theta + 1)$

$$\cot\theta(\csc\theta + 1)\left(\frac{\csc\theta - 1}{\cot\theta}\right) = \left(\frac{\cot\theta}{\csc\theta + 1}\right)\cot\theta(\csc\theta + 1)$$

$$(\csc\theta + 1)(\csc\theta - 1) = \cot^2\theta$$

$$\csc^2\theta - 1 = \cot^2\theta \quad ✓$$

109. Explain how to verify a trigonometric identity.

Expanding Your Skills

For Exercises 110–115, simplify each expression to an integer.

110. $(1 + \cot^2 x)(\sin^2 x + 1) - \cot^2 x$

111. $\ln|\tan x| + \ln|\csc x| - \ln|\sec x|$

112. $\tan x - \dfrac{\cos x(1 - \tan^2 x)}{\cos x - \sin x}$

113. $2(\sin^6 x + \cos^6 x) - 3(\cos^4 x + \sin^4 x)$

114. $\dfrac{4\sec^2 x(1 + \cos x \tan x)}{(\tan x + \sec x)^2 + 1}$

115. $(\sin 2x \cos 2z + \cos 2x \sin 2z)^2 +$
$(\cos 2x \cos 2z - \sin 2x \sin 2z)^2$

For Exercises 116–122, verify that the equation is an identity.

116. $\sqrt{\dfrac{1 + \cos x}{1 - \cos x}} = \csc x + \cot x$, for $0 < x < \frac{\pi}{2}$.

117. $\sqrt{\dfrac{\csc x - \cot x}{\csc x + \cot x}} = \dfrac{1 - \cos x}{\sin x}$, for $0 < x < \frac{\pi}{2}$.

118. $(\sin x + \cos y)^2 + (\sin x - \cos y)(\sin x + \cos y) =$
$2\sin x(\sin x + \cos y)$

119. $(\tan x + \tan y)(\tan x - \tan y) +$
$2\tan y(\tan y - \tan x) = (\tan x - \tan y)^2$

120. $(A\cot\theta - B)(A\cot\theta + B) -$
$(B\cot\theta - A)(B\cot\theta + A) = (A^2 - B^2)\csc^2\theta$

121. $\dfrac{\sec^2 x \tan x - 25\tan x}{\sec^3 x + 5\sec^2 x - \sec x - 5} = \csc x - 5\cot x$

122. $\sin^6 x - \cos^6 x =$
$(\sin^2 x - \cos^2 x)(1 - \sin^2 x \cos^2 x)$

For Exercises 123–124, replace x by the given expression and solve for y. Assume a and b are nonzero constants, and simplify your answer completely.

123. $x^2 + y^2 = a^2$; $x = a\cos\theta$

124. $b^2x^2 - a^2y^2 = a^2b^2$; $x = a\sec\theta$

Technology Connections

125. An equation of the form $f(x) = g(x)$ is given. Use a graphing utility to graph $y = f(x)$ and $y = g(x)$ on the same screen on the interval $[0, 2\pi]$ (be sure to use radian mode). Make a conjecture as to whether the equation is an identity, a conditional equation, or a contradiction.

a. $\sin 2x = 2\sin x$

b. $\cos 2x = 2\cos^2 x - 1$

c. $\dfrac{\sec x \sin x}{\tan x} = 2$

126. Graph the function in the standard viewing window to determine the constant value it might be equivalent to. Confirm your conjecture algebraically.

$$y = (2\cos x - \sin x)^2 + (\cos x + 2\sin x)^2$$

Go online for more practice problems.

Lesson 5-2

Sum and Difference Formulas

What Will You Learn?

After completing this lesson, you should be able to:

- Apply the Sum and Difference Formulas for Sine and Cosine
- Apply the Sum and Difference Formulas for Tangent
- Use Sum and Difference Formulas to Verify Identities
- Write a Sum of $A\sin x$ and $B\cos x$ as a Single Term

Learn: Apply the Sum and Difference Formulas for Sine and Cosine

In Exercise 32 from Lesson 4.3, we evaluated the expressions $\sin 60°$ and $\sin 30° + \sin 30°$ and found that they are not equal. That is,

$$\sin 60° = \frac{\sqrt{3}}{2}, \text{ whereas } \sin 30° + \sin 30° = \frac{1}{2} + \frac{1}{2} = 1.$$

So the question arises, can we express $\sin(u \pm v)$ and $\cos(u \pm v)$ in terms of trigonometric functions of u and v alone?

We begin by investigating $\cos(u - v)$. Let points P and Q be the points on the unit circle associated with the real numbers u and v, measured from the point $S(1, 0)$. For simplicity, we show the case where u and v are positive and $u - v > 0$ (Figure 5-2). Let R be the point on the unit circle associated with the real number $u - v$ (Figure 5-3).

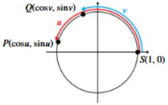


Figure 5-2

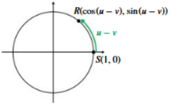


Figure 5-3

The lengths of the arcs between P and Q and between R and S are the same. Consequently, the lengths of the chords $\overline{PQ}$ and $\overline{RS}$ are also equal.

Insights A chord of a circle is a line segment with endpoints on the circle.

By the distance formula, we have:

$$\sqrt{(\cos u - \cos v)^2 + (\sin u - \sin v)^2} = \sqrt{[\cos(u - v) - 1]^2 + [\sin(u - v) - 0]^2}$$

Squaring both sides and expanding the squared expressions yields

LHS: $\cos^2 u - 2\cos u\cos v + \cos^2 v + \sin^2 u - 2\sin u\sin v + \sin^2 v$

$= \cos^2 u + \sin^2 u - 2\cos u\cos v + \cos^2 v + \sin^2 v - 2\sin u\sin v$ Group the squared terms.

$= 1 - 2\cos u\cos v + 1 - 2\sin u\sin v$ Substitute: $\sin^2 u + \cos^2 u = 1$, $\sin^2 v + \cos^2 v = 1$

$= 2 - 2\cos u\cos v - 2\sin u\sin v$

RHS: $\cos^2(u - v) - 2\cos(u - v) + 1 + \sin^2(u - v)$

$= \cos^2(u - v) + \sin^2(u - v) - 2\cos(u - v) + 1$ Group the squared terms.

$= 1 - 2\cos(u - v) + 1$ Substitute: $\cos^2(u - v) + \sin^2(u - v) = 1$

$= 2 - 2\cos(u - v)$

So,

$2 - 2\cos u\cos v - 2\sin u\sin v = 2 - 2\cos(u - v)$

$-2\cos u\cos v - 2\sin u\sin v = -2\cos(u - v)$ Subtract 2.

$\cos u\cos v + \sin u\sin v = \cos(u - v)$ Divide by -2.

Similar identities for $\cos(u + v)$, $\sin(u \pm v)$, and $\tan(u \pm v)$ are given in the box below. The derivations for these identities are addressed in Exercises 73–76.

Sum and Difference Formulas

- Sine Formulas $\sin(u + v) = \sin u\cos v + \cos u\sin v$
  $\sin(u - v) = \sin u\cos v - \cos u\sin v$
- Cosine Formulas $\cos(u + v) = \cos u\cos v - \sin u\sin v$
  $\cos(u - v) = \cos u\cos v + \sin u\sin v$
- Tangent Formulas $\tan(u + v) = \dfrac{\tan u + \tan v}{1 - \tan u\tan v}$
  $\tan(u - v) = \dfrac{\tan u - \tan v}{1 + \tan u\tan v}$

If an angle θ is the sum or difference of two angles for which the trigonometric function values are known, we can use the identities above to find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$. For example, consider the angles 15°, 105°, and $\frac{7\pi}{12}$ as a sum or difference of other "special" angles.

$15° = 45° - 30°$ $105° = 135° - 30°$ $\frac{7\pi}{12} = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$

Example 1: Applying the Addition and Subtraction Formulas

Find the exact values. **a.** $\cos 15°$ **b.** $\sin\frac{11\pi}{12}$

Solution:

a. $\cos 15° = \cos(45° - 35°)$ Write 15° as the sum or difference of angles for which the sine and cosine functions are known (that is, integer multiples of 30° or 45°).

$= \cos 45°\cos 30° + \sin 45°\sin 30°$ Apply $\cos(u - v) = \cos u\cos v + \sin u\sin v$ with $u = 45°$ and $v = 30°$.

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$ Substitute the known function values.

$= \frac{\sqrt{6} + \sqrt{2}}{4}$ Multiply radicals and write the result over the common denominator.

Insights

The combination 45° − 30° is not the only way to make a sum or difference of 15° from our "special" angles. For example, the same result is obtained from

cos(−45° + 60°) and cos(135° − 120°).

b. First note that $\frac{11\pi}{12} = \frac{3\pi}{12} + \frac{8\pi}{12} = \frac{\pi}{4} + \frac{2\pi}{3}$. Look for a combination of angles that are integer multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.

$\sin\frac{11\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)$

$= \sin\frac{\pi}{4}\cos\frac{2\pi}{3} + \cos\frac{\pi}{4}\sin\frac{2\pi}{3}$ Apply $\sin(u + v) = \sin u \cos v + \cos u \sin v$ with $u = \frac{\pi}{4}$ and $v = \frac{2\pi}{3}$.

$= \frac{\sqrt{2}}{2} \cdot \left(-\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{\sqrt{3}}{2}\right)$ Substitute the known function values.

$= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$

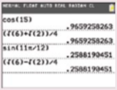
$= \frac{\sqrt{6} - \sqrt{2}}{4}$ Add the fractions.

Check Your Work

You can support your results from Example 1 by using a calculator. Confirm that the decimal approximations agree to the level of accuracy provided by the calculator.

Degree mode {

Radian mode {



```
cos(15)
                    .9659258263
(√(6)+√(2))/4
                    .9659258263
sin(11π/12)
                    .2588190451
(√(6)−√(2))/4
                    .2588190451
```

Apply the Skills

1. Find the exact values.

   a. sin 195°

   b. $\cos\frac{5\pi}{12}$

Example 2: Applying the Sum Formula for Sine

Find the exact value of the expression. $\sin 25° \cos 35° + \cos 25° \sin 35°$

Solution:

$\sin 25° \cos 35° + \cos 25° \sin 35°$ Recognize that this expression is in the form $\sin u \cos v + \cos u \sin v$ for $u = 25°$ and $v = 35°$.

$= \sin(25° + 35°)$ Apply the formula $\sin(u + v) = \sin u \cos v + \cos u \sin v$.

$= \sin 60°$

$= \frac{\sqrt{3}}{2}$

Apply the Skills

2. Find the exact value of the expression.

$\cos 99^\circ \cos 36^\circ - \sin 99^\circ \sin 36^\circ$

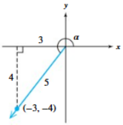
Example 3: Applying the Difference Formula for Cosine

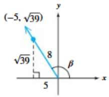
Find the exact value of $\cos(\alpha - \beta)$ given that $\sin\alpha = -\frac{4}{5}$ and $\cos\beta = -\frac{5}{8}$ for α in Quadrant III and β in Quadrant II.

Solution:

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

To find the value of $\cos(\alpha - \beta)$, we need to know the values of the factors on the right side of the identity. We can set up a representative triangle for each angle.





$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$	Apply the difference formula for cosine.
$= -\frac{3}{5} \cdot \left(-\frac{5}{8}\right) + \left(-\frac{4}{8}\right)\left(\frac{\sqrt{39}}{8}\right)$	Substitute values for the sine and cosine of α and β.
$= \frac{15 - 4\sqrt{39}}{40}$	Simplify.

Apply the Skills

3. Find the exact value of $\cos(\alpha + \beta)$ given that $\sin\alpha = \frac{5}{13}$ and $\cos\beta = \frac{5}{6}$ for α in Quadrant II and β in Quadrant IV.

Example 4: Evaluating the Sine of the Sum of Two Angles Given as Inverse Functions

Find the exact value of $\sin\left[\tan^{-1}\left(-\frac{9}{40}\right) + \cos^{-1}\frac{8}{17}\right]$.

Solution:

Let $u = \tan^{-1}\left(-\frac{9}{40}\right)$ and $v = \cos^{-1}\frac{8}{17}$.

Draw angles u and v in standard position and set up representative triangles with the hypotenuse on the terminal sides of u and v (Figures 5-4 and 5-5).

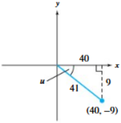


Figure 5-4

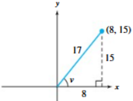


Figure 5-5

It follows that $\tan u = -\frac{9}{40}$ and $\cos v = \frac{8}{17}$.

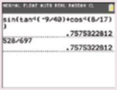
$\sin\left[\tan^{-1}\left(-\frac{9}{40}\right) + \cos^{-1}\frac{8}{17}\right]$

Apply the formula for the sine of a sum.

$$\begin{aligned} &= \sin(u + v) = \sin u \cos v + \cos u \sin v \\ &\qquad = -\frac{9}{41}\cdot\frac{8}{17} + \frac{40}{41}\cdot\frac{15}{17} \\ &\qquad = \frac{528}{697} \end{aligned}$$

Check Your Work

The result checks on a calculator.



```
NORMAL FLOAT AUTO REAL RADIAN CL
sin(tan⁻¹(⁻9/40)+cos⁻¹(8/17)
)
                    .7575322812
528/697
                    .7575322812
```

Apply the Skills

4. Find the exact value of $\cos\left[\sin^{-1}\left(-\frac{12}{37}\right) + \tan^{-1}\left(\frac{5}{12}\right)\right]$.

Insights

In Example 4, the values of $\sin u$, $\cos u$, $\sin v$, and $\cos v$ can also be found by using trigonometric identities.

Because $\tan u = -\frac{9}{40}$ for u in Quadrant IV, $\sec^2 u = 1 + \tan^2 u$ and

$$\sec u = \sqrt{1 + \left(-\frac{9}{40}\right)^2} = \frac{41}{40}$$

Therefore, $\cos u = \frac{40}{41}$

Using $\tan u = \frac{\sin u}{\cos u}$, we have $\sin u = \tan u \cos u$.

So, $\sin u = -\frac{9}{40} \cdot \frac{41}{40} = -\frac{9}{41}$

We also know that $\cos v = \frac{8}{17}$ for v in Quadrant I.

Using the Pythagorean identity $\sin^2 v + \cos^2 v = 1$, we find that $\sin v = \frac{15}{17}$.

Learn: Apply the Sum and Difference Formulas for Tangent

We can use the identities for $\sin(u \pm v)$ and $\cos(u \pm v)$ to derive similar identities for $\tan(u \pm v)$.

$$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)} = \frac{\sin u\cos v + \cos u\sin v}{\cos u\cos v - \sin u\sin v}$$

$$= \frac{\frac{1}{\cos u\cos v}\cdot(\sin u\cos v + \cos u\sin v)}{\frac{1}{\cos u\cos v}\cdot(\cos u\cos v - \sin u\sin v)}$$

To write this identity in terms of $\tan u$ and $\tan v$, multiply numerator and denominator by $\frac{1}{\cos u\cos v}$.

$$= \frac{\frac{\sin u\cancelto{1}{\cos v}}{\cos u\cancel{\cos v}} + \frac{\cancelto{1}{\cos u}\sin v}{\cancel{\cos u}\cos v}}{\frac{\cancel{\cos u\cos v}}{\underset{1}{\cancel{\cos u\cos v}}} - \frac{\sin u\sin v}{\cos u\cos v}}$$

Simplify common factors within each term of the complex fraction.

$$= \frac{\tan u + \tan v}{1 - \tan u\tan v}$$

Therefore, $\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u\tan v}$.

Applying this result to $\tan(u - v) = \tan[u + (-v)]$ and applying the odd-function identity $\tan(-x) = -\tan x$, we can derive the identity $\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u\tan v}$.

Example 5: Applying the Subtraction Formula for Tangent

Find the exact value of tan 255°.

Solution:

$\tan 255° = \tan(300° - 45°)$ Write 255° as the sum or difference of integer multiples of 30° or 45°.

$= \frac{\tan 300° - \tan 45°}{1 + \tan 300° \tan 45°}$ Apply the formula $\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u\tan v}$ with $u = 300°$ and $v = 45°$.

$= \frac{-\sqrt{3} - 1}{1 + (-\sqrt{3})(1)}$ Substitute known values for tan 300° and tan 45°.

$= -\frac{(\sqrt{3} + 1)}{(1 - \sqrt{3})} \cdot \frac{(1 + \sqrt{3})}{(1 + \sqrt{3})}$ Rationalize the denominator by multiplying numerator and denominator by the conjugate of the denominator.

$= -\frac{4 + 2\sqrt{3}}{1 - 3}$ Multiply.

$= -\frac{2(2 + \sqrt{3})}{-2}$ Factor the numerator and simplify the denominator.

$= 2 + \sqrt{3}$ Simplify common factors.

Apply the Skills

5. Find the exact value of $\tan 165°$.

Learn: Use Sum and Difference Formulas to Verify Identities

The sum and difference formulas for sine, cosine, and tangent can be used to verify the cofunction identities and periodic identities of the six trigonometric functions.

Cofunction Identities*

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\quad$ $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\quad$ $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\quad$ $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

*All statements can be made using 90° for $\frac{\pi}{2}$.

Periodic Identities*

$\sin(\theta + 2\pi) = \sin\theta$ $\quad$ $\cos(\theta + 2\pi) = \cos\theta$

$\tan(\theta + \pi) = \tan\theta$ $\quad$ $\cot(\theta + \pi) = \cot\theta$

$\csc(\theta + 2\pi) = \csc\theta$ $\quad$ $\sec(\theta + 2\pi) = \sec\theta$

*All statements can be made using 180° for π and 360° for 2π.

Example 6: Verifying a Cofunction Identity

Verify the cofunction identity $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

Solution:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta$$

Apply $\sin(u - v) = \sin u\cos v - \cos u\sin v$ with $u = \frac{\pi}{2}$ and $v = \theta$.

$$= (1)\cos\theta - (0)\sin\theta$$

$$= \cos\theta$$

Apply the Skills

6. Verify the identity. $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

Example 7: Verifying an Identity

Verify the identity. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

Solution:

$\cos(x+y)\cos(x-y)$

$= (\cos x\cos y - \sin x\sin y)(\cos x\cos y + \sin x\sin y)$

Apply the sum and difference formulas for cosine.

$= \cos^2 x\cos^2 y + \cos x\cos y\sin x\sin y - \sin x\sin y\cos x\cos y - \sin^2 x\sin^2 y$

$= \cos^2 x\cos^2 y - \sin^2 x\sin^2 y$

$= \cos^2 x(1-\sin^2 y) - (1-\cos^2 x)\sin^2 y$

$\cos^2 y = 1 - \sin^2 y$ and $\sin^2 x = 1 - \cos^2 x$.

$= \cos^2 x - \cos^2 x\sin^2 y - \sin^2 y + \cos^2 x\sin^2 y$

$= \cos^2 x - \sin^2 y$

Apply the distributive property and simplify.

Apply the Skills

7. Verify the identity. $\sin(x+y) - \sin(x-y) = 2\cos x\sin y$

Learn: Write a Sum of $A\sin x$ and $B\cos x$ as a Single Term

Consider an expression of the form $A\sin x + B\cos x$ for real numbers A and B. We can write this in terms of a single trigonometric function of the form $k\sin(x+\alpha)$, where k is a positive real number. To begin, apply the sum formula for the expression $\sin(x+\alpha)$.

$\sin(x+\alpha) = \sin x\cos\alpha + \cos x\sin\alpha$

Let $\cos\alpha = \frac{A}{k}$ and $\sin\alpha = \frac{B}{k}$ for some positive real number k.

Figure 5-6 shows the case where $0 \le \alpha \le \frac{\pi}{2}$.

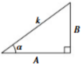


Figure 5-6

$\sin(x+\alpha) = \sin x\left(\frac{A}{k}\right) + \cos x\left(\frac{B}{k}\right).$

Multiplying both sides by k gives

$$k\sin(x+\alpha) = A\sin x + B\cos x \qquad (1)$$

To write the constant k in terms of A and B, use a Pythagorean identity.

$$\begin{array}{ccccc} \sin^2\alpha & + & \cos^2\alpha & = & 1 \\ \left(\frac{B}{k}\right)^2 & + & \left(\frac{A}{k}\right)^2 & = & 1 \\ B^2 & + & A^2 & = & k^2 \end{array}$$

Since k is assumed to be positive, we have $k = \sqrt{A^2+B^2}$.

So, equation (1) becomes $\sqrt{A^2+B^2}\sin(x+\alpha) = A\sin x + B\cos x$.

Sum of $A\sin x$ and $B\cos x$

For the real numbers A, B, and x,

$$A\sin x + B\cos x = k\sin(x + \alpha)$$

where $k = \sqrt{A^2 + B^2}$, and α satisfies $\cos\alpha = \frac{A}{k}$ and $\sin\alpha = \frac{B}{k}$.

The formula presented here to simplify a sum of $A\sin x$ and $B\cos x$ is also called a reduction formula because it “reduces” the sum of two trigonometric functions into one term.

Example 8: Writing a Sum of Sine and Cosine as a Single Term

Write $5\sin x - 12\cos x$ in the form $k\sin(x + \alpha)$.

Solution:

$5\sin x - 12\cos x$ is in the form $A\sin x + B\cos x$ with $A = 5$ and $B = -12$.

$$k = \sqrt{A^2 + B^2} = \sqrt{(5)^2 + (-12)^2} = \sqrt{169} = 13$$

$$\cos\alpha = \frac{A}{k} = \frac{5}{13} \quad \text{and} \quad \sin\alpha = \frac{B}{k} = \frac{-12}{13} = -\frac{12}{13}$$

Since $\cos\alpha > 0$ and $\sin\alpha < 0$, α is in Quadrant IV. Taking the inverse cosine, we have the first quadrant angle

$$\cos^{-1}\left(\frac{5}{13}\right) \approx 1.1760 \quad \text{or} \quad \approx 67.4°.$$

To find a fourth quadrant angle, subtract from 2π(or 360°).

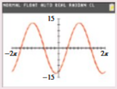
$$\alpha = 2\pi - \cos^{-1}\frac{5}{13} \approx 5.1072 \quad \text{or} \quad \alpha = 360° - \cos^{-1}\frac{5}{13} \approx 292.6°$$

Therefore, $5\sin x - 12\cos x \approx 13\sin(x + 5.1072)$ or

$$5\sin x - 12\cos x \approx 13\sin(x + 292.6°).$$

Check Your Work

You can support your results from Example 8 by using a calculator to graph $y = 5\sin x - 12\cos x$ and $y = 13\sin(x + 5.1072)$. Confirm that the two graphs coincide.



Apply the Skills

8. Write $-4\sin x + 3\cos x$ in the form $k\sin(x + \alpha)$.

Practice 5-2

Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, use reference angles to find the exact value.

R.1. $\sin 240°$

R.2. $\cos \frac{3\pi}{4}$

R.3. Given that $\sin\theta < 0$ and $\cos\theta > 0$, identify the quadrant in which θ lies.

R.4. Find the difference quotient and simplify. $f(x) = -5x^2 - 8x + 3$

Concept Connections

1. Fill in the blanks with the correct operation + or −.

$\sin(u + v) = \sin u \cos v$ ______ $\cos u \sin v$

$\cos(u + v) = \cos u \cos v$ ______ $\sin u \sin v$

2. Fill in the blanks to complete each identity.

$\sin(u - v) =$ ____________ − ____________

$\cos(u - v) =$ ____________ + ____________

3. Fill in the boxes to complete each identity.

$\tan(u + v) = \dfrac{\boxed{}}{1 - \tan u \tan v}$;

$\tan(u - v) = \dfrac{\tan u - \tan v}{\boxed{}}$

4. If $0° < \alpha < 90°$ and $90° < \beta < 180°$, then ______ $< \alpha + \beta <$ ______ and ______ $< \beta - \alpha <$ ______.

5. Write $\frac{13\pi}{12}$ as a sum or difference of angles that are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.

6. Write 285° as a sum or difference of angles that are multiples of 30° or 45°.

Learn: Apply the Sum and Difference Formulas for Sine, Cosine, and Tangent.

For Exercises 7–18, use an addition or subtraction formula to find the exact value. (See Examples 1 and 5 pp. 612 and 616)

7. $\cos 165°$

8. $\sin \frac{5\pi}{12}$

9. $\sin\left(-\frac{\pi}{12}\right)$

10. $\tan \frac{19\pi}{12}$

11. $\cos\left(-\frac{19\pi}{12}\right)$

12. $\sin(-15°)$

13. $\tan \frac{7\pi}{12}$

14. $\cos \frac{7\pi}{12}$

15. $\sin 105°$

16. $\tan(-105°)$

17. $\tan 15°$

18. $\cos(-195°)$

For Exercises 19–26, use an addition or subtraction formula to find the exact value. (See Examples 2 and 5, pp. 613 and 616)

19. $\sin 140° \cos 20° - \cos 140° \sin 20°$

20. $\cos \frac{35\pi}{18} \cos \frac{5\pi}{18} + \sin \frac{35\pi}{18} \sin \frac{5\pi}{18}$

21. $\dfrac{\tan \frac{5\pi}{4} + \tan \frac{\pi}{12}}{1 - \tan \frac{5\pi}{4} \tan \frac{\pi}{12}}$

22. $\sin \frac{35\pi}{36} \cos \frac{13\pi}{18} - \cos \frac{35\pi}{36} \sin \frac{13\pi}{18}$

23. $\cos \frac{2\pi}{3} \cos \frac{7\pi}{6} - \sin \frac{2\pi}{3} \sin \frac{7\pi}{6}$

24. $\dfrac{\tan 15° - \tan 45°}{1 + \tan 15° \tan 45°}$

25. $\cos 200° \cos 25° - \sin 200° \sin 25°$

26. $\sin \frac{10\pi}{9} \cos \frac{7\pi}{18} + \cos \frac{10\pi}{9} \sin \frac{7\pi}{18}$

For Exercises 27–32, find the exact value for the expression under the given conditions. (See Examples 3 and 5, pp. 614 and 616)

27. $\cos(\alpha + \beta)$; $\sin\alpha = -\frac{3}{5}$ for α in Quadrant III and $\cos\beta = -\frac{3}{4}$ for β in Quadrant II.

28. $\sin(\alpha - \beta)$; $\sin\alpha = \frac{3}{8}$ for α in Quadrant II and $\cos\beta = \frac{12}{13}$ for β in Quadrant IV.

29. $\tan(\alpha - \beta)$; $\sin\alpha = \frac{8}{17}$ for α in Quadrant II and $\cos\beta = -\frac{9}{41}$ for β in Quadrant III.

30. $\cos(\alpha-\beta)$; $\sin\alpha=\frac{2}{3}$ for α in Quadrant II and $\cos\beta=-\frac{1}{4}$ for β in Quadrant III.

31. $\sin(\alpha+\beta)$; $\cos\alpha=\frac{3}{7}$ for α in Quadrant IV and $\sin\beta=\frac{7}{25}$ for β in Quadrant II.

32. $\tan(\alpha+\beta)$; $\cos\alpha=-\frac{13}{85}$ for α in Quadrant III and $\sin\beta=-\frac{63}{65}$ for β in Quadrant IV.

For Exercises 33–40, find the exact value. (See Example 4, p. 615)

33. $\sin\left(\arcsin\frac{1}{2}-\arccos\frac{\sqrt{2}}{2}\right)$

34. $\cos\left(\tan^{-1}\sqrt{3}-\sin^{-1}\frac{\sqrt{3}}{2}\right)$

35. $\tan\left[\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)+\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

36. $\sin\left[\arccos\left(-\frac{1}{2}\right)+\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right]$

37. $\cos\left(\tan^{-1}\frac{8}{15}-\cos^{-1}\frac{3}{5}\right)$

38. $\sin\left(\cos^{-1}\frac{12}{13}+\tan^{-1}\frac{4}{3}\right)$

39. $\cos\left(\sin^{-1}\frac{4}{5}+\tan^{-1}2\right)$

40. $\tan\left(\arccos\frac{1}{5}-\arcsin\frac{3}{5}\right)$

Learn: Use Sum and Difference Formulas to Verify Identities

For Exercises 41–62, verify the identity. (See Examples 6–7, pp. 617–618)

41. $\cos\left(\frac{\pi}{2}+x\right)=-\sin x$

42. $\sin\left(\frac{\pi}{2}+x\right)=\cos x$

43. $\tan(\pi+x)=\tan x$

44. $\cos(x-\pi)=-\cos x$

45. $\sin\left(\frac{3\pi}{2}+x\right)=-\cos x$

46. $\tan(\pi-x)=-\tan x$

47. $\sin(x+y)+\sin(x-y)=2\sin x\cos y$

48. $\cos(x-y)-\cos(x+y)=2\sin x\sin y$

49. $\frac{\cos(x+y)}{\sin x\cos y}=\cot x-\tan y$

50. $\frac{\sin(x+y)}{\cos x\sin y}=\tan x\cot y+1$

51. $\sin\left(x+\frac{\pi}{4}\right)+\sin\left(x-\frac{\pi}{4}\right)=\sqrt{2}\sin x$

52. $\cos\left(x+\frac{\pi}{4}\right)-\cos\left(x-\frac{\pi}{4}\right)=-\sqrt{2}\sin x$

53. $\tan\left(x-\frac{\pi}{4}\right)=\frac{\sin x-\cos x}{\sin x+\cos x}$

54. $\tan\left(x+\frac{\pi}{4}\right)=\frac{\cos x+\sin x}{\cos x-\sin x}$

55. $\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}=\frac{\cos\alpha\cot\beta+1}{\cos\alpha\cot\beta-1}$

56. $\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}=\frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}$

57. $\csc(x+y)=\frac{\sec x\sec y}{\tan x+\tan y}$

58. $\sec(x-y)=\frac{\csc x\sec y}{\cot x+\tan y}$

59. $\cos(A+B)\cos(A-B)=\cos^2A+\cos^2B-1$

60. $\sin(A+B)\sin(A-B)=\sin^2A-\sin^2B$

61. $\tan(x+y)-\tan(x-y)=\frac{2\tan y\sec^2x}{1-\tan^2x\tan^2y}$

62. $\tan(x+y)+\tan(x-y)=\frac{2\tan x\sec^2y}{1-\tan^2x\tan^2y}$

Learn: Write a Sum of $A\sin x$ and $B\cos x$ as a Single Term

For Exercises 63–66,

a. Write the given expression in the form $k\sin(x+\alpha)$ for $0\le\alpha<2\pi$. Round α to 3 decimal places. **(See Example 8, p. 619)**

b. Verify the result from part (a) by applying the sum formula for sine.

63. $8\sin x-6\cos x$

64. $-15\sin x+8\cos x$

65. $-\sin x-3\cos x$

66. $2\sin x+9\cos x$

Often graphing a function of the form $y=A\sin x+B\cos x$ is easier by using its reduction formula $y=k\sin(x+\alpha)$. For Exercises 67–70,

a. Use the reduction formula to write the given function as a sine function.

b. Graph the function.

67. $y=\sqrt{3}\sin x+\cos x$

68. $y=\sqrt{2}\sin x-\sqrt{2}\cos x$

69. $y=-\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x$

70. $y=-\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x$

Mixed Exercises

71. a. Is it true that $\sin(2\cdot30^\circ)=2\sin30^\circ$?

b. Expand $\sin(2\cdot30^\circ)$ as $\sin(30^\circ+30^\circ)$ using the sum formula for sine. Then simplify the result.

72. **a.** Is it true that $\cos(2 \cdot 45^\circ) = 2\cos 45^\circ$?

b. Expand $\cos(2 \cdot 45^\circ)$ as $\cos(45^\circ + 45^\circ)$ using the sum formula for cosine. Then simplify the result.

73. Derive $\cos(u + v) = \cos u \cos v - \sin u \sin v$ by using the identity for $\cos(u - v)$ and the odd and even function identities for sine and cosine.

74. Derive $\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$ by using the identity for $\tan(u + v)$ and the odd function identity for tangent.

75. Derive $\sin(u + v) = \sin u \cos v + \cos u \sin v$. {*Hint*: Write $\sin(u + v) = \cos\left[\frac{\pi}{2} - (u + v)\right]$ and apply the cofunction identities.}

76. Derive $\sin(u - v) = \sin u \cos v - \cos u \sin v$.

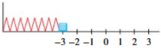
77. For $f(x) = \sin x$, show that $\frac{f(x + h) - f(x)}{h} = -\sin x\left(\frac{1 - \cos h}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$

78. For $f(x) = \cos x$, show that $\frac{f(x + h) - f(x)}{h} = -\cos x\left(\frac{1 - \cos h}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)$

79. A human ear detects sound as a result of fluctuations in air pressure against the eardrum, which are in turn transmitted into nerve impulses that the brain translates as sound. Suppose that the pressure $P(t)$ of a sound wave on a person's eardrum from a source 20 ft away is given by $P(t) = 0.02\cos(1000t - 10\pi)$. In this model $P(t)$ is in pounds per square foot and t is the time in seconds after the sound begins. Simplify the right side by applying the difference formula for cosine.

80. The raised part of a sundial, called a *gnomon*, casts a shadow of length l when the angle of elevation of the Sun is θ. The length of the shadow is given by $l = \frac{h\sin(90^\circ - \theta)}{\sin\theta}$, where h is the height of the gnomon. Simplify the right side of this equation.

For Exercises 81–82, consider a 1-kg object oscillating at the end of a horizontal spring. The horizontal position $x(t)$ of the object is given by $x(t) = \frac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t)$ where v_0 is the initial velocity, x_0 is the initial position, and ω is the number of back-and-forth cycles that the object makes per unit time t.



81. At time $t = 0$ sec, the object is moved 3 ft to the left of the equilibrium position and then given a velocity of 4 ft/sec to the right ($v_0 = 4$ ft/sec).

a. If the object completes 1 cycle in 1 sec ($\omega = 1$), write a model of the form $x(t) = \frac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t)$ to represent the horizontal motion of the spring.

b. Use the reduction formula to write the function in part (a) in the form $x(t) = k\sin(t + \alpha)$. Round α to 2 decimal places.

c. What is the maximum displacement of the object from its equilibrium position?

82. At time $t = 0$ sec, the object is moved 2 ft to the right of the equilibrium position and then given a velocity of 3 ft/sec to the left ($v_0 = -3$ ft/sec).

a. If the block completes 1 cycle in 1 sec ($\omega = 1$), write a model of the form $x(t) = \frac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t)$ to represent the horizontal motion of the spring.

b. Use the reduction formula to write the function in part (a) in the form $x(t) = k\sin(t + \alpha)$. Round α to 2 decimal places.

c. What is the maximum displacement of the object from its equilibrium position? Round to 2 decimal places.

Write About It

83. Explain why we cannot use $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\tan\frac{\pi}{2} - \tan\theta}{1 - \tan\frac{\pi}{2} \cdot \tan\theta}$ to prove the cofunction identity $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$.

84. Describe the pattern for the expansions of $\cos(u + v)$ and $\cos(u - v)$.

85. Describe the pattern for the expansions of $\sin(u + v)$ and $\sin(u - v)$.

86. The sum and difference formulas for cosine can be written in a single statement as $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$. Explain why the symbol $\mp$ is used on the right.

Expanding Your Skills

For Exercises 87–90, write the trigonometric expression as an algebraic expression in *x* and *y*. Assume that *x* and *y* are Quadrant I angles.

87. $\cos(\sin^{-1}x + \tan^{-1}y)$ **88.** $\sin(\cos^{-1}x + \tan^{-1}y)$

89. $\sin(\sin^{-1}x + \sin^{-1}y)$ **90.** $\cos(\sin^{-1}x - \cos^{-1}y)$

For Exercises 91–96, verify the identity.

91. $\cos(a + b + c) = \cos a \cos b \cos c - \sin a \sin b \cos c - \sin a \cos b \sin c - \cos a \sin b \sin c$

92. $\sin(a + b + c) = \sin a \cos b \cos c + \cos a \sin b \cos c + \cos a \cos b \sin c - \sin a \sin b \sin c$

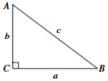
93. $\cos(x + y)\cos y + \sin(x + y)\sin y = \cos x$

94. $\cos(x - y)\sin y + \sin(x - y)\cos y = \sin x$

95. $\sec(x + y) = \dfrac{\cos(x - y)}{\cos^2 x - \sin^2 y}$

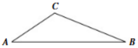
96. $\sec(x - y) = \dfrac{\cos(x + y)}{\cos^2 x - \sin^2 y}$

97. Suppose that $\triangle ABC$ is a right triangle.



Show that $\sin(A - B) = \dfrac{a^2 - b^2}{c^2}$ and $\cos(A - B) = \dfrac{2ab}{c^2}$.

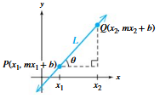
98. Suppose that $\triangle ABC$ contains no right angle.



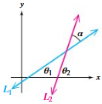
Show that $\tan A + \tan B + \tan C = (\tan A)(\tan B)(\tan C)$.

(*Hint*: $A + B + C = 180°$, which implies that $A + B = 180° - C$. Then take the tangent of both sides of the equation.)

99. Let *L* be a line defined by $y = mx + b$ with a positive slope, and let θ be the acute angle formed by *L* and the horizontal. Let $P(x_1, mx_1 + b)$ and $Q(x_2, mx_2 + b)$ be arbitrary points on *L*. Show that $m = \tan\theta$.



100. Let L_1 and L_2 be nonparallel lines with positive slopes m_1 and m_2, respectively, where $m_2 > m_1$.



a. Show that the acute angle α formed by L_1 and L_2 must satisfy $\tan\alpha = \dfrac{m_2 - m_1}{1 + m_2 m_1}$.

b. Find the measure of the acute angle formed by the lines $y = \frac{3}{4}x + 2$ and $y = 2x - 3$. Round to the nearest tenth of a degree.

Technology Connections

101. a. Graph $y = \cos\left(x + \frac{\pi}{2}\right) - \cos\left(x - \frac{\pi}{2}\right)$ on the interval $[0, 2\pi]$. What simpler function $y = f(x)$ does this graph appear to represent?

b. Simplify $\cos\left(x + \frac{\pi}{2}\right) - \cos\left(x - \frac{\pi}{2}\right)$ to confirm your response to part (a).

102. a. Graph $y = \sin 3x \cos 2x - \cos 3x \sin 2x$ on the interval $[0, 2\pi]$. What simpler function $y = f(x)$ does this graph appear to represent?

b. Simplify $\sin 3x \cos 2x - \cos 3x \sin 2x$ to confirm your response to part (a).

GO ONLINE **Go online for more practice problems.**

Lesson 5-3

Double-Angle, Power-Reducing, and Half-Angle Formulas

What Will You Learn?

After completing this lesson, you should be able to:

- Apply the Double-Angle Formulas
- Apply the Power-Reducing Formulas
- Apply the Half-Angle Formulas

Learn: Apply the Double-Angle Formulas

A projectile launched from ground level at an angle of elevation θ has a maximum horizontal range given by $x_{\max} = \dfrac{v_0^2 \sin 2\theta}{g}$ (see Exercises 89–90).

This expression involves the double angle 2θ. We can write a trigonometric function of a double angle 2θ in terms of $\sin\theta$, $\cos\theta$, and $\tan\theta$ by applying the sum and difference formulas.

$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta \\ &= 2\sin\theta\cos\theta\end{aligned} \qquad \sin(u + v) = \sin u\cos v + \cos u\sin v$$

$$\begin{aligned}\cos 2\theta &= \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta \\ &= \cos^2\theta - \sin^2\theta\end{aligned} \qquad \cos(u + v) = \cos u \cos v - \sin u \sin v$$

$$\begin{aligned}\tan 2\theta &= \tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} \\ &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned} \qquad \tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u\tan v}$$

Notice that the double-angle formula for cosine contains the expressions $\cos^2\theta$ and $\sin^2\theta$. From the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we can express the double-angle formula for $\cos 2\theta$ in alternative forms.

$$\begin{aligned}&\cos^2\theta - \sin^2\theta \\ &\quad = (1 - \sin^2\theta) - \sin^2\theta && \text{Substitute } 1 - \sin^2\theta \text{ for } \cos^2\theta. \\ &\quad = 1 - 2\sin^2\theta\end{aligned}$$

$$\begin{aligned}&\cos^2\theta - \sin^2\theta \\ &\quad = \cos^2\theta - (1 - \cos^2\theta) && \text{Substitute } 1 - \cos^2\theta \text{ for } \sin^2\theta. \\ &\quad = \cos^2\theta - 1 + \cos^2\theta \\ &\quad = 2\cos^2\theta - 1\end{aligned}$$

Blend Images/SuperStock

Double-Angle Formulas

- Sine Formula $\sin 2\theta = 2\sin\theta\cos\theta$
- Cosine Formulas $\cos 2\theta = \cos^2\theta - \sin^2\theta$
  $= 1 - 2\sin^2\theta$
  $= 2\cos^2\theta - 1$
- Tangent Formula $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Example 1: Using the Double-Angle Formulas

Given that $\sin\theta = \frac{2}{3}$ for θ in Quadrant II, find the exact function values.

a. $\sin 2\theta$ **b.** $\cos 2\theta$ **c.** $\tan 2\theta$

Solution:

To find $\sin 2\theta = 2\sin\theta\cos\theta$, we need to know the value of $\cos\theta$. Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we have $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$.

$\cos\theta = -\sqrt{1 - \sin^2\theta}$ The cosine function is negative in Quadrant II.

$= -\sqrt{1 - \left(\frac{2}{3}\right)^2}$

$= -\sqrt{1 - \frac{4}{9}}$ Substitute $\sin\theta = \frac{2}{3}$.

$= -\sqrt{\frac{5}{9}} = -\frac{\sqrt{5}}{3}$

a. $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right) = -\frac{4\sqrt{5}}{9}$

b. $\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(-\frac{\sqrt{5}}{3}\right)^2 - \left(\frac{2}{3}\right)^2 = \frac{5}{9} - \frac{4}{9} = \frac{1}{9}$ Use any of the three formulas for $\cos 2\theta$.

c. With the values of $\sin 2\theta$ and $\cos 2\theta$ known, the easiest way to find $\tan 2\theta$ is to divide.

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{4\sqrt{5}}{9}}{\frac{1}{9}} = -\frac{4\sqrt{5}}{9} \cdot \frac{9}{1} = -4\sqrt{5}$$

Alternatively, from Figure 5-7,

$$\tan\theta = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}.$$

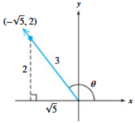


Figure 5-7

Therefore,

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2\left(-\frac{2\sqrt{5}}{5}\right)}{1-\left(\frac{2\sqrt{5}}{5}\right)^2} = \frac{-\frac{4\sqrt{5}}{5}}{1-\frac{4}{5}}\\ &= \frac{-\frac{4\sqrt{5}}{5}}{\frac{1}{5}} = -\frac{4\sqrt{5}}{5}\cdot\frac{5}{1}\\ &= -4\sqrt{5}\end{aligned}$$

Apply the Skills

1. Given that $\sin\theta = \frac{4}{5}$ for θ in Quadrant II, find the exact function values.

a. $\sin 2\theta$ **b.** $\cos 2\theta$ **c.** $\tan 2\theta$

The double-angle formulas can be used with angles other than 2θ. For example:

$$\begin{aligned}\sin(4\theta) &= \sin(2\cdot 2\theta)\\ &= 2\sin(2\theta)\cos(2\theta)\\ \cos(6x) &= \cos(2\cdot 3x)\\ &= \cos^2(3x)-\sin^2(3x)\\ \sin(\alpha-1) &= \sin\left(2\cdot\frac{\alpha-1}{2}\right)\\ &= 2\sin\left(\frac{\alpha-1}{2}\right)\cos\left(\frac{\alpha-1}{2}\right)\\ \tan x &= \tan\left(2\cdot\frac{x}{2}\right)\\ &= \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}\end{aligned}$$

In Example 2, we apply the sum formula for sine as well as the double-angle formulas for sine and cosine to prove an identity.

Example 2: Applying the Sum and Double-Angle Formulas

Verify the identity. $\sin 3x = 3\sin x - 4\sin^3 x$

Solution:

$$\begin{aligned}
\sin 3x &= \sin(2x + x) && \text{Apply } \sin(u+v) = \sin u \cos v + \cos u \sin v \text{ with } u = 2x \text{ and } v = x.\\
&= \sin 2x \cos x + \cos 2x \sin x && \\
&= (2\sin x \cos x)\cos x + (\cos^2 x - \sin^2 x)\sin x && \text{Simplify } \sin 2x = 2\sin x \cos x \text{ and } \cos 2x = \cos^2 x - \sin^2 x.\\
&= 2\sin x \cos^2 x + \sin x \cos^2 x - \sin^3 x && \\
&= 3\sin x \cos^2 x - \sin^3 x && \text{Combine like terms.}\\
&= 3\sin x(1 - \sin^2 x) - \sin^3 x && \text{Substitute } 1 - \sin^2 x \text{ for } \cos^2 x.\\
&= 3\sin x - 3\sin^3 x - \sin^3 x && \text{Apply the distributive property.}\\
&= 3\sin x - 4\sin^3 x &&
\end{aligned}$$

Apply the Skills

2. Verify the identity. $\cos 3x = 4\cos^3 x - 3\cos x$

Learn: Apply the Power-Reducing Formulas

The double-angle formulas for cosine can be used to express even powers of sine and cosine in terms of the first power of cosine only. For example, solving $\cos 2\theta = 1 - 2\sin^2\theta$ for $\sin^2\theta$ yields

RHS expressed in terms of first power of cosine

$$\cos 2\theta = 1 - 2\sin^2\theta \quad \Rightarrow \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2} \qquad (1)$$

Equation (1) is called a **power-reducing** formula. The derivations of the power-reducing formulas in the following box are addressed in Exercises 57–58.

Power-Reducing Formulas

- Sine Formula $\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$
- Cosine Formula $\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$
- Tangent Formula $\tan^2\theta = \dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}$

Insights The power-reducing formulas are important in integral calculus.

Example 3: Applying the Power-Reducing Formulas

Write $\sin^4 x + \cos^2 x$ in terms of first powers of cosine.

Solution:

$\sin^4 x + \cos^2 x = (\sin^2 x)^2 + \cos^2 x$	Write each term as a power of $\sin^2 x$ and $\cos^2 x$.
$= \left(\frac{1-\cos 2x}{2}\right)^2 + \frac{1+\cos 2x}{2}$	Apply the power-reducing formulas.
$= \frac{1-2\cos 2x+\cos^2 2x}{4} + \frac{1+\cos 2x}{2}$	Square the first term.
$= \frac{1-2\cos 2x+\cos^2 2x}{4} + \frac{1+\cos 2x}{2}\cdot\frac{2}{2}$	Write each term with a common denominator.
$= \frac{1-2\cos 2x+\cos^2 2x}{4} + \frac{2+2\cos 2x}{4}$	
$= \frac{1}{4}[1-2\cos 2x+\cos^2 2x+2+2\cos 2x]$	Factor out $\frac{1}{4}$.
$= \frac{1}{4}[\cos^2 2x+3]$	Combine like terms.
$= \frac{1}{4}\left[\left(\frac{1+\cos 4x}{2}\right)+3\right]$	Apply the power-reducing formula for cosine: $\cos^2 2x = \frac{1+\cos(2\cdot 2x)}{2}$ $= \frac{1+\cos 4x}{2}$ Notice that the angle on the RHS is doubled.
$= \frac{1}{4}\left[\frac{1}{2}+\frac{\cos 4x}{2}+\frac{6}{2}\right]$	Write each term with a common denominator.
$= \frac{1}{4}\left(\frac{\cos 4x}{2}+\frac{7}{2}\right)$	Combine like terms.
$= \frac{1}{8}\cos 4x + \frac{7}{8}$	Apply the distributive property.

Apply the Skills

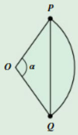
3. Write $\cos^4 x$ in terms of first powers of cosine.

Point of Interest

In ancient times, trigonometric relationships were based on the chord function, denoted crd. The chord of a central angle $\angle POQ$ is the length of the line segment with endpoints P and Q where the angle intercepts the circle. The chord function is related to the modern sine function by

$$\text{crd}\,\alpha = d\sin\frac{\alpha}{2}$$

where d is the diameter of the circle.



Learn: Apply the Half-Angle Formulas

Another set of formulas called the half-angle formulas can be derived from the power-reducing formulas. Substituting $\frac{\alpha}{2}$ for θ in the power-reducing formulas given on page 627 results in the half-angle formulas given in the box below.

Half-Angle Formulas

- Sine Formula $\quad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$
- Cosine Formula $\quad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$
- Tangent Formula $\quad \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$

The sign + or − is determined by the quadrant in which the angle $\frac{\alpha}{2}$ lies.

To derive the formula for $\sin\frac{\alpha}{2}$, start with the power-reducing formula for $\sin^2\theta$.

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\left[2\left(\frac{\alpha}{2}\right)\right]}{2}$$

Substitute $\frac{\alpha}{2}$ for θ.

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Simplify the argument and take the square root of both sides. The sign is chosen based on the quadrant of $\frac{\alpha}{2}$ in standard position.

Example 4: Using the Half-Angle Formula for Cosine

Use the half-angle formula to find the exact value of $\cos 157.5°$.

Solution:

$$\cos 157.5° = \cos\frac{315°}{2}$$

$\frac{\alpha}{2} = 157.5°$ which means that $\alpha = 2(157.5°) = 315°$.

In Quadrant II, $\cos\theta < 0$.

$$= -\sqrt{\frac{1+\cos 315°}{2}}$$

Apply the formula $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$ for $\alpha = 315°$. $\cos\frac{\alpha}{2} < 0$ because 157.5° is a second-quadrant angle.

$$= -\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}$$

$\cos 315° = \frac{\sqrt{2}}{2}$

$$= -\sqrt{\frac{2\cdot\left(1+\frac{\sqrt{2}}{2}\right)}{2\cdot 2}}$$

Multiply numerator and denominator by 2 to simplify the complex fraction.

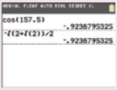
$$= -\sqrt{\frac{2+\sqrt{2}}{4}}$$

$$= -\frac{\sqrt{2+\sqrt{2}}}{2}$$

Simplify the radical in the denominator.

Check Your Work

The results from Example 4 can be supported on a calculator.



```
NORMAL FLOAT AUTO REAL DEGREE CL
cos(157.5)
                 ⁻.9238795325
⁻√(2+√(2))/2
                 ⁻.9238795325
```

Apply the Skills

4. Use the half-angle formula to find the exact value of sin 165°.

On page 629, notice that we present three forms for the half-angle tangent formula. The formula $\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$ follows directly from $\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$. However, in practice we prefer to apply the more convenient alternative forms without the radical sign or the ± sign. In Example 5, we derive $\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$.

Example 5: Verifying an Identity for the Tangent of a Half-Angle

Show that $\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha}$.

Solution:

$$\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \pm\sqrt{\frac{(1-\cos\alpha)}{(1+\cos\alpha)}\cdot\frac{(1-\cos\alpha)}{(1-\cos\alpha)}}$$

Rationalize the denominator by multiplying numerator and denominator by the conjugate of the denominator.

$$= \pm\sqrt{\frac{(1-\cos\alpha)^2}{1-\cos^2\alpha}}$$

Multiply.

$$= \pm\sqrt{\frac{(1-\cos\alpha)^2}{\sin^2\alpha}}$$

From the Pythagorean identity $\sin^2\alpha + \cos^2\alpha = 1$, substitute $\sin^2\alpha$ for $1-\cos^2\alpha$.

$$= \pm\frac{|1-\cos\alpha|}{|\sin\alpha|}$$

Take the principal square root.

$$= \frac{1-\cos\alpha}{\sin\alpha}$$

- Since $-1 \le \cos\alpha \le 1$, the value $1-\cos\alpha \ge 0$ for all α.
- With the numerator positive or zero, the denominator controls the sign of the quotient. The value of $\sin\alpha$ has the same sign as $\tan\frac{\alpha}{2}$ for α in any quadrant (see Exercise 45). Therefore, the absolute value bars and ± are not needed.

So, $\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$.

Apply the Skills

5. Show that $\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sin\alpha}{1+\cos\alpha}$.

Example 6: Using the Half-Angle Formulas

If $\sin\alpha = -\frac{4}{5}$ and $\pi < \alpha < \frac{3\pi}{2}$, find the exact values of each expression.

a. $\sin\frac{\alpha}{2}$ **b.** $\cos\frac{\alpha}{2}$ **c.** $\tan\frac{\alpha}{2}$

Solution:

To apply the half-angle formulas, we need to know the value of $\cos\alpha$. The angle α is in the third quadrant with $\sin\alpha = -\frac{4}{5}$. From Figure 5-8, we have $\cos\alpha = -\frac{3}{5}$.

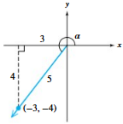


Figure 5-8

Next, since α is on the interval $\left(\pi, \frac{3\pi}{2}\right)$, angle $\frac{\alpha}{2}$ is on the interval $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. Therefore $\frac{\alpha}{2}$ is in the second quadrant.

In Quadrant II, $\sin\frac{\alpha}{2} > 0$.

a. $\sin\frac{\alpha}{2} = +\sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{\frac{8}{5}}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

In Quadrant II, $\sin\frac{\alpha}{2} < 0$.

b. $\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}} = -\sqrt{\frac{\frac{2}{5}}{2}} = -\sqrt{\frac{2}{5}\cdot\frac{1}{2}}$

$= -\sqrt{\frac{1}{5}} = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$

c. Use any of the three half-angle formulas for tangent.

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\left(-\frac{3}{5}\right)}{-\frac{4}{5}} = \frac{\frac{8}{5}}{-\frac{4}{5}} = \frac{8}{5}\cdot\left(-\frac{5}{4}\right) = -2$$

Insights

Alternatively,

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\frac{2\sqrt{5}}{5}}{-\frac{\sqrt{5}}{5}}$$

$$= \frac{2\sqrt{5}}{5}\cdot\left(-\frac{5}{\sqrt{5}}\right) = -2$$

Apply the Skills

6. If $\sin\alpha = -\frac{5}{13}$ and $\frac{3\pi}{2} < \alpha < 2\pi$, find the exact values of each expression.

a. $\sin\frac{\alpha}{2}$ **b.** $\cos\frac{\alpha}{2}$ **c.** $\tan\frac{\alpha}{2}$

Practice 5-3

Practice Exercises

Prerequisite Review

R.1. Simplify the radical expression. $\sqrt{(a+3)^2}$

R.2. Simplify the radical expression. $\sqrt[4]{u^4v^8}$

R.3. Rationalize the denominator. Assume the variable represents a positive number.

$$\sqrt{\frac{4}{w}}$$

R.4. Rationalize the denominator. $\frac{-14}{\sqrt{2}-3}$

Concept Connections

1. $\cos 2\theta =$ ________ or ________ or ________

2. $\sin 2\theta =$ ________

3. From the relationship $\cos 2\theta = 2\cos^2\theta - 1$, it follows that $\cos^2\theta =$ ________.

4. From the relationship $\cos 2\theta = 1 - 2\sin^2\theta$, it follows that $\sin^2\theta =$ ________.

5. From the relationship $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, it follows that $\sin\frac{u}{2} =$ ________.

6. From the relationship $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, it follows that $\cos\frac{u}{2} =$ ________.

Learn: Apply the Double-Angle Formulas

For Exercises 7–14, use the given information to find the exact function values. (See Example 1)

a. $\sin 2\theta$ b. $\cos 2\theta$ c. $\tan 2\theta$

7. $\sin\theta = -\frac{12}{13}$, θ in Quadrant IV

8. $\sin\theta = \frac{4}{7}$, $\tan\theta > 0$

9. $\cos\theta = -\frac{\sqrt{5}}{5}$, $\sin\theta > 0$

10. $\cos\theta = -\frac{5}{8}$, θ in Quadrant III

11. $\tan\theta = -\frac{5}{2}$, θ in Quadrant II

12. $\cot\theta = \frac{\sqrt{2}}{6}$, $\cos\theta < 0$

13. $\sec\theta = \frac{37}{12}$, θ in Quadrant IV

14. $\csc\theta = \frac{25}{24}$, $\cos\theta > 0$

For Exercises 15–20, find the exact value of the expression.

15. $\frac{2\tan 67.5^\circ}{1-\tan^2 67.5^\circ}$

16. $\frac{2\tan 15^\circ}{1-\tan^2 15^\circ}$

17. $2\sin\frac{\pi}{12}\cos\frac{\pi}{12}$

18. $2\sin(-22.5^\circ)\cos(-22.5^\circ)$

19. $\cos^2\frac{\pi}{8} - \sin^2\frac{\pi}{8}$

20. $2\cos^2(-165^\circ) - 1$

For Exercises 21–34, verify the identity. (See Example 2)

21. $\frac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$

22. $\frac{\sin 2\theta}{\sin 4\theta} = \frac{1}{2}\sec 2\theta$

23. $\cos^4 x - \sin^4 x = \cos 2x$

24. $(\sin x - \cos x)^2 = 1 - \sin 2x$

25. $\frac{\cos 3x}{\sin 2x} = \frac{1}{2}\csc x - 2\sin x$

26. $\frac{\sin 3x}{\sin 2x} = 2\cos x - \frac{1}{2}\sec x$

27. $\sin 2A - \tan A = \tan A\cos 2A$

28. $\tan 2A - \tan A = \tan A\sec 2A$

29. $\frac{2\tan x}{2-\sec^2 x} = \tan 2x$

30. $\cot x - \tan x = 2\cot 2x$

31. $\frac{\cos 2x}{\sin x} - \frac{\sin 2x}{\cos x} = \csc x - 4\sin x$

32. $\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x} = \sec x$

33. $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\sin^3\theta\cos\theta$

34. $\cos 4\theta = \sin^4\theta - 6\sin^2\theta\cos^2\theta + \cos^4\theta$

Learn: Apply the Power-Reducing Formulas

For Exercises 35–38, write the expression in terms of first powers of cosine. (See Example 3)

35. $\sin^4 x$

36. $\tan^4 x$

37. $\cos^2 x\sin^2 x$

38. $\cos^4 x + \sin^2 x$

Learn: Apply the Half-Angle Formulas

For Exercises 39–44, use the half-angle formula to find the exact value. (See Example 4)

39. $\sin 165°$

40. $\tan\frac{7\pi}{8}$

41. $\cos\frac{7\pi}{12}$

42. $\cos 112.5°$

43. $\tan 75°$

44. $\sin\frac{\pi}{8}$

45. Fill in the table for α in the given quadrant. What do you notice about the signs of $\sin\alpha$ and $\tan\frac{\alpha}{2}$? **(See Example 5)**

Quadrant for α	I	II	III	IV
Quadrant for $\frac{\alpha}{2}$				
Sign of $\sin\alpha$				
Sign of $\tan\frac{\alpha}{2}$				

46. What are the advantages to using the formula $\tan\frac{\alpha}{2}=\frac{1-\cos\alpha}{\sin\alpha}$ or $\tan\frac{\alpha}{2}=\frac{\sin\alpha}{1+\cos\alpha}$ as opposed to $\tan\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$?

For Exercises 47–50, use the given information to find the exact function values. (See Example 6)

a. $\sin\frac{\alpha}{2}$ **b.** $\cos\frac{\alpha}{2}$ **c.** $\tan\frac{\alpha}{2}$

47. $\sin\alpha=\frac{11}{61}$, $0<\alpha<\frac{\pi}{2}$

48. $\cos\alpha=\frac{12}{13}$, $\frac{3\pi}{2}<\alpha<2\pi$

49. $\cos\alpha=-\frac{12}{37}$, $\pi<\alpha<\frac{3\pi}{2}$

50. $\sin\alpha=\frac{33}{65}$, $\frac{\pi}{2}<\alpha<\pi$

For Exercises 51–56, verify the identity.

51. $\sin^2\frac{\theta}{2}=\frac{\csc\theta-\cot\theta}{2\csc\theta}$

52. $\cos^2\frac{\theta}{2}=\frac{\sec\theta+1}{2\sec\theta}$

53. $\tan\frac{x}{2}+\cot\frac{x}{2}=2\csc x$

54. $\tan\frac{x}{2}=\csc x-\cot x$

55. $2\sin^2\frac{x}{2}+4\cos^2\frac{x}{2}-3=\cos x$

56. $\cos^2\frac{x}{2}-3\sin^2\frac{x}{2}+1=2\cos x$

Mixed Exercises

57. Show that $\cos^2\theta=\frac{1+\cos 2\theta}{2}$.

58. Show that $\tan^2\theta=\frac{1-\cos 2\theta}{1+\cos 2\theta}$.

For Exercises 59–62, verify the identity.

59. $\frac{\sin 2\theta}{\sin\theta}-\frac{\cos 2\theta}{\cos\theta}=\sec\theta$

60. $\frac{\sin 2A+\sin A}{1+\cos A+\cos 2A}=\tan A$

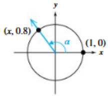
61. $\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\cos\alpha$

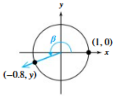
62. $\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2-1=\sin x$

For Exercises 63–68, given restrictions on x, determine whether Expression 1 is less than, greater than, or equal to Expression 2 or whether the relationship cannot be determined.

	x	Expression 1	Expression 2
63.	$\frac{5\pi}{4}<x<\frac{3\pi}{2}$	$\tan x$	$\tan\frac{x}{2}$
64.	$\frac{\pi}{4}<x<\frac{\pi}{2}$	$\cos x$	$\cos 2x$
65.	$\frac{\pi}{2}<x<\frac{3\pi}{4}$	$\cos x$	$\cos\frac{x}{2}$
66.	$\frac{\pi}{2}<x<\frac{3\pi}{4}$	$\tan x$	$\sin\frac{x}{2}$
67.	$\frac{5\pi}{4}<x<\frac{3\pi}{2}$	$\tan x$	$\tan 2x$
68.	$0<x<\frac{\pi}{4}$	$\sin x$	$\sin 2x$

For Exercises 69–74, use the information about α and β given in the figures to find the exact function values. Write the answers as decimals or as square roots of decimals.





69. a. $\sin 2\alpha$ **b.** $\cos 2\alpha$

70. a. $\sin 2\beta$ **b.** $\cos 2\beta$

71. a. $\sin\frac{\alpha}{2}$ **b.** $\cos\frac{\alpha}{2}$

72. a. $\sin\frac{\beta}{2}$ **b.** $\cos\frac{\beta}{2}$

73. a. $\sin(\alpha+\beta)$ **b.** $\cos(\alpha+\beta)$

74. a. $\sin(\alpha-\beta)$ **b.** $\cos(\alpha-\beta)$

For Exercises 75–78,

a. Rewrite the function as a single trigonometric function raised to the first power.

b. Graph the result over one period.

75. $y = 4\cos^2 x - 4\sin^2 x$

76. $y = -2\sin 2x\cos 2x$

77. $y = -6\sin^2 x$

78. $y = 4\cos^2 x$

For Exercises 79–84, find the exact value.

79. $\cos\left(2\sin^{-1}\frac{4}{5}\right)$

80. $\cos\left(\frac{1}{2}\arcsin\frac{3}{8}\right)$

81. $\sin\left(\frac{1}{2}\cos^{-1}\frac{2}{5}\right)$

82. $\sin\left(2\tan^{-1}\frac{12}{5}\right)$

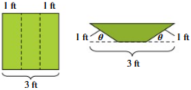
83. $\tan\left[2\arcsin\left(-\frac{1}{3}\right)\right]$

84. $\tan\left[\frac{1}{2}\cos^{-1}\left(-\frac{15}{17}\right)\right]$

85. Find an algebraic expression representing $\cos(2\sin^{-1}x)$ for $0 < x \leq 1$.

86. Find an algebraic expression representing $\sin(2\cos^{-1}x)$ for $0 < x \leq 1$.

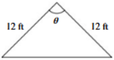
87. A feeding trough for cattle is made from a metal sheet 3 ft wide. The sides of the trough are made by folding up a flap 1 ft wide from each end of the strip. Each flap makes an angle θ with the horizontal.



a. Write an expression representing the area of a cross section of the trough in terms of θ.

b. Show that the area can be written as $A = \frac{1}{2}\sin 2\theta + \sin\theta$.

88. Consider the triangular area of the roof truss.



a. Write the area as a function of $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$.

b. Write the area as a function of $\sin\theta$.

89. Consider an object launched from an initial height h_0 with an initial velocity v_0 at an angle θ from the horizontal. The path of the object is given by

$$y = -\frac{g}{2v_0^2\cos^2\theta}x^2 + (\tan\theta)x + h_0$$

where x (in ft) is the horizontal distance from the launching point, y (in ft) is the height above ground level, and g is the acceleration due to gravity ($g = 32$ ft/sec² or 9.8 m/sec²). Show that the horizontal distance traveled by a soccer ball kicked from ground level with velocity v_0 at angle θ is $x = \frac{v_0^2\sin 2\theta}{g}$.

90. Refer to Exercise 89. Suppose that you kick a soccer ball from ground level with an initial velocity of 80 ft/sec.

a. Can you make the ball travel 200 horizontal feet?

b. Determine the smallest positive angle at which the ball can be kicked to make it travel 100 ft downfield to a teammate.

91. Write $\cos 3x$ as a third-degree polynomial in $\cos x$. [*Hint*: Write $\cos 3x$ as $\cos(2x + x)$.]

92. Write $\cos 4x$ as a fourth-degree polynomial in $\cos x$.

93. Write $\cos 5x$ in terms of powers of $\cos x$.

94. Write $\sin 5x$ in terms of powers of $\sin x$.

Write About It

95. When using the half-angle formulas, explain how to determine whether the + or − sign is used.

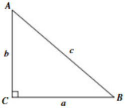
96. Explain the process to derive the double-angle formulas for $\sin 2\theta$ and $\cos 2\theta$.

97. Explain the process to derive the power-reducing formulas for $\sin^2\theta$ and $\cos^2\theta$.

98. Explain the process to derive the half-angle formulas for $\sin\frac{\alpha}{2}$ and $\cos\frac{\alpha}{2}$.

Expanding Your Skills

For Exercises 99–104, verify the identities for a right $\triangle ABC$.



99. $\sin 2A = \sin 2B$

100. $\cos 2A = -\cos 2B$

101. $\tan 2A = \frac{2ab}{b^2 - a^2}$

102. $\cos 2A = \frac{b^2 - a^2}{c^2}$

103. $\cos\frac{A}{2} = \sqrt{\frac{c + b}{2c}}$

104. $\tan\frac{A}{2} = \frac{c - b}{a}$

GO ONLINE Go online for more practice problems.

Lesson 5-4

Product-to-Sum and Sum-to-Product Formulas

What Will You Learn?

After completing this lesson, you should be able to:

- Apply the Product-to-Sum Formulas
- Apply the Sum-to-Product Formulas

Learn: Apply the Product-to-Sum Formulas

We now introduce a set of identities that enables us to write a product of factors of sine and cosine as a sum. These are called the product-to-sum formulas.

Product-to-Sum Formulas

- Product of Sine Functions $\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$ (1)
- Product of Cosine Functions $\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$ (2)
- Product of Sine and Cosine $\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$ (3)

  $\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$ (4)

Insights

In applications of integral calculus, often working with a sum of terms involving sine or cosine is easier than working with a product.

The derivations of the product-to-sum formulas are straightforward and follow from the sum and difference formulas for sine and cosine. To verify any of the formulas (1)–(4), simplify the right-hand side. For example, to verify (1), we have:

$$\begin{aligned}
\text{RHS: } & \tfrac{1}{2}[\cos(u - v) - \cos(u + v)] \\
&= \tfrac{1}{2}[\cos u \cos v + \sin u \sin v - (\cos u \cos v - \sin u \sin v)] \\
&= \tfrac{1}{2}(\cos u \cos v + \sin u \sin v - \cos u \cos v + \sin u \sin v) \\
&= \tfrac{1}{2}(2 \sin u \sin v) \\
&= \sin u \sin v \text{ (LHS)}
\end{aligned}$$

The remaining product-to-sum formulas are verified in a similar way (see Exercises 37–38). Although the product-to-sum formulas are difficult to memorize, they are easy to derive. Each is $\frac{1}{2}$ the sum or difference of $\cos(u - v)$ and $\cos(u + v)$ or $\frac{1}{2}$ the sum or difference of $\sin(u + v)$ and $\sin(u - v)$.

Example 1: Writing a Product of Sine and Cosine as a Sum

Write the product as a sum or difference.

a. $\sin 7x \cos 3x$

b. $\sin(-x) \sin 4x$

Solution:

a. $\sin 7x \cos 3x$

$= \frac{1}{2}[\sin(7x + 3x) + \sin(7x - 3x)]$	Apply product-to-sum formula (3) with $u = 7x$ and $v = 3x$.
$= \frac{1}{2}[\sin 10x + \sin 4x]$	Combine terms within the arguments.
$= \frac{1}{2}\sin 10x + \frac{1}{2}\sin 4x$	Apply the distributive property.

b. $\sin(-x)\sin 4x$

$= \frac{1}{2}(\cos[(-x) - 4x] - \cos[(-x) + 4x])$	Apply product-to-sum formula (1) with $u = -x$ and $v = 4x$.
$= \frac{1}{2}(\cos[-5x] - \cos 3x)$	Combine terms within the arguments.
$= \frac{1}{2}(\cos 5x - \cos 3x)$	Apply the even function property of cosine, $\cos(-x) = \cos x$.
$= \frac{1}{2}\cos 5x - \frac{1}{2}\cos 3x$	Apply the distributive property.

Insights

By convention, when we apply the product-to-sum formulas, we simplify as much as possible. In Example 1(b), we simplified $\cos(-5x)$ as $\cos 5x$ by using the even function property of the cosine function.

Apply the Skills

1. Write the product as a sum or difference.

a. $\cos 6x \sin 2x$

b. $\cos 4x \cos 5x$

Example 2: Using a Product-to-Sum Formula to Find an Exact Value

Use a product-to-sum formula to find the exact value of $\cos 75° \sin 15°$.

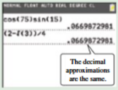
Solution:

$\cos 75° \sin 15°$

$= \frac{1}{2}[\sin(75° + 15°) - \sin(75° - 15°)]$	Apply product-to-sum formula (4) with $u = 75°$ and $v = 15°$.
$= \frac{1}{2}(\sin 90° - \sin 60°)$	Combine terms within the arguments.
$= \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right)$	Evaluate $\sin 90° = 1$ and $\sin 60° = \frac{\sqrt{3}}{2}$.
$= \frac{1}{2}\left(\frac{2 - \sqrt{3}}{2}\right)$	Write the terms inside parentheses with a common denominator and subtract.
$= \frac{2 - \sqrt{3}}{4}$	Multiply.

Insights

The result of Example 2 can be supported on a calculator.



```
NORMAL FLOAT AUTO REAL DEGREE CL
cos(75)sin(15)
                .0669872981
(2-√(3))/4
                .0669872981
```

The decimal approximations are the same.

Apply the Skills

2. Use a product-to-sum formula to find the exact value of $\sin 165° \cos 75°$.

Learn: Apply the Sum-to-Product Formulas

The product-to-sum formulas can also be used in reverse to write a sum or difference of sine and cosine terms as a product. We call these the sum-to-product formulas.

Sum-to-Product Formulas

- Difference of Cosine Functions $\quad \cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} \quad (5)$
- Sum of Cosine Functions $\quad \cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2} \quad (6)$
- Sum of Sine Functions $\quad \sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} \quad (7)$
- Difference of Sine Functions $\quad \sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} \quad (8)$

Insights

The sum-to-product formulas are often helpful when we solve trigonometric equations (Lesson 5.5).

The sum-to-product formulas follow directly from the product-to-sum formulas. For example, to derive formula (5), start with the product-to-sum formula (1) and let $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$.

$$\sin u \sin v = \tfrac{1}{2}[\cos(u - v) - \cos(u + v)] \qquad (1)$$

$$\begin{aligned}\sin\frac{x+y}{2}\sin\frac{x-y}{2} &= \frac{1}{2}\left[\cos\left(\frac{x+y}{2} - \frac{x-y}{2}\right) - \cos\left(\frac{x+y}{2} + \frac{x-y}{2}\right)\right] \\ &= \frac{1}{2}(\cos y - \cos x) \\ &= -\frac{1}{2}(\cos x - \cos y) \qquad \text{Factor out } -1.\end{aligned}$$

Therefore, $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$. (5)

Example 3: Writing a Sum or Difference as a Product

Write each expression as a product.

a. $\sin 7\theta + \sin 4\theta$ **b.** $\cos\alpha - \cos 3\alpha$

Solution:

a. $\sin 7\theta + \sin 4\theta$

$= 2\sin\frac{7\theta + 4\theta}{2}\cos\frac{7\theta - 4\theta}{2}$	Apply sum-to-product formula (7) with $x = 7\theta$ and $y = 4\theta$.
$= 2\sin\frac{11\theta}{2}\cos\frac{3\theta}{2}$	Simplify the arguments.

b. $\cos\alpha - \cos 3\alpha$

$= -2\sin\frac{\alpha + 3\alpha}{2}\sin\frac{\alpha - 3\alpha}{2}$	Apply sum-to-product formula (5) with $x = \alpha$ and $y = 3\alpha$.
$= -2\sin 2\alpha\sin(-\alpha)$	Simplify the arguments.
$= -2\sin 2\alpha(-\sin\alpha)$	Apply the odd function property
$= 2\sin 2\alpha\sin\alpha$	$\sin(-\alpha) = -\sin\alpha$.

Good Practices

By convention, when we apply the sum-to-product formulas, we simplify as much as possible. In Example 3(b), we simplified $\sin(-\alpha)$ as $-\sin\alpha$ by using the odd function property of the sine function.

Apply the Skills

3. Write each expression as a product.

a. $\cos 5x + \cos 2x$

b. $\sin\beta - \sin 5\beta$

Example 4: Using a Sum-to-Product Formula to Find an Exact Value

Use a sum-to-product formula to find the exact value of $\cos 255° + \cos 195°$.

Solution:

$\cos 255° + \cos 195°$

$= 2\cos\frac{255° + 195°}{2}\cos\frac{255° - 195°}{2}$	Apply sum-to-product formula (6) with $x = 255°$ and $y = 195°$.
$= 2\cos 225°\cos 30°$	Simplify the arguments.
$= 2\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$	Evaluate $\cos 225° = -\frac{\sqrt{2}}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$.
$= -\frac{\sqrt{6}}{2}$	Simplify.

Apply the Skills

4. Use a sum-to-product formula to find the exact value of $\sin 75° - \sin 15°$.

In Example 5, we use the sum-to-product formulas to verify an identity.

Example 5: Using Sum-to-Product Formulas to Verify an Identity

Verify the identity. $\frac{\sin 5x+\sin x}{\cos 5x+\cos x}=\tan 3x$

Solution:

$$\frac{\sin 5x+\sin x}{\cos 5x+\cos x}=\frac{2\sin\frac{5x+x}{2}\cos\frac{5x-x}{2}}{2\cos\frac{5x+x}{2}\cos\frac{5x-x}{2}}$$

In the numerator apply formula (7), $\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$.

In the denominator, apply formula (6), $\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$.

$$=\frac{2\sin\frac{6x}{2}\cos\frac{4x}{2}}{2\cos\frac{6x}{2}\cos\frac{4x}{2}}$$

Combine like terms and simplify.

$$=\frac{2\sin 3x\cos 2x}{2\cos 3x\cos 2x}$$

$$=\frac{\sin 3x}{\cos 3x}$$

$$=\tan 3x$$

Apply the Skills

5. Verify the identity. $\frac{\cos 4x-\cos 2x}{\sin 2x-\sin 4x}=\tan 3x$

Point of Interest

Before the introduction of logarithms, the product-to-sum formulas (historically known as the "prosthapheretic formulae") were used in an algorithm to approximate the products of many-digit numbers. Each factor was scaled to a number between –1 and 1 by dividing the factor by a power of 10. Then using a table of cosines, two angles *u* and *v* were found whose cosines were closest to the scaled factors. Taking the sum and difference of *u* and *v* and applying the formula $\cos u\cos v=\frac{1}{2}[\cos(u-v)+\cos(u+v)]$ resulted in the product of the two scaled factors. At this point, the product was multiplied by an appropriate power of 10 to reverse the original scaling factor in the first step.

This process is very similar to the process using logarithms to approximate the product of two numbers and is the basis of multiplication using a slide rule: scale down each factor, add the logarithms of the scaled-down factors, take the inverse logarithm, and reverse the scaling factor.

Practice 5-4

Practice Exercises

Concept Connections

1. The formula $\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$ converts a product of sines to a __________ of __________.

2. The formula $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$ converts a difference of cosines to a __________ of __________.

Learn: Apply the Product-to-Sum Formulas

For Exercises 3–10, write the product as a sum or difference. (See Example 1, p. 636)

3. $\sin 5x \sin 8x$

4. $\sin(-2\theta)\sin 4\theta$

5. $\cos(-3t)\cos 7t$

6. $\cos\frac{x}{2}\cos\frac{3x}{2}$

7. $\sin\frac{x}{4}\cos\frac{5x}{4}$

8. $\sin 4\alpha \cos 5\alpha$

9. $\cos(-x)\sin 2x$

10. $\cos 5x \sin 2x$

In Exercises 11–14, use a product-to-sum formula to find the exact value. (See Example 2, p. 637)

11. $\sin\frac{17\pi}{12}\sin\frac{\pi}{12}$

12. $\sin 82.5^\circ \cos 37.5^\circ$

13. $\cos 97.5^\circ \cos 37.5^\circ$

14. $\cos\frac{25\pi}{24}\sin\frac{17\pi}{24}$

For Exercises 15–18, verify the identities.

15. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$

16. $\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$

17. $2\sin\left(\frac{\pi}{4} + \frac{x-y}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x+y}{2}\right)$
$= \cos y + \sin x$

18. $2\sin\left(\frac{\pi}{4} + \frac{x-y}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x+y}{2}\right)$
$= \cos x - \sin y$

Learn: Apply the Sum-to-Product Formulas

For Exercises 19–22, write each expression as a product. (See Example 3, p. 638)

19. $\sin 7x - \sin 9x$

20. $\sin 3\theta + \sin 10\theta$

21. $\cos 5\beta + \cos 6\beta$

22. $\cos 20x - \cos 14x$

For Exercises 23–26, use a sum-to-product formula to find the exact value. (See Example 4, p. 639)

23. $\sin\frac{11\pi}{12} + \sin\frac{5\pi}{12}$

24. $\sin 165^\circ - \sin 105^\circ$

25. $\cos 285^\circ - \cos 15^\circ$

26. $\cos\frac{23\pi}{12} + \cos\frac{5\pi}{12}$

For Exercises 27–34, verify the identity. (See Example 5, p. 640)

27. $\frac{\cos 3x - \cos x}{\cos 3x + \cos x} = -\tan 2x \tan x$

28. $\frac{\cos x - \cos 3x}{\sin x - \sin 3x} = -\tan 2x$

29. $\sin 2x + \sin 4x + \sin 6x = 4\cos x \cos 2x \sin 3x$

30. $\cos 2x + \cos 4x + \cos 6x$
$= 4\cos x \cos 2x \cos 3x - 1$

31. $\sin 5x - \sin 3x + 2\sin x \cos 2x$
$= 2\sin 2x \cos 3x$

32. $\sin t + \sin 3t + \sin 5t + \sin 7t$
$= 4\cos t \cos 2t \sin 4t$

33. $\frac{\cos 3x - \cos 2x + \cos x}{\sin 3x - \sin 2x + \sin x} = \cot 2x$

34. $\frac{\sin 4t + \sin 3t + \sin 2t}{\cos 4t + \cos 3t + \cos 2t} = \tan 3t$

Mixed Exercises

35. Change $\sin(x + h) - \sin x$ to a product.

36. Change $\cos(x + h) - \cos x$ to a product.

37. Derive formula (2) on page 636.
$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$

38. Derive formula (3) on page 636.
$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$

39. Derive formula (6) on page 628.
$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

40. Derive formula (7) on page 638.

$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

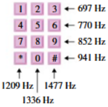
Touch-tone telephones use DTMF (dual-tone multi-frequency) signaling over analog telephone lines. The DTMF keypad is a four-row, three-column grid in which each row represents a low frequency and each column represents a high frequency. Pressing a single key sends a signal formed by the sum of two sine waves. The tone is then decoded by the switching center to determine the key pressed by the user.

The signal produced by pressing any key is given by

$$y = \sin(2\pi \ell t) + \sin(2\pi h t)$$

where ℓ and h represent the low and high frequencies for that key. Frequencies are measured in cycles per second or hertz (Hz).

For Exercises 41–42, use the information given in the figure.



41. Suppose a user makes a long distance call and presses the number 1 on the keypad.

a. Write a function representing this sound as the sum of two sine waves.

b. Write a function representing this sound as the product of sines and cosines.

c. Use a graphing utility to graph the signal on the interval $0 \le t \le 0.01$.

42. Suppose that the number 9 is pressed on a phone keypad.

a. Write a function representing this sound as the sum of two sine waves.

b. Write a function representing this sound as the product of sines and cosines.

c. Use a graphing utility to graph the signal on the interval $0 \le t \le 0.01$.

Write About It

43. Explain how the formulas (2) and (6) on pages 636 and 638 are related.

44. In Lesson 5.2, we derived the reduction formula $A\sin x + B\cos x = k\sin(x + \alpha)$ to write a linear combination of $\sin x$ and $\cos x$ as a single term. Explain how this is different from the sum-to-product formulas.

Expanding Your Skills

For Exercises 45–46, verify the identity.

45. $\cos A\sin(B - C) + \cos B\sin(C - A) + \cos C\sin(A - B) = 0$

46. $\cos A\sin(B - C) - \cos B\sin(C - A) - \cos C\sin(A - B) = 2\cos A\sin(B - C)$

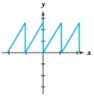
For Exercises 47–48, let *A*, *B*, and *C* represent the angles in $\triangle ABC$. Use the fact that $C = \pi - (A + B)$ to prove the identities.

47. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

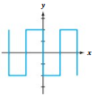
48. $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Technology Connections

In this lesson, we used the product-to-sum formulas to write sums and differences of trigonometric functions. Another type of expression involving the sum or difference of sine and cosine terms is a Fourier series, named after Jean-Baptiste Joseph Fourier (1768–1830). A Fourier series is an expression of the form $A_0 + (A_1\cos x + B_1\sin x) + (A_2\cos 2x + B_2\sin 2x) + (A_3\cos 3x + B_3\sin 3x) + \ldots$ where $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$ are constants. Fourier proved that any continuous function can be represented by a Fourier series. In Exercises 49–50, we use Fourier polynomials (a finite number of terms from a Fourier series) to model a "sawtooth" wave and a "square" wave.



Sawtooth wave



Square wave

49. A "sawtooth" wave is a periodic wave that rises linearly upward and then drops sharply. In music, such waves are generated in digital synthesizers to give high-quality sound without distortion. Given,

$$f(x)=\frac{1}{2}-\frac{1}{\pi}\left(\frac{\sin\pi x}{1}+\frac{\sin 2\pi x}{2}+\frac{\sin 3\pi x}{3}+\cdots\right)$$

a. Graph the first three terms of the function on the window [−4, 4, 1] by [−1, 1.5, 0.5].

b. Graph the first five terms of the function on the window [−4, 4, 1] by [−1, 1.5, 0.5].

50. A "square" wave is a periodic wave that alternates between two fixed values with equal time spent at each value and with negligible transition time between them. Square waves have practical uses in electronics and music, and because of their rectangular pattern, they are used in timing devices to synchronize circuits. Given,

$$f(x)=\frac{4}{\pi}\left(\frac{1}{1}\sin\pi x+\frac{1}{3}\sin 3\pi x+\frac{1}{5}\sin 5\pi x+\cdots\right)$$

a. Graph the first three terms of the Fourier series on the window [−4, 4, 1] by [−2, 2, 1].

b. Graph the first five terms of the Fourier series on the window [−4, 4, 1] by [−2, 2, 1].

51. The sound wave generated by the note middle C on a piano is modeled by $y_C=k\sin 524\pi t$, and the sound wave for A above middle C is modeled by $y_A=k\sin 880\pi t$. In each case, y is the pressure of the sound wave, and the amplitude k is related to the loudness of sound. When two keys on a piano are struck simultaneously, a sound is made called a musical chord. The chord C-A is represented by $y=k(\sin 524\pi t+\sin 880\pi t)$.

a. Use the sum-to-product formula to represent this function as a product of factors for $k=1$.

b. Graph the function $y=\sin 524\pi t+\sin 880\pi t$ and your result from part (a) on the window [0, 0.01, 0.001] by [−2, 2, 1] and verify that the graphs are the same.

Problem Recognition Exercises

Verifying Trigonometric Identities

For Exercises 1–14, verify the identities.

1. $\frac{\cot x\cos x}{\cot x+\cos x}=\frac{1-\sin x}{\cos x}$

2. $\frac{1}{2}\cot\frac{\theta}{2}-\frac{1}{2}\tan\frac{\theta}{2}=\cos\theta$

3. $\frac{\sin(A-B)}{\sin A\sin B}=\cot B-\cot A$

4. $\frac{\sin 7x-\sin 5x}{\cos 7x+\cos 5x}=\tan x$

5. $\frac{\cos\theta}{1-\sin\theta}=\frac{1-\sin\theta}{\cos\theta}=2\tan\theta$

6. $1+\tan x\tan\frac{x}{2}=\sec x$

7. $\sin\left(\frac{\pi}{3}+\beta\right)-\sin\left(\frac{\pi}{3}-\beta\right)=\sin\beta$

8. $\sin\theta\cot\theta+\cos^2\theta\sec\theta=2\cos\theta$

9. $1-4\sin^4 x-2\sin^2 x\cos 2x=\cos 2x$

10. $\frac{\cos 3x-\cos 7x}{\sin 2x}=2\sin 5x$

11. $\tan\left(x+\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)=\frac{4\tan x}{1-\tan^2 x}$

12. $\tan x+\sec x\csc x(1-2\sin^2 x)=\cot x$

13. $\left(\frac{\tan B+1}{\tan B-1}\right)^2=\frac{1+\sin 2B}{1-\sin 2B}$

14. $\frac{\cos(2x+y)+\cos(2x-y)}{\cos 2x}=2\cos y$

Go online for more practice problems.

Lesson 5-5

Trigonometric Equations

What Will You Learn?

After completing this lesson, you should be able to:

- Solve Trigonometric Equations in Linear Form
- Solve Trigonometric Equations Involving Multiple or Compound Angles
- Solve Higher-Degree Trigonometric Equations
- Use Identities to Solve Trigonometric Equations
- Use Inverse Functions to Express Solutions to Trigonometric Equations
- Approximate the Solutions to Trigonometric Equations on a Calculator

Learn: Solve Trigonometric Equations in Linear Form

In this chapter we have worked extensively with trigonometric identities. These are equations that are true for all values of the variable for which the expressions within the equation are defined. Now we look at conditional trigonometric equations. These are equations that are true for some values of the variable but not for others.

For example, from the unit circle we know that the equation $\cos x = \frac{1}{2}$ has solutions $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$ on the interval $[0, 2\pi)$ (Figure 5-9).

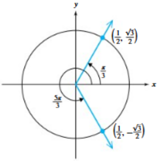

y
x
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{3}$
$\frac{5\pi}{3}$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Figure 5-9

However, because the cosine function is periodic, any integer multiple of 2π added to $\frac{\pi}{3}$ or $\frac{5\pi}{3}$ also has a cosine value of $\frac{1}{2}$. Therefore, there are infinitely many solutions to the equation. For any integer n, the solutions are:

$$x = \frac{\pi}{3} + 2n\pi \quad \left(\frac{\pi}{3} \text{ plus any integer multiple of } 2\pi\right)$$

$$x = \frac{5\pi}{3} + 2n\pi \quad \left(\frac{5\pi}{3} \text{ plus any integer multiple of } 2\pi\right)$$

The graphs of $y = \cos x$ and $y = \frac{1}{2}$ illustrate several such solutions (Figure 5-10).

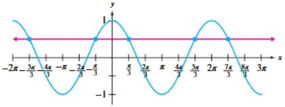


Figure 5-10

We can write the solution set to the equation $\cos x = \frac{1}{2}$ over a specific interval such as $[0, 2\pi)$, or we can write the solution set for the general solution.

The solution set over the interval $[0, 2\pi)$ is $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$.

The general solution set is $\left\{x \,\middle|\, x = \frac{\pi}{3} + 2n\pi,\ x = \frac{5\pi}{3} + 2n\pi\right\}$.

Insights Solutions to trigonometric equations such as $\cos x = \frac{1}{2}$ are in radians.

In Example 1, we solve a trigonometric equation that is linear in the variable $\tan x$.

Example 1: Solving a Trigonometric Equation in Linear Form

Solve $2\tan x = \sqrt{3} - \tan x$

a. Over $[0, 2\pi)$. **b.** Over the set of real numbers.

Solution:

a. This equation is linear in the variable $\tan x$. Therefore, isolate $\tan x$.

$2\tan x = \sqrt{3} - \tan x$	
$3\tan x = \sqrt{3}$	Add $\tan x$ to both sides.
$\tan x = \frac{\sqrt{3}}{3}$	Divide by 3.
$x = \frac{\pi}{6}, x = \frac{7\pi}{6}$	On the interval $[0, 2\pi)$, $\tan x$ equals $\frac{\sqrt{3}}{3}$ for $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$.
The solution set is $\left\{\frac{\pi}{6}, \frac{7\pi}{6}\right\}$.	Write the solution set.

b. Any x of the form

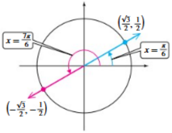
$$x = \frac{\pi}{6} + n\pi \quad \text{or} \quad x = \frac{7\pi}{6} + n\pi$$

will provide a solution for any integer n.

The period of the tangent function is π. Therefore, adding $n\pi$ to either solution for any integer n also gives a solution.

However, notice that for $n = 1$, $\frac{\pi}{6} + (1)\pi = \frac{7\pi}{6}$. This means that the formula $x = \frac{\pi}{6} + n\pi$ is sufficient to generate all solutions to the equation.

Therefore, the general solution is $\left\{x \,\middle|\, x = \frac{\pi}{6} + n\pi\right\}$.


$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$x = \frac{7\pi}{6}$
$x = \frac{\pi}{6}$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

Apply the Skills

1. Solve $\sin x = \sqrt{2} - \sin x$

a. Over $[0, 2\pi)$. **b.** Over the set of real numbers.

Learn: Solve Trigonometric Equations Involving Multiple or Compound Angles

On the interval $[0, 2\pi)$, there are two solutions to the equation $\cos x = \frac{1}{2}$ (Figure 5-11).

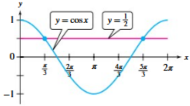


Figure 5-11

Now consider the equation $\cos 4x = \frac{1}{2}$. The factor of 4 compresses the graph of $y = \cos 4x$ horizontally, and the period is $\frac{\pi}{2}$. Therefore, the interval $[0, 2\pi)$ contains four complete cycles of $y = \cos 4x$ and eight solutions to the equation $\cos 4x = \frac{1}{2}$ (Figure 5-12).

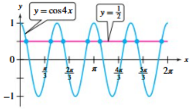


Figure 5-12

This discussion illustrates that the period plays an important role in determining the number of solutions to a trigonometric equation containing a multiple angle.

Example 2: Solving an Equation Containing a Multiple Angle

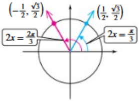
Given $2 \sin 2x - \sqrt{3} = 0$,

a. Write the solution set for the general solution.

b. Write the solution set on the interval $[0, 2\pi)$.

Solution:

a. To solve $2 \sin 2x - \sqrt{3} = 0$, begin by isolating $\sin 2x$.


$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$2x = \frac{2\pi}{3}$
$2x = \frac{\pi}{3}$

$$\sin 2x = \frac{\sqrt{3}}{2}$$

From the unit circle, $\sin 2x = \frac{\sqrt{3}}{2}$ for

$$2x = \frac{\pi}{3} \quad \text{and} \quad 2x = \frac{2\pi}{3}.$$

$$2x = \frac{\pi}{3} + 2n\pi \qquad 2x = \frac{2\pi}{3} + 2n\pi$$

Any integer multiple of 2π added to $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ will provide a solution.

$$\frac{1}{2} \cdot 2x = \frac{1}{2} \cdot \left(\frac{\pi}{3} + 2n\pi\right) \qquad \frac{1}{2} \cdot 2x = \frac{1}{2} \cdot \left(\frac{2\pi}{3} + 2n\pi\right)$$

Solve for x.

$$x = \frac{\pi}{6} + n\pi \qquad x = \frac{\pi}{3} + n\pi$$

General solution: $\left\{x \middle| x = \frac{\pi}{6} + n\pi,\ x = \frac{\pi}{3} + n\pi\right\}$

Insights

As an alternative approach, we can use a substitution to help solve an equation with a compound argument. In Example 2, letting $u = 2x$ gives the simpler equation $2\sin u - \sqrt{3} = 0$.

The solutions are $u = \frac{\pi}{3} + 2n\pi$ and $u = \frac{2\pi}{3} + 2n\pi$.

Then using back substitution we can solve for x.

$$2x = \frac{\pi}{3} + 2n\pi \quad \text{and} \quad 2x = \frac{2\pi}{3} + 2n\pi$$

b. To find solutions on the interval $[0, 2\pi)$, substitute $n = \ldots, -1, 0, 1, 2, \ldots$

n	$x = \frac{\pi}{6} + n\pi$	$x = \frac{\pi}{3} + n\pi$
-1	$x = \frac{\pi}{6} + (-1)\pi = -\frac{5\pi}{6}$ Not on $[0, 2\pi)$	$x = \frac{\pi}{3} + (-1)\pi = -\frac{2\pi}{3}$ Not on $[0, 2\pi)$
0	$x = \frac{\pi}{6} + (0)\pi = \frac{\pi}{6}$	$x = \frac{\pi}{3} + (0)\pi = \frac{\pi}{3}$
1	$x = \frac{\pi}{6} + (1)\pi = \frac{7\pi}{6}$	$x = \frac{\pi}{3} + (1)\pi = \frac{4\pi}{3}$
2	$x = \frac{\pi}{6} + (2)\pi = \frac{13\pi}{6}$ Not on $[0, 2\pi)$	$x = \frac{\pi}{3} + (2)\pi = \frac{7\pi}{3}$ Not on $[0, 2\pi)$

The solution set on the interval $[0, 2\pi)$ is $\left\{\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}\right\}$ (Figure 5-13).

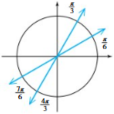


Figure 5-13

The solutions to the equation $2\sin 2x - \sqrt{3} = 0$ are the x-coordinates of the points of intersection of the graphs of $y = 2\sin 2x - \sqrt{3}$ and $y = 0$. See Figure 5-14.

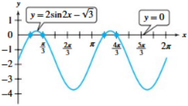


Figure 5-14

Apply the Skills

2. Solve the equation $1 + \cos 3x = 0$.

a. Write the solution set for the general solution.

b. Write the solution set on the interval $[0, 2\pi)$.

Example 3: Solving an Equation Involving a Half-Angle

Given $-1 + \sin\frac{x}{2} = 0$,

- **a.** Write the solution set for the general solution.
- **b.** Write the solution set on the interval $[0, 2\pi)$.

Solution:

a. $-1 + \sin\frac{x}{2} = 0$

$\sin\frac{x}{2} = 1$ Isolate $\sin\frac{x}{2}$.

$\frac{x}{2} = \frac{\pi}{2} + 2n\pi$ From the unit circle, $\sin\frac{x}{2} = 1$ for $\frac{x}{2} = \ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$.

$2 \cdot \frac{x}{2} = 2 \cdot \left(\frac{\pi}{2} + 2n\pi\right)$ Solve for x.

$x = \pi + 4n\pi$

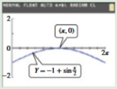
The general solution is $\{x \mid x = \pi + 4n\pi\}$.

b. Substituting integer values of n gives $x = \ldots -3\pi, \pi, 5\pi, 9\pi, \ldots$.

The solution set on the interval $[0, 2\pi)$ is $\{\pi\}$.

Insights

When solving a trigonometric equation with a compound argument, sometimes it is helpful to use a graphing utility to illustrate the number of solutions on a given interval. In Example 3, we can graph $Y = -1 + \sin\frac{x}{2}$ on the interval $[0, 2\pi)$. From the graph it appears that only one solution exists on the interval.



Apply the Skills

3. Solve the equation $1 + 2\cos\frac{x}{2} = 0$.

- **a.** Write the solution set for the general solution.
- **b.** Write the solution set on the interval $[0, 2\pi)$.

In Example 4, we solve a trigonometric equation where the argument to the trigonometric function has more than one term.

Example 4: Solving an Equation Involving a Compound Angle

Given $\sin\left(x - \frac{\pi}{3}\right) = -\frac{\sqrt{2}}{2}$,

- **a.** Write the solution set for the general solution.
- **b.** Write the solution set on the interval $[0, 2\pi)$.

Solution:

a. $\sin\left(x - \frac{\pi}{3}\right) = -\frac{\sqrt{2}}{2}$

$\sin u = -\frac{\sqrt{2}}{2}$ Let $u = x - \frac{\pi}{3}$.

$u = \frac{5\pi}{4} + 2n\pi,$ $u = \frac{7\pi}{4} + 2n\pi$ From the unit circle, recognize that $\sin u = -\frac{\sqrt{2}}{2}$ for $u = \frac{5\pi}{4}$ and $u = \frac{7\pi}{4}$.

$x - \frac{\pi}{3} = \frac{5\pi}{4} + 2n\pi,$ $x - \frac{\pi}{3} = \frac{7\pi}{4} + 2n\pi$ Back substitute.

$x = \frac{5\pi}{4} + \frac{\pi}{3} + 2n\pi,$ $x = \frac{7\pi}{4} + \frac{\pi}{3} + 2n\pi$ Add $\frac{\pi}{3}$ to both sides.

$x = \frac{15\pi}{12} + \frac{4\pi}{12} + 2n\pi,$ $x = \frac{21\pi}{12} + \frac{4\pi}{12} + 2n\pi$ Write terms with a common denominator.

$x = \frac{19\pi}{12} + 2n\pi,$ $x = \frac{25\pi}{12} + 2n\pi$

The general solution is $\left\{x \middle| x = \frac{19\pi}{12} + 2n\pi,\ x = \frac{25\pi}{12} + 2n\pi\right\}$.

b. To find solutions on the interval $[0, 2\pi)$, substitute $n = ..., -1, 0, 1, ...$

n	$x = \frac{19\pi}{12} + 2n\pi$	$x = \frac{25\pi}{12} + 2n\pi$
-1	$x = \frac{19\pi}{12} + 2(-1)\pi = -\frac{5\pi}{12}$ Not on $[0, 2\pi)$	$x = \frac{25\pi}{12} + 2(-1)\pi = \frac{\pi}{12}$
0	$x = \frac{19\pi}{12} + 2(0)\pi = \frac{19\pi}{12}$	$x = \frac{25\pi}{12} + 2(0)\pi = \frac{25\pi}{12}$ Not on $[0, 2\pi)$
1	$x = \frac{19\pi}{12} + 2(1)\pi = \frac{43\pi}{12}$ Not on $[0, 2\pi)$	$x = \frac{25\pi}{12} + 2(1)\pi = \frac{49\pi}{12}$ Not on $[0, 2\pi)$

The solution set on the interval $[0, 2\pi)$ is $\left\{\frac{\pi}{12}, \frac{19\pi}{12}\right\}$.

The graphs of $y = \sin\left(x - \frac{\pi}{3}\right)$ and $y = -\frac{\sqrt{2}}{2}$ are shown in Figure 5-15.

The x-coordinates of the points of intersection of the graphs make up the solution set to the equation $\sin\left(x - \frac{\pi}{3}\right) = -\frac{\sqrt{2}}{2}$.

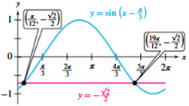


Figure 5-15

Apply the Skills

4. Solve the equation $\cot\left(x - \frac{\pi}{4}\right) = -1$.

a. Write the solution set for the general solution.

b. Write the solution set on the interval $[0, 2\pi)$.

Learn: Solve Higher-Degree Trigonometric Equations

In Examples 5 and 6, we solve trigonometric equations that are quadratic in $\sin x$, $\cos x$, or $\tan x$. As with standard algebraic quadratic equations, we set one side equal to zero and factor the other side. If the nonzero side is not factorable, we can use the square root property or apply the quadratic formula.

Example 5: Solving a Quadratic Trigonometric Equation

Solve the equation on the interval $[0, 2\pi)$. $2\sin^2 x + 7\sin x - 4 = 0$

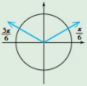
Solution:

$2\sin^2 x + 7\sin x - 4 = 0$	This equation is quadratic in form. $2u^2 + 7u - 4 = 0$, where $u = \sin x$.
$(2\sin x - 1)(\sin x + 4) = 0$	Factor as $(2u - 1)(u + 4) = 0$.
$2\sin x - 1 = 0$ or $\sin x + 4 = 0$	Set each factor equal to zero and solve the resulting equation.
$\sin x = \frac{1}{2}$ or $\sin x = -4$	The only solutions are those for which $\sin x = \frac{1}{2}$.
$x = \frac{\pi}{6}, x = \frac{5\pi}{6}$ No solution here because $-1 \le \sin x \le 1$.	From the unit circle, recognize that $\sin x = \frac{1}{2}$ for $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ on the interval $[0, 2\pi)$.

The solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$.

Insights

If we were asked to find the general solution, it would consist of
$x = \frac{\pi}{6} + 2n\pi$ and $x = \frac{5\pi}{6} + 2n\pi$.



Apply the Skills

5. Solve the equation on the interval $[0, 2\pi)$. $2\cos^2 x - \cos x - 3 = 0$

In Example 6, we apply the square root property to solve a trigonometric equation of the form $u^2 = k$.

Example 6: Solving a Quadratic Trigonometric Equation

Solve the equation on the interval $[0, 2\pi)$. $\cot^2 x - 3 = 0$

Solution:

$\cot^2 x - 3 = 0$

$\cot^2 x = 3$ Isolate $\cot^2 x$ to write the equation in the form $u^2 = k$.

$\cot x = \pm\sqrt{3}$ Apply the square root property.

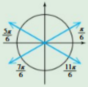
$\cot x = \sqrt{3}$ $\cot x = -\sqrt{3}$ From the unit circle, recognize that $\cot x = \sqrt{3}$ for $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$ and that $\cot x = -\sqrt{3}$ for $x = \frac{5\pi}{6}$ and $x = \frac{11\pi}{6}$.

$x = \frac{\pi}{6}, x = \frac{7\pi}{6}$ $x = \frac{5\pi}{6}, x = \frac{11\pi}{6}$

The solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$.

Insights

If we were asked to find the general solution, notice that each pair of solutions differs by π radians (180°). The general solution consists of $x = \frac{\pi}{6} + n\pi$ and $x = \frac{5\pi}{6} + n\pi$.



Apply the Skills

6. Solve the equation on the interval $[0, 2\pi)$.

$$4\cos^2 x - 3 = 0$$

For trigonometric equations containing more than one trigonometric function, sometimes we can set one side equal to zero and factor the other side to separate factors.

Example 7: Solving a Trigonometric Equation by Factoring

Solve the equation on the interval $[0, 2\pi)$. $\tan x \sin^2 x = \tan x$

Solution:

$\tan x \sin^2 x = \tan x$

$\tan x \sin^2 x - \tan x = 0$ Set one side equal to zero.

$\tan x(\sin^2 x - 1) = 0$ Factor.

$\tan x = 0$ or $\sin^2 x - 1 = 0$ Set each factor equal to zero.

$x = 0, x = \pi$ $\sin^2 x = 1$ The equation $\sin^2 x - 1 = 0$ is quadratic.

$\sin x = \pm 1$ Isolate $\sin^2 x$ and apply the square root property.

$x = \frac{\pi}{2}, x = \frac{3\pi}{2}$

At this point, particular care must be taken before stating the solution set.

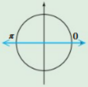
Recall that the tangent function is undefined at odd multiples of $\frac{\pi}{2}$. Therefore, the values $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ do not check in the original equation and must be excluded from the solution set.

The solution set is $\{0, \pi\}$.

Insights

Notice that the solutions differ by π. Therefore, we can generate the general solution with one statement.

$$x = n\pi$$



Apply the Skills

7. Solve the equation on the interval $[0, 2\pi)$.

$$\cot x \cos^2 x = \cot x$$

Learn: Use Identities to Solve Trigonometric Equations

The identities studied earlier in this chapter can be of great service when solving a trigonometric equation as shown in Examples 8–10.

Example 8: Using Identities to Solve an Equation

Solve the equation on the interval $[0, 2\pi)$. $\sec^2 x - \tan x = 1$

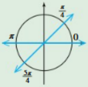
Solution:

$\sec^2 x - \tan x = 1$ $(\tan^2 x + 1) - \tan x = 1$	Use the Pythagorean identity $\tan^2 x + 1 = \sec^2 x$ and substitute $\tan^2 x + 1$ for $\sec^2 x$. The left side is now quadratic in the variable $\tan x$.
$\tan^2 x - \tan x = 0$	Set one side of the equation equal to zero.
$\tan x(\tan x - 1) = 0$	Factor.
$\tan x = 0$ or $\tan x - 1 = 0$	Set each factor equal to zero.
$x = 0, x = \pi$ $\quad$ $\tan x = 1$ $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$	From the unit circle, recognize that $\tan x = 0$ for $x = 0$ and $x = \pi$, and that $\tan x = 1$ for $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

The solution set is $\left\{0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}\right\}$.

Insights

If we were asked to find the general solution, notice that the pairs of solutions differ by π radians (180°) and can be combined into one statement. The general solution consists of $x = n\pi$ and $x = \frac{\pi}{4} + n\pi$.



Apply the Skills

8. Solve the equation on the interval $[0, 2\pi)$.

$$\csc^2 x - 1 = \cot x$$

Example 9: Using Identities to Solve an Equation

Solve the equation on the interval $[0, 2\pi)$. $\cos 3x + \cos x = 0$

Solution:

$$\cos 3x + \cos x = 0$$

$$2\cos\frac{3x + x}{2}\cos\frac{3x - x}{2} = 0$$

Apply the sum-to-product formula $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$.

$$2\cos 2x \cos x = 0$$

$$\cos 2x = 0 \quad \text{or} \quad \cos x = 0$$

Set each trigonometric factor equal to zero.

$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots \quad \text{or} \quad x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

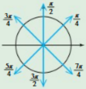
$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots \quad \text{or} \quad x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

From the unit circle, recognize that $\cos 2x = 0$ for $2x$ equal to odd multiples of $\frac{\pi}{2}$, and that $\cos x = 0$ for x equal to odd multiples of $\frac{\pi}{2}$.

The solution set on the interval $[0, 2\pi)$ is $\left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}\right\}$.

Insights

Notice that the odd multiples of $\frac{\pi}{4}$ differ by $\frac{\pi}{2}$. The odd multiples of $\frac{\pi}{2}$ differ by π. Therefore, the general solution consists of $x = \frac{\pi}{4} + \frac{n\pi}{2}$ and $x = \frac{\pi}{2} + n\pi$.



Apply the Skills

9. Solve the equation on the interval $[0, 2\pi)$.

$$\sin 3x + \sin x = 0$$

In Example 10, we encounter a situation where squaring both sides of an equation results in a quadratic equation that is easily factored and solved. However, we must remember to check all potential solutions to eliminate any extraneous solutions.

Example 10: Squaring Both Sides of a Trigonometric Equation

Solve the equation on the interval $[0, 2\pi)$. $\sin x + 1 = \cos x$

Solution:

$\sin x + 1 = \cos x$

$(\sin x + 1)^2 = (\cos x)^2$ Squaring both sides of the equation will result in $\sin^2 x$ and $\cos^2 x$ terms.

$\sin^2 x + 2\sin x + 1 = \cos^2 x$

$\sin^2 x + 2\sin x + 1 = 1 - \sin^2 x$ Replacing $\cos^2 x$ by $1 - \sin^2 x$ results in a quadratic equation in the variable $\sin x$.

$2\sin^2 x + 2\sin x = 0$

$2\sin x(\sin x + 1) = 0$ Set one side equal to zero and factor the other side.

$\sin x = 0$ or $\sin x + 1 = 0$ Set each factor equal to zero.

$x = 0, x = \pi$ $\sin x = -1$ From the unit circle, $\sin x = 0$ for $x = 0, x = \pi$.

$x = \frac{3\pi}{2}$ $\sin x = -1$ for $x = \frac{3\pi}{2}$.

Check Your Work

After squaring both sides of an equation, we must check all potential solutions to eliminate any extraneous solutions.

Check: $x = 0$	Check: $x = \pi$	Check: $x = \frac{3\pi}{2}$
$\sin(0) + 1 \overset{?}{=} \cos(0)$	$\sin(\pi) + 1 \overset{?}{=} \cos(\pi)$	$\sin\left(\frac{3\pi}{2}\right) + 1 \overset{?}{=} \cos\left(\frac{3\pi}{2}\right)$
$0 + 1 \overset{?}{=} 1$ ✓(True)	$0 + 1 \overset{?}{=} -1$ (False)	$-1 + 1 \overset{?}{=} 0$ ✓(True)

The solution set is $\left\{0, \frac{3\pi}{2}\right\}$.

The graphs of $y = \sin x + 1$ and $y = \cos x$ are shown in Figure 5-16. The solutions to the equation $\sin x + 1 = \cos x$ are the x-coordinates of the points of intersection.

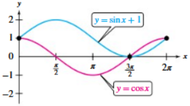


Figure 5-16

Apply the Skills

10. Solve the equation on the interval $[0, 2\pi)$.

$$\tan x + 1 = \sec x$$

In Example 10, we could have taken an alternative approach to solving $\sin x + 1 = \cos x$ by first writing the equation as $\sin x - \cos x = -1$. The left side of the equation can then be written as a single term of the form $k\sin(x + \alpha)$ by using the technique offered on page 618. This approach is examined in Exercise 119, on page 661.

Learn: Use Inverse Functions to Express Solutions to Trigonometric Equations

In Example 11, we encounter equations in which the solutions are not "special" angles (multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$). Therefore, we will use inverse functions to express the exact values of the solutions.

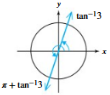
Example 11: Using Inverse Functions to Solve Trigonometric Equations

Solve the equation on the interval $[0, 2\pi)$. Give the exact solutions in radians and give approximations in degrees rounded to 1 decimal place.

a. $\tan x = 3$ **b.** $\sin x = -0.8$

Solution:

a. No multiples of the "special" angles $\left(\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \cdots\right)$ have a tangent value equal to 3. Therefore, our strategy is to find one solution to the equation by using the inverse tangent function. Then use the periodicity of the tangent function to find other solutions.



Given $\tan x = 3$, two solutions are

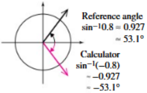
$x = \tan^{-1}3 \approx 1.249$ $(\approx 71.6°)$ The value $\tan^{-1}3$ is a first quadrant angle.

$x = \pi + \tan^{-1}3 \approx 4.391$ $(\approx 251.6°)$ The corresponding angle in Quadrant III is $\pi + \tan^{-1}3$.

The solution set is $\{\tan^{-1}3, \pi + \tan^{-1}3\}$ or approximately $\{71.6°, 251.6°\}$.

b. Given $\sin x = -0.8$, one solution is $x = \sin^{-1}(-0.8) \approx -0.927$ $(\approx -53.1°)$.

Notice that $x = \sin^{-1}(-0.8) \approx -0.927$ is a negative fourth quadrant angle.

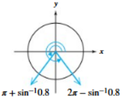


The reference angle is

$|\sin^{-1}(-0.8)| = \sin^{-1}0.8 \approx 0.927$.

Therefore, the solutions on the interval $[0, 2\pi)$ are

$\pi + \sin^{-1}0.8 \approx 4.069$ $(\approx 233.1°)$
$2\pi - \sin^{-1}0.8 \approx 5.356$ $(\approx 306.9°)$



The solution set is $\{\pi + \sin^{-1}0.8, 2\pi - \sin^{-1}0.8\}$ or approximately $\{233.1°, 306.9°\}$.

Apply the Skills

11. Solve the equation on the interval $[0, 2\pi)$. Give the exact solutions in radians and give approximations in degrees rounded to 1 decimal place.

a. $\cos x = 0.6$ **b.** $\tan x = -5$

Learn: Approximate the Solutions to Trigonometric Equations on a Calculator

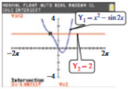
Sometimes finding the solutions to a trigonometric equation is not possible using analytical methods. In particular, this happens when there is a combination of trigonometric expressions and variable algebraic expressions within the equation. In Example 12, we will solve the equation $x^2 - \sin 2x = 2$ by graphing $Y_1 = x^2 - \sin 2x$ and $Y_2 = 2$. The x-coordinates of the points of intersection represent solutions to the equation.

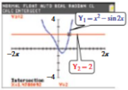
Example 12: Using a Graphing Utility to Solve an Equation

Use a graphing: cdf utility to solve the equation $x^2 - \sin 2x = 2$. Round the solutions to 4 decimal places.

Solution:

Graph $Y_1 = x^2 - \sin 2x$ and $Y_2 = 2$.





The x-coordinates of the points of intersection are approximately −1.0821, and 1.4780. Since $-1 \leq \sin 2x \leq 1$, the end behavior of the graph of $Y_1 = x^2 - \sin 2x$ is dictated by the term x^2. Furthermore, the graph of $y = x^2$ does not cross the line $y = 2$ more than two times, so we know that these are the only solutions.

The solution set is $\{-1.0821, 1.4780\}$.

Insights

As an alternative method to solve the equation in Example 12, set one side of the equation equal to zero.

$$x^2 - \sin 2x - 2 = 0$$

Then set Y_1 equal to the nonzero expression on the left.

$$Y_1 = x^2 - \sin 2x - 2$$

Use the ZERO function on a graphing utility to find the zeros of the function Y_1. These are the solutions to the equation.

Apply the Skills

12. Use a graphing utility to solve the equation $\cos 3x = 0.6x$. Round the solutions to 4 decimal places.

Practice 5-5

Practice Exercises

Prerequisite Review

R.1. Factor by grouping. $15tx + 10t + 12x + 8$

R.2. Factor the trinomial. $39x + 10x^2 + 35$

For Exercises R.3–R.6, solve the equation.

R.3. $4v^2 - 3v - 7 = 0$

R.4. $p^3 = 9p$

R.5. $36x^3 + 9x^2 - 16x - 4 = 0$

R.6. $4y^4 - 29y^2 + 7 = 0$

Concept Connections

1. How many solutions to the equation $\cos x = \frac{\sqrt{3}}{2}$ exist?

How many solutions exist on the interval $[0, 2\pi)$?

2. The solutions to the equation $\tan^2 x = 1$ on the interval $[0, 2\pi)$ are __________. The formula $x =$ __________ can be used to generate *all* solutions to the equation over the set of real numbers.

3. Given $\sin 2x = \frac{\sqrt{2}}{2}$, the general solution consists of $\frac{\pi}{8} + n\pi$ and $\frac{3\pi}{8} + n\pi$. What are the solutions on the interval $[0, 2\pi)$?

4. Given the equation $\sin x = 0.2$, one solution is $x = \sin^{-1}(0.2)$. What is the other solution on the interval $[0, 2\pi)$?

Learn: Solve Trigonometric Equations in Linear Form

For Exercises 5–8, determine if the given value is a solution to the equation.

5. $2\sin x + \frac{\sqrt{3}}{2} = \sin x$

a. $\frac{4\pi}{3}$ **b.** $-\frac{\pi}{3}$

6. $4(\cot x - \sqrt{3}) = \cot x - \sqrt{3}$

a. $\frac{5\pi}{6}$ **b.** $\frac{7\pi}{6}$

7. $10\cot x - \sqrt{2} = 8\cos x$

a. $\frac{7\pi}{4}$ **b.** $\frac{3\pi}{4}$

8. $3\tan x - 2\sqrt{3} = 2\tan x - \sqrt{3}$

a. $\frac{\pi}{3}$ **b.** $\frac{4\pi}{3}$

For Exercises 9–12, solve the equation over the interval $[0, 2\pi)$.

9. a. $\sin\theta = \frac{\sqrt{2}}{2}$ **b.** $\sin\theta = -\frac{\sqrt{2}}{2}$

10. a. $\cos\alpha = -\frac{\sqrt{3}}{2}$ **b.** $\cos\alpha = \frac{\sqrt{3}}{2}$

11. a. $\cot\beta = -\frac{\sqrt{3}}{3}$ **b.** $\cot\beta = \frac{\sqrt{3}}{3}$

12. a. $\tan x = 1$ **b.** $\tan x = -1$

For Exercises 13–18, solve the equation

a. over the interval $[0, 2\pi)$. (**See Example 1, p. 645**)

b. over the set of real numbers.

13. $2\sin x + 5 = 6$

14. $3\sqrt{2} + 6\cos x = 0$

15. $5\sec x + 10 = 3\sec x + 14$

16. $3(2\csc x - \sqrt{3}) = \sqrt{3}$

17. $3\tan x + 1 = -2(2 + \tan x)$

18. $5\cot x + 2\sqrt{3} = -2\cot x - 5\sqrt{3}$

Learn: Solve Trigonometric Equations Involving Multiple or Compound Angles

For Exercises 19–20, identify the number of solutions to the equation on the interval $[0, 2\pi)$.

19. a. $\cos x = \frac{1}{2}$ **b.** $\cos 4x = \frac{1}{2}$ **c.** $\cos\frac{x}{2} = \frac{1}{2}$

20. a. $\sin x = -1$ **b.** $\sin 2x = -1$ **c.** $\sin\frac{x}{2} = -1$

21. a. Write the solution set to the equation $\cos u = 0$ on the interval $[0, 2\pi)$.

b. Write the solution set to the equation $\cos 2x = 0$ on the interval $[0, 2\pi)$.

22. a. Write the solution set to the equation $\sin u = 1$ on the interval $[0, 2\pi)$.

b. Write the solution set to the equation $\sin\left(x - \frac{\pi}{2}\right) = 1$ on the interval $[0, 2\pi)$.

For Exercises 23–32, solve the equation.

a. Write the solution set for the general solution.

b. Write the solution set on the interval $[0, 2\pi)$. **(See Examples 2–4, pp. 646–650)**

23. $2\cos 2x = \sqrt{3}$

24. $5 + \sin 3x = 4$

25. $\tan(-3x) = \sqrt{3}$

26. $\sec(-2x) = -\sqrt{2}$

27. $\sqrt{2}\sin\frac{x}{2} - 1 = 0$

28. $\sqrt{5}\cot\frac{x}{2} = \sqrt{5}$

29. $\csc\left(x - \frac{\pi}{3}\right) = 2$

30. $\cos\left(x + \frac{\pi}{4}\right) = \frac{1}{2}$

31. $\tan\left(x - \frac{\pi}{2}\right) + 1 = 0$

32. $\cot\left(x - \frac{\pi}{2}\right) + \sqrt{3} = 0$

Learn: Solve Higher-Degree Trigonometric Equations

For Exercises 33–42, solve the equation on the interval $[0, 2\pi)$. (See Examples 5–7)

33. $(\tan\theta + 1)(\sec\theta - 2) = 0$

34. $(\cot x - 1)(2\sin x + 1) = 0$

35. $\sec^2\theta - 2 = 0$

36. $2\sin^2 x - 1 = 0$

37. $\cos^2 x + 2\cos x - 3 = 0$

38. $2\csc^2 x - 5\csc x + 2 = 0$

39. $2\sin^2\alpha + \sin\alpha - 1 = 0$

40. $2\cos^2 x + 5\cos x + 2 = 0$

41. $\cos x\tan^2 x = 3\cos x$

42. $4\sin^2 x\cot x - 3\cot x = 0$

Learn: Use Identities to Solve Trigonometric Equations

For Exercises 43–56, solve the equation on the interval $[0, 2\pi)$. (See Examples 8–10)

43. $6\cos^2 x - 7\sin x - 1 = 0$

44. $2\sin^2 x - \cos x + 1 = 0$

45. $2\cos x - 5 = 3\sec x$

46. $4\csc x + \sin x + 5 = 0$

47. $2\cot x = \csc x$

48. $2\tan x + \sec x = 0$

49. $\cos 4x - \cos 2x = 0$

50. $\sin\theta - \sin 3\theta = 0$

51. $\sin 3x + \sin x = \cos x$

52. $\cos 3\theta - \cos\theta = \sin 2\theta$

53. $1 - \sin x = \cos x$

54. $\sec x + 1 = \tan x$

55. $\tan 2x + 1 = \sec 2x$

56. $\sin 2x = 1 - \cos 2x$

Learn: Use Inverse Functions to Express Solutions to Trigonometric Equations

For Exercises 57–62, solve the equations on the interval $[0, 2\pi)$. Give the exact solutions in radians and give approximations in degrees rounded to 1 decimal place. (See Example 11, p. 655)

57. $\cos x = \frac{3}{11}$

58. $\tan x = -3$

59. $4\tan x = -7$

60. $6\sin x = 1$

61. $(\sin x - 1)(4\sin x - 3) = 0$

62. $(3\cos x + 2)(\cos x - 1) = 0$

Learn: Approximate the Solutions to Trigonometric Equations on a Calculator

For Exercises 63–68, use a graphing utility to solve the equation. Round the solutions to 4 decimal places. (See Example 12, p. 656)

63. $\sin x = 1 - x^3$

64. $\cos 2x = x^2 - 2$

65. $\cos x = \ln x^2$

66. $\sin x = (\ln x)^2$

67. $3^{\sin x} = 0.5x + 1$

68. $3^{\cos x} = -0.5x + 1$

Mixed Exercises

For Exercises 69–88, solve the equation on the interval $[0, 2\pi)$.

69. $\sin 2x = 2\sin x$

70. $\cos 2x = \cos x$

71. $\cos 6x - 5\sin 3x + 2 = 0$

72. $\cos 4x - 3\cos 2x - 1 = 0$

73. $\cos 2x - 3\sin x + 1 = 0$

74. $4\sin^2 x = 1 - 4\cos 2x$

75. $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = 1$

76. $\dfrac{\cos 3x + \cos x}{2\cos 2x} = -1$

77. $\sin 5x + \sin x + \cos 4x - \cos 2x = 0$

78. $\sin 4x - \sin 2x - \cos 3x = 0$

79. $\sin 3\theta = \sin 2\theta$

80. $\tan 2\theta \tan\theta = 1$

81. $15\cos^2 x - 7\cos x - 2 = 0$

82. $20\sin^2 x - 7\sin x - 3 = 0$

83. $16\cos^2 x - 8\cos x - 1 = 0$

84. $16\sin^2 x - 12\sin x + 1 = 0$

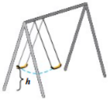
85. $\sin^2 x - 4\sin x + 1 = 0$

86. $\cos^2 x - 4\cos x - 1 = 0$

87. $\sin(\cos x) = 0$

88. $\cos(\sin x) = 0$

89. The height $h(t)$ (in feet) of the seat of a child's swing above ground level is given by $h(t) = -1.1\cos\left(\frac{2\pi}{3}t\right) + 3.1$ where t is the time in seconds after the swing is set in motion.

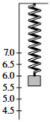


a. Find the maximum and minimum height of the swing.

b. When is the first time after $t = 0$ that the swing is at a height of 3 ft? Round to 1 decimal place.

c. When is the second time after $t = 0$ that the swing is at a height of 3 ft? Round to 1 decimal place.

90. A vertical spring is attached to the ceiling. The height h of a block attached to the spring relative to ground level is given by $h(t) = -0.8\cos\left(\frac{10\pi}{3}t\right) + 5.8$ where t is the time in seconds and $h(t)$ is in feet.



a. Find the maximum and minimum height of the block.

b. When is the first time after $t = 0$ that the block is at a height of 6 ft? Round to 3 decimal places.

c. When is the second time after $t = 0$ that the block is at a height of 6 ft? Round to 3 decimal places.

91. The monthly sales of winter coats follow a periodic cycle with sales peaking late in the year and before the winter holidays. The monthly sales total S (in \$1000) is given by

$$S(t) = 350\cos\left(\frac{\pi}{6}t + \frac{\pi}{6}\right) + 500$$

where t is the month number ($t = 1$ corresponds to January).

a. Write the range of this function in interval notation.

b. During which months ($0 \leq t \leq 12$) are sales at their maximum?

c. During which months ($0 \leq t \leq 12$) are sales at their minimum?

d. In which months, between 1 and 12 inclusive, are the sales \$675,000?

92. With each heartbeat, blood pressure increases as the heart contracts, then decreases as the heart rests between beats. The maximum blood pressure is called the systolic pressure and the minimum blood pressure is called the diastolic pressure. When a doctor records an individual's blood pressure such as "120 over 80," it is understood as "systolic over diastolic." Suppose that the blood pressure for a certain individual is approximated by $p(t) = 90 + 20\sin(140\pi t)$ where p is the blood pressure in mmHg (millimeters of mercury) and t is the time in minutes after recording begins.

a. Find the period of the function and interpret the results.

b. Find the maximum and minimum values and interpret this as a blood pressure reading.

c. Find the times at which the blood pressure is at its maximum.

The refractive index n of a substance is a dimensionless measure of how much light bends (refracts) when passing from one medium to another.

$$n = \frac{c}{v}$$

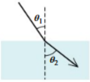
where c is the speed of light in a vacuum (a constant) and v is the speed of light in the medium. For example, the refractive index of diamond is 2.42, which means that light travels 2.42 times as fast in a vacuum as it does in a diamond.

Snell's law is an equation that relates the indices of refraction of two different mediums to the angle of incidence θ_1 and angle of refraction θ_2. Angles θ_1 and θ_2 are measured from a line perpendicular to the boundary between the two mediums. Use this relationship and the table of refraction indices to complete Exercises 93–94.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Index of Refraction	
Medium	**Index**
Air	1.00029
Water	1.33
Acetone	1.36
Human cornea	1.38
Glycerin	1.47
Sodium chloride	1.54
Sapphire	1.77
Cubic zirconia	2.15
Diamond	2.42

93. Assume that a beam of light travels from air into water.



a. If the incidence angle θ_1 is 40°, what is the angle of refraction θ_2? Round to the nearest tenth of a degree.

b. Alina stands on the deck of a boat and tries to spear a lobster by aiming the spear directly at the lobster. Will Alina hit the lobster? Explain your answer.

94. If a beam of light traveling through air enters a sapphire at an angle of incidence of 23°, what is the angle of refraction? Round to the nearest tenth of a degree.

Write About It

95. Explain why $\cos^2 x - \cos x - 12 = 0$ has no solution.

96. Explain why $x = \sin^{-1}(-0.4)$ is not a solution to the equation $\sin x = -0.4$ on the interval $[0, 2\pi)$.

97. What is the difference between the general solution to a trigonometric equation and the solution over the interval $[0, 2\pi)$?

98. Explain two different methods to solve the equation $\sin 3x + x = 1$ by using a graphing utility.

Expanding Your Skills

For Exercises 99–104, simplify the expression on the left to a single trigonometric function, then solve the equation on $[0, 2\pi)$.

99. $\dfrac{1-\cos^2 x}{\sin x \cos x} = -\sqrt{3}$

100. $\dfrac{\sin x \cot x}{\cos x \tan x} = -1$

101. $\dfrac{\sec x - \cos x}{\tan x} = \dfrac{\sqrt{3}}{2}$

102. $\dfrac{\csc x}{\cot x + \tan x} = -\dfrac{1}{2}$

103. $\cos^4 x - \sin^4 x = -1$

104. $\dfrac{2\cos 2x}{\cot x - \tan x} = \dfrac{1}{\sqrt{2}}$

For Exercises 105–118, solve the equation on the interval $[0, 2\pi)$.

105. $\sqrt{3}\tan^2 x - 2\tan x - \sqrt{3} = 0$

106. $\sqrt{2}\sin^2 x - \sin x - \sqrt{2} = 0$

107. $4\sin^2 x \cos x - 2\cos x + 2\sin^2 x - 1 = 0$

108. $\tan^2 x \sec x - 2\tan^2 x - 3\sec x + 6 = 0$

109. $4\sin^4 x - 7\sin^2 x + 3 = 0$

110. $8\cos^4 x - 6\cos^2 x + 1 = 0$

111. $2\sin^3 x + 7\sin^2 x + 2\sin x - 3 = 0$

112. $6\sec^3 x - 7\sec^2 x - 11\sec x + 2 = 0$

113. $\sqrt{\sin x + 1} + \sqrt{2\sin x + 3} = 1$

114. $\sqrt{5\cos x + 4} - 2 = \sqrt{\cos x}$

115. $4\sin^{-1}(x-2) + \pi = 2\pi$

116. $\frac{3}{2}\cos^{-1}(x+1) = \frac{\pi}{2}$

117. $2\sin\left(2x - \frac{\pi}{2}\right) - \sqrt{2} = 0$

118. $2\cos(2x - \pi) - \sqrt{3} = 0$

119. Consider the equation $\sin x + 1 = \cos x$ from Example 10.

a. Write the equation in the form $\sin x - \cos x = -1$. Then write the equation with the left side of the equation in the form $k\sin(x + \alpha)$. (*Hint*: See page 618.)

b. Solve the equation over the interval $[0, 2\pi)$ using the form from part (a).

For Exercises 120–122, use the technique from Exercise 119 to solve the equation over the interval $[0, 2\pi)$.

120. $\sqrt{3}\sin x + \cos x = 1$

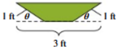
121. $\sqrt{2}\sin x - \sqrt{2}\cos x = -1$

122. $-\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = 1$

123. In a six-cylinder engine, the height of the pistons above the crankshaft can be modeled by sine or cosine functions. There are three piston pairs, and the firing sequence is 1, 5, 3, 2, 6, 4. Therefore the height of the piston pairs differs by a phase shift. Suppose that the height of piston 1 is given by $h_1 = 3.5\sin(100t) + 7$ and the height of piston 2 is given by $h_2 = 3.5\sin\left(100t + \frac{4000\pi}{3}\right) + 7$. The values of h_1 and h_2 are measured in inches and t is measured in seconds.

Determine the times for which pistons 1 and 2 are at the same height. (*Hint*: Set $h_1 = h_2$. Then use the sum formula for the sine function to expand the right side of the equation.)

124. From Exercise 87 in Lesson 5.3, the cross-sectional area of a feeding trough in the figure is given by $A = \frac{1}{2}\sin 2\theta + \sin\theta$. In calculus, you will learn that to find the value of θ to maximize the area, we need to solve the equation $\cos 2\theta + \cos\theta = 0$ for $0 < \theta < \frac{\pi}{2}$.



a. Find the value of θ to maximize the area.

b. Find the maximum area.

c. If the trough is 20 ft long, what is the maximum volume that it can hold?

125. Consider an isosceles triangle with two sides of length x and angle θ included between them. We have already shown that the area of the triangle is given by $A = \frac{1}{2}x^2\sin\theta$. If an isosceles triangle has two 6-in. sides, what angle is required to make the area 10 in.^{2}? Give the exact value and approximate to the nearest one-hundredth of a degree.

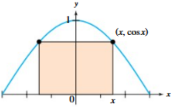
For Exercises 126–127, consider a projectile launched from ground level at an angle of elevation θ with an initial velocity v_0. The maximum horizontal range is given by $x_{\max} = \frac{v_0^2\sin 2\theta}{g}$, where g is the acceleration due to gravity ($g = 32$ ft/sec^2 or $g = 9.8$ m/sec^2).

126. If a soccer ball is kicked from ground level with an initial velocity of 28 m/sec, what is the smallest positive angle at which the player should kick the ball to reach a teammate 48 m down the field? Assume that the ball reaches the teammate at ground level on the fly. Round to the nearest tenth of a degree.

127. A quarterback throws a football with an initial velocity of 62 ft/sec to a receiver 40 yd (120 ft) down the field. At what angle could the ball be released so that it hits the receiver's hands at the same height that it left the quarterback's hand? Round to the nearest tenth of a degree.

Technology Connections

128. Suppose that a rectangle is bounded by the x-axis and the graph of $y = \cos x$.



a. Write a function that represents the area $A(x)$ of the rectangle for $0 < x < \frac{\pi}{2}$.

b. Complete the table.

x	$A(x)$
0.2	
0.4	
0.6	
0.8	
1.0	
1.2	
1.4	

c. Graph the function from part (a) on the viewing window: $\left[0, \frac{\pi}{2}, \frac{\pi}{6}\right]$ by $[-3, 3, 1]$ and approximate the values of x for which the area is 1 square unit. Round to 2 decimal places.

d. In calculus, we can show that the maximum value of the area of the rectangle will occur at values of x for which $2\cos x - 2x\sin x = 0$. Confirm this result by graphing $y = 2\cos x - 2x\sin x$ and the function from part (a) on the same viewing window. What do you notice?

Problem Recognition Exercises

Trigonometric Identities and Trigonometric Equations

For Exercises 1–6, prove the identity in part (a), and solve the equation in part (b) on the interval $[0, 2\pi)$.

1. **a.** Prove that $\sin^4 x - \cos^4 x = -\cos 2x$.

b. Solve $\sin^4 x - \cos^4 x = 1$.

2. **a.** Prove that $\tan x \sin x + \cos x = \sec x$.

b. Solve $\tan x \sin x + \cos x = 2$.

3. **a.** Prove that $\cot x - \sec x \csc x \cos 2x = \tan x$.

b. Solve $\cot x - \sec x \csc x \cos 2x = 1$.

4. **a.** Prove that $\frac{\sec x + \tan x}{\cos x + \cot x} = \sec x \tan x$.

b. Solve $\frac{\sec x + \tan x}{\cos x + \cot x} = 0$.

5. **a.** Prove that $\frac{2\cot x}{1 + \cot^2 x} = \sin 2x$.

b. Solve $\frac{2\cot x}{1 + \cot^2 x} = 0$.

6. **a.** Prove that $\cos 2x - \cos 4x = 2\sin 3x \sin x$.

b. Solve $\cos 2x - \cos 4x = 0$.

7. **a.** Compute $\sin 75°$ by using $\sin(u + v) = \sin u \cos v + \cos u \sin v$.

b. Compute $\sin 75°$ by using the half-angle formula $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}}$.

c. Show that the results for parts (a) and (b) are equivalent.

8. **a.** Compute $\cos\frac{\pi}{12}$ by applying a sum or difference formula for cosine.

b. Use the result of part (a) to find $\sin\frac{\pi}{24}$ and $\cos\frac{\pi}{24}$.

Go online for more practice problems.

Chapter 5 Review

Chapter 5 Key Concepts

Lesson 5-1 Fundamental Trigonometric Identities

Key Concepts	Reference
The fundamental trigonometric identities are used to simplify trigonometric expressions and to prove other trigonometric relationships. **Reciprocal and Quotient Identities:** $\csc x = \frac{1}{\sin x}$ $\sec x = \frac{1}{\cos x}$ $\cot x = \frac{1}{\tan x}$ $\tan x = \frac{\sin x}{\cos x}$ $\sin x = \frac{1}{\csc x}$ $\cos x = \frac{1}{\sec x}$ $\tan x = \frac{1}{\cot x}$ $\cot x = \frac{\cos x}{\sin x}$	p. 598
Even and Odd Identities: $\sin(-x) = -\sin x$ $\csc(-x) = -\csc x$ $\tan(-x) = -\tan x$ $\cos(-x) = \cos x$ $\sec(-x) = \sec x$ $\cot(-x) = -\cot x$	p. 598
The Pythagorean Identities can be used in several forms. $\sin^2 x + \cos^2 x = 1$ $\tan^2 x + 1 = \sec^2 x$ $1 + \cot^2 x = \csc^2 x$ $1 - \cos^2 x = \sin^2 x$ $\sec^2 x - 1 = \tan^2 x$ $\csc^2 x - 1 = \cot^2 x$ $1 - \sin^2 x = \cos^2 x$ $\sec^2 x - \tan^2 x = 1$ $\csc^2 x - \cot^2 x = 1$	p. 598
To verify a trigonometric identity, work on one side of the equation (usually the more complicated side) until it matches the other side of the equation. Although there is no explicit procedure that works for every identity, helpful guidelines are offered on page 598.	p. 601

Lesson 5-2 Sum and Difference Formulas

Key Concepts	Reference
The sine, cosine, or tangent of a sum or difference of angles can be expressed in terms of the sine, cosine, or tangent of the individual angles. **Sum and Difference Formulas:** $\sin(u + v) = \sin u \cos v + \cos u \sin v$ $\cos(u + v) = \cos u \cos v - \sin u \sin v$ $\sin(u - v) = \sin u \cos v - \cos u \sin v$ $\cos(u - v) = \cos u \cos v + \sin u \sin v$ $\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$ $\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$	p. 612
The cofunction and periodic properties were first introduced in Chapter 4 and can be verified using the sum and difference formulas. **Cofunction Identities:** $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$ $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$	p. 617
Periodic Identities: $\sin(\theta + 2\pi) = \sin\theta$ $\cos(\theta + 2\pi) = \cos\theta$ $\tan(\theta + \pi) = \tan\theta$ $\csc(\theta + 2\pi) = \csc\theta$ $\sec(\theta + 2\pi) = \sec\theta$ $\cot(\theta + \pi) = \cot\theta$	p. 617
A linear combination of sin x and cos x can be written as a sine function involving a phase shift. For the real numbers A, B, and x, $A\sin x + B\cos x = k\sin(x + \alpha)$ where $k = \sqrt{A^2 + B^2}$ and α satisfies $\cos\alpha = \frac{A}{\sqrt{A^2 + B^2}}$ and $\sin\alpha = -\frac{B}{\sqrt{A^2 + B^2}}$.	p. 619

Lesson 5-3 Double-Angle, Power-Reducing, and Half-Angle Formulas

Key Concepts	Reference
The sine, cosine, or tangent of a double angle follows directly from the sum and difference formulas.	p. 625
Double-Angle Formulas: $\sin 2\theta = 2\sin\theta\cos\theta$ $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $= 1 - 2\sin^2\theta$ $= 2\cos^2\theta - 1$	
The double-angle cosine formulas are used to derive the power-reducing formulas. These convert the square of a sine or cosine term to a first power of a cosine term. **Power-Reducing Formulas:** $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ $\quad$ $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ $\quad$ $\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$	p. 627
Substituting $\frac{\alpha}{2}$ for θ in the power-reducing formulas results in the half-angle formulas. **Half-Angle Formulas:** $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$ $\quad$ $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$ The + or − sign is determined by the quadrant of angle $\frac{\alpha}{2}$. $\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$	p. 629

Lesson 5-4 Product-to-Sum and Sum-to-Product Formulas

Key Concepts		Reference
Product-to-Sum Formulas:	**Sum-to-Product Formulas:**	pp. 636, 638
$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$	$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$	
$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$	$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$	
$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$	$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$	
$\cos u \sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$	$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$	

Lesson 5-5 Trigonometric Equations

Key Concepts	Reference
To solve a conditional trigonometric equation, we manipulate both sides of the equation into a simpler form to find the values of the variable that make the equation true. For example, this equation is linear in the variable sin *x*. $2\sin x + 1 = 0$ $\sin x = -\frac{1}{2}$ is an equivalent equation. The solution set on the interval $[0, 2\pi)$ is $\left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$. The general solution is $\left\{x \mid x = \frac{7\pi}{6} + 2n\pi, x = \frac{11\pi}{6} + 2n\pi \text{ for integers } n\right\}$	pp. 644–645

This equation is quadratic in the variable cos *x*. $2\cos^2 x + 5\cos x - 3 = 0$ $(2\cos x - 1)(\cos x + 3) = 0$ $\cos x = \frac{1}{2}$ or $\cos x = -3$ Setting each factor equal to zero results in two linear equations in the variable cos *x*. The second of the two equations, $\cos x = -3$, has no solution because -3 is not in the range of the cosine function. The solution set over the interval $[0, 2\pi)$ is $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$.	p. 650
An equation involving a multiple angle may have numerous solutions on the interval $[0, 2\pi)$. $\tan 3x = 1$ The solution set over $[0, 2\pi)$ is $\left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}\right\}$.	p. 646
Trigonometric identities are also used to manipulate an equation into a more manageable form.	p. 652
$\sin^2 x = 2\cos x + 2$ Use the Pythagorean identity $\sin^2 x = 1 - \cos^2 x$. $1 - \cos^2 x = 2\cos x + 2$ $\cos^2 x + 2\cos x + 1 = 0$ Combine like terms. $(\cos x + 1)^2 = 0$ Factor. $\cos x + 1 = 0$ Set each factor equal to zero $\cos x = -1$ The solution set over $[0, 2\pi)$ is $\{\pi\}$.	p. 652
Other algebraic techniques such as factoring, squaring both sides of the equation (being sure to check the potential solutions), applying the square root property, and applying the quadratic formula may also be used.	p. 654
Often the solutions to a trigonometric equation are not multiples of the special angles $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, or $\frac{\pi}{2}$. In such a case, use the inverse trigonometric functions to find the exact value. $\tan x = 5$ The solution set over $[0, 2\pi)$ is $\{\tan^{-1} 5, \pi + \tan^{-1} 5\}$.	p. 655

Go online for a detailed chapter summary.

Chapter 5 Test

For Exercises 1–2, simplify the expression.

1. $\dfrac{1}{\cot^2\theta + 1} + \dfrac{1}{\tan^2\theta + 1}$

2. $\cos\left(x + \frac{\pi}{6}\right) + \cos\left(x - \frac{\pi}{6}\right)$

For Exercises 3–8, verify the identity.

3. $(\sin\theta - \cos\theta)(\sec\theta + \csc\theta) = \tan\theta - \cot\theta$

4. $\dfrac{\cot x - \tan x}{\cot x + \tan x} = \cos 2x$

5. $\dfrac{\sin 7x - \sin 5x}{\cos 7x + \cos 5x} = \tan x$

6. $\dfrac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = \frac{1}{2}\sin 2x + 1$

7. $4\cos^2\frac{x}{2} + 2\sin^2\frac{x}{2} - 3 = \cos x$

8. $\dfrac{1 - \cos x}{1 + \cos x} = (\csc x - \cot x)^2$

9. Write $\sqrt{144 - x^2}$ as a function of θ by making the substitution $x = 12\sin\theta$ for $0 < \theta < \frac{\pi}{2}$.

10. Write $8\cos x - 15\sin x$ in the form $k\sin(x + \alpha)$ for $0 \le \alpha < 2\pi$. Round α to 3 decimal places.

11. Write $\cos^2 x + 3\sin^4 x$ in terms of first powers of cosine.

For Exercises 12–17, find the exact value.

12. $\sin 250^\circ \cos 10^\circ - \cos 250^\circ \sin 10^\circ$

13. $\tan\frac{\pi}{12}$

14. $\cos\left(\arctan 3 + \arcsin\frac{4}{5}\right)$

15. $\cos 105^\circ$

16. $\sin\frac{17\pi}{24}\sin\frac{\pi}{24}$

17. $\sin 555^\circ + \sin 105^\circ$

18. Find the exact value for $\tan(\alpha + \beta)$ given $\sin\alpha = \frac{24}{25}$ for α in Quadrant II and $\cos\beta = \frac{3\sqrt{73}}{73}$ for β in Quadrant IV.

19. Given $\sin\theta = \frac{2\sqrt{5}}{5}$ and θ in Quadrant II, find the exact function values.

a. $\sin 2\theta$ **b.** $\cos 2\theta$ **c.** $\tan 2\theta$

20. Given $\tan\theta = \frac{15}{8}$ and $\pi < \theta < \frac{3\pi}{2}$, find the exact function values.

a. $\sin\frac{\theta}{2}$ **b.** $\cos\frac{\theta}{2}$ **c.** $\tan\frac{\theta}{2}$

For Exercises 21–22,

a. Write the solution set for the general solution.

b. Write the solution set on the interval $[0, 2\pi)$.

21. $\sqrt{3}\tan x + 5 = 6$

22. $2\cos 2x = \sqrt{2}$

For Exercises 23–30, solve the equation on the interval [0, 2π).

23. $2\cos^3 x - \cos x = 0$

24. $6\csc^2 x - 17\csc x + 10 = 0$

25. $6\sin^2 x - 5\cos x - 2 = 0$

26. $2\sin^2 2x - \sin 2x - 1 = 0$

27. $18\cos^2 x + 3\cos x - 4 = 0$

28. $\cot\left(x - \frac{\pi}{4}\right) = \sqrt{3}$

29. $8\sin x + 5 = 0$

30. $\sin x + \sin 3x - \sin 2x = 0$

31. Use a graphing utility to solve $\sin 2x = 3x^3$. Round the solutions to 4 decimal places.

32. For a projectile launched from ground level at an angle of elevation θ with an initial velocity v_0, the maximum horizontal range is given by $x_{\max} = \frac{v_0^2 \sin 2\theta}{g}$, where g is the acceleration due to gravity ($g = 32$ ft/sec² or $g = 9.8$ m/sec²). If a toy rocket is launched from the ground with an initial velocity of 50 ft/sec and lands 73 ft from the launch point, find the angle of elevation of the rocket at launch. Round to the nearest tenth of a degree.

Go online for more practice problems.

Chapter 5 Cumulative Review Exercises

For Exercises 1–8, solve the equations and inequalities. Write the solution set to inequalities in interval notation. Solve trigonometric equations on the interval $[0, 2\pi)$.

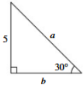
1. $5 + 3|2x - 1| = 6$
2. $4x^4 + 3x^2 - 1 = 0$
3. $4\sin^4 x + 3\sin^2 x - 1 = 0$
4. $4x^2 + 3x - 1 \geq 0$
5. $4x^2 + 3x - 1 < 0$
6. $\sqrt{x+2} - 2x = 1$
7. $16^{x^2-1} = \left(\frac{1}{8}\right)^{x+1}$
8. $\ln 2x + \ln(2x + 1) - \ln(1 - x) = 0$
9. Given $P(x) = 4x^4 + 3x^3 + 15x^2 + 12x - 4$
   - **a.** Use synthetic division to show that $x = 2i$ is a zero of the polynomial.
   - **b.** Use the result from part (a) to factor the polynomial completely.
10. Given $L(x) = -\frac{2}{3}x + 5$
    - **a.** Find the equation of a line perpendicular to L that passes through the point $(-1, 3)$
    - **b.** Find the equation of a line parallel to L that has an undefined slope.
11. Determine if the function is even, odd, or neither.
    - **a.** $f(x) = 6x^4 + 7|x|$
    - **b.** $g(x) = 6x^4 + 7x$
    - **c.** $h(x) = 6x^3 + 7x$
    - **d.** $r(x) = \cos x + \sec x$
    - **e.** $t(x) = x^2 \sin x$

For Exercises 12–15, write the domain and range of each function in interval notation.

12. $f(x) = (x - 2)^2(x + 2)^2$
13. $g(x) = -2\sin 4x$
14. $h(x) = 3 + \sqrt{x - 2}$
15. $r(x) = \sin^{-1} x$
16. Given $y = \dfrac{2x^2}{2x^2 - x - 3}$
    - **a.** Identify the asymptotes.
    - **b.** Determine if the graph crosses the horizontal asymptote.
17. Given $f(x) = \log_3 x$,
    - **a.** Write the domain and range in interval notation.
    - **b.** Write an equation of the asymptote.
    - **c.** Find the intercepts.
    - **d.** Determine the intervals over which f is increasing, decreasing, and constant.

For Exercises 18–19, verify the identity.

18. $\cot^2\theta - \cos^2\theta = \cot^2\theta\cos^2\theta$
19. $\sin 2x\cos x - \cos 2x\sin x = \sin x$
20. Given the right triangle, find a and b.



Go online for more practice problems.

Chapter 6

Applications of Trigonometric Functions

Outline

6-1 Applications of Right Triangles p. 672
6-2 The Law of Sines p. 689
6-3 The Law of Cosines p. 707
Problem Recognition Exercises: Solving Triangles Using a Variety of Tools p. 720
armonic Motion p. 721
Review p. 735

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Launch Activity: Critique the Reasoning of Others

Properties of Right Triangles

Robert and Keshia are sectioning off an area on school grounds to create a sustainable vegetable garden as part of their agriculture class. They were told that the plot of land provided by the school is in the shape of a right triangle with the other angles measuring 30° and 60°. Robert and Keshia were asked to determine the area of the triangle so their agriculture class could purchase the appropriate amount of soil and seeds for the area. However, the only information they were given other than the shape of the land was that the longest side of the triangle measured 32 feet. Knowing they need the other side lengths to determine the area of the triangle, Robert suggested using the Pythagorean Theorem to find the missing lengths. Keshia suggested using special right triangle properties for a 30-60-90 triangle.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer this question: To determine the area of the triangular plot of land, can you use both Robert and Keshia's method? If not, why?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

GO ONLINE to answer these questions.

Lesson 6-1

Applications of Right Triangles

What Will You Learn?

After completing this lesson, you should be able to:

- Solve a Right Triangle
- Solve Applications of Right Triangles
- Compute the Bearing of an Object

Learn: Solve a Right Triangle

In Lesson 4.3, we introduced the trigonometric functions as functions of acute angles of a right triangle. We saw immediate applications in finding the measures of the angles and sides of a right triangle when two sides or a side and an angle are known. This is called **solving a right triangle**. In this section, we revisit this concept. Then in Lessons 6.2 and 6.3, we expand our study to solve *oblique* triangles (triangles that do not contain a right angle).

As a labeling convention, we will use capital letters to label the vertices of a triangle. The same letters in lowercase will be used to denote the lengths of the sides opposite the corresponding vertices. So, $\triangle ABC$ has sides of length a, b, and c, opposite angles A, B, and C, respectively (Figure 6-1).

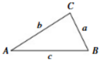


Figure 6-1

Example 1: Solving a Right Triangle

Solve the right triangle given in Figure 6-2. Round the lengths of the sides to 1 decimal place.

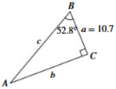


Figure 6-2

Solution:

We begin by noting that $B = 52.8°$, $C = 90°$, and $A + B + C = 180°$.

Therefore, $A + 52.8° + 90° = 180°$

$$A = 180° - 52.8° - 90°$$

$$A = 37.2°$$

To find b, we need to establish a relationship between a known angle and the known side a.

$$\tan 52.8^\circ = \frac{b}{10.7} \Rightarrow b = 10.7\tan 52.8^\circ \approx 14.1$$

To find c, we establish a relationship between a known angle and side a.

$$\cos 52.8^\circ = \frac{10.7}{c} \Rightarrow c = \frac{10.7}{\cos 52.8^\circ} \approx 17.7$$

In summary, $A = 37.2^\circ$, $B = 52.8^\circ$, $C = 90^\circ$, $a = 10.7$, $b \approx 14.1$, and $c \approx 17.7$.

Check Your Work

Since $\triangle ABC$ is a right triangle, its sides must satisfy the Pythagorean theorem. Therefore to double-check, we have

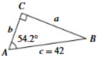
$$a^2 + b^2 = c^2$$

$$(10.7)^2 + (14.1)^2 \overset{?}{=} (17.7)^2$$

$$313.3 \approx 313.29 \checkmark$$

Apply the Skills

1. Solve the right triangle. Round the lengths of the sides to 1 decimal place.



Good Practices

In Example 1, after finding the length of side b, you can use it to find the length of c by applying the relationship $\sin 52.8^\circ = \frac{14.1}{c}$. However, the length of side b was rounded. Using a rounded value in further calculations potentially compounds the error. Therefore, we recommend using relationships involving the given information (in this case, angle B and side a) rather than the computed information.

Learn: Solve Applications of Right Triangles

Historically, trigonometry was developed to measure distances that were difficult or impossible to measure directly. We examine such scenarios in Examples 2 and 3.

Example 2: Computing Distances in an Application

A beachgoer flies a kite on a windy day. When the spool of string is held at a height of 4 ft above the ground, the string makes an angle of 42° from the horizontal. If 120 ft of string is out and the sag in the string is negligible, find the height of the kite to the nearest foot.

Solution:

A sketch of this scenario is given in Figure 6-3.

$\triangle ABC$ is a right triangle, and the height of the kite is 4 ft $+ a$. To solve for a, notice that a is the side opposite the given angle, and that $c = 120$ ft is the length of the hypotenuse. Therefore,

$$\sin 42° = \frac{a}{120} \Rightarrow a = 120 \sin 42° \approx 80$$

The height of the kite is 4 ft + 80 ft = 84 ft.

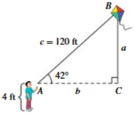


Figure 6-3

Apply the Skills

2. A 40-ft ladder on a fire truck leans against a burning building such that the top of the ladder makes an angle of 20° with the building. If the bottom of the ladder is anchored to the top of the fire truck 6 ft off the ground, how far above ground does the ladder reach? Round to the nearest tenth of a foot.

Example 3: Computing Distances in an Application

A mathematics enthusiast visiting London wants to know the diameter of the clock face on Big Ben. She stands 200 ft down the road from the tower. Then from an eye level of 5 ft, she measures the angle of elevation to the bottom of the clock face at 42° and the angle of elevation to the top of the clock face at 45.4°. Find the diameter of the clock face to the nearest foot.

Solution:

Figure 6-4 reveals that the diameter of the clock face is given by $h_2 - h_1$.

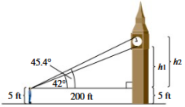


Figure 6-4

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From the lower right triangle,

$$\tan 42° = \frac{h_1}{200} \Rightarrow h_1 = (200)\tan 42° \approx 180$$

From the larger right triangle,

$$\tan 45.4° = \frac{h_2}{200} \Rightarrow h_2 = (200)\tan 45.4° \approx 203$$

Therefore, the diameter of the clock face is $h_2 - h_1 \approx 203 \text{ ft} - 180 \text{ ft} \approx 23$ ft.

Apply the Skills

3. From an eye level of 6 ft, an observer standing 30 ft away from a building sights the bottom of a window at an angle of elevation of 6.1°. The observer sights the top of the window at an angle of elevation of 20.1°. Find the vertical dimension of the window. Round to the nearest tenth of a foot.

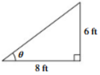
In Example 4, we use the inverse trigonometric functions to find the measure of an unknown angle in an application.

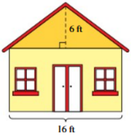
Example 4: Computing an Angle in a Construction Application

Find the pitch angle of the roof to the nearest tenth of a degree.

Solution:

The pitch angle is the angle that the roof makes with the horizontal. Using either right triangle formed by the 6-ft altitude and half the width of the house we have





$$\tan\theta = \frac{6}{8}$$

$$\theta = \tan^{-1}\frac{6}{8} \approx 36.9°$$

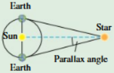
The pitch angle of the roof is approximately 36.9°.

Apply the Skills

4. A straight hiking trail is 2500 ft long with a gain in altitude of 310 ft. To the nearest degree, what is the average angle of ascent?

Point of Interest

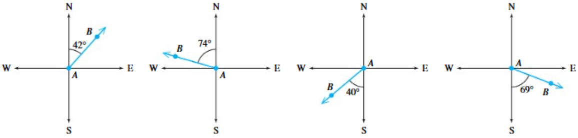
In Examples 2–4, we used trigonometry to determine distances and angles that are difficult to measure directly. One distance impossible for us to measure directly is the distance between the Earth and a star. Astronomers use an indirect method called the method of parallax. They measure the angle represented by the apparent displacement of a star caused by the change in the position of an observer on Earth. This is done at a 6-month interval when the Earth is at opposite points in its orbit around the Sun. Then using the right triangle formed by the star, Earth, and Sun, astronomers can calculate the distance from the Earth to the star (see Exercises 31–32).



You can simulate parallax by holding your index finger at arm's length in front of your face. View your finger with one eye open and the other eye closed. Then alternate. Your finger will seem to move relative to objects in the background. This demonstrates parallax.

Learn: Compute the Bearing of an Object

In navigation and surveying applications, the term *bearing* is used to convey the angular relationship from one point to another. The **bearing** from point A to point B is the degree measure of the acute angle formed by ray $\overrightarrow{AB}$ and the north-south line. Figure 6-5 provides four examples of the bearing from A to B.



(a) N42°E (b) N74°W (c) S40°W (d) S69°E

Figure 6-5

Notice that a bearing is denoted by the letter N or S (north or south), followed by the measure of the angle in degrees, followed by the letter E or W (east or west). For example, a ship bearing S40°W is heading in a southwesterly direction. Specifically, the ship is traveling 40° to the west of due south.

Example 5: Finding and Interpreting Bearing

Refer to Figure 6-6.

- **a.** Find the bearing from *O* to *C*.
- **b.** Find the bearing from *O* to *B*.
- **c.** Find the bearing from *A* to *O*.

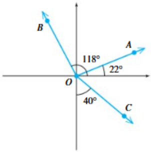


Figure 6-6

Solution:

a. Ray $\overrightarrow{OC}$ is directed 40° to the east of due south.

Therefore, the bearing from *O* to *C* is S40°E.

b. Ray $\overrightarrow{OB}$ is directed $118° - 90° = 28°$ to the west of due north.

Therefore, the bearing from *O* to *B* is N28°W.

c. To find the bearing from *A* to *O*, we need to find the acute angle between ray $\overrightarrow{AO}$ and the north-south line.

It is helpful to sketch a coordinate system with origin at the initial point of the ray (Figure 6-7).

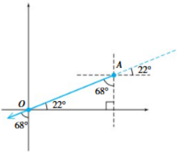


Figure 6-7

Ray $\overrightarrow{AO}$ is directed $90° - 22° = 68°$ to the west of due south.

Therefore, the bearing from *A* to *O* is S68°W.

Good Practices

Remember that bearing is always measured from the north-south line. Therefore, the letter N or S is written first.

Apply the Skills

5. Refer to Figure 6-6.

- **a.** Find the bearing from *C* to *O*.
- **b.** Find the bearing from *B* to *O*.

Example 6: Interpreting Bearing

A plane leaves an airport heading S32° W at 320 knots (nautical miles per hour). Determine the number of nautical miles (nmi) south and the number of nautical miles west that the plane travels in 1.2 hr. Round to the nearest nautical mile.

Solution:

First sketch the path of the plane from the airport A to a point P, 1.2 hr after take-off (Figure 6-8). After 1.2 hr, the number of nautical miles traveled from A to P is

$AP = (320\text{ knots})(1.2\text{ hr}) = 384\text{ nmi}.$

The distance that the plane flies south is the vertical leg of $+\triangle APB$, and the distance that the plane flies west is the horizontal leg of $\triangle APB$.

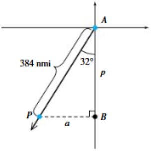


Figure 6-8

Distance south: $\cos 32° = \frac{p}{384} \Rightarrow p = 384\cos 32° \approx 326\text{ nmi}$

Distance west: $\sin 32° = \frac{a}{384} \Rightarrow a = 384\sin 32° \approx 203\text{ nmi}$

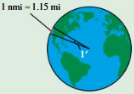
Apply the Skills

6. A boat leaves a dock heading N27°W at 6 knots (nautical miles per hour). Determine the number of nautical miles (nmi) north and the number of nautical miles west that the boat travels in 0.5 hr. Round to the nearest tenth of a nautical mile.

Insights

One nautical mile (1 nmi) is defined as the distance along any meridian through 1′ of latitude. A circle has 360° and 60′ per degree. So, 1 nmi is the circumference of the Earth divided by (360)(60).

$$1\text{ nmi} = \frac{24{,}860\text{ mi}}{(360)(60)}$$
$$\approx 1.15\text{ mi}$$



The unit of nautical miles is often used in navigation for ships and planes. The related unit of speed is 1 nautical mile per hour or 1 knot.

Example 7: Computing Bearing

A boat leaves a marina heading N52°E at 6 mph. After 1 hr, the boat makes a 90° counterclockwise turn to a new bearing of N38°W. If the boat travels this new course for 2 hr,

a. Find the boat's distance from the marina to the nearest tenth of a mile.

b. To a tenth of a degree, find the bearing required for the boat to return to the marina.

Solution:

a. The distance that the boat travels during the first hour is (6 mph)(1 hr) = 6 mi.

The distance that the boat travels for the last 2 hr is (6 mph)(2 hr) = 12 mi.

Since the boat makes a 90° turn, the path of the boat forms two legs of a right triangle. The length of the hypotenuse is the distance a from the boat to the marina (Figure 6-9). Using the Pythagorean theorem,

$$\begin{aligned} a^2 &= (6)^2 + (12)^2 \\ a &= \sqrt{(6)^2 + (12)^2} \\ &= \sqrt{180} \\ &= 6\sqrt{5} \\ &\approx 13.4 \end{aligned}$$

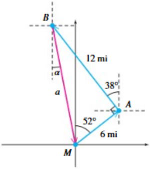


Figure 6-9

The boat is approximately 13.4 mi from the marina.

b. To find the bearing from the boat to the marina we need to find the acute angle formed by ray $\overrightarrow{BM}$ and the north-south line. This is labeled α. To find α, our strategy is to find the measure of angle M in $\triangle BMA$ and subtract 52°. That is, $\alpha = M - 52°$.

We have $\tan M = \frac{12}{6} \Rightarrow M = \tan^{-1} 2 \approx 63.4°$.

Therefore, $\alpha = 63.4° - 52° = 11.4°$.

The boat should follow a bearing of S11.4°E to return to the marina.

Apply the Skills

7. A plane leaves an airport heading S50°E at 450 mph. After 2 hr, the plane makes a 90° clockwise turn to a new bearing of S40°W. If the plane travels this new course for $\frac{1}{2}$ hr,

a. Find the plane's distance from the airport to the nearest mile.

b. To the nearest degree, find the bearing from the airport to the plane.

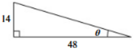
Practice 6-1

Practice Exercises

Prerequisite Review

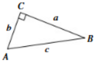
R.1. In a right triangle, one leg measures 15 ft and the hypotenuse measures 25 ft. Find the length of the missing side.

R.2. Find the exact values of the six trigonometric functions for the angle θ.



Concept Connections

For Exercises 1–2, refer to triangle *ABC*.



1. If the lengths of sides a and b are known, the inverse __________ or __________ function can be used directly to find the value of angle A.

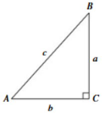
2. The measure of angle B and the length of side c can be used along with which trigonometric functions to find the length of side a?

3. If the bearing from A to C is N12°E, what is the bearing from C to A?

4. The acute angle used to express the bearing between two points is measured relative to the (north-south or east-west) line.

Learn: Solve a Right Triangle

For Exercises 5–12, given right triangle *ABC*, determine if the expression is true or false.



5. $\cot B = \frac{a}{b}$

6. $\sin^{-1}\frac{a}{b} = A$

7. $\csc B = \frac{c}{a}$

8. $\sec A = \frac{c}{b}$

9. $\cos^{-1}\frac{a}{c} = B$

10. $A + B = 90°$

11. $c^2 - b^2 = a^2$

12. $\tan A = \frac{b}{a}$

For Exercises 13–18, solve the right triangle for the unknown sides and angles. Round values to 1 decimal place if necessary. (See Example 1, p. 672)

13. $B = 24°, c = 18$

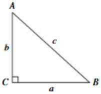
14. $A = 28°, c = 15$

15. $a = 10, b = 7$

16. $a = 2, b = 8.6$

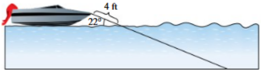
17. $A = 38.2°, b = 17.8$

18. $A = 25.6°, a = 34$



Learn: Solve Applications of Right Triangles

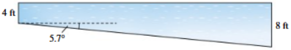
19. A boat is anchored off Elliot Key, Florida. From the bow of the boat, 36 ft of anchor line is out with 4 ft of line above the water. The angle that the line makes with the water line is 22°. **(See Example 2, p. 674)**



a. How deep is the water? Round to the nearest foot.

b. What is the horizontal distance between the anchor and the bow of the boat? Round to the nearest foot.

20. A swimming pool has a depth of 4 ft at the shallow end and 8 ft at the deep end. The bottom of the pool slopes downward at an angle of 5.7°. How long is the pool? Round to the nearest foot.



21. An airplane climbs at an angle of 11° at an average speed of 420 mph. How long will it take for the plane to reach its cruising altitude of 6.5 mi? Round to the nearest minute.

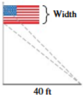
22. An airplane begins its descent with an average speed of 240 mph at an angle of depression of 4°. How much altitude will the plane lose in 5 min? Round to the nearest tenth of a mile.

23. A railroad bridge over New Scotland Road in Slingerlands, New York, has a low clearance for trucks. An engineer standing 20 ft away measures a 15.4° angle of elevation from an eye level of 5.5 ft to the bottom of the bridge. If the road is flat between the engineer and the bridge, how high over the roadway is the bottom of the bridge? Round to the nearest inch.

24. At 11:30 A.M. the angle of elevation of the sun for Washington, D.C., is 55.7°. If the height of the Washington Monument is approximately 555 ft, what is the length of the shadow it will cast at that time? Round to the nearest foot.

25. The suggested size for a United States flag is often determined by the height of the flagpole from which it will fly. Federal recommendations on flag size for flagpole heights are provided in the table. (www.publications.usa.gov)

Flagpole height (ft)	Flag size (ft)
25	5 by 8
40	6 by 10
60	10 by 15
125	20 by 30



Measurements taken 40 ft from the base of a flagpole show the angle of elevation to the top of the flagpole to be 56.3° and the angle of elevation to the bottom of the flag to be 51.3°. **(See Example 3, p. 674)**

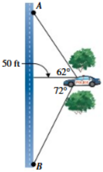
a. Determine the height of the flagpole. Round to the nearest foot.

b. Determine the vertical width of the flag. Round to the nearest foot.

c. What other information is needed to determine if the display conforms to the federal guidelines?

26. Roberto has plans to build a tree house in a large oak tree in his backyard. He takes measurements from the ground and uses trigonometry to estimate how tall the tree house can be. Twenty feet from the base of the tree, Roberto measures a 26.6° angle of elevation to the proposed bottom of the tree house and a 42° angle of elevation to the proposed top of the tree house. If Roberto is 6 ft tall, will there be enough room for him to stand inside the tree house?

27. A police officer hiding between two bushes 50 ft from a straight highway sights two points *A* and *B*. The angle from the police car to *A* is 62°, and the angle to point *B* is 72°.

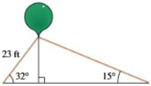


a. Find the distance between *A* and *B*. Round to the nearest foot.

b. Suppose that a motorist takes 2.7 sec to pass from *A* to *B*. Using the rounded distance from part (a), find the motorist's speed in ft/sec. Round to 1 decimal place.

c. Determine the motorist's speed in mph. Round to the nearest mph.

28. A large weather balloon is tethered by two ropes. One rope measures 23 ft and attaches to the balloon at an angle of 32° from the ground. The second rope attaches to the base of the balloon at an angle of 15° with the ground.



a. How far from the ground is the balloon floating? Round to the nearest tenth of a foot.

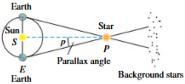
b. Find the length of the second rope. Round to the nearest tenth of a foot.

c. If both ropes suddenly detach and the balloon rises straight up at a rate of 3 ft/sec, how long will it take the balloon to reach a height of 50 ft from the ground? Round to the nearest tenth of a second.

29. The Duquesne Incline is a century-old railroad scaling Mt. Washington in Pittsburgh's south side neighborhood. The incline is 244 m long and rises 120 m in elevation. What is the angle of the incline to the nearest tenth of a degree? **(See Example 4, p. 675)**

30. The Jackson Hole Aerial Tram takes passengers from an elevation of 6311 ft to an elevation of 10,450 ft up the slope of Rendezvous Mountain in Wyoming. If the tram runs on a cable measuring 12,463 ft in length, what is the angle of incline of the tram to the nearest tenth of a degree?

To approximate the distance from the Earth to stars relatively close by, astronomers often use the method of parallax. Parallax is the apparent displacement of an object caused by a change in the observer's point of view. As the Earth orbits the Sun, a nearby star will appear to move against the more distant background stars. Astronomers measure a star's position at times exactly 6 months apart when the Earth is at opposite points in its orbit around the Sun. The Sun, Earth, and star form the vertices of a right triangle with $\angle PSE = 90°$. The length of $\overline{SE}$ is the distance between the Earth and Sun, approximately 92,900,000 mi. The parallax angle (or simply parallax) is denoted by p. Use this information for Exercises 31–32.



31. a. Find the distance between the Earth and Proxima Centauri (the closest star to the Earth beyond the Sun) if the parallax angle is 0.772″ (arcseconds). Round to the nearest hundred billion miles.

b. Write the distance in part (a) in light-years. Round to 1 decimal place. (*Hint*: 1 light-year is the distance that light travels in 1 yr and is approximately 5.878×10^{12} mi.)

32. a. Find the distance between the Earth and Barnard's Star if the parallax angle is 0.547 arcseconds. Round to the nearest hundred billion miles.

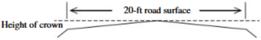
b. Write the distance in part (a) in light-years. Round to 1 decimal place. (*Hint*: 1 light-year is the distance that light travels in 1 yr and is approximately 5.878×10^{12} mi.)

33. A sewer line must have a minimum slope of 0.25 in. per horizontal foot but not more than 3 in. per horizontal foot. A slope less than 0.25 in. per foot will cause drain clogs, and a slope of more than 3 in. per foot will allow water to drain without the solids.

a. To the nearest tenth of a degree, find the angle of depression for the minimum slope of a sewer line.

b. Find the angle of depression for the maximum slope of a sewer line. Round to the nearest tenth of a degree.

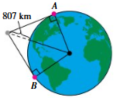
34. In order for gravel roads to have proper drainage, the highest point on the road (the crown) should slope downward on either side to the shoulders of the road. EPA guidelines for maintaining gravel roads with a low volume of traffic suggest that a 20-ft wide road should have a centerline crown that is 5 to 7 in. high. (*Source*: http://water.epa.gov)



a. To the nearest tenth of a degree, find the angle of depression for a 5-in. centerline crown.

b. To the nearest tenth of a degree, find the angle of depression for a 7-in. centerline crown.

35. The weather satellite NOAA-15 orbits the Earth every 101 min at an altitude of 807 km as shown in the figure. The cameras on the satellite have a maximum range defined by the set of lines tangent to the Earth from the satellite. Using the radius of the Earth as 6357 km, approximate the length of the arc from *A* to *B* to the nearest kilometer. This is the maximum distance along the surface of the Earth "seen" by the satellite's cameras.



36. A communications satellite is in geosynchronous orbit. This means that its orbit coincides with the rotation period of the Earth so that it remains above a fixed point on the surface of the Earth at all times. The altitude of the satellite is 35,786 km. The sender and receiver of a signal must both be within the line of sight of the satellite. What is the maximum distance along the surface of the Earth for which the sender and receiver can communicate? Take the radius of the Earth to be 6357 km and round to the nearest kilometer.

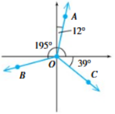
Learn: Compute the Bearing of an Object

37. a. Find the bearing from *O* to *A*. **(See Example 5, p. 677)**

b. Find the bearing from *O* to *B*.

c. Find the bearing from *O* to *C*.

d. Find the bearing from *C* to *O*.

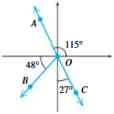


38. a. Find the bearing from *O* to *A*.

b. Find the bearing from *O* to *B*.

c. Find the bearing from *O* to *C*.

d. Find the bearing from *A* to *O*.



For Exercises 39–42, make a sketch to illustrate the bearing.

39. N45°W

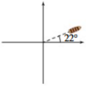
40. S10°W

41. S67°E

42. N75°E

For Exercises 43–46, find the bearing of the ship. Round to the nearest tenth of a degree if necessary.

43.



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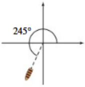
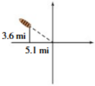
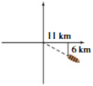


45.



46.



For Exercises 47–50, find the bearing to the nearest tenth of a degree.

47. A runner jogs 5 mi east then 2 mi south. What is the bearing from the starting point?

48. A cyclist rides west for 7 mi and then north 10 mi. What is the bearing from the starting point?

49. A boat leaves Matheson Hammock Marina at a constant speed of 3.5 mph. The boat travels south for 36 min and then east for 24 min to a favorite fishing spot. After a day of fishing, find the bearing that the captain should use to travel back to the marina.

50. A plane leaves Atlanta's Hartsfield Airport and flies north for 1 hr and west for 2 hr at an average speed of 400 mph. Find the bearing that the plane should take for the return trip.

For Exercises 51–54, round answers to the nearest unit.

51. A fishing boat leaves a dock at a bearing of N34°E. After running for 1.4 hr at 25 mph, how far north and east has it traveled? (**See Example 6, p. 678**)

52. A helicopter takes off from the roof of a building and travels at 100 mph on a bearing of S53°E. The flight takes 3.2 hr. How far south and how far east has the helicopter traveled?

53. An airplane flying 310 knots has a bearing of S44°W. After flying for 2.4 hr, how many nautical miles (nmi) south and west has it traveled?

54. A ship traveling 6.4 knots has a bearing of N11°W. After 3 hr, how many nautical miles (nmi) north and west has it traveled?

55. A ship leaves port at 8 knots heading N27°W. After 2 hr it makes a 90° clockwise turn to a new bearing of N63°E and travels for 1.4 hr. (**See Example 7, p. 679**)

a. Find the ship's distance from port to the nearest tenth of a nautical mile.

b. Find the bearing required for the ship to return to port. Round to the nearest degree.

56. A plane leaves an airport heading S82°E at 320 mph. After 1.2 hr, it makes a 90° counterclockwise turn to a new bearing of N8°E and travels for 45 min.

a. Find the plane's distance from the airport to the nearest mile.

b. Find the bearing required for the plane to return to the airport. Round to the nearest degree.

57. A boat leaves a dock at a bearing of N42°W and travels 24 mph for 1 hr. The boat then makes a 90° left turn and travels 8 mi to its destination.

a. Find the exact distance from the dock to the boat's destination.

b. Find the bearing required to locate the boat from the dock. Round to the nearest degree.

58. A runner leaves the starting point of a race and heads N10°W at 6.3 mph. After running for 20 min, the runner makes a 90° right turn and runs 3 more miles at a speed of 6.8 mph.

a. Find the total time of the run to the nearest minute.

b. Find the bearing required to locate the runner from the starting point. Round to the nearest degree.

Mixed Exercises

For Exercises 59–60, find the lengths of the sides to the nearest tenth of a unit and the measures of the angles to the nearest tenth of a degree if necessary.

59. $A =$ ________

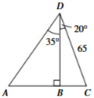
$C =$ ________

$AB =$ ________

$BC =$ ________

$AD =$ ________

$DB =$ ________

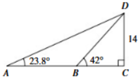


60. $\angle ADC =$ ________ $\angle ADB =$ ________

$\angle BDC =$ ________ $BC =$ ________

$AC =$ ________ $AD =$ ________

$BD =$ ________ $AB =$ ________



61. Find the measures of the sides and angles. Round angle measures to the nearest tenth of a degree. Give exact answers for the lengths of sides.

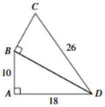
$\angle ADC =$ ________,

$\angle CBA =$ ________,

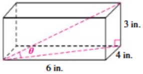
$\angle DCB =$ ________,

$BD =$ ________,

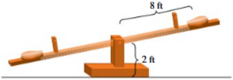
$BC =$ ________



62. A rectangular prism measures 6 in. by 4 in. by 3 in. Find the angle θ formed by the diagonal of the prism and the diagonal of the base, as shown. Round to 1 decimal place.



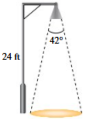
63. Two children sit on either end of an 18-ft teeter totter. The center of the teeter totter is 2 ft off the ground, and the position of each child is 8 ft from the center.



a. When one end of the teeter totter is on the ground, find the angle that the teeter totter makes with the ground. Round to the nearest tenth of a degree.

b. When one end of the teeter totter is on the ground, how high is the child's seat at the other end? Round to 1 decimal place.

64. A light fixture mounted 24 ft above the ground illuminates a cone of light with an angle of 42° at the top.



a. What is the radius of the circle of light on the ground? Round to 1 decimal place.

b. How many square feet will be lit on the ground? Round to the nearest whole unit.

For Exercises 65–66, refer to the table giving the angle of the Sun at noon at the first day of each season based on the latitude of the observer in the northern hemisphere. The term 23.5° represents the tilt of the Earth to the ecliptic plane (plane of the Earth's orbit around the Sun).

Date	Sun's angle at noon
First day of spring: March equinox	90° – latitude
First day of fall: September equinox	90° – latitude
First day of summer: Summer solstice	90° – latitude + 23.5°
First day of winter: Winter solstice	90° – latitude – 23.5°

65. In 1987, a group of protesters led by Jacqueline Kennedy Onassis defeated a plan to build two high rises that would have blocked sunlight to a significant swath of New York City's Central Park. Eventually the design was scaled down and the buildings' height reduced from approximately 1020 ft to what is now the Time Warner Center at 750 ft.

a. The latitude of New York City is 40.7° N. Find the angle of elevation of the Sun at noon on the winter solstice (shortest day of the year).

b. At noon on the winter solstice, how long is the shadow cast by a 1020-ft building?

c. At noon on the winter solstice, how long is the shadow cast by a 750-ft building?

66. The Earth is tilted 23.5° on its axis. Therefore, at noon on the summer solstice, the Earth is tilted 23.5° toward the Sun, and all points on the Earth with latitude 23.5° N will see the Sun directly overhead at noon. Points on the Earth with latitude 23.5° N are on the Tropic of Cancer. Key West, Florida, has latitude 24.6° N, just above the Tropic of Cancer.

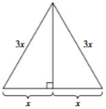
a. Find the angle of elevation of the Sun at noon in Key West on the first day of summer and on the first day of winter.

b. Find the length of a shadow cast by a 105-ft building at noon on the first day of summer and on the first day of winter. Round to the nearest tenth of a foot.

c. Find the length of a shadow cast by a 40-ft tree at noon on the first day of fall or spring.

67. OSHA safety regulations require that the base of a ladder be placed 1 ft from the wall for every 4 ft of ladder length. To the nearest tenth of a degree, find the angle that the ladder forms with the ground and the angle that it forms with the wall.

68. An A-frame storage shed is constructed such that the slanted roof line is always 3 times as long as the distance from the bottom corner of the roof to the centerline of the shed. To the nearest degree, what is the angle at the top of the roof?

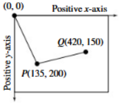


69. An airplane needs to fly to an airfield located 189 mi east and 568 mi north of its current location. To the nearest tenth of a degree, determine the bearing at which the plane should fly.

70. Software used to program video games often uses an origin at the top left of the display canvas. The positive *x*-axis is to the right and the positive *y*-axis is downward. Suppose that a player moves on a direct path from the origin to a point *P* with pixel location (135, 200). Then the player moves directly to point *Q* at a pixel location of (420, 150).

a. Find the player's bearing from the origin to point *P*. Round to one-hundredth of a degree.

b. Find the player's bearing from point *P* to point *Q*. Round to one-hundredth of a degree



71. Two ships leave port at the same time. One travels N72° E at 20 knots and the other travels S18° E at a rate of 30 knots.

a. After 2 hr, how far apart are the two ships? Give an exact answer in nautical miles.

b. Find the bearing from the ship farther to the north to the ship farther to the south. Round to the nearest tenth of a degree.

72. Deshauna lives 17 mi due west of her friend Jasmine. They both like to bicycle and decide to meet for lunch at a restaurant 13 mi from Deshauna's house. If the bearing from Deshauna's house to the restaurant is N20.2° E, how far does Jasmine have to ride? Round to the nearest tenth of a mile.

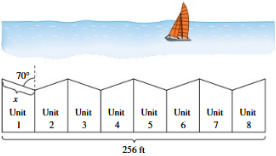
Write About It

73. Explain how to represent the bearing from point *P* to point *Q*.

74. What is meant by solving a triangle?

Expanding Your Skills

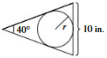
75. A contractor building eight ocean-front condominiums wants to maximize the view of the ocean for each unit. The side of the building facing the ocean is not built in a straight line parallel to the ocean, but instead is built in a zigzag pattern, as shown in the figure. Each condo has a window of length *x* facing the ocean at an angle of 70° from a line perpendicular to the ocean.



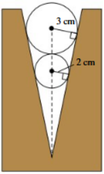
a. Find the length *x* of each window. Round to the nearest foot.

b. The windows facing the ocean are 8 ft high and *x* feet wide. By using the zigzag pattern, how much more ocean-front viewing area does each window provide than if the windows were parallel to the ocean? Round to the nearest square foot.

76. A circle is inscribed within an isosceles triangle such that the circle is tangent to all three sides as shown in the figure. Using the fact that the radius *r* of the circle is perpendicular to any line tangent to the circle, find the value of *r* to the nearest tenth of an inch.

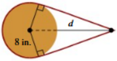


77. A tool requires a V-shaped block to hold two rollers of radius 3 cm and 2 cm. Determine the angle of the "V" cut to the nearest tenth of a degree.

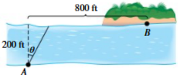


Technology Connections

78. A 5-ft (60-in.) belt is to be looped around a rotating drum of radius 8 in. and a small spindle located *d* inches from the rim of the drum. Find the distance *d* between the spindle and the rim of the drum to the nearest tenth of an inch. (*Hint*: Set up an equation and use a graphing utility to approximate the solution.).



79. Pipe for a water line must be installed from a main water line at point *A* to a building on Hontoon Island State Park at point *B* as shown in the figure. The cost to install water pipe over land is \$10 per foot, and the cost to install pipe under water is \$20 per foot.



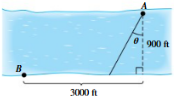
a. Write an expression in terms of θ to represent the total cost *c* (in dollars) to lay pipe from point *A* to point *B*.

b. Use the TABLE function on a calculator to find the cost for $\theta = 20°, 25°, 30°, 35°$, and $40°$. Round to 1 decimal place.

c. Which angle from part (b) yields the least cost?

d. Using calculus, we can show that the angle needed to minimize the total cost is a solution to the equation $4000\sec\theta\tan\theta - 2000\sec^2\theta = 0$. Solve the equation for θ, where $0° < \theta < 90°$.

80. An athlete participating in a triathlon can run 11 ft/sec and swim 3 ft/sec. Starting at point *A*, 900 ft from a straight shoreline, the athlete must swim to shore and then run to point *B*, 3000 ft down the beach.



a. Write an expression representing the total time *t* (in seconds) needed to get from point *A* to point *B* as a function of θ.

b. Use the TABLE function on a calculator to find the time *t* for $\theta = 0°, 5°, 10°, 15°$, $20°$, and $25°$. Round to 1 decimal place.

c. Which angle from part (b) gives the shortest total time?

d. Using calculus, we can show that the angle needed to minimize the total time is a solution to the equation $300\sec\theta\tan\theta - \frac{900}{11}\sec^2\theta = 0$. Solve the equation for θ, where $0° < \theta < 90°$. Round to the nearest tenth of a degree.

GO ONLINE **Go online for more practice problems.**

Lesson 6-2

The Law of Sines

What Will You Learn?

After completing this lesson, you should be able to:

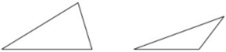
- Solve a Triangle Using the Law of Sines (SAA or ASA)
- Solve a Triangle Using the Law of Sines (SSA) Ambiguous Case
- Compute the Area of a Triangle Given SAS
- Apply the Law of Sines

Learn: Solve a Triangle Using the Law of Sines (SAA or ASA)

A mechanical engineer designing a new tool will use computer software for digital prototyping. It is imperative that the precise measure of each line segment, angle, and curve is entered and represented correctly before manufacturing takes place. The software uses mathematical relationships among the sides and angles in 2- and 3-dimensional figures as a means to check the engineer's measurements.

As a simple example, suppose that an engineer wants to draw a triangle with sides of lengths 3 cm, 5 cm, and 15 cm. No such triangle is possible because the length of the longest side exceeds the sum of the lengths of the other two sides. The law of sines and law of cosines presented in this chapter enable us to determine conditions under which a triangle cannot be constructed.

A triangle that does not contain a right angle is called an **oblique triangle.** An oblique triangle will have either three acute angles or one obtuse angle and two acute angles (Figure 6-10).



(a) Acute Triangle
All angles acute

(b) Obtuse Triangle
One angle obtuse

Figure 6-10

For an oblique triangle, given the length of a side and any other two parts of the triangle we can solve the triangle. To organize this discussion, we outline four distinct cases involving known angles (A) and known sides (S).

1. Two angles and any side are known (ASA or SAA).
2. Two sides and an angle opposite one of them are known (SSA).
3. Three sides are known (SSS).
4. Two sides and the angle between them are known (SAS).

Insights

Notation such as ASA is a shortcut used to convey what parts of an oblique triangle are known. For example, ASA stands for "Angle-Side-Angle."

The first two cases can be solved by using the law of sines (presented in this section), whereas cases 3 and 4 require the law of cosines (presented in Lesson 6.3).

Law of Sines

If $\triangle ABC$ is a triangle with sides of lengths a, b, and c opposite angles A, B, and C, respectively, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

In words: The ratio of the length of a side of a triangle to the sine of the angle opposite that side is the same for all three side-angle combinations.

To prove the law of sines, refer to Figure 6-11. The altitude h drawn from vertex C to side c is a common leg of two right triangles.

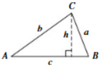


Figure 6-11

From the right triangles, we have

$$\sin A = \frac{h}{b} \Rightarrow h = b\sin A$$

$$\sin B = \frac{h}{a} \Rightarrow h = a\sin B$$

Equating the expressions for h gives

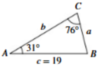
$$b\sin A = a\sin B \Rightarrow \frac{b}{\sin B} = \frac{a}{\sin A}$$

Drawing an altitude from vertex A to side a and applying similar reasoning can be used to show that $\frac{b}{\sin B} = \frac{c}{\sin C}$, completing the law of sines.

To apply the law of sines, we need to have one complete ratio and one other piece of information about the triangle. That is, we need to know the measure of one side and the angle opposite it, along with one other part of the triangle.

Example 1: Solving an Oblique Triangle Given SAA

Solve the triangle with $A = 31°$, $C = 76°$, and $c = 19$. Round the lengths of the sides to 1 decimal place.



Solution:

In this situation, side c and angle C provide a complete ratio and angle A is a third piece of information. To find the length of side a, we have

$$\frac{c}{\sin C}=\frac{a}{\sin A}\Rightarrow\frac{19}{\sin 76^\circ}=\frac{a}{\sin 31^\circ}\Rightarrow a=\frac{19\sin 31^\circ}{\sin 76^\circ}\approx 10.1$$

The measure of angle B is given by $B=180^\circ-(31^\circ+76^\circ)=73^\circ$.

To find the length of side b, we can use $\frac{c}{\sin C}=\frac{b}{\sin B}$ or $\frac{a}{\sin A}=\frac{b}{\sin B}$.

However, we choose the equation on the left because it does not include any values that were rounded in previous calculations.

$$\frac{c}{\sin C}=\frac{a}{\sin B}\Rightarrow\frac{19}{\sin 76^\circ}=\frac{b}{\sin 73^\circ}\Rightarrow b=\frac{19\sin 73^\circ}{\sin 76^\circ}\approx 18.7$$

In summary, $A=31^\circ$, $B=73^\circ$, $C=76^\circ$, $a\approx 10.1$, $b\approx 18.7$, and $c=19$.

Apply the Skills

1. Solve $\triangle ABC$ with $B=70^\circ$, $C=48^\circ$, and $b=15$ cm. Round the lengths of the sides to 1 decimal place.

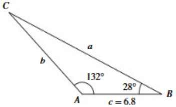
Insights

Notice that the units of measurement were left off the calculations in Example 1. Since $\sin\theta$ is unitless, the values of a and b will have the same units of measurement as c. For simplicity, when applying the law of sines or cosines, the units are left off in the equations and applied later once a solution is found.

In Example 2, we solve an oblique triangle with two given angles and the side between them known (ASA).

Example 2: Solving an Oblique Triangle Given ASA

Solve the triangle with $A=132^\circ$, $B=28^\circ$, and $c=6.8$. Round the lengths of the sides to 1 decimal place.



Solution:

First note that $C=180^\circ-(132^\circ+28^\circ)=20^\circ$.

With the measure of angle C known, angle C and side c provide a complete ratio. To find the length of side a, we have

$$\frac{c}{\sin C}=\frac{a}{\sin A}\Rightarrow\frac{6.8}{\sin 20^\circ}=\frac{a}{\sin 132^\circ}\Rightarrow a=\frac{6.8\sin 132^\circ}{\sin 20^\circ}\approx 14.8$$

To find the length of side b, we have

$$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{6.8}{\sin 20^\circ} = \frac{b}{\sin 28^\circ} \Rightarrow b = \frac{6.8 \sin 28^\circ}{\sin 20^\circ} \approx 9.3$$

In summary, $A = 132^\circ$, $B = 28^\circ$, $C = 20^\circ$, $a \approx 14.8$, $b \approx 9.3$, and $c = 6.8$.

Apply the Skills

2. Solve $\triangle ABC$ with $A = 33^\circ$, $B = 112^\circ$, and $c = 16$. Round the lengths of the sides to 1 decimal place.

Learn: Solve a Triangle Using the Law of Sines (SSA) Ambiguous Case

In Examples 1 and 2, the measures of two angles and a side of a triangle were given, and we found a unique solution for the measures of the remaining sides and angle. Now we address the case where the lengths of two sides and the measure of an angle opposite one of the known sides are given. A great deal of care must be given to this scenario because the given information may define a unique triangle, two different triangles, or no triangle.

Suppose that the measure of angle A and the lengths of sides a and b of a triangle are given (Figure 6-12). The altitude h of the triangle is given by

$$\sin A = \frac{h}{b} \Rightarrow h = b \sin A$$

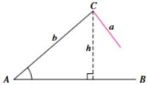
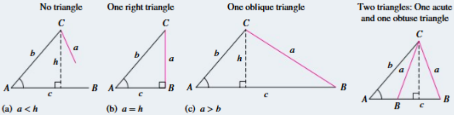


Figure 6-12

Depending on the length of side a (shown in red) relative to the length of the altitude h, we have four different scenarios.

Law of Sines Ambiguous Case



(a) $a < h$ Side a is too short to meet side c.

(b) $a = h$ Side a exactly equals the altitude.

(c) $a > b$ One triangle is possible.

(d) $h < a < b$ Side a intersects side c at an obtuse angle or acute angle.

Example 3: Solving a Triangle Given SSA (One Solution)

Solve the triangle with $A = 57°$, $a = 62$, and $b = 50$. Round the measures of the unknown angles and side to 1 decimal place.

Solution:

We begin with a sketch. For convenience we draw side c as a horizontal line segment and angle A in standard position relative to side c (Figure 6-13).

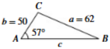


Figure 6-13

Since $a > b$, side a is long enough to meet side c. Furthermore, side a must extend to the right as shown in Figure 6-13.

From the law of sines,

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{62}{\sin 57°} = \frac{50}{\sin B} \Rightarrow \sin B = \frac{50 \sin 57°}{62} \approx 0.6763$$

Using a calculator, we determine that $\sin^{-1}\left(\frac{50 \sin 57°}{62}\right) \approx 42.6°$. However, there are two angles between 0° and 180° for which $\sin B \approx 0.6763$.

$$B \approx 42.6° \text{ or } B \approx 180° - 42.6° \approx 137.4°$$

However, angle B cannot equal 137.4° because taken together with $A = 57°$, the sum exceeds 180° $(137.4° + 57° = 194.4°)$. This tells us that $B \approx 42.6°$, and consequently, there is one unique triangle that can be formed from the given information.

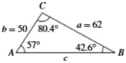
Insights

Here, since $a > b$, side a must extend to the right of the altitude. This means that angle B must be acute. Therefore, angle B is the first quadrant angle equal to $\sin^{-1}(50 \sin 57°/62)$.

With $A = 57°$ and $B \approx 42.6°$, we have $C = 180° - (57° + 42.6°) \approx 80.4°$.

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{62}{\sin 57°} = \frac{c}{\sin 80.4°}$$

So, $c \approx \frac{62 \sin 80.4°}{\sin 57°} \approx 72.9$



In summary, $A = 57°$, $B \approx 42.6°$, $C \approx 80.4°$, $a = 62$, $b = 50$, and $c \approx 72.9$.

Apply the Skills

3. Solve $\triangle ABC$ with $A = 49°$, $a = 32$, and $b = 28$. Round the measures of the unknown angles and side to 1 decimal place.

Example 4: Solving a Triangle Given SSA (No Solution)

Solve the triangle with $C = 39°$, $c = 6$, and $a = 15$. Round the measures of the unknown angles and side to 1 decimal place.

Solution:

We begin with a sketch. For convenience we draw side b as a horizontal line segment and angle C in standard position relative to side b (Figure 6-14). Intuitively, we might question whether the length of side c is long enough to intersect side b.

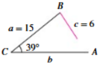


Figure 6-14

From the law of sines,

$$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{6}{\sin 39°} = \frac{15}{\sin A} \Rightarrow \sin A = \frac{15 \sin 39°}{6} \approx 1.5733$$

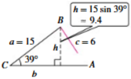
This is not possible because $-1 \le \sin A \le 1$ for any angle A. No triangle can be constructed from the given information.

Alternative Approach:

The altitude h drawn from vertex B to side b can be computed by

$$\sin 39° = \frac{h}{15} \Rightarrow h = 15 \sin 39°$$

$$h \approx 9.4$$



The length of side c would need to be at least 9.4 units long to reach side b. Therefore, no triangle is possible.

Apply the Skills

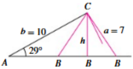
4. Solve $\triangle ABC$ with $B = 31°$, $b = 3$, $a = 6.2$. Round the measures of the unknown angles and side to 1 decimal place.

Example 5: Solving a Triangle Given SSA (Two Solutions)

Solve the triangle with $A = 29°$, $a = 7$, and $b = 10$. Round the measures of the unknown angles and side to 1 decimal place.

Solution:

From a preliminary sketch, we do not know if side a is long enough to reach side c, or if side a makes up the third side of one or two triangles.



To determine whether the given information defines zero, one, or two triangles, we can apply the law of sines to determine the possible values for angle B. Or we can compare the length of side a to the altitude h.

Using the law of sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{7}{\sin 29^\circ} = \frac{10}{\sin B}$$

$$\sin B = \frac{10\sin 29^\circ}{7} \approx 0.6926$$

Using a calculator we determine that $\sin^{-1}\left(\frac{10\sin 29^\circ}{7}\right) \approx 43.8^\circ$. However, there are two angles between 0° and 180° for which $\sin^{-1} 0.6926$.

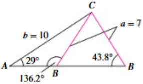
$B \approx 43.8^\circ$ or $B \approx 180^\circ - 43.8^\circ \approx 136.2^\circ$

Finding the altitude h:

$$\sin A = \frac{h}{10}$$

$$h = 10\sin 29^\circ \approx 4.8$$

Since $h < a < b$ (that is, $4.8 < 7 < 10$), there are two possible triangles.



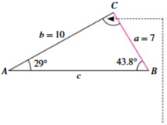
Adding either value for angle B to the given angle results in a sum less than 180°.

$$A + B = 29^\circ + 43.8^\circ = 72.8^\circ < 180^\circ$$

$$A + B = 29^\circ + 136.2^\circ = 165.2^\circ < 180^\circ$$

With two viable values for angle B, we can construct two triangles.

Triangle 1: $\angle B$ acute



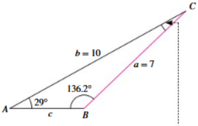
$C \approx 180^\circ - (29^\circ + 43.8^\circ) \approx 107.2^\circ$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{7}{\sin 29^\circ} = \frac{c}{\sin 107.2^\circ}$$

$$c = \frac{7\sin 107.2^\circ}{\sin 29^\circ} \approx 13.8$$

Triangle 2: $\angle B$ obtuse



$C \approx 180^\circ - (29^\circ + 136.2^\circ) \approx 14.8^\circ$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{7}{\sin 29^\circ} = \frac{c}{\sin 14.8^\circ}$$

$$c = \frac{7\sin 14.8^\circ}{\sin 29^\circ} \approx 3.7$$

In summary we have

Triangle 1: $A = 29°$, $B \approx 43.8°$, $C \approx 107.2°$, $a = 7$, $b = 10$, $c \approx 13.8$

Triangle 2: $A = 29°$, $B \approx 136.2°$, $C \approx 14.8°$, $a = 7$, $b = 10$, $c \approx 3.7$

Check Your Work

After solving a triangle, confirm that the longest side is opposite the largest angle and that the shortest side is opposite the smallest angle. To further confirm your results, you can double check that the three ratios $a/\sin A$, $b/\sin B$, and $c/\sin C$ are the same. For example, for Triangle 1 in Example 5, $7/\sin 29° \approx 14.4$, $10/\sin 43.8° \approx 14.4$, and $13.8/\sin 107.2° \approx 14.4$.

Apply the Skills

5. Solve $\triangle ABC$ with $A = 40°$, $a = 22$, and $b = 25$. Round the measures of the unknown angles and the third side to 1 decimal place.

Learn: Compute the Area of a Triangle Given SAS

The approach we used to prove the law of sines can be used to find a formula for the area of a triangle given the lengths of two sides and the angle between them.

In Figure 6-15, we have the relationship

$\sin A = \frac{h}{b} \Rightarrow h = b \sin A$

The area of $\triangle ABC$ is $\frac{1}{2}$(base)(height) $= \frac{1}{2} cb \sin A$.

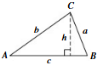


Figure 6-15

The area is one-half the product of the lengths of the two sides times the sine of the angle between them.

Area of a Triangle Given Two Sides and the Angle Between Them

For a triangle $\triangle ABC$, with sides of lengths a, b, and c across from angles A, B, and C, respectively, the area of the triangle is given by

$$\text{Area} = \frac{1}{2} bc \sin A \quad \text{or} \quad \text{Area} = \frac{1}{2} ab \sin C \quad \text{or} \quad \text{Area} = \frac{1}{2} ac \sin B$$

The proofs of the second and third relationships are found in a similar way by drawing the altitude from vertex B and vertex A, respectively.

Example 6: Finding the Area of an Oblique Triangle (SAS)

A triangle has sides of 5 cm and 8 cm, and the angle between them is 121°. Find the area of the triangle. Round to 1 decimal place.

Solution:

Let A denote the vertex of the given angle, and let sides b and c represent the lengths of the two given sides (Figure 6-16).

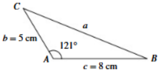


Figure 6-16

$$\text{Area} = \frac{1}{2}bc\sin A = \frac{1}{2}(5\text{ cm})(8\text{ cm})\sin 121°$$

$$\approx 17.1\text{ cm}^2$$

The area is approximately 17.1 cm^2.

Apply the Skills

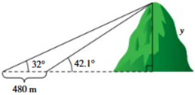
6. A triangle has sides of 12 m and 6 m, and the angle between them is 76°. Find the area of the triangle. Round to 1 decimal place.

Learn: Apply the Law of Sines

The law of sines can be used in applications in which the value of an angle in a triangle and the length of the side opposite it are known, along with one other part of the triangle.

Example 7: Apply the Law of Sines to Find Height

To estimate the height of a hill, the angle of elevation is measured at two points 480 m apart on a direct line to the hill. The angle of elevation from the closer point is 42.1°. If the angle of elevation from the farther point is 32°, find the height of the hill. Round to the nearest meter.



Solution:

A sketch is provided with the relevant angles and sides labeled (Figure 6-17).

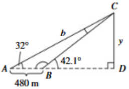


Figure 6-17

$\angle CBA$ is supplementary to $\angle DBC$. Therefore,

$$\angle CBA = 180° - 42.1° = 137.9°$$

From $\triangle ABC$, $C = 180° - (32° + 137.9°) = 10.1°$.

Using $\triangle ABC$ and the law of sines,

$$\frac{480}{\sin 10.1°} = \frac{b}{\sin 137.9°}$$

$$480 \sin 137.9° = b \sin 10.1°$$

$$b = \frac{480 \sin 137.9°}{\sin 10.1°} \approx 1835 \text{ m}$$

Finally, we can use the right triangle ADC to find the value of y (Figure 6-18).

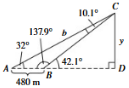


Figure 6-18

$$\frac{y}{b} = \sin 32° = \frac{y}{1835} \Rightarrow y = 1835 \sin 32° \approx 972$$

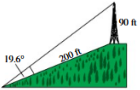
The height of the hill is approximately 972 m.

Apply the Skills

7. To measure the maximum height of the water in a fountain, a trigonometry student measures the angle of elevation of the maximum point to be 25°. At a point 30 ft farther away along the same line of sight, the angle of elevation is 18°. If the eye level of the student is 5.5 ft, find the maximum height of the water. Round to the nearest tenth of a foot.

Example 8: Apply the Law of Sines to Find Angles

A 90-ft cell tower is located at the top of a hill. An observer 200 ft down the hill measures the angle between the bottom and top of the tower to be 19.6°. Determine the angle of inclination of the hill to the nearest tenth of a degree.



Solution:

To begin, we can solve $\triangle ABC$ for angles B and C (Figure 6-19).

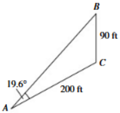


Figure 6-19

$$\frac{\sin B}{200} = \frac{\sin 19.6^\circ}{90} \Rightarrow \sin B = \frac{200 \sin 19.6^\circ}{90}$$

Since the observer is looking uphill, angle C must be greater than 90°. Therefore, angle B must be acute.

$$B = \sin^{-1}\left(\frac{200 \sin 19.6^\circ}{90}\right) \approx 48.2^\circ \text{ and}$$
$$C \approx 180^\circ - (19.6^\circ + 48.2^\circ) \approx 112.2^\circ$$

Let α represent the angle of inclination of the hill.

From Figure 6-20,

$\beta \approx 180^\circ - 112.2^\circ \approx 67.8^\circ$ and

$\alpha = 90^\circ - \beta \approx 90^\circ - 67.8^\circ \approx 22.2^\circ$

The angle of the hill is approximately 22.2°.

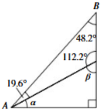
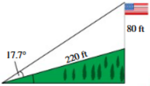


Figure 6-20

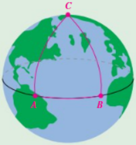
Apply the Skills

8. An 80-ft flagpole is located at the top of a hill. An observer 220 ft down the hill measures the angle formed by the bottom of the pole to the top of the pole to be 17.7°. Determine the angle of inclination of the hill to the nearest tenth of a degree.



Point of Interest

In addition to the six trigonometric functions, other related functions were used by early mathematicians and astronomers to address specific applications. For example, the *haversine* function is given by haversin $\theta = \sin^2 \frac{\theta}{2}$. The haversine function is used in navigation to compute great-circle distances between two points on a sphere based on the latitude and longitude of each point. Furthermore, the *law of haversines* is used to relate the sides and angles of a triangle whose vertices are on a spherical surface rather than a flat surface.



Practice 6-2

Practice Exercises

Prerequisite Review

R.1. Use a calculator to approximate the value of $\sin^{-1}0.76$ to the nearest tenth of a degree.

R.2. Use a calculator to approximate the degree measure of θ for $\sin\theta = \frac{3}{4}$ and $90° \leq \theta \leq 180°$.

Concept Connections

1. To apply the law of sines, an angle must be given and a side (opposite/adjacent) the given angle must be given, along with one other known side or angle.

2. In which cases can the law of sines be used to solve a triangle? Choose from ASA, SAA, SSA, SSS, SAS.

3. Suppose that $\triangle ABC$ is a triangle with sides of length a, b, and c opposite angles A, B, and C, respectively. Write the law of sines.

4. Suppose that the lengths of two sides a and b in a triangle are known along with the measure of the angle C between them. Then the area is given by Area = ____________.

5. Suppose that we know the measure of angle A and the lengths of sides a and b of a triangle. If $a > b$, how many possible triangles can be formed?

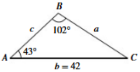
6. Suppose that we know the measure of angle A and the lengths of sides a and b of a triangle. If $b\sin A < a < b$, how many possible triangles can be formed?

Learn: Solve a Triangle Using the Law of Sines (SAA or ASA)

Note: Due to round-off error, the answers to exercises involving calculations to determine unknown angles and sides may differ slightly from those given.

For Exercises 7–12, solve the triangle. For the sides, give an expression for the exact value of the length and an approximation to 1 decimal place. (See Examples 1–2, pp. 690–691)

7.



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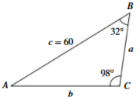

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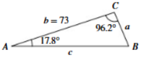

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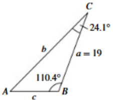

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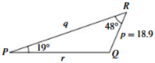
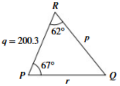


12.



For Exercises 13–20, solve $\triangle ABC$ subject to the given conditions. Round the lengths of sides and measures of the angles to 1 decimal place if necessary. (See Examples 1–2, pp. 690–691)

13. $A = 127°, B = 34°, a = 42$

14. $A = 27°, B = 71°, b = 186$

15. $A = 122.1°, B = 24.3°, c = 102$

16. $A = 10.6°, C = 98.8°, b = 59$

17. $A = 34.7°, C = 68.1°, c = 43.9$

18. $A = 112.7°, C = 50.2°, a = 12.3$

19. $B = 28.1°, C = 45°, a = \frac{2}{3}$

20. $A = 53.8°, B = 60°, c = \frac{17}{9}$

Learn: Solve a Triangle Using the Law of Sines (SSA) Ambiguous Case

For Exercises 21–28, information is given about $\triangle ABC$. Determine if the information gives one triangle, two triangles, or no triangle. Solve the resulting triangle(s). Round the lengths of sides and measures of the angles to 1 decimal place if necessary. (See Examples 3–5, pp. 693–694)

21. $b = 33, c = 25, B = 38°$

22. $b = 5, c = 12, C = 73°$

23. $a = 185, c = 132, C = 63°$

24. $b = 6, c = 12, B = 38°$

25. $a = 13, b = 18, A = 45°$

26. $a = 3, b = 1, B = 17°$

27. $a = 132.5, b = 108.2, B = 13.1°$

28. $a = 48.8, c = 39.9, C = 22°$

Learn: Compute the Area of a Triangle Given SAS

For Exercises 29–34, find the area of a triangle with the given measurements. Round to 1 decimal place. (See Example 6, p. 697)

29. $A = 107°, b = 17$ ft, $c = 3$ ft

30. $A = 143°, b = 20$ m, $c = 16$ m

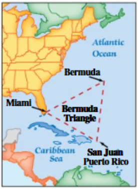
31. $B = 98.8°, a = 2.1$ in., $c = 5.3$ in.

32. $B = 2.6°, a = 3.5$ cm, $c = 10.8$ cm

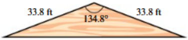
33. $C = 74.6°, a = \frac{1}{3}$ mm, $b = \frac{2}{3}$ mm

34. $C = 125.7°, a = \frac{17}{8}$ mi, $b = \frac{23}{9}$ mi

35. The area in the Atlantic Ocean known as the Bermuda Triangle is defined by an imaginary triangle connecting Miami, Florida; San Juan, Puerto Rico; and the island of Bermuda. Measuring on a map, the distance from both Miami to San Juan and from Miami to Bermuda is approximately 1033 mi. Assuming that the angle from Bermuda to Miami to San Juan is approximately 65°, what is the area of the Bermuda Triangle? Round to the nearest square mile.



36. A gabled roof has a traditional triangular shape that can vary in pitch (the slope of the sides) from a medium pitch in warmer regions to a steep pitch in regions that experience considerable snow in the winter.



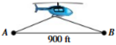
a. If the triangular face of a gabled roof measures 33.8 ft on each sloping side with an angle of 134.8° at the top of the roof, what is the area of the face? Round to the nearest square foot.

b. A homeowner plans to apply aluminum facing made from recycled cans to the face of the gable. If the cost of installation is $11.50 per square foot, what is the price for the new facing? Round to the nearest dollar.

Learn: Apply the Law of Sines

37. Two fire lookout towers at points *A* and *B* are 2 mi apart. Find the distances from points *A* and *B* to a fire at point *C* if $\angle ABC$ is $78°48'$ and $\angle BAC$ is $84°36'$. Round to the nearest tenth of a mile.

38. A news helicopter hovers directly over a large group of demonstrators on a city street. The demonstrators are concerned that the helicopter is hovering too close and want to estimate the height of the helicopter. Luckily, two nearby trigonometry students take simultaneous observations from locations at points *A* and *B* separated by 900 ft on the street. They find that the angle of elevation to the helicopter from *A* is $42°30'$ and the angle of elevation from *B* is $30°12'$.



a. What is the distance from each location to the helicopter? Round to the nearest foot.

b. How high is the helicopter from the ground at the moment of observation? Round to the nearest foot.

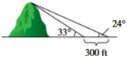
39. A surveyor wants to measure the distance across a lake from a point *A* due north to a point *B*. The surveyor runs a 1500-ft line from point *A* to point *C* in the direction N52°E. Then, the surveyor runs a second line from *C* to *B* in the direction N41°24′ W.

a. To the nearest foot, how long is the second line?

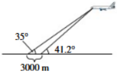
b. To the nearest foot, what is the distance between *A* and *B*?

40. A catamaran sails due north parallel to a straight shoreline at a constant speed of 8 mph. The captain records a bearing of N30°E to a lighthouse on shore. Then, 1.5 hr later, the captain records the bearing to the lighthouse as S42°E. To the nearest tenth of a mile, what is the distance from the catamaran to the lighthouse at the time the second bearing is recorded?

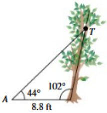
41. From a point along a straight road, the angle of elevation to the top of a hill is 33°. From 300 ft farther down the road, the angle of elevation to the top of the hill is 24°. How high is the hill? Round to the nearest foot. **(See Example 7, p. 697)**



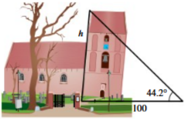
42. An observer on the ground sights a plane at an angle of elevation of 41.2°. At the same time, a second observer 3000 m farther away along the same line of sight measures the angle of elevation as 35°. How high is the plane? Round to the nearest 100 m.



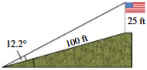
43. A wire is fastened to a point *T* on a tree and to point *A* located 8.8 ft from the base of the tree along level ground (see figure). The angle that the wire makes with level ground is 44°, and the tree leans 12° from vertical away from point *A*. How high off the ground is the point where the wire is fastened to the tree? Round to the nearest tenth of a foot.



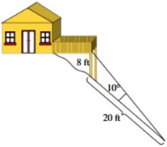
44. The Leaning Tower of Suurhusen is a medieval steeple in Suurhusen, Germany. The tower leans at an angle of 5.1939° from the vertical. (In comparison, the Leaning Tower of Pisa leans at an angle of 3.97°.) The angle of elevation to the top of the tower is 44.2° when measured 100 ft from the base of the tower. Find the distance h from the base of the tower to the top of the tower. Round to the nearest tenth of a foot.



45. A 25-ft flagpole is oriented vertically at the top of a hill. An observer standing 100 ft downhill measures the angle formed between the top and bottom of the pole as 12.2°. To the nearest degree, determine the angle of inclination of the hill. (**See Example 8, p. 698**)



46. A mountain cabin is built on the side of a hill with a porch extending over a portion of the downhill slope. An observer 20 ft downhill from one of the 8-ft vertical support beams measures the angle formed from the top of the beam to the bottom as 10°. Find the angle of inclination of the hill to the nearest tenth of a degree.



47. A citizen advisory board is recommending an expansion of affordable housing for workers in a metropolitan region. Two residential buildings will be constructed with a grassy recreational area equipped for outdoor exercising between them. The taller building is 700 ft high. From the roof of the shorter building, the angle of elevation to the top of the taller building is 78° and the angle of depression to the base of the taller building is 62°.

a. How tall is the shorter building? Round to the nearest foot.

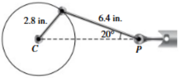
b. What is the distance between the buildings? Round to the nearest foot.

48. A hiker wants to estimate the height of a mountain before attempting a climb to the top. The first measurement shows an angle of elevation to the top of the mountain as 63.4°. The second measurement, taken 950 ft closer to the base of the mountain, yields an angle of elevation of 75.3°. From these measurements, estimate the height of the mountain. Round to the nearest hundred feet.

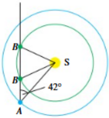
49. A surveyor fixes a 420-ft baseline between points A and B, where B is due east of A. The bearing from A to a third point P is N52°E and the bearing from B to P is N27°W. Find the perimeter and area of the triangular plot of land formed by A, B, and P. Round the perimeter to the nearest foot and the area to the nearest hundred square feet.

50. A surveyor fixes a 200-ft baseline between points A and B, where B is due east of A. The bearing from A to a third point P is S37°W and the bearing from B to P is S62°W. Find the perimeter and area of the triangular plot of land formed by A, B, and P. Round the perimeter to the nearest foot and the area to the nearest 100 square feet.

51. The connecting rod from the piston to the crankshaft in a certain 2.0-L engine is 6.4 in. The radius of the crank circle is 2.8 in. If the angle made by the connecting rod with the horizontal at the wrist pin P is 20°, how far is the wrist pin from the center C of the crankshaft? Round to the nearest tenth of an inch.



52. Two planets follow a circular orbit around a central star in the same plane. The distance between the star at point S and one planet at point A is 135 million miles. The distance between the star and the other planet at point B is 100 million miles. If an observer on the first planet at point A sights the second planet such that $\angle SAB = 42°$, find the distance between the planets. Round to the nearest million miles.



Mixed Exercises

For Exercises 53–66, solve $\triangle ABC$. Round the lengths of sides and measures of the angles to 1 decimal place if necessary.

53. $A = 117°, B = 32°, b = 8.2$

54. $a = 42, c = 60, C = 17°$

55. $b = 6, c = 8, B = 24°$

56. $A = 103.1°, B = 13°, c = 3.8$

57. $B = 13.3°, C = 68.4°, c = 12$

58. $a = 11, c = 3, C = 142°$

59. $a = 118, c = 112, C = 24.6°$

60. $B = 47.8°, C = 99.3°, a = 183$

61. $a = 16, b = 15.1, A = 113°$

62. $b = 15.1, c = 18.6, B = 113°$

63. $A = 4.6°, C = 1.2°, a = 23$

64. $B = 12°, C = 136°, b = 800$

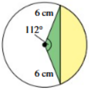
65. $a = 325, c = 221, C = 78.8°$

66. $A = 153°, C = 2°, b = 2$

67. After a hurricane, two homeowners examine the trees on their property for damage. They think a 20-ft palm tree is leaning slightly from its original vertical position. From a point 23 ft away, they measure the angle of elevation to the top of the palm tree as 43°. Is the palm tree leaning? If so, by how many degrees from the vertical? Round to the nearest tenth of a degree.

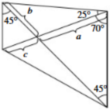
68. A company manufactures pennants in the shape of an isosceles triangle. The long sides of each triangle are 18 in., and the angle between the long sides is 40°. The weatherproof fabric from which the pennants are made costs \$6.95/yd². How much will it cost the company to make 10,000 pennants? Round to the nearest dollar.

69. Consider a sector of a circle of radius 6 cm with central angle 112°.



- **a.** Find the area of the triangular region shaded in green. Round to the nearest hundredth of a square centimeter.
- **b.** Find the area of the region in yellow bounded by the circle and the chord of the circle.

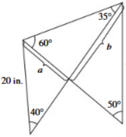
70. Refer to the figure.



a. Write expressions for the exact values of a, b, and c.

b. Approximate the values of a, b, and c to the nearest tenth of a centimeter.

71. Refer to the figure.



a. Write expressions for the exact values of a, b, and c.

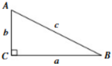
b. Approximate the values of a, b, and c to the nearest tenth of an inch.

72. Given $\triangle ABC$ with sides of lengths a, b, and c opposite angles A, B, and C, respectively, use the law of sines to show that

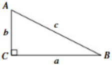
$$\frac{a+b}{b}=\frac{\sin A+\sin B}{\sin B}.$$

73. Fire station A is located 10 mi due west of fire station B. Smoke is spotted at a bearing of S37°20′E from fire station A and at a bearing of S47°30′W from fire station B. How far is the fire from each station? Round to the nearest tenth of a mile.

74. Show that the law of sines holds true for a right triangle.



75. Show that the equation $c^2=a^2+b^2-2ab\cos C$ holds true for a right triangle and reduces to the Pythagorean theorem when solving for the hypotenuse.



Write About It

76. Explain why the law of sines can be used to solve a triangle if the measure of an angle is given along with the length of its opposite side and the measure of one other side or angle.

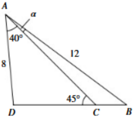
77. Given SSA (the *ambiguous case*), outline the four possible scenarios regarding the number of triangles that are possible.

78. List various methods to confirm your answers after solving a triangle using the law of sines.

79. A triangle $\triangle ABC$ has sides a, b, and c across from angles A, B, and C, respectively. Explain the relationship given by $\text{Area}=\frac{1}{2}bc\sin A$.

Expanding Your Skills

80. From the figure, show that $\sin(45°-\alpha)=\frac{2}{3}\sin 95°$.



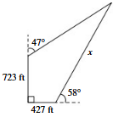
81. Use the law of sines in the form $\frac{\sin A}{\sin B}=\frac{a}{b}$ (1) to prove the law of tangents: $\frac{a-b}{a+b}=\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$.

[*Hint*: Subtract 1 from both sides of equation (1) and simplify to get a second equation. Then add 1 to both sides of (1) and simplify to get a third equation. Then divide the results.]

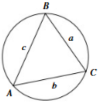
82. Use the relationships $\frac{a}{c} = \frac{\sin A}{\sin C}$ and $\frac{b}{c} = \frac{\sin B}{\sin C}$ from the law of sines to show that $\frac{a+b}{c} = \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}$.

This is called a Mollweide equation, named after German mathematician and astronomer Karl Mollweide (1774–1825). Notice that the equation uses all six parts of a triangle, and for this reason, is sometimes used to check the solution to a triangle.

83. When given two sides and the included angle, SAS, a triangle may be solved using the law of tangents and the law of sines. The law of tangents will give the angles opposite the given sides, and the law of sines can be used to find the third side. Mollweide's equation may be used to check the answer. Find the value of x in the figure. Round to the nearest foot.



84. Given $\triangle ABC$ circumscribed by a circle, show that the diameter d of the circle is $d = \frac{b}{\sin B}$. (*Hint*: Draw two radii from the center of the circle to points A and C. Use a theorem from geometry relating the measure of an inscribed angle of a circle to the corresponding central angle.) Note that it follows from the law of sines that $d = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.



Go online for more practice problems.

Lesson 6-3

The Law of Cosines

What Will You Learn?

After completing this lesson, you should be able to:

- Solve a Triangle Using the Law of Cosines (SAS)
- Solve a Triangle Using the Law of Cosines (SSS)
- Apply the Law of Cosines
- Use Heron's Formula to Compute the Area of a Triangle

Learn: Solve a Triangle Using the Law of Cosines (SAS)

Given the measure of an angle in a triangle, the side opposite the angle, and one other part of the triangle, we can use the law of sines to solve the triangle. The law of sines applies in such a case because the known angle and opposite side form a complete ratio such as $\frac{a}{\sin A}$. We now look at the cases in which a complete ratio is not given.

- Two sides and the angle between them are known (SAS).
- Three sides are known (SSS).

In these cases, we use the law of cosines.

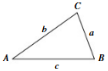
Law of Cosines

If $\triangle ABC$ has sides of lengths a, b, and c opposite vertices A, B, and C, respectively, then

Law of Cosines	Alternative Form
(1) $a^2 = b^2 + c^2 - 2bc\cos A$	$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
(2) $b^2 = a^2 + c^2 - 2ac\cos B$	$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$
(3) $c^2 = a^2 + b^2 - 2ab\cos C$	$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

The formulas for the law of cosines looks daunting, but they all share the same pattern.

$$\overbrace{a^2}^{\text{Square of one side}} = \overbrace{b^2 + c^2}^{\text{Sum of squares of the other sides}} - \overbrace{2bc\cos A}^{\text{Twice the product of the other sides times the cosine of the included angle}}$$



The alternative forms of equations (1)–(3) have the cosine factor isolated for the purpose of computing the measure of an unknown angle in a triangle given the lengths of the sides.

We will prove the first of the three formulas given by the law of cosines. Formulas (2) and (3) are proved likewise.

First, place $\triangle ABC$ in a rectangular coordinate system with vertex A at the origin, side c on the positive x-axis, and vertex B at $(c, 0)$. Let (x, y) represent point C (Figure 6-21).

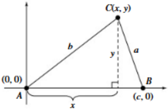


Figure 6-21

From the right triangle on the left,

$$\cos A = \frac{x}{b} \Rightarrow x = b\cos A$$

$$\sin A = \frac{y}{b} \Rightarrow y = b\sin A$$

Therefore, the ordered pair representing point C is $(b\cos A, b\sin A)$. The distance from C to B is the length of side a.

$a = \sqrt{(x-c)^2 + (y-0)^2}$	Apply the distance formula.
$a^2 = (x-c)^2 + y^2$	Square both sides.
$a^2 = (b\cos A - c)^2 + (b\sin A)^2$	Substitute $x = b\cos A$ and $y = b\sin A$.
$a^2 = b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A$	Square each term.
$a^2 = b^2\cos^2 A + b^2\sin^2 A - 2bc\cos A + c^2$	Group terms with b^2.
$a^2 = b^2(\cos^2 A + \sin^2 A) - 2bc\cos A + c^2$	Factor out b^2.
$a^2 = b^2(1) - 2bc\cos A + c^2$	Replace $\cos^2 A + \sin^2 A$ with 1.
$a^2 = b^2 + c^2 - 2bc\cos A$	Law of cosines

In Example 1, we solve an oblique triangle given two sides and the angle between them (SAS). We use the following guidelines.

Guidelines for Solving an Oblique Triangle Given SAS

1. Find the length of the side opposite the known angle by using the law of cosines.
2. Use the law of sines to find the measure of the angle opposite the shorter of the two given sides. This angle will always be acute.

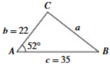
Alternatively, use the law of cosines (alternative form) to find either remaining angle. The rationale for step 2 is as follows:

- The angles opposite the shorter two sides in a triangle are always acute angles. Therefore, the measure of the smaller of the two given angles must be acute. Applying the law of sines to find this angle will not involve the ambiguous case.
- The law of cosines can be used to find the measure of *either* remaining angle because the inverse cosine will return an acute angle or an obtuse angle, depending on whether the argument is positive or negative.

3. Find the measure of the third angle by subtracting the sum of the measures of the other two angles from 180°.

Example 1: Solving an Oblique Triangle Given SAS

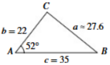
Solve the triangle with $A = 52°$, $b = 22$, and $c = 35$. Round the measures of the unknown side and angles to 1 decimal place.



Solution:

We begin by finding the length of the side opposite the known angle.

$a^2 = b^2 + c^2 - 2bc\cos A$	Apply the law of cosines.
$a^2 = (22)^2 + (35)^2 - 2(22)(35)\cos 52°$	Substitute $b = 22$, $c = 35$, $A = 52°$.
$a = \sqrt{(22)^2 + (35)^2 - 2(22)(35)\cos 52°}$	Take the principal square root.
≈ 27.6	



Of the two remaining unknown angles, angle B is opposite the shortest side. Using the law of sines,

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{27.6}{\sin 52°} = \frac{22}{\sin B}$$

$$\sin B \approx \frac{22 \sin 52°}{27.6}$$

$$B = \sin^{-1}\left(\frac{22\sin 52°}{27.6}\right) \approx 38.9°$$

With the measures of angles A and B known,

$$C = 180° - (A + B) \approx 180° - (52° + 38.9°) \approx 89.1°.$$

In summary $A = 52°$, $B \approx 38.9°$, $C \approx 89.1°$, $a \approx 27.6$, $b = 22$, and $c = 35$.

Good Practices

In Example 1, we used a calculator to approximate the length of side *a*. Notice that no rounding was done until all calculations were completed. Then we used the value of side *a* to find the measure of angle *B*. The level of rounding used for side *a* can affect the result of subsequent calculations. For example,

$$B = \sin^{-1}\left(\frac{22\sin 52^\circ}{27.6}\right) \approx 38.91188771 \text{ and } B = \sin^{-1}\left(\frac{22\sin 52^\circ}{27.58407744}\right) \approx 38.93859082^\circ$$

The result of your calculations may differ slightly from those given in the text. This may be due to round-off error or the order in which you solve for the sides and angles in a triangle.

Apply the Skills

1. Solve $\triangle ABC$ with $B = 61^\circ$, $c = 19$, and $a = 28$. Round the measures of the unknown side and angles to 1 decimal place.

Learn: Solve a Triangle Using the Law of Cosines (SSS)

In Example 2, we apply the law of cosines to find the measures of the angles of a triangle where the lengths of the three sides are known. We recommend the following guidelines.

Guidelines for Solving an Oblique Triangle Given SSS

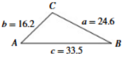
1. Use the alternative formulas for the law of cosines to find the largest angle of the triangle (angle opposite the longest side).
2. Find either of the remaining two angles using either the law of sines or the law of cosines. Since the largest angle was found in step 1, the two remaining angles are guaranteed to be acute.
    - The law of sines may be easier for you to remember and apply, but the drawback is that you will have to use a rounded value from a prior calculation.
    - The benefit of using the law of cosines is that it involves using the lengths of the three given sides, rather than a rounded value from a prior calculation.
3. Find the measure of the third angle by subtracting the sum of the measures of the other two angles from 180°.

Check Your Work

You can also use the law of cosines to find the measure of the third angle as a check.

Example 2: Solving an Oblique Triangle Given SSS

Solve the triangle with $a = 24.6$, $b = 16.2$, and $c = 33.5$. Round the measures of the angles to the nearest tenth of a degree.



Solution:

We begin by finding angle C because it is opposite the longest side.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Apply the alternative form of the law of cosines.

$$\cos C = \frac{(24.6)^2 + (16.2)^2 - (33.5)^2}{2(24.6)(16.2)}$$

Substitute $a = 24.6$, $b = 16.2$, $c = 33.5$.

$$C = \cos^{-1}\left[\frac{(24.6)^2 + (16.2)^2 - (33.5)^2}{2(24.6)(16.2)}\right]$$

$$\approx 108.6^\circ$$

Take the inverse cosine.

Now solve for either of the other two angles. We will solve for angle B.

Using the law of cosines:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{(24.6)^2 + (33.5)^2 - (16.2)^2}{2(24.6)(33.5)}$$

$$B = \cos^{-1}\left[\frac{(24.6)^2 + (33.5)^2 - (16.2)^2}{2(24.6)(33.5)}\right]$$

$$\approx 27.3^\circ$$

Using the law of sines:

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{33.5}{\sin 108.6^\circ} = \frac{16.2}{\sin B}$$

$$\sin B = \frac{16.2 \sin 108.6^\circ}{33.5}$$

$$B = \sin^{-1}\left(\frac{16.2 \sin 108.6^\circ}{33.5}\right) \approx 27.3^\circ$$

Insights

In Example 2, using the law of sines to find the second angle requires that we use the rounded value 108.6° from the previous calculation. Using the law of cosines does not require the use of a rounded value.

With the measures of angles B and C known,

$$A = 180^\circ - (B + C) \approx 180^\circ - (27.3^\circ + 108.6^\circ) \approx 44.1^\circ$$

In summary $A \approx 44.1^\circ$, $B \approx 27.3^\circ$, $C \approx 108.6^\circ$, $a = 24.6$, $b = 16.2$, and $c = 33.5$.

Apply the Skills

2. Solve $\triangle ABC$ with $a = 62$, $b = 48$, and $c = 41$. Round the measures of the angles to the nearest tenth of a degree.

Learn: Apply the Law of Cosines

Example 3: Applying the Law of Cosines

A Coast Guard ship on a routine training run leaves its base at 16 knots (nautical miles per hour) on a course of N82°E. Three hours later, a distress call comes in from a cargo ship located 180 nmi (nautical miles) from base at a bearing of N48°E.

a. How far is the Coast Guard ship from the cargo ship? Round to the nearest tenth of a nautical mile.

b. What bearing should the Coast Guard ship follow to reach the cargo ship? Round to the nearest tenth of a degree.

Solution:

a. Let *A* represent the location of the base, let *C* represent the location of the Coast Guard ship after 3 hr, and let *B* be the location of the cargo ship (Figure 6-22). The distance *c* between the base and the cargo ship is given as 180 nmi. The distance *b* between the base and the Coast Guard ship is $b = (16 \text{ knots})(3 \text{ hr}) = 48$ nmi.

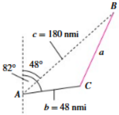


Figure 6-22

From Figure 6-22, $\angle BAC = 82° - 48° = 34°$.

In $\triangle ABC$ we know the lengths of sides *b* and *c* and the angle between them, so we can apply the law of cosines SAS case (Figure 6-23).

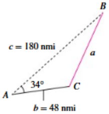


Figure 6-23

To find the length of side *a*, we have

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a^2 = (48)^2 + (180)^2 - 2(48)(180)\cos 34°$$

$$a^2 = \sqrt{(48)^2 + (180)^2 - 2(48)(180)\cos 34°}$$

$$\approx 142.8 \text{ nmi}$$

The Coast Guard ship is approximately 142.8 nmi from the cargo ship.

b. To find the bearing α from the Coast Guard ship to the cargo ship, we will first find the measure of angle C (Figure 6-24).

$$\cos C = \frac{a^2 + b^2 + c^2}{2ab} = \frac{(142.8)^2 + (48)^2 - (180)^2}{2(142.8)(48)}$$

$$C = \cos^{-1}\left[\frac{(142.8)^2 + (48)^2 - (180)^2}{2(142.8)(48)}\right]$$

$$\approx 135.1°$$

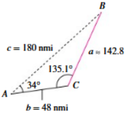


Figure 6-24

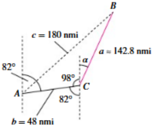


Figure 6-25

From Figure 6-25,

$\alpha = C - 98°$

$\alpha \approx 135.1° - 98° \approx 37.1°$

The bearing from the Coast Guard ship to the cargo ship is N37.1°E.

Apply the Skills

3. Repeat Example 3 with the initial bearing of the Coast Guard ship N17°W at a speed of 18 mph for 3 hr.

Learn: Use Heron's Formula to Compute the Area of a Triangle

Suppose that the lengths of three sides of a triangle are known (SSS). We can use the law of cosines to find the measures of the three angles. Then from Lesson 6.2, we can use any of the expressions $\frac{1}{2}ab\sin C, \frac{1}{2}bc\sin A$, or $\frac{1}{2}ac\sin B$ to find the area of $\triangle ABC$. The question then arises, is there a more direct route to finding the area? The answer is yes. We can use Heron's formula, named in honor of its discoverer, Greek mathematician Heron of Alexandria (c. a.d. 62), to find the area of a triangle given the lengths of its sides.

Heron's Formula for the Area of a Triangle

Given a triangle with sides of lengths a, b, and c, the area is given by

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter: $s = \frac{1}{2}(a + b + c)$.

Point of Interest

In addition to his formula for finding the area of a triangle and many other contributions in mathematics, Heron was an accomplished engineer. He is credited with inventing the first steam engine, referred to as Hero's ball, the first known device that could transform steam into rotary motion.

The proof of Heron's formula uses the law of cosines as outlined in Exercise 64.

Example 4: Finding the Area of a Triangle with Heron's Formula

Find the area of a triangle with sides of length $a = 12.2$ cm, $b = 16$ cm, and $c = 9.8$ cm. Round to the nearest tenth of a square centimeter.

Solution:

The semi-perimeter is given by:

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(12.2 + 16 + 9.8) = 19$$

The area is given by:

$$\begin{aligned} \text{Area} &= \sqrt{s(s - a)(s - b)(s - c)} \\ &= \sqrt{19(19 - 12.2)(19 - 16)(19 - 9.8)} \\ &\approx 59.7 \end{aligned}$$

The area is approximately 59.7 cm².

Apply the Skills

4. Find the area of a triangle with sides of length $a = 15.4$ m, $b = 22.6$ m, and $c = 26$ m. Round to the nearest square meter.

Practice 6-3

Practice Exercises

Prerequisite Review

R.1. Find the exact distance between the points (–7, 2) and (–6, –5).

R.2. Determine if the points (–3, 7), (–1, 5) and (–7, 3) form the vertices of a right triangle.

R.3. Evaluate $\cos^{-1}\left(\frac{5}{8}\right)$ in degrees to 2 decimal places.

Concept Connections

1. In which situation(s) can the law of cosines be used to solve for the remaining angles and sides of an oblique triangle? Choose from AAS, SAS, ASS, SSS.

2. Given $\triangle ABC$ with sides a, b, and c opposite vertices A, B, and C, write an expression for a^2 in terms of the lengths of the other sides and opposite angle.

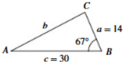
3. Given $\triangle ABC$ with sides a, b, and c opposite vertices A, B, and C, write an expression for $\cos B$ in terms of the lengths of the sides of the triangle.

4. Given a triangle of sides of lengths a, b, and c, Heron's formula gives the area of the triangle as ____________, where s represents half the perimeter of the triangle. That is $s =$ ____________.

Learn: Solve Triangles Using the Law of Cosines

For Exercises 5–22, solve $\triangle ABC$ subject to the given conditions if possible. Round the lengths of sides and measures of the angles (in degrees) to 1 decimal place if necessary. (See Examples 1–2, pp. 709–711)

5.



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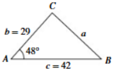

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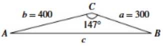

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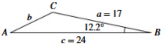

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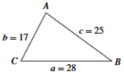

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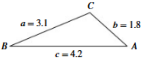


11. $a = 28.3$, $c = 17.4$, $B = 11.3°$

12. $b = 89.2$, $c = 23.1$, $A = 108°$

13. $a = 15$, $b = 12$, $c = 15$

14. $a = 25$, $b = 30$, $c = 35$

15. $a = 27$, $c = 26$, $B = 67.8°$

16. $a = 40.5$, $b = 38.1$, $C = 73.2°$

17. $a = 2.3$, $b = 10.8$, $c = 9.7$

18. $a = 500$, $b = 200$, $c = 400$

19. $b = 146.8$, $c = 122.7$, $A = 110.4°$

20. $b = 802.5$, $c = 436.1$, $A = 103.7°$

21. $a = 4.4$, $b = 6.2$, $c = 11.1$

22. $a = 18$, $b = 32$, $c = 10$

Learn: Apply the Law of Cosines

23. A boat leaves port and follows a course of N77°E at 9 knots for 3 hr and 20 min. Then, the boat changes to a new course of S28°E at 12 knots for 5 hr. (**See Example 3, p. 712**)

a. How far is the boat from port?

b. Suppose that the boat becomes disabled. How long will it take a rescue boat to arrive if the rescue boat leaves from port and travels 18 knots? Round to the nearest minute.

c. What bearing should the rescue boat follow?

24. A fishing boat leaves a marina and follows a course of S62°W at 6 knots for 20 min. Then the boat changes to a new course of S30°W at 4 knots for 1.5 hr.

a. How far is the boat from the marina?

b. What course should the boat follow for its return trip to the marina?

25. Two planes leave the same airport. The first plane leaves at 1:00 p.m. and averages 480 mph at a bearing of S62°E. The second plane leaves at 1:15 p.m. and averages 410 mph at a bearing of N12°W.

a. How far apart are the planes at 2:45 P.M.?

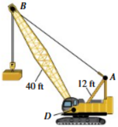
b. What is the bearing from the first plane to the second plane at that time? Round to the nearest degree.

26. Two boats leave a marina at the same time. The first boat travels 6 knots at a bearing of N39°E, and the second boat travels 4 knots at a bearing of S87°W.

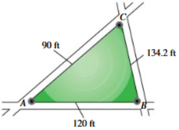
a. How far apart are the boats at the end of 2 hr?

b. What is the bearing from the first boat to the second boat at that time?

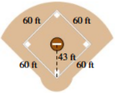
27. A 40-ft boom on a crane is attached to the crane platform at point *D*. A cable is attached to the end of the boom at point *B* and to a 12-ft A-frame at point *A* anchored by the counterweight. If $\angle BDA$ is 85.2°, find the length of the cable to the nearest tenth of a foot.



28. In a public park, a triangular region of grass is bounded by walkways. Sprinkler heads are placed at the vertices of the triangle as shown in the figure. Each sprinkler head can be set to limit its angle of rotation. For each sprinkler head, how large should the angle of rotation be set so as to keep the walkways dry and to conserve water? Round to the nearest degree.



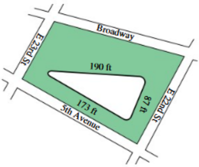
29. A regulation fast-pitch softball diamond for high school competition is a square, 60 ft on a side. The pitcher's mound is collinear with home plate and second base. Furthermore, the distance from the back of home plate to the center of the pitcher's mound is 43 ft. To the nearest tenth of a foot, find the distance between



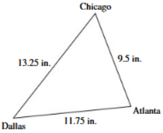
a. The pitcher's mound and first base.

b. The pitcher's mound and second base.

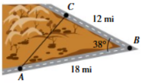
30. The Flatiron Building in New York City is said to have gotten its name because its cross section resembles the shape of the household appliance of the same name. The perimeter of the building is bounded by Broadway, 5th Avenue, and East 22nd and 23rd Streets. Using side lengths of approximately 87 ft, 190 ft, and 173 ft, determine the angle at the "point" of the Flatiron Building at the corner of 5th Avenue and East 23rd Street. Round to the nearest tenth of a degree.



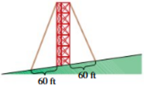
31. The distance between Dallas, Texas, and Atlanta, Georgia, on a map is 11.75 in. The distance from Atlanta to Chicago, Illinois, is 9.5 in., and the distance from Chicago to Dallas is 13.25 in. If the bearing from Dallas to Atlanta is N85°E, find the bearing from Chicago to Dallas. Round to the nearest tenth of a degree.



32. A bicyclist is at point A on a paved road and must ride to point C on another paved road. The two roads meet at an angle of 38° at point B. The distance from A to B is 18 mi, and the distance from B to C is 12 mi (see the figure). If the bicyclist can ride 22 mph on the paved roads and 6.8 mph off-road, would it be faster for the bicyclist to ride from A to C on the paved roads or to ride a direct line from A to C off-road? Explain.



33. A 150-ft tower is anchored on a hill by two guy wires. The angle of elevation of the hill is 8°. Each guy wire extends from the top of the tower to a ground anchor 60 ft from the base of the tower and in line with the tower. Find the length of each guy wire. Round to the nearest foot.



34. A 75-ft tower is located on the side of a hill that is inclined 26° to the horizontal. A cable is attached to the top of the tower and anchored uphill a distance of 35 ft from the base of the tower. Find the length of the cable. Round to the nearest foot.

Learn: Use Heron's Formula to Compute the Area of a Triangle

In Exercises 35–38, use Heron's formula to find the area of the triangle with sides of the given lengths. Round to the nearest tenth of a square unit. (See Example 4, p. 714)

35. $a = 13$ in., $b = 7$ in., $c = 8$ in.

36. $a = 10$ mi, $b = 17$ mi, $c = 9$ mi

37. $a = 18.6$ cm, $b = 12.3$ cm, $c = 25.9$ cm

38. $a = 7.9$ yd, $b = 12.1$ yd, $c = 19.3$ yd

39. A triangular sail for a Bolger Tortise sailboat (named after boat designer Phil Bolger) has sides of length 8′1″, 10′1″, and 9′10″. What is the sail area?

40. Find the area of the grassy region from Exercise 28. Round to the nearest square foot.

41. A farmer wants to fertilize a triangular field with sides of length 300 yd, 220 yd, and 180 yd. Fertilizer costs $180 per acre (1 acre = 4840 yd^2). Furthermore, the time required to fertilize 1 acre is approximately 2.8 hr with combined labor and equipment costs of $23.00 per hour.

a. To the nearest acre, how big is the field?

b. Using the number of acres from part (a), estimate the cost to fertilize the field. Round to the nearest dollar.

42. An artist lays down a white triangular background on which to paint a Penrose triangle. The triangular area measures 20 ft on each side and will be covered with three layers of primer paint.

a. To the nearest square foot, what is the area of the triangular region to be primed?

b. If primer paint covers 200 ft^2/gal, how many gallons of primer must be purchased?

Mixed Exercises

For Exercises 43–46, the vertices of a triangle are defined by the given points. To the nearest tenth, determine

a. the perimeter of the triangle.

b. the area of the triangle.

c. the measure of the angles in the triangle.

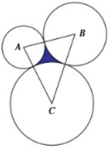
43. $A(2, 2)$, $B(4, 7)$, $C(8, 1)$

44. $A(1, 5)$, $B(5, 8)$, $C(10, 3)$

45. $A(-3, -2)$, $B(-1, 5)$, $C(6, 1)$

46. $A(-2, -1)$, $B(10, 2)$, $C(5, -4)$

47. Three mutually tangential circles with radii located at A, B, and C have radii of 2 ft, 3 ft, and 4 ft, respectively.

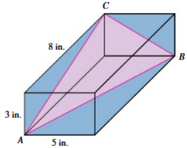


a. Find the area of $\triangle ABC$. Round to the nearest tenth of a square foot.

b. Find the measures of angles A, B, and C. Round to the nearest tenth of a degree.

c. Find the area of the shaded region. Round to the nearest tenth of a square foot.

48. The rectangular box shown in the figure is 8 in. long, 5 in. wide, and 3 in. high. Line segment $\overline{AC}$ connects opposite vertices on the left side of the box. Line segment $\overline{CB}$ connects opposite vertices at the back of the box, and line segment $\overline{AB}$ connects opposite vertices on the bottom of the box.



a. Find the measure of $\angle ABC$. Round to the nearest degree (°).

b. Find the area of the triangle.

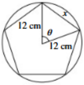
49. A parallelogram has adjacent sides of 12.2 cm and 9.6 cm, and the included angle is 62.5°. To the nearest tenth of a centimeter,

a. Find the length of the shorter diagonal.

b. Find the length of the longer diagonal.

50. A regular octagon (8-sided figure with sides of equal length) is inscribed in a circle of radius 10 in. Find the perimeter of the octagon to the nearest inch.

51. A regular pentagon is inscribed in a circle of radius 12 cm. Find the length of one side of the pentagon. Round to the nearest tenth of a centimeter.



52. A truck leaves an intersection at an average speed of 54 mph. Five minutes later, a car leaves the same intersection and travels on a street that makes an angle of 110° with the truck. If the car travels at an average speed of 68 mph, and the roads are straight, how far apart are the two vehicles 15 minutes after the car leaves the intersection? Round to the nearest tenth of a mile.

53. Three mobile phone towers A, B, and C are situated as follows: The bearing from tower A to tower B is S70°E and the distance between towers A and B is 36 km. The bearing from tower C to tower B is N15°W and the distance between towers C and B is 45 km.

a. Find the distance between towers C and A to the nearest tenth of a kilometer.

b. Find the bearing from tower A to tower C. Round to the nearest tenth of a degree.

Write About It

54. Why is the law of sines not an option to solve a triangle given SAS or SSS?

55. To solve a triangle given SAS, the law of cosines is used first to find the length of the side opposite the known angle. Then, if you choose to use the law of sines to solve for one of the two remaining angles, the guidelines suggest solving for the angle opposite the *shorter* of the two given sides. Why is this the case?

56. Given the lengths of the three sides of a triangle (SSS), why is the law of cosines used to find the measure of the largest angle first?

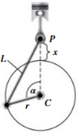
57. When applying the law of cosines (or law of sines) how can you minimize round-off error?

58. Suppose that $\triangle ABC$ is a right triangle with right angle at vertex C. What is the result when the law of cosines is used to find the length of side c?

59. If the measures of the three angles in a triangle are known, can you solve for the lengths of the sides?

Expanding Your Skills

60. A piston in an engine is attached to a connector rod of length L at the wrist pin at point P. As the piston travels back and forth, the connector cap at point Q travels counterclockwise along a circular path of radius r. If $L > 2r$, and α represents $\angle PCQ$,



a. Use the law of cosines to show that the distance x from the wrist pin to the crank circle is given by $x = r\cos\alpha + \sqrt{r^2\cos^2\alpha - r^2 + L^2} - r$.

b. Find the distance between the wrist pin and crank circle for a truck engine with connecting rod of length 5.9 in. and crank radius of 1.7 in. when $\alpha = 105°$. Round to the nearest tenth of an inch.

c. Find the maximum and minimum distances that the wrist pin is to the crank circle, using the values from part (b).

61. Two sides of a parallelogram are 8 ft and 12 ft. The length of the shorter diagonal is 6 ft. Find the exact length of the long diagonal.

62. Use the law of cosines to show that $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$.

63. Use the law of cosines to show that $a = b\cos C + c\cos B$

64. Complete the proof of Heron's formula.

a. Begin with the formula Area $= \frac{1}{2}bc\sin A$ and square both sides.

b. In the equation from part (a), replace $\sin^2 A$ by $1 - \cos^2 A$.

c. In the equation from part (b), factor $1 - \cos^2 A$ as a difference of squares.

d. Take the square root of both sides and write the area as

$$\text{Area} = \sqrt{\left[\frac{1}{2}bc(1+\cos A)\right]\left[\frac{1}{2}bc(1-\cos A)\right]} \quad (1)$$

e. Use the law of cosines to show that

$$\frac{1}{2}bc(1+\cos A) = \frac{a+b+c}{2}\cdot\frac{-a+b+c}{2} \quad (2)$$

Note: Using similar logic, we can also show that

$$\frac{1}{2}bc(1-\cos A) = \frac{a-b+c}{2}\cdot\frac{a+b-c}{2} \quad (3)$$

f. Use the substitution $s = \frac{1}{2}(a+b+c)$ to rewrite equation (2) as

$$\frac{1}{2}bc(1+\cos A) = s(s-a) \quad (4)$$

Note: Using similar logic, we can also show that $\frac{1}{2}bc(1-\cos A) = (s-b)(s-c)$ (5)

g. Substitute equations (4) and (5) into equation (1).

Problem Recognition Exercises

Solving Triangles Using a Variety of Tools

For Exercises 1–12, solve $\triangle ABC$ subject to the given conditions, if possible. Round the lengths of the sides and the measures of the angles to 2 decimal places, if necessary.

1. $a = 20, b = 29, B = 90°$

2. $b = 16.2, c = 12, A = 113°$

3. $a = 16.2, b = 18.5, c = 28.6$

4. $a = 19, b = 26, A = 72°$

5. $c = 20, b = 17, A = 90°$

6. $a = 14.1, A = 28°, B = 110°$

7. $a = 7, b = 8.8, A = 40°$

8. $b = 18, c = 26, C = 56°$

9. $b = 108, B = 62°, C = 54°$

10. $a = 12, c = 48, B = 62°$

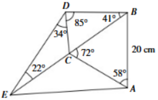
11. $a = 106, b = 58, c = 172$

12. $a = 46, b = 61, B = 136°$

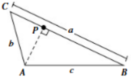
13. Refer to the figure. Find the indicated values to the nearest hundredth of a centimeter or hundredth of a degree.

a. Find the lengths of BC, AC, DB, DC, ED, EC, and EA.

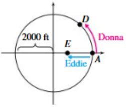
b. Find the measure of $\angle AEC$ and $\angle CAE$.



14. Use the right triangles $\triangle APB$ and $\triangle APC$ to show that for $\triangle ABC$, the length of side a is given by $a = b\cos C + c\cos B$.



15. A hiking trail is roughly in the shape of a circle with radius 2000 ft. Suppose that a rectangular coordinate system is set up with the origin at the center of the circle. Donna and Eddie start walking from point $A(2000, 0)$ at 220 ft/min. Eddie goes off-trail and walks directly toward the center of the circle and Donna walks along the trail counterclockwise.



a. Find the coordinates of each person after 5 min. Round the coordinates to the nearest foot.

b. How far apart are Eddie and Donna after 5 min? Round to the nearest foot.

c. If Eddie can no longer walk because he sprains his ankle after foolishly leaving the trail, at what bearing would Donna walk to find him? Round to the nearest tenth of a degree.

Go online for more practice problems.

Lesson 6-4

Harmonic Motion

What Will You Learn?

After completing this lesson, you should be able to:

- Model Simple Harmonic Motion
- Interpret Damped Harmonic Motion

Learn: Model Simple Harmonic Motion

The wavelike appearance of the graphs of the sine and cosine functions suggests that they would be good models to describe the motion of an oscillating or rotating object. For example, suppose that a block is attached to a spring hanging from the ceiling. If the block is pushed upward to compress the spring or pulled downward to stretch the spring, the spring applies a force on the block in the opposite direction. This is called a *restorative force* because it acts to return the block toward the equilibrium position (natural resting state). The greater the displacement d of the block from its equilibrium position, the more force F the spring applies to the block. In the absence of other forces called damping forces such as internal friction in the spring, the block will travel down and up indefinitely in a motion we call **simple harmonic motion**.

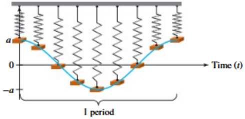


Figure 6-26 Spring-Mass System in Simple Harmonic Motion

In Figure 6-26, a block at the end of the spring is displaced a units upward from equilibrium position and then released. In the absence of damping, the model $d = a\cos\omega t$ describes the vertical displacement d as a function of the time t after release. The constant ω (omega) is generally taken to be positive and is related to the **period** P (time required to complete one up-and-down cycle). From our discussion in Lesson 4.5, $P = \frac{2\pi}{\omega}$. The **frequency** f with which the block oscillates up and down is the number of complete cycles per unit time. Therefore, the frequency is the reciprocal of the period.

Insights

In physics, you will often see the period of an object in simple harmonic motion represented by the letter T. This is because the period is a value of time.

Simple Harmonic Motion

If the displacement d of an object at time t is given by

$$d = a\sin\omega t \quad \text{or} \quad d = a\cos\omega t$$

then the object is said to have **simple harmonic motion.**

- The amplitude of motion is $|a|$.
- The period is given by $P = \frac{2\pi}{\omega}$ (time required for one complete cycle).
- The frequency f is the reciprocal of the period, $f = \frac{1}{P}$ or equivalently, $f = \frac{\omega}{2\pi}$ and is the number of cycles per unit time.
- The relationship $\omega = \frac{2\pi}{P}$ implies that $\omega = 2\pi f$. Therefore, simple harmonic motion can also be modeled by $d = a\cos 2\pi ft$ and $d = a\sin 2\pi ft$.

The difference between the models $d = a\cos\omega t$ and $d = a\sin\omega t$ is the position of the object at $t = 0$. Initially, if $d = 0$, then the motion "starts" with a displacement of 0 which is modeled most conveniently by the sine function. If $d = a$ or $d = -a$ initially, then the motion "starts" at a high or low point in the curve and is best modeled by the cosine function. If d is something other than 0, a, or $-a$ initially, then either function can be used with a phase shift.

Example 1: Writing a Model for Simple Harmonic Motion

Suppose that the block at the end of the spring in Figure 6-26 is originally compressed 10 in. upward. After release, the block makes one complete down-and-up cycle in 0.5 sec.

a. What is the period of motion?

b. What is the frequency of motion?

c. What is the amplitude?

d. Write a function of the form $d = a\cos\omega t$ to model the displacement d(in inches) of the block as a function of the time t(in seconds) after release.

e. Find the position of the block 6.25 sec after release.

Solution:

a. The period P is given as 0.5 sec.

This means that one up-and-down cycle is completed in $\frac{1}{2}$ sec.

b. To find the frequency, we have $f = \frac{1}{P} = \frac{1}{0.5} = 2$.

A frequency of 2 means that 2 cycles are completed in 1 sec. The unit of measurement assigned to frequency is "cycles per second" or Hertz (abbreviated Hz). Therefore, $f = 2$ Hz.

c. The amplitude is the maximum deviation from the equilibrium position.

$|a| = |10| = 10$. The amplitude is 10 in.

d. The starting position is positive 10 in. above equilibrium. Therefore, we will use $d = a\cos\omega t$ with $a = 10$.

Since $P = \frac{2\pi}{\omega}$, we have $\omega = \frac{2\pi}{P} = \frac{2\pi}{0.5} = 4\pi$.

The model is $d = 10\cos 4\pi t$.

e. When $t = 6.25$ sec, we have: $d = 10\cos 4\pi(6.25)$

$$= 10\cos 25\pi = 10(-1) = -10$$

Therefore, 6.25 sec after release, the block is 10 in. *below* the equilibrium position.

Apply the Skills

1. Suppose the block from Example 1 is initially pulled 6 in. *below* the equilibrium position. Repeat Example 1 if it takes the block 0.25 sec to return to its starting position.

In Example 2, we demonstrate that the position of an object rotating at a uniform speed can also be modeled by simple harmonic motion.

Example 2: Writing and Interpreting an Application of Simple Harmonic Motion

A Ferris wheel is 150 ft in diameter with its lowest point 3 ft off the ground. Once all the passengers have been loaded, the wheel makes one full rotation counterclockwise in 2.5 min. Suppose that two children are seated at the lowest point on the wheel and are the last passengers to be loaded. Choose a coordinate system with the origin at ground level below the center of the wheel.

a. Write a model representing the children's horizontal position x (in feet) at a time t min after the ride starts.

b. Write a model representing the children's height y (in feet) above ground level, t min after the ride starts.

c. Describe the children's position 2 min into the ride.

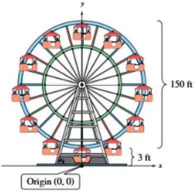


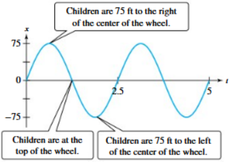
Figure 6-27

Solution:

a. Figure 6-27 shows the Ferris wheel and a coordinate system with origin 3 units below the lowest point on the wheel. The children's seat on the Ferris wheel is initially at (0, 3).

Notice that the x-coordinate of the children's position is initially zero and then turns positive as the wheel begins its counterclockwise motion.

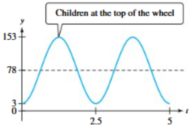
Therefore, we will use the sine function $x = a\sin 2\pi ft$ with $a > 0$ to model the children's position to the right or left of the center of the wheel.



The maximum deviation from $x = 0$ is equal to the radius of the wheel. Therefore, $a = 75$.

The period is 2.5 min, and the frequency is $f = \frac{1}{2.5} = 0.4$ rpm. Therefore, the horizontal position of the children is $x = 75\sin(2\pi \cdot 0.4t)$ or simply $x = 75\sin 0.8\pi t$.

b. The children's height starts at the lowest point $y = 3$ at $t = 0$ and increases to the highest point $y = 153$ at the half-way point during the revolution. Therefore, we will use the cosine function $y = a\cos 2\pi ft$ with a vertical shift of $\frac{(3 + 153)}{2} = 78$.



The amplitude is $\frac{(153 - 3)}{2} = 75$, but the value of a is taken to be -75 because the starting point is at the lowest point on the wheel. Therefore, the height is $y = -75\cos(2\pi \cdot 0.4t) + 78$ or simply $y = 78 - 75\cos 0.8\pi t$.

c. At $t = 2$ min,

$$x = 75\sin[0.8\pi(2)] \approx -71.3$$

$$y = 78 - 75\cos[0.8\pi(2)] \approx 54.8$$

Two minutes into the ride, the children have completed $\frac{2}{2.5} = 0.8$ revolution. They are on the way down from the high point to the low point at a position 71.3 ft to the left of center and 54.8 ft above ground. This is $(-71.3, 54.8)$.

Insights

Recall from Lesson 4.5 that the vertical shift is the midpoint of the range of a sine or cosine function.

The amplitude is half the difference between the maximum and minimum values.

Apply the Skills

2. Repeat Example 2 with a Ferris wheel 120 ft in diameter that completes one revolution in 1.25 min.

Learn: Interpret Damped Harmonic Motion

The spring-mass system in Example 1 would realistically be subject to internal friction within the spring and a small degree of air resistance. As a result, if the block were pushed up or pulled down from its natural resting position and then released, the distance it moves with each recovery will decrease until the block ultimately comes to rest. In such a case, the amplitude decreases with time and we say that the block undergoes **damped harmonic motion.** Air resistance and the force of friction are called **damping forces** because they dissipate energy in the form of heat, causing a decrease of movement.

Damped Harmonic Motion

The displacement d of an object oscillating under damped har"monic motion at time t is given by

$$d = ae^{-ct}\cos \omega t \quad \text{or} \quad d = ae^{-ct}\sin \omega t \quad \text{for } c > 0$$

- a is the displacement of the object at $t = 0$.
- c is a constant called the **damping factor** and is related to the damping force opposing motion.*
- The value of ω is related to the period P and the frequency f by $P = 2\pi/\omega$ and $\omega = 2\pi f$.**

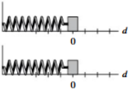
*If the damping factor $c = 0$, then there is no damping, and the model simplifies to that of simple harmonic motion: $d = a\cos \omega t$ or $d = a\sin \omega t$.

**Damped harmonic motion is not actually periodic because the amplitude decreases over time. However, we use the word "period" in this context as the time required for one complete back-and-forth cycle.

Insights

The negative factor in the model for damped harmonic motion is a result of the damping force acting in the opposite direction of motion.

Example 3: Comparing Models Involving Damped Harmonic Motion



Suppose that two blocks of equal mass are attached to two horizontal springs and pulled 8 cm to the right of the natural rest positions of the springs. Due to internal friction in the springs and friction on the surface of movement, the blocks move in damped harmonic motion. The damping factor for the first spring is 0.6 and the damping factor for the second spring is 0.2.

a. If each spring-mass system oscillates at 1 Hz, write a function to model the displacement for each system t seconds after release. Assume that a displacement to the right of equilibrium is positive.

b. Graph the functions from part (a) on a graphing utility.

c. Describe the differences in the graphs.

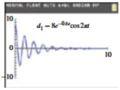
Solution:

a. The frequency f is given as 1 Hz. Therefore, $\omega = 2\pi f = 2\pi(1) = 2\pi$. Since the initial displacement is 8 cm from the equilibrium position, we choose the cosine model $d = ae^{-ct}\cos\omega t$ with $a = 8$.

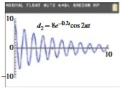
Spring 1: $d_1 = 8e^{-0.6t}\cos 2\pi t$

Spring 2: $d_2 = 8e^{-0.2t}\cos 2\pi t$

b. Spring 1


NORMAL FLOAT AUTO a+bi RADIAN MP
10
$d_1 = 8e^{-0.6t}\cos 2\pi t$
0
10
-10

Spring 2


NORMAL FLOAT AUTO a+bi RADIAN MP
10
$d_2 = 8e^{-0.2t}\cos 2\pi t$
0
10
-10

c. The graphs show that the "amplitude" decreases more rapidly for a function with a greater damping factor (graph of Spring 1).

Apply the Skills

3. Two spring-mass systems have an initial displacement of 6 cm and then oscillate at 0.25 Hz. The damping factors are 0.4 and 0.1, for spring 1 and spring 2, respectively. Write a function to model the displacement for each system t seconds after release.

Insights

If the restorative force of the spring (the force that causes recoil) is greater than the damping forces, the system is said to be *lightly damped*. In such a case the motion of the object oscillates, but with progressively smaller amplitudes (as in Example 3). If the damping forces are greater than the restorative force of the spring, then motion is said to be *overdamped* and follows a path of exponential decay.

From Example 3, the graphs of $d_1 = 8e^{-0.6t}\cos 2\pi t$ and $d_2 = 8e^{-0.2t}\cos 2\pi t$ both show a decrease in the "amplitude" for increasing time. In fact, plotting points at the relative maxima reveals a pattern of exponential decay. In Exercise 42, we show that the displacement of an object in damped harmonic motion is bounded above and below by $d = ae^{-ct}$ and $d = -ae^{-ct}$ (Figures 6-28 and 6-29).

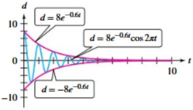


Figure 6-28

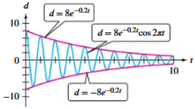
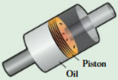


Figure 6-29

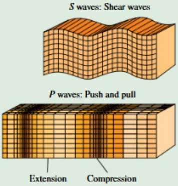
Point of Interest



Harmonic motion is in evidence in the world all around us. For example, a shock absorber mounted in the suspension system of a vehicle consists of a piston in a reservoir of hydraulic fluid. When a vehicle encounters vibration or a bump in the road, the piston moves through the fluid, which introduces a damping force. This turns the kinetic energy of suspension movement into heat energy that is dissipated through the hydraulic fluid. Ultimately, the shock absorber keeps the vehicle from bouncing up and down on its springs.

When a guitar string is plucked, the string vibrates and produces sound. But the sound fades away as internal friction in the string causes a decrease in the amplitude of vibration.

Scientists study earthquake waves as a means to build stronger structures in areas prone to earthquakes. *P* waves and *S* waves are the fundamental waves that result after an earthquake. *P* waves (primary waves) alternatively push (compress) and pull (dilate) the rock through which they travel. *S* waves (secondary waves) shear rock at right angles to the direction of travel. Areas affected by earthquakes feel the *P* waves first because they travel faster than *S* waves. However, *S* waves tend to cause more damage because they have greater amplitude, and because the undulation of the earth caused by *S* waves is particularly stressful to structures.



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Practice 6-4

Practice Exercises

Prerequisite Review

R.1. Identify the amplitude and period.

$y = 2\cos 4x$

R.2. Identify the amplitude and period.

$y = 3\sin\frac{\pi}{2}x$

R.3. Solve the equation. Write the solution set with the exact value given in terms of common or natural logarithms. Also give the approximate solution to 4 decimal places.

$2600 = 20{,}800e^{-0.2r}$

Concept Connections

1. The __________ of an object in simple harmonic motion is the amount of time required for one complete cycle.

2. For an object in simple harmonic motion, the number of cycles per unit time is called the __________.

3. Given an object in damped harmonic motion, the amplitude (increases/decreases) with time.

4. The term "ultrasound" refers to sound waves of a frequency greater than those detectable by the human ear (> 20,000 Hz). Suppose that a medical imaging device produces ultrasound with a period of 5 μs (microseconds: note that 5 μs = 0.000005 sec). What is the frequency?

5. The frequency of middle C on a piano is 264 Hz. What is the time required for one complete cycle?

6. If the displacement d of an object moving under simple harmonic motion is maximized at time $t = 0$, which model would be most convenient? $d = a\sin\omega t$ or $d = a\cos\omega t$

Learn: Model Simple Harmonic Motion

For Exercises 7–14, suppose that an object moves in simple harmonic motion with displacement *d* (in centimeters) at time *t* (in seconds). Determine the

a. Amplitude

b. Period

c. Frequency

d. Phase shift

e. Least positive value of t for which $d = 0$

7. $d = 6\cos 80\pi t$

8. $d = -2\cos 120\pi t$

9. $d = -0.25\sin\frac{\pi}{4}t$

10. $d = 0.125\sin\frac{\pi}{2}t$

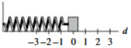
11. $d = -\cos\left(t - \frac{\pi}{6}\right)$

12. $d = -\cos\left(t + \frac{\pi}{4}\right)$

13. $d = \frac{1}{4}\sin\left(\frac{1}{2}t + 1\right)$

14. $d = \frac{1}{3}\sin\left(\frac{1}{4}t + \frac{1}{2}\right)$

For Exercises 15–22, suppose that an object is attached to a horizontal spring subject to the given conditions. Find a model for the displacement *d* as a function of the time *t*.



	Initial Displacement (d at $t = 0$)	Amplitude	Period or Frequency
15.	2.5 in.	2.5 in.	1.2 sec
16.	4.1 cm	4.1 cm	2 sec
17.	–6 ft	6 ft	2 Hz
18.	–12 m	12 m	4 Hz
19.	0 cm Initially moving to the right	8.5 cm	0.2 sec
20.	0 cm Initially moving to the right	15 cm	0.125 sec
21.	0 ft Initially moving to the left	0.25 ft	0.1 Hz
22.	0 m Initially moving to the left	1 m	0.01 Hz

23. A block hangs on a spring attached to the ceiling and is pulled down 5 in. below its equilibrium position. After release, the block makes one complete up-and-down cycle in 1 sec and follows simple harmonic motion. (**See Example 1, p. 722**)

a. What is the period of motion?

b. What is the frequency?

c. What is the amplitude?

d. Write a function to model the displacement *d* (in inches) as a function of the time *t* (in seconds) after release. Assume that a displacement above the equilibrium position is positive.

e. Find the displacement of the block and direction of movement at $t = 0.125$ sec.

24. The bob on a simple pendulum is pulled to the left 4 in. from its equilibrium position. After release, the pendulum makes one complete back-and-forth cycle in 2 sec and follows simple harmonic motion.

a. What is the period of motion?

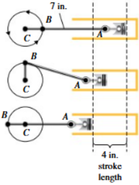
b. What is the frequency?

c. What is the amplitude?

d. Write a function to model the displacement *d* (in inches) of the bob as a function of the time *t* (in seconds) after release. Assume that a displacement to the right of the equilibrium position is positive.

e. Find the position and direction of movement of the bob at $t = 1.25$ sec.

25. A 7-in. connecting rod in an engine connects the wrist pin of a piston at point *A* to the end of a second rod at point *B* on the crankshaft. As the piston moves forward and back, the crankshaft is turned counterclockwise around point *C*. The stroke length of the engine is the distance that the piston travels (one way) within its cylinder. If the stroke length is 4 in., and the engine turns at 1800 rpm, write a model representing the distance *d* (in inches) from *A* to the center of the crankshaft *C* as a function of the time *t* (in minutes). Assume that at $t = 0$, the piston is at its farthest point from the crankshaft.



26. The blood pressure *p* for a certain individual follows a pattern of simple harmonic motion during the pumping cycle between heartbeats. The minimum pressure is 80 mmHg (millimeters of mercury) and the maximum pressure is 120 mmHg. The individual's pulse is 60 beats per minute or equivalently 1 beat per second. Write a model representing the blood pressure *p* at a time *t* seconds into the cycle. Assume that at $t = 0$, the blood pressure is 100 mmHg and is initially increasing.

27. A ferris wheel at a county fair is 180 ft in diameter with its lowest point 2.5 ft off the ground. Once all the passengers have been loaded, the wheel makes one full rotation counterclockwise in 1.2 min. Suppose that two children are seated at the lowest point on the wheel and are the last passengers to be loaded when the wheel starts. **(See Example 2, p. 723)**

a. Write a model representing the children's horizontal position *x* (in feet) relative to the center of the Ferris wheel, *t* minutes after the ride starts.

b. Write a model representing the children's height *y* (in feet) above ground level, *t* minutes after the ride starts.

c. Give the coordinates of the children's position 1 min into the ride, and describe the location.

28. The 30-ft-diameter waterwheel at Deleon Springs State Park is an old sugar mill that turns at a rate of 2 rev/min, counterclockwise. Suppose that a coordinate system is chosen with the origin at the bottom of the wheel (at water level). At $t = 0$, suppose that a point on the wheel is located at (0, 0).

a. Write a function that represents the *x* position (in feet) of the point as a function of the time *t* in minutes.

b. Write a function that represents the *y* position (in feet) of the point as a function of the time *t* in minutes.

c. Give the coordinates of the point at $t = 40$ sec, and describe the location.

29. Refer to the piston and crankshaft from Exercise 25. The stroke length of the engine is 4 in. and the engine turns at 1800 rpm. The connecting rod cap at point *B* rotates around the crank circle centered at point *C*. Assume that at $t = 0$, the piston is at its farthest point from the center of the crank circle.

a. How many revolutions per second is the engine turning?

b. Write a model representing the horizontal position *x* (in inches) of the connecting rod cap (point *B*) relative to the center of the crank circle (point *C*) at time *t* (in seconds).

c. Write a model representing the vertical position *y* (in inches) of the connecting rod cap relative to the center of the crank circle at time *t* (in seconds).

30. A laboratory centrifuge is a piece of equipment that spins blood samples at a high speed to separate substances within the blood of greater or lesser density. Suppose that a test tube is placed at a point 10 cm to the right of the center of rotation and that the centrifuge spins at 3600 rpm counterclockwise.

a. How many revolutions does the centrifuge make per second?

b. Write a model representing the horizontal position *x* (in centimeters) of the test tube relative to the center of the centrifuge at time *t* (in seconds).

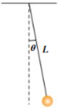
c. Write a model representing the vertical position *y* (in centimeters) of the test tube relative to the center of the centrifuge at time *t* (in seconds).

31. An alternating current generator generates current with a frequency of 60 Hz. Suppose that initially, the current is at its maximum of 5 amperes. If the current varies in simple harmonic motion over time, write a model for the current *I* (in amperes) as a function of the time *t* (in seconds).

32. An alternating current generator in the United Kingdom generates current with a frequency of 50 Hz. Suppose that initially, the current is at its maximum of 12 amperes. If the current varies in simple harmonic motion over time, write a model for the current *I* (in amperes) as a function of the time *t* (in seconds).

The angular displacement θ (in radians) of a pendulum swinging through a small angle with limited air resistance and internal friction can be approximated by the simple harmonic motion models $\theta = a\sin\omega t$ or $\theta = a\cos\omega t$. The period *T* (time required for one back-and-forth swing) is given by $T = 2\pi\sqrt{\frac{L}{g}}$, where *L* is the length of the pendulum and *g* is the acceleration due to gravity. The maximum angular displacement is $|a|$, and positive angles are taken to the right of the center position. Use this information for Exercises 33–34.

33. A 4-ft pendulum is initially at its rightmost position of 12°.



a. Determine the period for one back-and-forth swing. Use $g = 32$ ft/sec^2.

b. Write a model for the angular displacement θ of the pendulum after *t* seconds. (*Hint*: Be sure to convert the initial position to radians.)

34. A 1.4-m pendulum is initially at its leftmost position of −9°.

a. Determine the period for one back-and-forth swing. Use $g = 9.8$ m/sec^2.

b. Write a model for the angular displacement θ of the pendulum after *t* seconds.

Noel Hendrickson/Digital Vision/Getty Images

35. The brightness of a "young" star sometimes increases and decreases as a result of regional areas of "hot" and "cold" on the star's surface as well as variations in the density of the star's planet-forming debris, which can obstruct light. Suppose that for a particular star, the average magnitude (measure of brightness) is 4.3 with a variation of ±0.31 (on the magnitude scale, brighter objects have a smaller magnitude than dimmer objects). Furthermore, the magnitude of a star is initially observed to be 4.61, and the time between minimum brightness and maximum brightness is 6.4 days. Write a simple harmonic motion model to describe the magnitude M of the star for day t.

36. The magnitude of a star named Delta Cephei varies from an apparent magnitude of 3.6 to an apparent magnitude of 4.3 with a period of 5.4 days. At $t = 0$ days, the star is at its brightest with a magnitude of 3.6 (on the magnitude scale, brighter objects have a smaller magnitude than dimmer objects). Write a simple harmonic motion model to describe the magnitude M of the star for day t.

For Exercises 37–38, the ordered pair (t, d) gives the displacement d (in centimeters) of an object undergoing simple harmonic motion at time t (in seconds).

37. Suppose that the object has a minimum at (24, 32) and next consecutive maximum at (48, 54).

- **a.** What is the period?
- **b.** What is the frequency?
- **c.** What is the amplitude?
- **d.** Write a model representing the displacement as a function of time.

38. Suppose that the object has a maximum at (4, 16) and then returns to its next equilibrium position at (6, 11).

- **a.** What is the period?
- **b.** What is the frequency?
- **c.** What is the amplitude?
- **d.** Write a model representing the displacement as a function of time.

39. Write a function $y = f(t)$ for simple harmonic motion whose graph has a minimum at $\left(\frac{\pi}{4}, -6\right)$ and next consecutive equilibrium at $\left(\frac{\pi}{2}, 2\right)$.

40. Write a function $y = f(t)$ for simple harmonic motion whose graph has a maximum at $\left(\frac{\pi}{3}, 8\right)$ and next consecutive minimum at $(\pi, -2)$.

Learn: Interpret Damped Harmonic Motion

41. Use a graphing utility to graph the functions of the form $d = ae^{-ct}\cos \omega t$ and comment on the role of the damping factor $c > 0$ in the equation. (**See Example 3, p. 725**)

- **a.** $d = 6e^{-0.1t}\cos \pi t$
- **b.** $d = 6e^{-0.3t}\cos \pi t$
- **c.** $d = 6e^{-0.5t}\cos \pi t$

42. Show that the graph of $d = ae^{-ct}\cos \omega t$ is bounded by $y = ae^{-ct}$ and $y = -ae^{-ct}$ for $a > 0$ and $c > 0$.

For Exercises 43–46, use a graphing utility to graph the function and bounding curves for $t \geq 0$.

43. $d = -10e^{-0.125t}\sin 1.25t$

44. $d = 2e^{-0.55t}\sin 16t$

45. $d = 8e^{-1.2t}\cos 4\pi t$

46. $d = -4e^{-0.008t}\cos \frac{\pi}{8}t$

47. Suppose that a guitar string is plucked such that the center of the string is initially displaced 10 mm and then vibrates under damped harmonic motion. The note produced has a frequency of 110 Hz. The note is no longer audible to a normal human ear once the displacement at the middle of the string is less than 0.1 mm. What is the damping factor if the sound is no longer audible after 2.5 sec? Round to 2 decimal places.

48. A tuning fork is struck, and the tips of the prongs (tines) are initially displaced 0.4 mm from their natural position at rest. The prongs oscillate at 440 Hz under damped harmonic motion. The sound is no longer audible once the displacement is less than 0.1 mm. What is the damping factor if the sound is no longer audible after 4.0 sec? Round to 2 decimal places.

Write About It

49. Explain the criteria under which you would choose the model $d = a\sin\omega t$ versus $d = a\cos\omega t$ to represent the displacement d of an object moving in simple harmonic motion.

50. Explain the difference between an object moving in simple harmonic motion versus damped harmonic motion.

Expanding Your Skills

51. Consider the function defined by $y = x + 2\sin x$.

- **a.** As $x \to \infty$, which term in the function will contribute most to the y value?
- **b.** Before graphing the function, speculate about its general behavior.
- **c.** Graph $y = x$ and $y = x + 2\sin x$ on a graphing utility.

52. Consider the function defined by $y = e^x + \cos x$.

- **a.** As $x \to \infty$, which term in the function will contribute most to the y value?
- **b.** Before graphing the function, speculate about its general behavior.
- **c.** Graph $y = e^x$ and $y = e^x + \cos x$ on a graphing utility.

For Exercises 53–54, use the model $d = ae^{-ct}\cos\omega t$ or $d = ab^{kt}\cos\omega t$ to represent damped harmonic motion.

53. Suppose that a mass at the end of a spring is initially pulled in the positive direction 8 in. from its equilibrium position. Upon release, the displacement of the mass follows a pattern of damped harmonic motion with each cycle lasting 1 sec. If the maximum displacement for each cycle decreases by 25%, find a function that models the displacement of the mass t sec after being released.

54. A pendulum is pulled $\frac{\pi}{18}$ radians to one side and then released. The angular displacement θ follows a pattern of damped harmonic motion with each cycle lasting 2 sec. If the maximum displacement for each cycle decreases by 20%, find a function that models the angular displacement t sec after being released.

In a course in differential equations or physics, a more specific model for a mass moving in damped harmonic motion is found as $d = ae^{-bt/(2m)}\cos\left(t\sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2}\right)$, where b is a damping constant unique to the physical system, m is the mass of the object, and $|a|$ is the displacement at time $t = 0$. The value $\frac{2\pi}{\omega}$ is the related period under simple harmonic motion. Use this model for Exercises 55–56.

55. a. Write a model for damped harmonic motion for a mass of 5 kg oscillating on a spring with initial displacement 0.8 m, period of 6 sec, and damping constant $b = 0.3\,\frac{\text{N}\cdot\text{s}}{\text{m}}$ (Newton-second/meter).

- **b.** Graph the model from part (a) on the window [0, 10, 1] by [–1, 1, 0.1].

56. a. Write a model for damped harmonic motion for the bob of a pendulum of mass of 2 kg originally pulled to the right of its equilibrium position 0.6 m. Upon release, the bob makes one complete swing back and forth in 2 sec with a damping constant $b = 0.2\,\frac{\text{N}\cdot\text{s}}{\text{m}}$.

- **b.** Graph the model from part (a) on the window [0, 10, 1] by [–0.8, 0.8, 0.1].

57. Given the model $d = 20e^{-0.6t}\cos 2\pi t$, for an object in damped harmonic motion,

- **a.** Determine the initial displacement, d (in centimeters).
- **b.** Graph the model on a graphing utility.
- **c.** Determine the time t (in seconds) between two consecutive relative maxima.
- **d.** By what percentage is the displacement of the object decreased with each successive oscillation between consecutive maxima? Round to the nearest tenth of a percent.

58. Given the model $d = -8e^{-0.4t}\cos \pi t$, for an object in damped harmonic motion,

- **a.** Determine the initial displacement, d (in centimeters).
- **b.** Graph the model on a graphing utility.
- **c.** Determine the time t (in seconds) between two consecutive relative maxima.
- **d.** By what percentage is the displacement of the object decreased with each successive oscillation between consecutive maxima? Round to the nearest tenth of a percent.

Given a point on the Earth of latitude α, the angle of elevation of the Sun is at its minimum of $90° - \alpha - 23.5°$ at noon at the winter solstice. The angle of elevation of the Sun is at its maximum of $90° - \alpha + 23.5°$ at noon on the summer solstice. The angle of elevation follows simple harmonic motion as it varies from its minimum to its maximum over the course of 365 days. Use this information for Exercises 59–60.

59. Dallas, Texas, has latitude 32.8°N.

- **a.** Find the angle of elevation of the Sun at noon on the winter solstice and summer solstice.
- **b.** Suppose that the winter solstice is 10 days before the end of a given year. Write a function representing the angle A (in degrees) of the Sun as a function of the day number t for the following year. (*Hint*: Use a phase shift of –10 days.)
- **c.** Graph the function from part (b) on a graphing utility.
- **d.** Determine the angle of elevation of the Sun at noon on February 1 for Dallas. Round to the nearest tenth of a degree.
- **e.** How many days into the year will the spring and fall equinoxes occur? Round to the nearest day. (*Hint*: The angle of elevation of the Sun at noon on an equinox will be halfway between its minimum and maximum values. This is the day of the year when the duration of daylight equals the duration of darkness.)

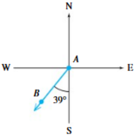
60. Denver, Colorado, has latitude 39.7°N.

- **a.** Find the angle of elevation of the Sun at noon on the winter solstice and summer solstice.
- **b.** Suppose that the winter solstice is 10 days before the end of a given year. Write a function representing the angle A (in degrees) of the Sun as a function of the day number t for the following year. (*Hint*: Use a phase shift of –10 days.)
- **c.** Graph the function from part (b) on a graphing utility.
- **d.** Determine the angle of elevation of the Sun at noon on January 15 for Denver. Round to the nearest tenth of a degree.

Chapter 6 Review

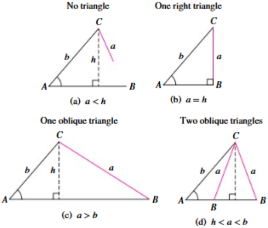
Chapter 6 Key Concepts

Lesson 6-1 Applications of Right Triangles

Key Concepts	Reference
Finding the measures of the angles and sides of a right triangle when two sides or a side and an angle are known is called **solving a right triangle.**	p. 672
The **bearing** from point *A* to point *B* is the degree measure of the acute angle formed by ray $\overrightarrow{AB}$ and the north-south line. In the figure, the bearing from *A* to *B* is S39°W and the bearing from *B* to *A* is N39°E.  	p. 676

Lesson 6-2 The Law of Sines

Key Concepts	Reference
A triangle that does not contain a right angle is called an **oblique triangle**. To solve an oblique triangle, • Use the law of sines if two angles and a side are known (ASA or SAA) or if two sides and the angle opposite one of them are known (SSA). • Use the law of cosines if three sides are known (SSS) or if two sides and the angle between them are known (SAS).	p. 689
Law of Sines If $\triangle ABC$ is a triangle with sides of lengths *a*, *b*, and *c* opposite angles *A*, *B*, and *C*, respectively, then $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or equivalently $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	p. 690

Law of Sines—Ambiguous Case When the lengths of two sides and the measure of an angle opposite one of the known sides are given, then four cases are possible regarding the number of triangles that can be constructed.  No triangle (a) $a < h$ One right triangle (b) $a = h$ One oblique triangle (c) $a > b$ Two oblique triangles (d) $h < a < b$ 	p. 692
Given two sides and the angle between them, the area of $\triangle ABC$ is given by Area $= \frac{1}{2}bc\sin A$ or Area $= \frac{1}{2}ab\sin C$ or Area $= \frac{1}{2}ac\sin B$	p. 696

Lesson 6-3 The Law of Cosines

Key Concepts	Reference
Use the law of cosines to solve an oblique triangle when • Two sides and the angle between them are known (SAS). • Three sides are known (SSS).	p. 707
Law of Cosines If $\triangle ABC$ has sides of lengths a, b, and c opposite vertices A, B, and C, respectively, then Law of Cosines — Alternative Form (1) $a^2 = b^2 + c^2 - 2bc\cos A$ — $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ (2) $b^2 = a^2 + c^2 - 2ac\cos B$ — $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$ (3) $c^2 = a^2 + b^2 - 2ab\cos C$ — $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$	p. 707
For a triangle with sides of lengths a, b, and c, the area is given by Area $= \sqrt{s(s-a)(s-b)(s-c)}$ where s is the semi-perimeter of the triangle: $s = \frac{1}{2}(a + b + c)$.	p. 713

Lesson 6-4 Harmonic Motion

Key Concepts	Reference
An object moving indefinitely in a sinusoidal manner with constant amplitude is said to be moving in **simple harmonic motion.** The **period** is the time required to complete one cycle, and the **frequency** is the number of cycles per unit time.	p. 721
The displacement d of an object undergoing simple harmonic motion is given by $d = a\sin\omega t$ or $d = a\cos\omega t$ or equivalently $d = a\sin 2\pi ft$ or $d = a\cos 2\pi ft$ where t is the time after recording begins. • The amplitude of motion is $\|a\|$. • The period is given by $p = \frac{2\pi}{\omega}$. The frequency f is the reciprocal of the period, $f = \frac{1}{p}$.	p. 722
An object undergoes **damped harmonic motion** if the amplitude of oscillation decreases with time. The displacement of an object under damped harmonic motion at time t is given by $d = ae^{-ct}\cos\omega t$ or $d = ae^{-ct}\sin\omega t$ for $c > 0$ • a is the displacement of the object at $t = 0$. • c is a constant called the **damping factor** and is related to the damping force opposing motion. • The value of ω is related to the period P and frequency f by $P = 2\pi/\omega$ and $\omega = 2\pi f$.	p. 725

Go online for a detailed chapter summary.

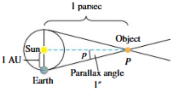
Chapter 6 Test

For Exercises 1–8, solve $\triangle ABC$ subject to the given conditions. Round the lengths of sides and measures of the angles to 1 decimal place if necessary.

1. $A = 62^\circ, C = 90^\circ, a = 17$
2. $a = 17.6, b = 20, c = 12.8$
3. $B = 97.3^\circ, C = 12.7^\circ, b = 77.1$
4. $B = 14.8^\circ, C = 90^\circ, a = 213$
5. $C = 81^\circ, a = 59, b = 44$
6. $A = 120^\circ, B = 30^\circ, a = 51$
7. $B = 13^\circ, a = 7, b = 2$
8. $B = 13^\circ, a = 7, b = 1$

For Exercises 9–12, find the area of $\triangle ABC$ with the given measurements. Round to 1 decimal place.

9. $A = 102^\circ$, $b = 230$ yd, $c = 188$ yd
10. $C = 90^\circ$, $c = 56$ cm, $a = 12$ cm
11. $a = 28$ m, $b = 46$ m, $c = 23$ m
12. $B = 43^\circ$, $a = 1.8$ ft, $c = 6.8$ ft
13. A ship travels west at 25 knots. At 7:30 p.m., the captain sees a floating beacon due south of the ship. By 9:15 p.m., the captain sights the beacon at a bearing of S32°E. What was the distance from the ship to the beacon at the time of the first sighting? Round to the nearest nautical mile.
14. Over the course of a day, as the angle of inclination of the Sun decreases from 64° to 49°, the length of the shadow of a building increases by 60 ft. How tall is the building? Round to the nearest foot.
15. To determine the height of a tower, a surveyor notes that the angle of elevation to the top of the tower is 48° when measured at a point on the ground 45 ft from the base of the tower. To the nearest tenth of a foot, how tall is the tower?
16. In addition to the light-year to quantify large distances in outer space, astronomers often use two other units of measurement.
    - the **astronomical unit** (AU), which is the average distance between the Earth and Sun (1 AU ≈ 93,000,000 mi).
    - the **parsec** (pc), which is the distance from the Earth to an object having a parallax angle of 1″ (1 second of arc).



    a. To the nearest whole unit, how many astronomical units are in 1 parsec?

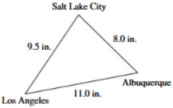
    b. How many light-years are in 1 parsec? Round to 2 decimal places. (*Hint*: 1 light-year ≈ 5.878×10^{12} mi.)

17. A triangular shade sail has sides of length $12'3''$ and $15'6''$, and the angle between them is $73^\circ 18'$.

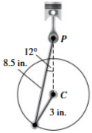
    a. If the cost of the sail material is \$8.75/ft^2, how much would this sail cost? Round to the nearest dollar.

    b. If the cost of wrapping the edges of the sail with a decorative border in a contrasting color is \$1.25/yd, what is the added cost to the sail? Round to the nearest dollar.

18. The distance between Los Angeles, California, and Salt Lake City, Utah, on a map is 9.5 in. The distance from Salt Lake City to Albuquerque, New Mexico, is 8.0 in., and the distance from Albuquerque to Los Angeles is 11.0 in. If the bearing from Salt Lake City to Los Angeles is S34°W, find the bearing from Albuquerque to Los Angeles. Round to the nearest tenth of a degree.

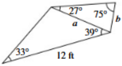


19. The connecting rod from the piston to the crankshaft in a certain engine is 8.5 in. The radius of the crank circle is 3 in. If the angle made by the connecting rod with the vertical at the wrist pin P is 12°, how far is the wrist pin from the center C of the crankshaft? Round to the nearest tenth of an inch.



20. Refer to the figure.

- **a.** Write expressions for the exact values of a and b.
- **b.** Approximate a and b to the nearest tenth of a foot.



21. Explain what the relationships $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ given in the law of sines represent.

22. The displacement of an object attached to a spring that moves in simple harmonic motion is given by $d = -1.8\cos 12\pi t$, where the displacement d is in centimeters at time t in seconds. Determine the amplitude and frequency for the motion of the object.

23. The displacement of an object attached to a spring that moves in simple harmonic motion is given by $d = 4\sin\left(2t - \frac{\pi}{4}\right)$, where the displacement d is in centimeters and time t is in seconds. Determine the period and phase shift for the motion of the object.

24. A block hangs on a spring attached to the ceiling and is pulled down 8 in. below its equilibrium position. After release, the block makes one complete up-and-down cycle in 1.2 sec and follows simple harmonic motion.

- **a.** Write a function to model the displacement d (in inches) as a function of the time t (in seconds) after release. Assume that a displacement above the equilibrium position is positive.
- **b.** Find the displacement of the block and direction of movement at $t = 0.3$ sec.

25. The angular displacement θ (in radians) for a simple pendulum is given by $\theta = 0.14\sin\frac{\pi}{2}t$.

- **a.** Determine the period of the pendulum.
- **b.** How many swings are completed in 1 sec?
- **c.** To the nearest degree, what is the maximum displacement of the pendulum?
- **d.** The length L of a pendulum is related to the period T by the equation $T = 2\pi\sqrt{\frac{L}{g}}$, where g is the acceleration due to gravity. Find the length of the pendulum to the nearest foot ($g = 32$ ft/sec^2).

26. A 20-ft-diameter waterwheel operates at a grain mill and turns at a rate of 1.5 rev/min, counterclockwise. Suppose that a coordinate system is chosen with the origin at the bottom of the wheel (at water level). At $t = 0$, suppose that a point on the wheel is located at (0, 0).

- **a.** Write a function that represents the x position (in feet) of the point as a function of the time t in minutes.
- **b.** Write a function that represents the y position (in feet) of the point as a function of the time t in minutes.
- **c.** Give the coordinates of the point at $t = 10$ sec and describe the location.

27. Use a graphing utility to graph the function $d = 4e^{-0.25t}\sin 1.3t$ and bounding curves for $t \geq 0$.

Go online for more practice problems.

Chapter 6 Cumulative Review Exercises

1. Write the equation of the line in slope-intercept form that passes through the point $\left(-3, \frac{1}{2}\right)$ with slope $m = \frac{2}{3}$.

2. Write the equation of a circle in standard form with a center of $(-8, 12)$ and that passes through the point $(-15, 12)$.

3. Write the function defined by $f(x) = -2x^2 - 12x$ in vertex form and identify the vertex of the parabola.

4. Write a polynomial $f(x)$ of degree 5, with integer coefficients, and with zeros -2 (multiplicity 2), 6, and $2i$.

5. Write equations for the asymptotes for the graph of $f(x) = \frac{2x^2 + 5x - 3}{x - 8}$.

6. Write equations for the asymptotes for the graph of $f(x) = \sec 2x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

7. Given $f(x) = \frac{x + 1}{x - 3}$, write an equation for the inverse function.

8. Write an equation for a sine function with an amplitude of 6, a period of π, and a phase shift of $\frac{\pi}{6}$.

9. Write an equation for a tangent function with a period of $\frac{1}{3}$ and a phase shift of $-\frac{4}{3}$.

10. A motorist drives on a toll road to and from work each day and pays $4.25 in tolls one-way.

a. Write a model for the cost C (in \$) for tolls in terms of the number of working days, x.

b. The department of transportation offers a prepaid toll pass for $150 a month. How many working days are required for the motorist to save money by buying the pass?

11. a. Write a variation model using k as the constant of variation if Z is directly proportional to the cube of A and inversely proportional to the square root of B.

b. If $Z = 100$ when $A = 2$ and $B = 25$, what is the constant of proportionality k?

12. a. Write an equation representing the fact that the product of two consecutive odd integers is 323.

b. Solve the equation from part (a) to find the two integers.

13. A salesperson makes a base salary of $2500 per month. After reaching $30,000 in total sales, the salesperson earns an additional 5% commission on the amount in sales over $30,000. Write a piecewise-defined function to model the salesperson's total monthly salary $S(x)$ (in \$) as a function of the amount in sales x.

14. The cost to buy tickets to a concert is $50 per ticket.

a. Write a function that represents the cost $C(x)$ (in \$) for x tickets to the concert.

b. There is a sales tax of 6% and a processing fee of $5 for a group of tickets. Write a function that represents the total cost $T(a)$ for a dollars spent on tickets.

c. Find $(T \circ C)(x)$.

d. Find $(T \circ C)(10)$ and interpret its meaning in the context of this problem.

15. Suppose that the population of a country in the year 2000 was 19.0 million and grew to 22.6 million in 2010. Write a model of the form $P(t) = P_0 e^{kt}$, where $P(t)$ is the population in millions, t years after the year 2000. Round the growth rate to 5 decimal places.

16. The bob on a simple pendulum is pulled to the left 6 in. from its equilibrium position. After release, the pendulum makes one complete back-and-forth cycle in 3 sec and follows simple harmonic motion. Write a function to model the displacement *d* (in inches) of the bob as a function of the time *t* (in seconds) after release. Assume that a displacement to the right of the equilibrium position is positive.

17. Simplify to a single trigonometric function. $\frac{1}{1-\cos x}+\frac{1}{1+\cos x}$

18. Find the exact value of $\cos 200° \cos 50° + \sin 200° \sin 50°$.

For Exercises 19–20, solve $\triangle ABC$ subject to the given conditions. Round the lengths of sides and measures of the angles to 1 decimal place if necessary.

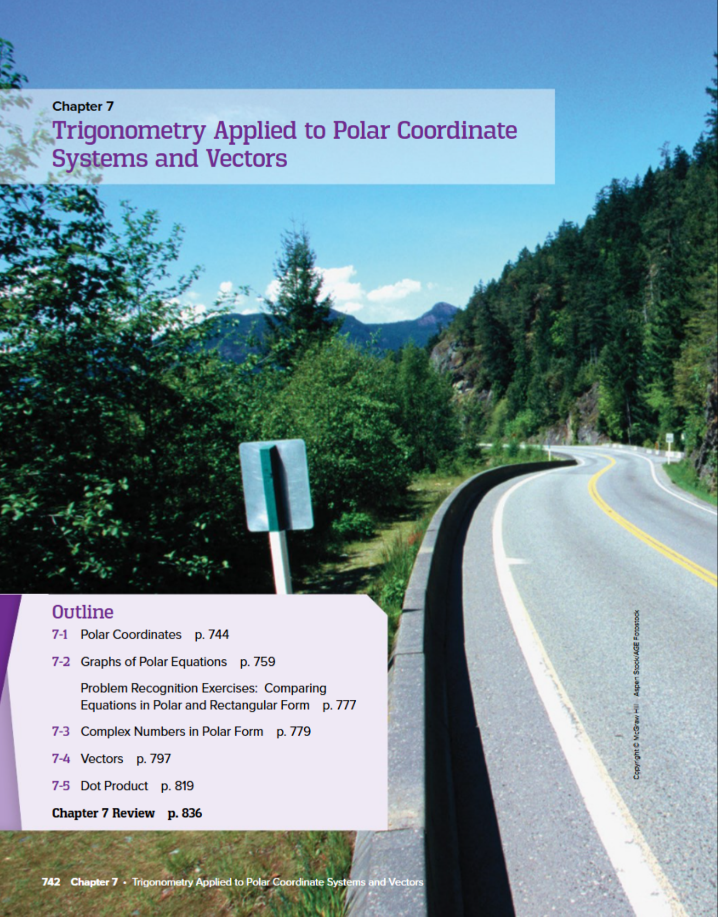
19. $a = 12.8$, $b = 20$, $c = 17.1$

20. $a = 12.8$, $b = 20$, $B = 98.7°$

Go online for more practice problems.

Chapter 7

Trigonometry Applied to Polar Coordinate Systems and Vectors



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Outline

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Problem Recognition Exercises: Comparing Equations in Polar and Rectangular Form p. 777

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Launch Activity: Mathematical Modeling

Getting from Point A to Point B

Chloe and Noah are part of their high school's math team and they are traveling to compete in their first competition. Chloe and Noah's coach is having them practice skills on the van ride to the competition. The van picks up students for the meet from the high school parking lot. First, they travel due south at 55 mph for 45 minutes. Then the van turns right and travels due west at 40 mph for 15 minutes and stops in the parking of the university hosting the meet. Chloe and Noah's coach ask them to gather information about their trip. Noah mentions that he could find the straight-line distance from their starting point, the high school parking lot, to their end point, the university parking lot. Chloe states that the linear path they traveled from their start point to the end point not only has a length but also a specific direction.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer this question: How could you determine the magnitude (or length) and direction of the vector that goes from the high school to the university meet?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

GO ONLINE to answer these questions

Lesson 7-1

Polar Coordinates

What Will You Learn?

After completing this lesson, you should be able to:

- Plot Points Using Polar Coordinates
- Convert Ordered Pairs Between Polar and Rectangular Coordinates
- Convert Between Equations in Polar and Rectangular Coordinates

Learn: Plot Points Using Polar Coordinates

Suppose that a ship leaves port and sends out a distress call after traveling 10 mi at a bearing of N30°E. Although the rescue team does not have the *x*- and *y*-coordinates of the ship relative to the port (the origin), the exact location of the ship is known. The ship's location is defined by its distance from port, 10 mi, and its direction, 30° east of north.

Defining a point in a plane by using distance and direction is the basis of a coordinate system called a polar coordinate system.

Polar Coordinate System

A **polar coordinate system** consists of a fixed point O called the **pole** (or origin), and a ray called the **polar axis**, with endpoint at the pole (Figure 7-1). Each point P in the plane is defined by an ordered pair of the form (r, θ), where r is the directed distance from the pole to P.

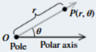


Figure 7-1

- If $r > 0$, point P is located r units from the pole in the direction of θ.
- If $r < 0$, point P is located $|r|$ units from the pole in the direction of $\theta + \pi$ (the direction *opposite* θ).
- If $r = 0$, point P is located at the pole.

θ is a directed angle from the polar axis to line $\overleftrightarrow{OP}$.

- $\theta > 0$ is measured counterclockwise from the polar axis.
- $\theta < 0$ is measured clockwise from the polar axis.

Insights

The polar axis in a polar coordinate system is usually aligned with the positive *x*-axis in a rectangular coordinate system.

Example 1: Plotting Points in a Polar Coordinate System

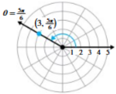
Plot the points whose polar coordinates are given.

a. $\left(3, \frac{5\pi}{6}\right)$ **b.** $\left(5, -\frac{\pi}{4}\right)$ **c.** $(3.5, 3\pi)$ **d.** $\left(-4, \frac{\pi}{3}\right)$

Solution:

Polar graph paper consists of concentric circles centered at the pole whose radii represent incremental distances from the pole.

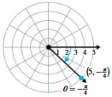
a. $\left(3, \frac{5\pi}{6}\right)$



$\theta = \frac{5\pi}{6}$
$\left(3, \frac{5\pi}{6}\right)$
1 2 3 4 5

To locate $\left(3, \frac{5\pi}{6}\right)$, draw the terminal side of angle $\theta = \frac{5\pi}{6}$. Then measure 3 units from the pole in that direction.

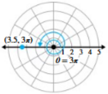
b. $\left(5, -\frac{\pi}{4}\right)$



1 2 3 4 5
$\left(5, -\frac{\pi}{4}\right)$
$\theta = -\frac{\pi}{4}$

To locate $\left(5, -\frac{\pi}{4}\right)$, draw the terminal side of angle $\theta = -\frac{\pi}{4}$. Then measure 5 units from the pole in that direction.

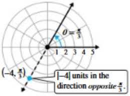
c. $(3.5, 3\pi)$



$(3.5, 3\pi)$
1 2 3 4 5
$\theta = 3\pi$

To locate $(3.5, 3\pi)$, draw the terminal side of angle $\theta = 3\pi$. Then measure 3.5 units from the pole in that direction.

d. $\left(-4, \frac{\pi}{3}\right)$



$\theta = \frac{\pi}{3}$
1 2 3 4 5
$\left(-4, \frac{\pi}{3}\right)$
$|-4|$ units in the direction *opposite* $\frac{\pi}{3}$.

To locate $\left(-4, \frac{\pi}{3}\right)$, draw the terminal side of angle $\theta = \frac{\pi}{3}$. Then measure $|-4| = 4$ units in the direction opposite $\theta = \frac{\pi}{3}$. That is, measure 4 units from the pole in the direction of $\frac{\pi}{3} + \pi$.

Apply the Skills

1. Plot the points whose polar coordinates are given.

a. $\left(3, \frac{5\pi}{4}\right)$ **b.** $\left(4, -\frac{7\pi}{6}\right)$ **c.** $(2.5, 4\pi)$ **d.** $\left(-3, \frac{4\pi}{3}\right)$

Insights

Polar coordinates are sometimes given with θ in degree measure. For example, the point $\left(3, \frac{5\pi}{6}\right)$ can also be represented by $(3, 150°)$.

The location of a point is uniquely described by an ordered pair (x, y) in a rectangular coordinate system. However, with polar coordinates, an infinite number of ordered pairs can represent a point in the plane. For example, the ordered pairs (r, θ) and $(r, \theta + 2n\pi)$ for any integer n represent the same point in polar coordinates. The coordinate $\theta + 2n\pi$ represents infinitely many angles coterminal to θ. One such example is shown in Figure 7-2.

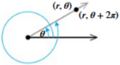

(r, θ)
$(r, \theta + 2\pi)$
θ

Figure 7-2

Likewise, (r, θ) and $(-r, \theta + (2n + 1)\pi)$ represent the same point. The coordinate $\theta \pm (2n + 1)\pi$ represents θ plus or minus an odd multiple of π. This is an angle with terminal side in the opposite direction of θ (Figure 7-3).

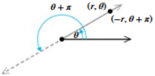

$\theta + \pi$
(r, θ)
$(-r, \theta + \pi)$
θ

Figure 7-3

Example 2: Finding Multiple Representations of a Point in Polar Coordinates

Given $\left(4, \frac{\pi}{6}\right)$ in polar coordinates, find two other polar coordinate representations.

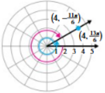
a. with $r > 0$. **b.** with $r < 0$.

Solution:

a. Using a positive value for r, we have $r = 4$. Since r is unchanged, any angle coterminal to $\theta = \frac{\pi}{6}$ will complete the ordered pair: $\left(4, \frac{\pi}{6} + 2n\pi\right)$.

For example: $\left(4, \frac{\pi}{6} + 2\pi\right) = \left(4, \frac{13\pi}{6}\right)$

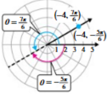
$\left(4, \frac{\pi}{6} - 2\pi\right) = \left(4, -\frac{11\pi}{6}\right)$


$\left(4, -\frac{11\pi}{6}\right)$
$\left(4, \frac{13\pi}{6}\right)$
1
2
3
4
5

b. Using a negative value for r, we have $r = -4$. Any angle coterminal to $\theta = \frac{\pi}{6} + \pi$ will complete the ordered pair.

For example: $\left(-4, \frac{\pi}{6} + \pi\right) = \left(-4, \frac{7\pi}{6}\right)$

$\left(-4, \frac{\pi}{6} - \pi\right) = \left(-4, -\frac{5\pi}{6}\right)$


$\theta = \frac{7\pi}{6}$
$\left(-4, \frac{7\pi}{6}\right)$
$\left(-4, -\frac{5\pi}{6}\right)$
1
2
3
4
5
$\theta = -\frac{5\pi}{6}$

Insights

Although an infinite number of ordered pairs can represent a point in polar coordinates, we often use the simplest representation with $r > 0$ and $0 \leq \theta < 2\pi$.

Apply the Skills

2. Given $\left(7, \frac{3\pi}{2}\right)$ in polar coordinates, find two other polar coordinate representations

a. with $r > 0$. **b.** with $r < 0$.

Learn: Convert Ordered Pairs Between Polar and Rectangular Coordinates

A point in a plane can be described by an ordered pair in rectangular coordinates or by an ordered pair in polar coordinates. It seems reasonable that we should be able to convert between the two coordinate systems.

To convert the coordinates of a point P in polar coordinates (r, θ) to rectangular coordinates (x, y), first consider the case with $r > 0$. From Figure 7-4.

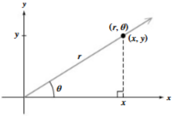


Figure 7-4

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{1}$$

If $r = 0$, then P is at the pole with rectangular coordinates $(0, 0)$ and equation (1) holds true.

If $r < 0$, then point P can be represented by $(-r, \theta + \pi)$, where $-r > 0$. Since $\cos(\theta + \pi) = -\cos\theta = \frac{x}{-r}$, we have $x = r\cos\theta$. Likewise, since $\sin(\theta + \pi) = -\sin\theta = \frac{y}{-r}$, we have $y = r\sin\theta$. So, equation (1) holds true for any real number r.

Converting Between Rectangular and Polar Coordinates

1. To convert the polar coordinates (r, θ) to rectangular coordinates (x, y), use $x = r\cos\theta$ and $y = r\sin\theta$.
2. To convert the rectangular coordinates (x, y) to polar coordinates (r, θ), use $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$ for $x \neq 0$.

Example 3: Converting from Polar to Rectangular Coordinates

Convert the ordered pairs in polar coordinates to rectangular coordinates.

a. $\left(4, \frac{3\pi}{2}\right)$ **b.** $\left(-2, -\frac{5\pi}{6}\right)$

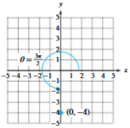
Solution:

a. Given $\left(4, \frac{3\pi}{2}\right)$,

$x = r\cos\theta = 4\cos\frac{3\pi}{2} = 4(0) = 0$

$y = r\sin\theta = 4\sin\frac{3\pi}{2} = 4(-1) = -4.$

The rectangular coordinates are $(0, -4)$.

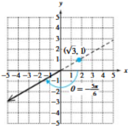


b. First note that $\left(-2, -\frac{5\pi}{6}\right)$ defines a point in Quadrant I. Therefore, we expect the *x*- and *y*-coordinates to be positive.

$x = r\cos\theta = -2\cos\left(-\frac{5\pi}{6}\right) = -2\left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

$y = r\sin\theta = -2\sin\left(-\frac{5\pi}{6}\right) = -2\left(-\frac{1}{2}\right) = 1$

The rectangular coordinates are $(\sqrt{3}, 1)$.



Apply the Skills

3. Convert the ordered pairs in polar coordinates to rectangular coordinates.

a. $(5, \pi)$ **b.** $\left(-10, -\frac{2\pi}{3}\right)$

Changing from polar coordinates to rectangular coordinates as in Example 3 is generally easier than the reverse process because the rectangular coordinates of a point are unique. In Example 4, we will convert from rectangular coordinates to polar coordinates, where we see that the value of *r* (whether taken positive or negative) depends on the choice of θ and vice versa.

Example 4: Converting from Rectangular to Polar Coordinates

Convert $(-3, 3)$ from rectangular coordinates to polar coordinates. Give two representations, one with $r > 0$ and one with $r < 0$.

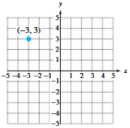
Solution:

The point $(-3, 3)$ is in Quadrant II with $x = -3$ and $y = 3$.

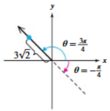
$r^2 = x^2 + y^2 = (-3)^2 + (3)^2 = 18,$

so $r = \pm\sqrt{18} = \pm 3\sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{3}{-3} = -1$



- Taking $r > 0$, we require $\tan\theta = -1$ for θ in Quadrant II. One possibility is $r = 3\sqrt{2}$ and $\theta = \frac{3\pi}{4}$.
- Taking $r < 0$, we require $\tan\theta = -1$ for θ in Quadrant IV. One possibility is $r = -3\sqrt{2}$ and $\theta = -\frac{\pi}{4}$.



In polar coordinates, we have $\left(3\sqrt{2}, \frac{3\pi}{4}\right)$ and $\left(-3\sqrt{2}, -\frac{\pi}{4}\right)$.

Apply the Skills

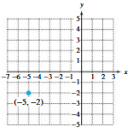
4. Convert $(-2, -2\sqrt{3})$ from rectangular coordinates to polar coordinates. Give two representations, one with $r > 0$ and one with $r < 0$.

Example 5: Converting from Rectangular to Polar Coordinates

Convert $(-5, -2)$ from rectangular coordinates to polar coordinates. Give two representations, one with $r > 0$ and one with $r < 0$.

Solution:

The point $(-5, -2)$ is in Quadrant III with $x = -5$ and $y = -2$.

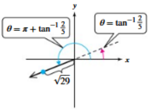


$r^2 = x^2 + y^2 = (-5)^2 + (-2)^2 = 29$, so $r = \pm\sqrt{29}$.

$\tan\theta = \frac{y}{x} = \frac{-2}{-5} = \frac{2}{5}$.

Therefore, $\theta = n\pi + \tan^{-1}\frac{2}{5}$.

- Taking $r > 0$, we require $\tan\theta = \frac{2}{5}$ for θ in Quadrant III.
  One possibility is $r = \sqrt{29}$ and $\theta = \pi + \tan^{-1}\frac{2}{5}$.
- Taking $r < 0$, we require $\tan\theta = \frac{2}{5}$ for θ in Quadrant I.
  One possibility is $r = -\sqrt{29}$ and $\theta = \tan^{-1}\frac{2}{5}$.


y
x
$\theta = \pi + \tan^{-1}\frac{2}{5}$
$\theta = \tan^{-1}\frac{2}{5}$
$\sqrt{29}$

In polar coordinates we have $\left(\sqrt{29}, \pi + \tan^{-1}\frac{2}{5}\right)$ and $\left(-\sqrt{29}, \tan^{-1}\frac{2}{5}\right)$.

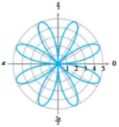
Apply the Skills

5. Convert $(2, -3)$ from rectangular coordinates to polar coordinates. Give two representations, one with $r > 0$ and one with $r < 0$.

Learn: Convert Between Equations in Polar and Rectangular Coordinates

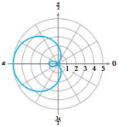
Why do we introduce a polar coordinate system when a rectangular coordinate system (at least to this point) seems to suffice? The answer is that many curves are more easily represented by a polar equation than by an equation in rectangular coordinates. A **polar equation** is an equation described in a polar coordinate system with variables r and θ rather than x and y.

The relationships $x = r\cos\theta$ and $y = r\sin\theta$ between polar and rectangular coordinates are based on trigonometric functions of the unit circle. Therefore, it seems reasonable that circles, ellipses, and other curves with loops such as those shown in Figures 7-5 and 7-6 may be easier to represent in polar form.



$r = 5\sin 4\theta$

Figure 7-5



$r = 2 - 3\cos\theta$

Figure 7-6

Insights

Notice that the curves in Figures 7-5 and 7-6 fail the vertical line test and do not define y as a function of x in a rectangular coordinate system.

We convert equations from rectangular coordinates to polar coordinates by substituting $x = r\cos\theta$ and $y = r\sin\theta$. This is demonstrated in Example 6.

Example 6: Converting an Equation from Rectangular to Polar Coordinates

Write an equivalent equation using polar coordinates.

a. $2x + y = 4$ **b.** $x^2 + (y - 2)^2 = 4$

Solution:

a. $2x + y = 4$	The equation represents the line $y = -2x + 4$ in rectangular coordinates.
$2r\cos\theta + r\sin\theta = 4$	Substitute $x = r\cos\theta$ and $y = r\sin\theta$.
$r(2\cos\theta + \sin\theta) = 4$	Factor out r.
$r = \dfrac{4}{2\cos\theta + \sin\theta}$	Isolate r to write r as a function of θ.

The polar equation is $r = \dfrac{4}{2\cos\theta + \sin\theta}$.

b. $x^2 + (y - 2)^2 = 4$	This equation in rectangular coordinates represents a circle with center (0, 2) and radius 2.
$(r\cos\theta)^2 + (r\sin\theta - 2)^2 = 4$	Substitute $x = r\cos\theta$ and $y = r\sin\theta$.
$r^2\cos^2\theta + r^2\sin^2\theta - 4r\sin\theta + 4 = 4$	Square the binomial $(a - b)^2 = a^2 - 2ab + b^2$.
$r^2(\cos^2\theta + \sin^2\theta) - 4r\sin\theta = 0$	Factor out r^2 and subtract 4 from both sides.
$r^2 - 4r\sin\theta = 0$	Substitute 1 for $\cos^2\theta + \sin^2\theta$.
$r(r - 4\sin\theta) = 0$	Factor out r.
$r = 0$ or $r = 4\sin\theta$	Set each factor equal to zero.

The graph of $r = 0$ is simply the pole. This point is included in the graph of $r = 4\sin\theta$ because $r = 0$ when $\theta = n\pi$. Therefore, the equation $r = 4\sin\theta$ by itself is sufficient to represent the equation $x^2 + (y - 2)^2 = 4$.

The polar equation is $r = 4\sin\theta$.

Apply the Skills

6. Write an equivalent equation using polar coordinates.

a. $y = -3$ **b.** $y = x^2$

Insights

In Example 6, notice that the linear equation $2x + y = 4$ is simpler in rectangular coordinates, but an equation of a circle such as $r = 4\sin\theta$ is simpler in polar coordinates.

TECHNOLOGY CONNECTIONS

Graphing a Polar Equation

Most graphing utilities have the capability to graph polar equations. To begin, hit the MODE button and verify that the calculator is in radian mode and that the POLAR setting is highlighted (Figure 7-7).

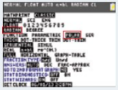
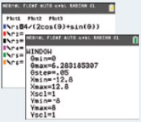


Figure 7-7

Then, in the graphing editor, enter the equation with r in terms of θ (Figure 7-8). In the WINDOW menu, give a range for θ, x, and y. For now, choose θ min $= 0$ and θ max to be 2π (Figure 7-8).



```
NORMAL FLOAT AUTO a+bi RADIAN CL
 Plot1  Plot2  Plot3
\r1=4/(2cos(θ)+sin(θ))
\r2=
\r3=
\r4=
\r5=
\r6=
```

```
NORMAL FLOAT AUTO a+bi RADIAN CL
WINDOW
 θmin=0
 θmax=6.283185307
 θstep=.05
 Xmin=-12.8
 Xmax=12.8
 Xscl=1
 Ymin=-8
 Ymax=8
 Yscl=1
```

Figure 7-8

More will be discussed about the range for θ in Lesson 7.2. The value of θ step gives the increment by which θ is increased to compute successive values of r. Small values of θ step produce more points and render a smoother curve than a graph rendered with a large value of θ step. A large value of θ step yields fewer points and a more jagged curve but takes less time to graph.

The graph of $r = \frac{4}{2\cos\theta + \sin\theta}$ from Example 6(a) is shown in Figure 7-9.

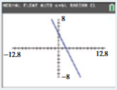


Figure 7-9

Notice that the graph of the polar equation is a line with slope −2 and y-intercept 4 as expected.

To convert an equation from polar coordinates to rectangular coordinates, the goal is to write an equation exclusively with x and y rather than r and θ. This is sometimes cumbersome, and to help with the task, we can use one or more of the following strategies.

1. Apply one or more of the relationships $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$, or $\tan\theta = \frac{y}{x}$ for ($x \neq 0$).
2. Multiply both sides by r.
3. Square both sides of the equation.

Example 7: Converting an Equation from Polar to Rectangular Coordinates

Write an equivalent equation using rectangular coordinates and identify the type of curve that the equation represents.

a. $r = 2$ **b.** $\theta = \frac{\pi}{3}$ **c.** $r = 4\csc\theta$ **d.** $r = -8\cos\theta$

Solution:

a.

$$r = 2$$

$$r^2 = 4$$

Square both sides to get r^2 on the left, which can be written as $x^2 + y^2$.

$$x^2 + y^2 = 4$$

This is an equation of a circle centered at the origin with radius 2.

b.

$$\theta = \frac{\pi}{3}$$

$$\tan\theta = \tan\frac{\pi}{3}$$

Take the tangent of both sides so that $\tan\theta$ can be written as $\frac{y}{x}$.

$$\frac{y}{x} = \sqrt{3}$$

Substitute $\frac{y}{x}$ for $\tan\theta$.

$$y = \sqrt{3}x$$

Write the equation in slope-intercept form. The equation is a line through the origin with a slope of $\sqrt{3}$.

c.

$$r = 4\csc\theta$$

$$r = \frac{4}{\sin\theta}$$

Write the expression on the right in terms of $\sin\theta$ for the possibility of applying the substitution $y = r\sin\theta$.

$$r\sin\theta = 4$$

Multiply both sides by $\sin\theta$.

$$y = 4$$

Replace $r\sin\theta$ by y. The equation is a horizontal line.

d.

$$r = -8\cos\theta$$

$$r^2 = -8r\cos\theta$$

$$x^2 + y^2 = -8x$$

Multiply both sides by r. By so doing, we have r^2 on the left, which can be replaced by $x^2 + y^2$. On the right, $r\cos\theta$ can be replaced by x.

$$x^2 + 8x + y^2 = 0$$

Collect variable terms on the left.

$$x^2 + 8x + 16 + y^2 = 16$$

Complete the square on the left. Add $\left[\frac{1}{2}(8)\right]^2 = 16$ to both sides.

$$(x + 4)^2 + y^2 = 16$$

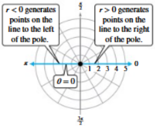
The equation is a circle centered at $(-4, 0)$ with radius 4.

Apply the Skills

7. Write an equivalent equation using rectangular coordinates, and identify the type of curve the equation represents.

a. $r = -3$ **b.** $\theta = -\frac{\pi}{4}$ **c.** $r = 6\sec\theta$ **d.** $r = 6\sin\theta$

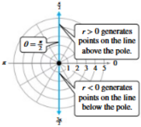
- The result of Example 7(a) suggests that a polar equation of the form $r = a$ for $a \neq 0$ is a circle centered at the pole with radius $|a|$ (see Exercise 72). The equation $r = a$ represents all points $|a|$ units from the pole regardless of the value of θ (regardless of the direction taken).
- The result of Example 7(b) suggests that a polar equation of the form $\theta = k$ is a line through the origin making an angle k with the polar axis (see Exercise 71). Of particular interest are the graphs of $\theta = 0$ and $\theta = \frac{\pi}{2}$. If the polar axis is positioned to coincide with the positive x-axis in a rectangular coordinate system, these equations represent the x- and y-axes, respectively (Figures 7-10 and 7-11).


$r < 0$ generates points on the line to the left of the pole.

$r > 0$ generates points on the line to the right of the pole.

$\theta = 0$

Figure 7-10


$\theta = \frac{\pi}{2}$

$r > 0$ generates points on the line above the pole.

$r < 0$ generates points on the line below the pole.

Figure 7-11

Practice 7-1

Practice Exercises

Prerequisite Review

R.1. Given a circle with center (–7, 3) and radius 3, write an equation of the circle in standard form.

R.2. Write the equation in slope-intercept form and determine the slope. $4x + 3y = 12$

R.3. Find the exact distance between the points (–9, 4) and (–4, –1).

Concept Connections

1. The origin of a polar coordinate system is called the ___________.

2. Given a point P represented by the ordered pair (r, θ) in polar coordinates, the distance from P to (0, 0) is ___________.

3. To convert an ordered pair (r, θ) in polar coordinates to rectangular coordinates, use the relationships $x =$ ___________ and $y =$ ___________.

4. To convert an ordered pair (x, y) from rectangular coordinates to polar coordinates, use the relationships $r^2 =$ ___________ and $\tan\theta =$ ___________.

5. Use an equation in polar coordinates to describe the set of points 4 units from the pole.

6. Write three ordered pairs in polar coordinates that represent the pole.

Learn: Plot Points Using Polar Coordinates

For Exercises 7–14, match each point in polar coordinates to the points *A*, *B*, *C*, or *D*.

$A\left(3, \frac{\pi}{6}\right)$ $B\left(3, \frac{3\pi}{4}\right)$ $C\left(3, -\frac{\pi}{6}\right)$ $D\left(-3, \frac{\pi}{4}\right)$

7. $\left(3, -\frac{5\pi}{4}\right)$

8. $\left(3, \frac{11\pi}{6}\right)$

9. $\left(3, -\frac{11\pi}{6}\right)$

10. $\left(3, \frac{5\pi}{4}\right)$

11. $\left(3, \frac{13\pi}{4}\right)$

12. $\left(-3, \frac{7\pi}{4}\right)$

13. $\left(-3, \frac{5\pi}{6}\right)$

14. $\left(-3, \frac{7\pi}{6}\right)$

For Exercises 15–18, plot the points whose polar coordinates are given. (See Example 1, p. 745)

15. a. $\left(5, \frac{\pi}{2}\right)$ **b.** $\left(2.5, \frac{8\pi}{3}\right)$
c. $\left(-2, \frac{\pi}{6}\right)$ **d.** $\left(-4, \frac{5\pi}{4}\right)$

16. a. $\left(3, \frac{\pi}{3}\right)$ **b.** $\left(4.5, \frac{11\pi}{4}\right)$
c. $\left(-5, \frac{5\pi}{6}\right)$ **d.** $\left(-3, \frac{\pi}{4}\right)$

17. a. $\left(4, -\frac{5\pi}{6}\right)$ **b.** $\left(3, -\frac{5\pi}{4}\right)$
c. $\left(-2, -\frac{\pi}{3}\right)$ **d.** $\left(-4.5, -\frac{2\pi}{3}\right)$

18. a. $\left(2, -\frac{2\pi}{3}\right)$ **b.** $(3, -\pi)$
c. $\left(-4, -\frac{\pi}{4}\right)$ **d.** $\left(-5, -\frac{5\pi}{6}\right)$

For Exercises 19–22, given a point in polar coordinates, find two other polar coordinate representations. (See Example 2, p. 746)

a. with $r > 0$. **b.** with $r < 0$.

19. $\left(7, \frac{5\pi}{6}\right)$

20. $\left(3.6, \frac{\pi}{2}\right)$

21. $\left(-8, \frac{7\pi}{4}\right)$

22. $\left(-2, \frac{4\pi}{3}\right)$

Learn: Convert Ordered Pairs Between Polar and Rectangular Coordinates

For Exercises 23–32, convert the ordered pair in polar coordinates to rectangular coordinates. (See Example 3, p. 748)

23. $\left(8, \frac{5\pi}{6}\right)$

24. $\left(6, \frac{2\pi}{3}\right)$

25. $\left(-1.2, \frac{5\pi}{4}\right)$

26. $\left(-\frac{3}{4}, \frac{7\pi}{4}\right)$

27. $\left(17, -\frac{\pi}{3}\right)$

28. $\left(5, -\frac{7\pi}{6}\right)$

29. $\left(-4, -\frac{11\pi}{6}\right)$

30. $\left(-8, -\frac{4\pi}{3}\right)$

31. $\left(7, \frac{\pi}{2}\right)$

32. $(16, \pi)$

For Exercises 33–38, convert the ordered pair in rectangular coordinates to polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$. (See Examples 4–5, pp. 748–749)

33. $(-3, 3\sqrt{3})$

34. $(-7\sqrt{3}, 7)$

35. $(8\sqrt{2}, -8\sqrt{2})$

36. $(5, -5\sqrt{3})$

37. $(2, -6)$

38. $(4, -10)$

For Exercises 39–44, convert the ordered pair in rectangular coordinates to polar coordinates with $r < 0$ and $0 \le \theta < 2\pi$. (See Examples 4–5, pp. 748–749)

39. $\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$

40. $\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

41. $(-1, \sqrt{3})$

42. $(10\sqrt{2}, 10\sqrt{2})$

43. $(-3, -8)$

44. $(-5, -1)$

Learn: Convert Between Equations in Polar and Rectangular Coordinates

For Exercises 45–56, write an equivalent equation using polar coordinates. (See Example 6, p. 750)

45. $x - y = 10$

46. $3x + 2y = 6$

47. $x = -6$

48. $y = 2$

49. $x^2 + y^2 = 16$

50. $2x^2 + 2y^2 = 50$

51. $x^2 = 3y$

52. $x = 2y^2$

53. $(x + 2)^2 + y^2 = 4$

54. $x^2 + (y - 1)^2 = 1$

55. $(x - 5)^5 + (y + 6)^2 = 61$

56. $(x + 3)^2 + (y - 4)^2 = 25$

For Exercises 57–70, convert the polar equation to rectangular form and identify the type of curve represented. (See Example 7, p. 753)

57. $r = 3$

58. $r = 7$

59. $r = 2\csc\theta$

60. $r = \sec\theta$

61. $r = 4\cos\theta$

62. $r = -2\sin\theta$

63. $\theta = \frac{3\pi}{4}$

64. $\theta = \frac{\pi}{6}$

65. $r = a\sin\theta \; (a \ne 0)$

66. $r = a\cos\theta \; (a \ne 0)$

67. $r = \frac{a}{\sin\theta}(a \ne 0)$

68. $r = \frac{a}{\cos\theta}(a \ne 0)$

69. $r = 2h\cos\theta + 2k\sin\theta$

70. $r = h\cos\theta + k\sin\theta$

71. Suppose that the polar axis is positioned to coincide with the positive x-axis in a rectangular coordinate system. Describe the graphs of the polar equations as they relate to a rectangular coordinate system.

a. $\theta = 0$

b. $\theta = \frac{\pi}{4}$

c. $\theta = \frac{\pi}{2}$

72. Show that a polar equation of the form $r = a$ $(a \ne 0)$ represents a circle centered at the pole with radius $|a|$.

Mixed Exercises

For Exercises 73–76, determine whether the two ordered pairs in polar coordinates represent the same point in the plane. If not, explain the change needed to make the two ordered pairs represent the same point. Assume that n is any integer.

73. (r, θ) and $(-r, -\theta)$

74. $(-r, \theta)$ and $(r, \theta + 2n\pi)$

75. $(-r, \theta)$ and $(r, \pi - \theta)$

76. (r, θ) and $(r, \theta + 2n\pi)$

For Exercises 77–78, determine whether the statement is true or false for two points $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ represented in polar coordinates. If the statement is false, give a counterexample.

77. If $|r_1| = |r_2|$, then P and Q are the same distance from the pole.

78. If $|\theta_1| = |\theta_2|$, then P and Q lie on the same line through the pole.

For Exercises 79–82, a point in polar coordinates is given. Let n be an integer and find *all* other polar representations of the given point.

79. $\left(3, \frac{\pi}{4}\right)$

80. $\left(-2, -\frac{4\pi}{3}\right)$

81. $\left(-4, \frac{5\pi}{6}\right)$

82. $\left(6, -\frac{7\pi}{4}\right)$

For Exercises 83–88, describe the graph of the set of points (r, θ) represented by the polar inequality. Assume that the polar axis is oriented to coincide with the positive x-axis in a rectangular coordinate system.

83. $r < 8$

84. $r \geq 5$

85. $2 \leq r \leq 4$

86. $1 < r < 2$

87. $0 < \theta < \frac{\pi}{4}$

88. $\frac{\pi}{2} < \theta < \pi$

89. Suppose that seating in a theater is in an area defined in polar coordinates where the pole is located at the front and center of the stage labeled as point A. The seating area is defined by $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $30 \leq r \leq 100$, and the values of r are in feet.

a. Sketch the seating area.

b. Determine the amount of area for seating. Write the exact answer in terms of π and give an approximation to the nearest square foot.

90. A rotating sprinkler spreads water over an area defined by $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$ and $4 \leq r \leq 24$ relative to the sprinkler head at the pole at point A and an imaginary line defined by $\theta = 0$. The values of r are in feet.

a. Sketch the area that is watered.

b. Determine the amount of area that is watered. Write the exact answer in terms of π and give an approximation to the nearest square foot.

91. A player in a video game must knock out a target located 84 pixels above and 156 pixels to the left of his position. Choose a polar coordinate system with the player at the pole and the polar axis extending to the player's right. Find the polar coordinates of the target (this determines the distance and angle at which the player should fire his gun). Find r to the nearest pixel and θ in degree measure to the nearest tenth of a degree.

92. A gunner on a naval ship sights a target located 2.1 mi north and 0.8 mi east of the ship's position. Choose a polar coordinate system with the gunner at the pole and the polar axis extending to the east. Find the polar coordinates of the target. Find r to the nearest hundredth of a mile and θ in degree measure to the nearest hundredth of a degree.

Write About It

93. Explain the significance of $r < 0$ in an ordered pair (r, θ) in polar coordinates.

94. Explain why a point in the plane can be represented by infinitely many ordered pairs in polar coordinates.

Expanding Your Skills

95. a. Given points $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ represented in polar coordinates, write the ordered pairs in rectangular coordinates.

b. Use the rectangular coordinates from part (a) and the distance formula to show that the distance d between P and Q is given by $d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$.

96. Use the results of Exercise 95.

a. If $\theta_1 = \theta_2$, what is the distance between P and Q? Explain the significance of the answer.

b. If $\theta_1 - \theta_2 = \frac{\pi}{2}$, what is the distance between P and Q? Explain the significance of the answer.

97. Given points $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ represented in polar coordinates with $\theta_1 \neq \theta_2$ and $r_1 > 0$ and $r_2 > 0$, use the law of cosines to show that the distance d between P and Q is given by $d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$.

98. Use the results of Exercise 97.

Find the distance between the points $\left(2, \frac{\pi}{6}\right)$ and $\left(4, \frac{\pi}{3}\right)$.

Technology Connections

For Exercises 99–102, use a graphing utility to graph the polar equation for the given intervals of θ, x, and y to confirm your answer to the given exercise. Use the window setting for $0 \le \theta \le \pi$, $-4.83 \le x \le 4.83$, and $-3 \le y \le 3$.

99. $r = 2\csc\theta$; Exercise 59

100. $r = \sec\theta$; Exercise 60

101. $r = 4\cos\theta$; Exercise 61

102. $r = -2\sin\theta$; Exercise 62

Go online for more practice problems.

Lesson 7-2

Graphs of Polar Equations

What Will You Learn?

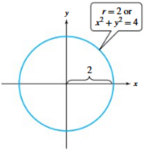
After completing this lesson, you should be able to:

- Graph Polar Equations by Plotting Points
- Test Polar Equations for Symmetry
- Categorize Polar Equations and Their Graphs

Learn: Graph Polar Equations by Plotting Points

Galaxies are clusters of millions of stars, planets, and space debris held together by gravitational attraction that when seen from a very large distance often appear in organized patterns. Perhaps the most beautiful pattern is the classic spiral galaxy. Interestingly, the graph of a spiral can be represented by a simple equation represented in a polar coordinate system (see Example 6)

A **solution** to a polar equation is an ordered pair (r, θ) that satisfies the equation. The graph of a polar equation is the graph of all solutions to the equation. In some cases, we can convert a polar equation to rectangular coordinates and then graph the equation from its rectangular form. For example, in Lesson 7.1, we determined that the equation $r = 2$ is equivalent to the rectangular equation $x^2 + y^2 = 4$. So, the graph of $r = 2$ is a circle with center at the pole and radius 2 units.



Converting a polar equation to an equation in rectangular coordinates does not always produce a rectangular equation that we recognize. As a result, it is sometimes necessary to make a table of solutions and plot the points to graph a polar equation.

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Example 1: Graph a Polar Equation by Plotting Points

Graph $r = 6\sin\theta$.

Solution:

The equation $r = 6\sin\theta$ is written with r as a function of θ. Therefore, we organize a table with arbitrary values of θ and solve for r (Table 7-1 on page 760).

TABLE 7-1 Values of $r = 6\sin\theta$ for Arbitrary Values of θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	3	$3\sqrt{2}$	$3\sqrt{3}$	6	$3\sqrt{3}$	$3\sqrt{2}$	3	0
$r \approx$	0	3	4.2	5.2	6	5.2	4.2	3	0

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r	-3	$-3\sqrt{2}$	$-3\sqrt{3}$	-6	$-3\sqrt{3}$	$-3\sqrt{2}$	-3	0
$r \approx$	-3	-4.2	-5.2	-6	-5.2	-4.2	-3	0

The graph of $r = 6\sin\theta$ is shown in Figure 7-12.

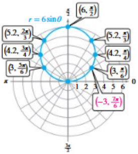


Figure 7-12

The graph is a circle with radius 3, centered at $\left(3, \frac{\pi}{2}\right)$ [or equivalently (0, 3) in rectangular coordinates]. Notice that the graph of the circle is traced completely by graphing solutions with $0 \le \theta \le \pi$. The points in Table 7-1 with $\pi < \theta \le 2\pi$ are redundant. For example, $\left(-3, \frac{7\pi}{6}\right)$ is the same point in the plane as $\left(3, \frac{\pi}{6}\right)$.

Apply the Skills

1. Graph $r = -4\cos\theta$.

The graph from Example 1 is consistent with the result obtained by converting the equation to rectangular form.

$$\begin{aligned}
r &= 6\sin\theta \\
r^2 &= 6r\sin\theta && \text{Multiply both sides by } r. \\
x^2 + y^2 &= 6y && \text{Replace } r^2 \text{ by } x^2 + y^2 \text{ and } r\sin\theta \text{ by } y. \\
x^2 + y^2 - 6y + 9 &= 9 && \text{Complete the square. Add } \left[\tfrac{1}{2}(-6)\right]^2 = 9 \text{ to both sides.} \\
x^2 + (y - 3)^2 &= 9 && \text{The equation is a circle centered at (0, 3) with radius 3.}
\end{aligned}$$

We also notice in Figure 7-12 that the graph of $r = 6\sin\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$ (y-axis). Knowing that the graph of $r = 6\sin\theta$ is traced completely for $0 \le \theta \le \pi$ and that the graph is symmetric to the line $\theta = \frac{\pi}{2}$ means that we can significantly reduce the number of points needed to construct the graph. We can graph $r = 6\sin\theta$ by plotting points for θ on the interval $\left[0, \frac{\pi}{2}\right]$ and using symmetry.

Learn: Test Polar Equations for Symmetry

Figures 7-13, 7-14, and 7-15 show a point P and a corresponding point Q that is symmetric to P with respect to the polar axis, the line $\theta = \frac{\pi}{2}$, and the pole, respectively.

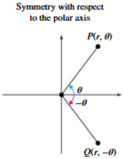

Symmetry with respect to the polar axis

Figure 7-13

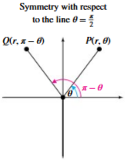

Symmetry with respect to the line $\theta = \frac{\pi}{2}$

Figure 7-14

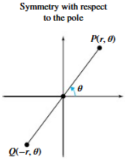

Symmetry with respect to the pole

Figure 7-15

For an equation in polar coordinates, if the substitution in Table 7-2 produces an equivalent equation, then the graph has the indicated type of symmetry.

TABLE 7-2 Tests for Symmetry in Polar Coordinates

Symmetry	Substitution Producing an Equivalent Equation
(1) Symmetry with respect to the polar axis	Replace (r, θ) by $(r, -\theta)$
(2) Symmetry with respect to the line $\theta = \frac{\pi}{2}$	Replace (r, θ) by $(r, \pi - \theta)$
(3) Symmetry with respect to the pole	Replace (r, θ) by $(-r, \theta)$

It is very important to note that the three tests for symmetry given in Table 7-2 are inconclusive if the given substitution fails to produce an equivalent equation. For example, a graph may be symmetric with respect to the polar axis, but a different representation of the point $(r, -\theta)$ satisfies the equation determining the graph. A rigorous test for symmetry would require that we test *all* representations of the point $(r, -\theta)$ in the original equation. That is, we would need to substitute $(r, -\theta + 2n\pi)$ and $(-r, (\pi - \theta) + 2n\pi)$ to determine if we reach an equivalent equation for some value of n.

For example, consider the graph of $r = 4\sin\frac{\theta}{2}$. From Figure 7-16, the graph is symmetric with respect to the polar axis. However, applying (1) from Table 7-2 results in a different equation, giving us no information about symmetry.

$$r = 4\sin\frac{\theta}{2}$$

$$r = 4\sin\frac{-\theta}{2}$$

$$r = -4\sin\frac{\theta}{2}$$

Different

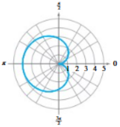


Figure 7-16

Although the graph is actually symmetric with respect to the polar axis, the test is inconclusive.

Insights

In the example on page 761, we apply the odd function property of the sine function, $\sin(-\theta) = -\sin\theta$, to conclude $4\sin\frac{-\theta}{2} = -4\sin\frac{\theta}{2}$.

However, substituting a *different* representation of the point $(r, -\theta)$ such as $(r, 2\pi - \theta)$ may result in an equivalent equation and indicate that symmetry exists.

$$r = 4\sin\frac{\theta}{2}$$

$$r = 4\sin\frac{2\pi - \theta}{2} = 4\sin\left(\pi - \frac{\theta}{2}\right)$$

Replace (r, θ) by $(r, 2\pi - \theta)$ and apply the difference formula for $\sin(a - b) = \sin a\cos b - \cos a\sin b$

$$= 4\left(\sin\pi\cos\frac{\theta}{2} - \cos\pi\sin\frac{\theta}{2}\right) = 4\left[0\cdot\cos\frac{\theta}{2} - (-1)\cdot\sin\frac{\theta}{2}\right] = 4\sin\frac{\theta}{2}$$

Since substituting $(r, 2\pi - \theta)$ into the original equation results in an equivalent equation, we confirm that the graph is symmetric with respect to the polar axis.

Example 2: Using Symmetry to Graph a Polar Equation

Graph $r = 2 - 2\cos\theta$.

Solution:

First test for symmetry.

Test for symmetry with respect to the polar axis:

$$\begin{aligned} r &= 2 - 2\cos\theta \\ r &= 2 - 2\cos(-\theta) \\ &= 2 - 2\cos\theta \end{aligned}$$

Same

Replace (r, θ) by $(r, -\theta)$.

The cosine function is even: $\cos\theta = \cos(-\theta)$.

The graph is symmetric with respect to the polar axis.

Test for symmetry with respect to the line $\theta = \frac{\pi}{2}$:

$$\begin{aligned} r &= 2 - 2\cos\theta \\ r &= 2 - 2\cos(\pi - \theta) \\ &= 2 - 2(-\cos\theta) \\ &= 2 + 2\cos\theta \end{aligned}$$

Different

Replace (r, θ) by $(r, \pi - \theta)$.

$\cos(\pi - \theta) = -\cos\theta$

The test is inconclusive.

Test for symmetry with respect to the pole:

$$\begin{aligned} r &= 2 - 2\cos\theta \\ -r &= 2 - 2\cos\theta \\ r &= -2 + 2\cos\theta \end{aligned}$$

Different

Replace (r, θ) by $(-r, \theta)$.

The test is inconclusive.

Since the graph of $r = 2 - 2\cos\theta$ is symmetric with respect to the polar axis, we can plot several points for $0 \le \theta \le \pi$ (Figure 7-17) and then use symmetry to plot additional points for $\pi < \theta < 2\pi$ (Figure 7-18).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	$2-\sqrt{3}$	$2-\sqrt{2}$	1	2	3	$2+\sqrt{2}$	$2+\sqrt{3}$	4
$r \approx$	0	0.27	0.59	1	2	3	3.41	3.73	4

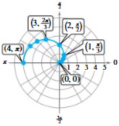


Figure 7-17

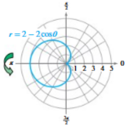


Figure 7-18

The graph of $r = 2 - 2\cos\theta$ is called a limaçon (pronounced "lee-ma-sohn") and is further categorized as a cardioid because it resembles the shape of a heart.

Apply the Skills

2. Graph $r = 3 + 3\sin\theta$.

Another tool in our analysis of polar equations is identifying the maximum value of $|r|$ if it exists. In Example 2, the value of $|r|$ in the equation $r = 2 - 2\cos\theta$ is maximized when $\cos\theta = -1$. Therefore, for $\theta = \pi$,

$$|r| = |2 - 2\cos(\pi)| = |2 - 2(-1)| = 4 \text{ (maximum value of } |r|\text{)}.$$

The maximum value of $|r|$ tells us the distance from the pole of the point or points on a graph that are farthest from the pole. This can be helpful when setting up a viewing window to graph a polar equation by hand or by using a graphing utility.

TECHNOLOGY CONNECTIONS

Graphing a Polar Equation

To graph a polar equation on a graphing utility, the calculator must be in RADIAN mode with the POLAR option highlighted (Figure 7-19).

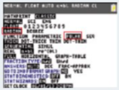
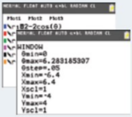


Figure 7-19

It is also important to set appropriate window parameters. To graph $r = 2 - 2\cos\theta$ from Example 2, we know that the maximum value of $|r|$ is 4. We need a window at least as large as $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$. However, we choose $-6.4 \leq x \leq 6.4$ and $-4 \leq y \leq 4$ to accommodate an aspect ratio of width (x) to height (y) of 1.6 to 1 (Figure 7-20).



```
NORMAL FLOAT AUTO a+bi RADIAN CL
Plot1  Plot2  Plot3
\r1=2-2cos(θ)
```

```
NORMAL FLOAT AUTO a+bi RADIAN CL
WINDOW
 θmin=0
 θmax=6.283185307
 θstep=.05
 Xmin=-6.4
 Xmax=6.4
 Xscl=1
 Ymin=-4
 Ymax=4
 Yscl=1
```

Figure 7-20

This ensures a viewing window without distortion (Figure 7-21). Check the user's manual to determine the correct aspect ratio for your device.

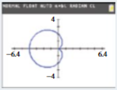


Figure 7-21

The value of θmin and θmax are the endpoints of an interval over which the graph is traced once, in this case $[0, 2\pi)$. The value of θstep is the incremental change in θ between consecutive points. Smaller values of θstep result in a smoother curve and a graph traced more slowly than graphs traced with larger values of θstep.

Example 3: Using Symmetry to Graph a Polar Equation

Graph $r = 2 + 3\sin\theta$.

Solution:

Test for symmetry with respect to the polar axis:

$r = 2 + 3\sin\theta$

$r = 2 + 3\sin(-\theta)$ Replace (r, θ) by $(r, -\theta)$.

$\quad = 2 + 3(-\sin\theta)$

$\quad = 2 - 3\sin\theta$ Different The sine function is odd: $\sin(-\theta) = -\sin\theta$.

The test is inconclusive.

Test for symmetry with respect to the line $\theta = \frac{\pi}{2}$:

$r = 2 + 3\sin\theta$

$r = 2 + 3\sin(\pi - \theta)$ Replace (r, θ) by $(r, \pi - \theta)$.

$= 2 + 3\sin\theta$ Same $\sin(\pi - \theta) = \sin\theta$

The graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

Test for symmetry with respect to the pole:

$r = 2 + 3\sin\theta$

$-r = 2 + 3\sin\theta$ Replace (r, θ) by $(-r, \theta)$.

$r = -2 - 3\sin\theta$ Different

The test is inconclusive.

Since the graph of $r = 2 + 3\sin\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$, we can plot points for $0 \le \theta \le \frac{\pi}{2}$ and $\frac{3\pi}{2} \le \theta \le 2\pi$ (Figure 7-22) and then use symmetry to plot additional points for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ (Figure 7-23).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r	2	$\frac{7}{2}$	$2 + \frac{3\sqrt{2}}{2}$	$2 + \frac{3\sqrt{3}}{2}$	5	-1	$2 - \frac{3\sqrt{3}}{2}$	$2 - \frac{3\sqrt{2}}{2}$	$\frac{1}{2}$	2
$r \approx$	2	3.5	4.1	4.6	5	−1	−0.6	−0.1	0.5	2

Shown in Figure 7-22 in blue

Shown in Figure 7-22 in red

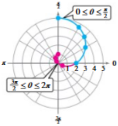


Figure 7-22

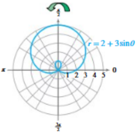


Figure 7-23

The graph of $r = 2 + 3\sin\theta$ (Figure 7-23) is another type of limaçon and is further categorized as a limaçon with a loop.

Apply the Skills

3. Graph $r = 1 + 4\cos\theta$.

In Example 3, the graph passes through the pole twice for θ on the interval $[0, 2\pi)$. The values of θ for which $r = 0$ sometimes tell us where a loop "begins" or "ends" and is helpful to graph a polar equation. We demonstrate this in Example 4.

Example 4: Using Symmetry to Graph a Polar Equation

Graph $r = 4\cos 2\theta$.

Solution:

The graph is at the pole when $r = 0$.

$$r = 4\cos 2\theta = 0 \text{ for } \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$$

Test for symmetry.

Polar axis:

$r = 4\cos 2\theta$

$r = 4\cos 2(-\theta)$

$r = 4\cos 2\theta$ Same

The graph is symmetric with respect to the polar axis.

Line $\theta = \frac{\pi}{2}$:

$r = 4\cos 2\theta$

$r = 4\cos[2(\pi - \theta)]$

$r = 4\cos(2\pi - 2\theta)$

$r = 4\cos 2\theta$ Same

The graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

Pole:

$r = 4\cos 2\theta$

$-r = 4\cos 2\theta$

$r = -4\cos 2\theta$ Different

The test for symmetry about the pole is inconclusive.

Since the graph of $r = 4\cos 2\theta$ is symmetric with respect to both the polar axis and the line $\theta = \frac{\pi}{2}$, we can plot points for $0 \le \theta \le \frac{\pi}{2}$ (Figure 7-24) and then use symmetry to plot additional points.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	4	2	0	-2	-4

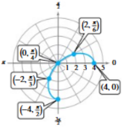


Figure 7-24

Applying symmetry with respect to the polar axis, we can sketch a mirror image of the part of the curve above the polar axis. Applying symmetry with respect to the line $\theta = \frac{\pi}{2}$, we can sketch a mirror image of the curve to the left of $\theta = \frac{\pi}{2}$ (Figure 7-25).

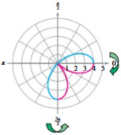


Figure 7-25

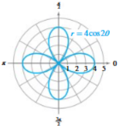


Figure 7-26

Applying symmetry with respect to the polar axis and line $\theta = \frac{\pi}{2}$ results in a graph of a "rose" with four petals (Figure 7-26).

Insights

From Figure 7-26, the graph of $r = 4\cos 2\theta$ is traced through the pole four times for $0 \le \theta < 2\pi$. This is consistent with our analysis that $r = 4\cos 2\theta = 0$ for $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, and so on.

Among other applications in calculus, we will find the area enclosed by polar graphs. Identifying symmetry and points on a graph that pass through the pole are important parts of this analysis.

Apply the Skills

4. Graph $r = 3\sin 2\theta$.

Learn: Categorize Polar Equations and Their Graphs

Throughout the text, we have categorized a number of different types of equations in a rectangular coordinate system along with their graphs. For example, the graph of a linear equation in two variables $Ax + By = C$ is a line, the graph of a quadratic equation $y = ax^2 + bx + c$ is a parabola, and so on. We now present a library of polar equations and their associated graphs (Table 7-3).

TABLE 7-3 Common Polar Equations and Their Graphs

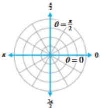
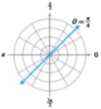
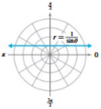
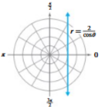
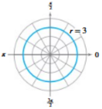
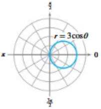
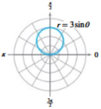
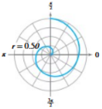
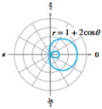
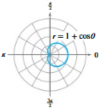
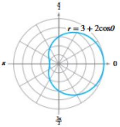
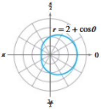
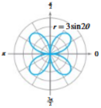
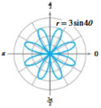
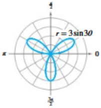
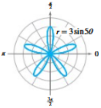
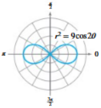
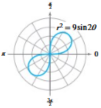
Graph Type	Common Polar Equations and Their Graphs			
Lines	 $\theta = 0$ *x*-axis $\theta = \frac{\pi}{2}$ *y*-axis	 $\theta = a$ Line through origin	 $r = \frac{a}{\sin\theta}$ Horizontal line ($y = a$)	 $r = \frac{a}{\cos\theta}$ Vertical line ($x = a$)
Circles and Spirals	 $r = a$ Circle: center at the pole, radius $\lvert a \rvert$	 $r = a\cos\theta$ Circle: center on polar axis, radius $\left\lvert \frac{a}{2} \right\rvert$	 $r = a\sin\theta$ Circle: center on $\theta = \frac{\pi}{2}$, radius $\left\lvert \frac{a}{2} \right\rvert$	 $r = a\theta$ Spiral

TABLE 7-3 continued

Graph Type	Common Polar Equations and Their Graphs			
Limaçons $r = a \pm b\cos\theta$ $r = a \pm b\sin\theta$ $(a > 0, b > 0)$ The orientation and symmetry depend on the trigonometric function and the sign of the sine or cosine term.	 $a < b$ limaçon with a loop	 $a = b$ cardioid	 $b < a < 2b$ dimpled limaçon	 $a \geq 2b$ convex limaçon
Roses $r = a\cos n\theta$ $r = a\sin n\theta$ n petals if n is odd $2n$ petals if n is even	 $r = a\sin 2\theta$ 4-petal rose	 $r = a\sin 4\theta$ 8-petal rose	 $r = a\sin 3\theta$ 3-petal rose	 $r = a\sin 5\theta$ 5-petal rose
Lemniscates $r^2 = a^2\cos 2\theta$ $r^2 = a^2\sin 2\theta$ Figure-eight symmetric to the polar axis (cosine) or symmetric to the line $\theta = \frac{\pi}{4}$(sine)	 $r^2 = a^2\cos 2\theta$ figure-eight	 $r^2 = a^2\sin 2\theta$ figure-eight		

Example 5: Identifying and Graphing a Polar Equation

Graph $r^2 = 16\sin 2\theta$.

Solution:

- The graph of $r^2 = 16\sin 2\theta$ is a lemniscate of the form $r^2 = a^2\sin 2\theta$. It is a figure-eight curve symmetric with respect to the line $\theta = \frac{\pi}{4}$ ($y = x$ in rectangular coordinates) and to the pole.
- The graph is at the pole when $r = 0$. We have $\pm\sqrt{16\sin 2\theta} = 0$ for $\theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$.
- The maximum value of $|r|$ is 4 and occurs when $\sin 2\theta = 1$ or $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$.

The value of $16\sin 2\theta$ is negative for $\frac{\pi}{2} < \theta < \pi$, and consequently, the equation has no solution on this interval (r^2 cannot be negative). Therefore, we can plot points for $0 \le \theta \le \frac{\pi}{2}$ and apply symmetry to graph the curve (Figure 7-27).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	$\pm 2\sqrt{2\sqrt{3}}$	± 4	$\pm 2\sqrt{2\sqrt{3}}$	0
$r \approx$	0	± 3.7	± 4	± 3.7	0

Since a lemniscate of the form $r^2 = a^2 \sin 2\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{4}$, we could have limited the table to $0 \le \theta \le \frac{\pi}{4}$ and then used symmetry to complete the curve.

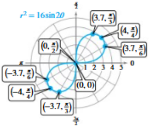


Figure 7-27

Insights

The graph of a lemniscate looks like a propeller on an airplane. The two loops meet at the pole for values of θ for which $r = 0$.

Apply the Skills

5. Graph $r^2 = 25\cos 2\theta$.

Example 6: Identifying and Graphing a Polar Equation

Graph $r = -0.4\theta$ for $\theta \ge 0$.

Solution:

- The graph of $r = -0.4\theta$ is a spiral and is not symmetric with respect to the polar axis, line $\theta = \frac{\pi}{2}$, or the pole.
- The value of $|r|$ increases for increasing values of $|\theta|$ and has no upper bound.
- The graph is at the pole only for $\theta = 0$.

To see several revolutions of the spiral, we take values of θ and r on the interval $0 \le \theta \le 4\pi$. The graph is shown in Figure 7-28.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$r \approx$	0	−0.63	−1.26	−1.88	−2.51	−3.14	−3.77	−4.40	−5.03

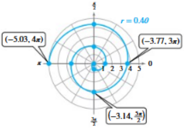


Figure 7-28

Apply the Skills

6. Graph $r = 0.8\theta$ for $\theta \geq 0$.

Point of Interest

The logarithmic spiral, represented by $r = ae^{b\theta}$, was referred to by the Swiss mathematician Jacob Bernoulli (1654–1705) as *spira mirabilis* (marvelous spiral). Bernoulli was so fascinated with the properties of the logarithmic spiral that he asked to have it inscribed on his tombstone. Ironically, the spiral found on Bernoulli's tomb was carved incorrectly. It is *not* a logarithmic spiral, but rather an Archimedean spiral given by $r = a\theta$.

Logarithmic spiral

Archimedean spiral

Seth Joel Photography/Cultura/Getty Images; Joachim Belaieff/Maskot/Getty Images

Practice 7-2

Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, determine whether the graph of the equation is symmetric with respect to the *x*-axis, *y*-axis, origin, or none of these.

R.1. $y = -|x| - 6$

R.2. $x = y^2 + 3$

R.3. $x^2 + y^2 = 7$

R.4. $y = x^2 + 2x + 5$

R.5. Solve $2\sin 2x = 1$ on the interval $(0, \pi]$.

R.6. Solve $\cos 3x = 1$ on the interval $[0, 2\pi)$.

Concept Connections

1. If replacing (r, θ) by $(r, -\theta)$ in a polar equation results in an equivalent equation, then the graph of the equation is symmetric with respect to the __________.
2. If replacing (r, θ) by $(r, \pi - \theta)$ in a polar equation results in an equivalent equation, then the graph of the equation is symmetric with respect to the __________.
3. If replacing (r, θ) by $(-r, \theta)$ in a polar equation results in an equivalent equation, then the graph of the equation is symmetric with respect to the __________.
4. The graph of a polar equation of the form $r = a\cos\theta$ is symmetric with respect to the __________, whereas the graph of a polar equation of the form $r = a\sin\theta$ is symmetric with respect to the __________.
5. Determining the ordered pairs for which $r = 0$ in a polar equation gives what information?
6. What is the maximum value of r for the equation $r = a\sin\theta + 2$? What is the minimum value of r?

Learn: Graph Polar Equations by Plotting Points

For Exercises 7–14, graph the polar equation. (See Example 1, p. 759)

7. $r = 3$

8. $r = 4$

9. $\theta = \frac{3\pi}{4}$

10. $\theta = \frac{\pi}{4}$

11. $r = 4\cos\theta$

12. $r = -4\sin\theta$

13. $r\sin\theta = -2$

14. $r\cos\theta = 3$

Learn: Test Polar Equations for Symmetry

For Exercises 15–22, determine whether the tests for symmetry in Table 7-2 detect symmetry with respect to

a. The polar axis. Replace (r, θ) by $(r, -\theta)$.

b. The line $\theta = \frac{\pi}{2}$. Replace (r, θ) by $(r, \pi - \theta)$.

c. The pole. Replace (r, θ) by $(-r, \theta)$.

Otherwise, indicate that the test is inconclusive.

15. $r^2 = 16\cos 2\theta$

16. $r = 5\cos 4\theta$

17. $r = 3\cos 3\theta$

18. $r = 6 - 4\cos\theta$

19. $r = \frac{3}{4 + \sin\theta}$

20. $r = 3 + 2\sin\theta$

21. $r = 2\theta$

22. $r^2 = 9\sin 2\theta$

For Exercises 23–30, use symmetry to graph the polar curve and identify the type of curve. (See Examples 2–4, pp. 762–766)

23. $r = -2 - 2\sin\theta$

24. $r = 3 + 3\cos\theta$

25. $r = 2 - 4\cos\theta$

26. $r = 1 + 7\sin\theta$

27. $r = 3\cos 5\theta$

28. $r = 8\sin 4\theta$

29. $r = 3 - 2\sin\theta$

30. $r = -3 + 2\cos\theta$

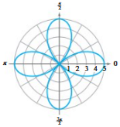
Learn: Categorize Polar Equations and Their Graphs

For Exercises 31–40,

a. Identify the type of polar curve from among those listed in Table 7-3.

b. Write a polar equation that represents the given graph.

31.



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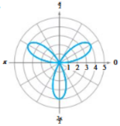

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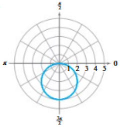

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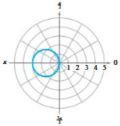

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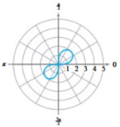

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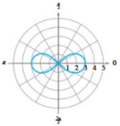

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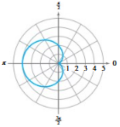

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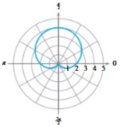

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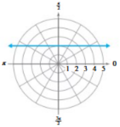

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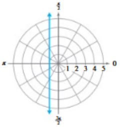


41. Identify the values of θ on the interval $[0, 2\pi)$ for which there are no points on the graph of $r^2 = 100\cos\theta$.

42. Identify values of θ on the interval $[0, 2\pi)$ for which there are no points on the graph of $r^2 = 49\sin\theta$.

For Exercises 43–46, graph the polar curve. (See Examples 5–6, pp. 768–769)

43. $r^2 = 81\sin 2\theta$

44. $r^2 = 36\cos 2\theta$

45. $r = 0.5\theta$ for $\theta \geq 0$

46. $r = -0.6\theta$ for $\theta \geq 0$

For Exercises 47–52, for the purpose of determining a suitable viewing window for the graph of the equation,

a. Find the maximum value of $|r|$.

b. Experiment with a graphing utility to find an interval for θ over which the graph is traced only once.

47. $r = 6\cos\theta$

48. $r = -4\sin\theta$

49. $r = 3 - \sin\theta$

50. $r = 5 + 2\cos\theta$

51. $r = 3\cos\frac{\theta}{2}$

52. $r = 7\sin\frac{\theta}{4}$

Mixed Exercises

For Exercises 53–56, use a graphing utility or construct a table of values to match each polar equation with a graph from the choices a–d.

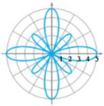
53. $r = 4 - \sin 4\theta$

54. $r = 1 - 4\sin 4\theta$

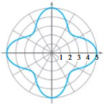
55. $r = 1 + 4\cos 4\theta$

56. $r = 4 + \cos 4\theta$

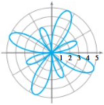
a.



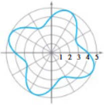
b.



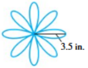
c.



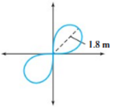
d.



57. A quilter wants to make a design in the shape of a daisy where the "petals" are 3.5 in. long and plans to use a computer to print a graph of the outline of the figure. Determine an equation that could be used.



58. The outline of a propeller on an airplane is in the shape of a lemniscate. Write an equation that could be used for the outline.



59. a. Given $r = 8\sin 4\theta$, determine the set of values of θ for which $r = 0$ on the interval $[0, 2\pi)$.

b. Use a graphing utility to graph $r = 8\sin 4\theta$ on the given intervals.

i. $0 \le \theta \le \frac{\pi}{2}$ **ii.** $\frac{\pi}{2} \le \theta \le \pi$

iii. $\pi \le \theta \le \frac{3\pi}{2}$ **iv.** $\frac{3\pi}{2} \le \theta \le 2\pi$

60. a. Given $r = 6\sin 3\theta$, determine the set of values of θ for which $r = 0$ on the interval $[0, \pi)$.

b. Use a graphing utility to graph $r = 6\sin 3\theta$ on the given intervals.

i. $0 \le \theta \le \frac{\pi}{3}$ **ii.** $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$

iii. $\frac{2\pi}{3} \le \theta \le \pi$

61. If $r = f(\sin\theta)$, that is, if r is expressed as a function of $\sin\theta$, what type of symmetry does the graph have? Explain.

62. If $r = f(\cos\theta)$, that is, if r is expressed as a function of $\cos\theta$, what type of symmetry does the graph have? Explain.

For Exercises 63–66, use the results of Exercises 61–62 to determine by inspection whether the graph of the given equation is symmetric with respect to the polar axis or to the line $\theta = \frac{\pi}{2}$.

63. $r = 2\sin\theta + 3$

64. $r = \sin^2\theta + \sin\theta - 1$

65. $r = 2\cos^2\theta - 3\cos\theta + 1$

66. $r = 1 - 5\cos\theta$

67. Given $r = 1 + \sec\theta$ (*conchoid*),

a. For what value(s) of θ on the interval $[0, 2\pi)$ is r undefined?

b. Use a graphing utility to graph $r = 1 + \sec\theta$ on the interval $[0, 2\pi)$.

c. Discuss the behavior of the graph for values of θ near those found in part (a).

68. Given $r = \frac{2}{\theta}$ (*hyperbolic spiral*),

a. For what value of θ is r undefined?

b. Use a graphing utility to graph $r = \frac{2}{\theta}$ on the interval $[-4\pi, 4\pi]$.

c. Discuss the behavior of the graph for values of θ near 0.

69. Use a graphing utility to graph the equations on the same viewing window for $0 \le \theta \le 2\pi$.

a. $r = 6\sin 2\theta$

b. $r = 6\sin\left[2\left(\theta + \frac{\pi}{4}\right)\right]$

c. How do the graphs of $r = 6\sin 2\theta$ and $r = 6\sin\left[2\left(\theta + \frac{\pi}{4}\right)\right]$ differ?

70. Use a graphing utility to graph the equations on the same viewing window for $0 \le \theta \le 2\pi$.

a. $r = 2 + 4\cos\theta$

b. $r = 2 + 4\cos\left(\theta - \frac{\pi}{4}\right)$

c. $r = 2 + 4\cos\left(\theta - \frac{\pi}{2}\right)$

d. Based on the results of parts (a)–(c), make a hypothesis about the effect of α on the graph of $r = f(\theta - \alpha)$.

71. A logarithmic spiral is represented by $r = ae^{b\theta}$.

a. Graph $r = 0.5e^{0.1\theta}$ and $r = 0.5e^{-0.1\theta}$ over the interval $0 \le \theta \le 4\pi$.

b. What appears to be the difference between the graphs for $b = 0.1$ and $b = -0.1$?

72. An Archimedean spiral is represented by $r = a\theta$.

a. Graph $r = 0.5\theta$ and $r = -0.5\theta$ over the interval $0 \le \theta \le 8\pi$ and use a ZOOM square viewing window.

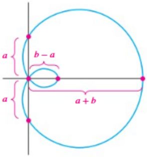
b. Archimedean spirals have the property that a ray through the origin will intersect successive turns of the spiral at a constant distance of $2\pi|a|$. What is the distance between each point of the spiral $r = 0.5\theta$ along the line $\theta = \frac{\pi}{2}$?

In Exercises 73–78, consider the graphs of polar equations of the following forms where a and b are positive real numbers.

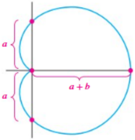
$r = a + b\cos\theta$ **(loop is to the right)** $r = a - b\cos\theta$ **(loop is to the left)**
$r = a + b\sin\theta$ **(loop is upward)** $r = a - b\sin\theta$ **(loop is downward)**

Depending on the values of a and b, the graph will be a limaçon with a loop, a cardioid, or a limaçon with a dimple. The sine or cosine term tells us whether the graph is symmetric to the vertical axis or the horizontal axis, respectively. With a positive cosine term, the loop is to the right, and with a negative cosine term, the loop is to the left. With a positive sine term, the loop is upward, and with a negative sine term, the loop is downward. In the figures below, the graphs were generated using a positive cosine term. The curves illustrate the relationship between a and b in the function.

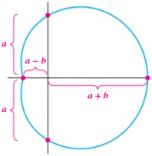
Limaçon with a loop
$a < b$ or $\frac{a}{b} < 1$



Cardioid
$a = b$ or $\frac{a}{b} = 1$

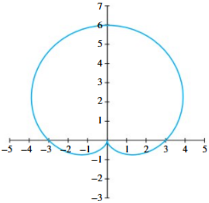


Limaçon with a dimple
$b < a < 2b$ or $1 < \frac{a}{b} < 2$

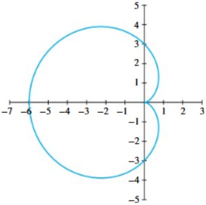


For Exercises 73–78, identify the type of curve (limaçon with a loop, cardioid, or limaçon with a dimple), and write a polar equation that might represent the curve. (Answers may vary.)

73.



74.



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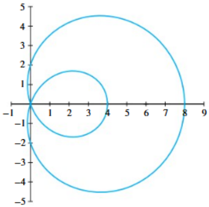

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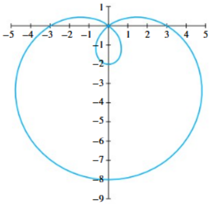

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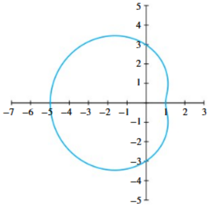

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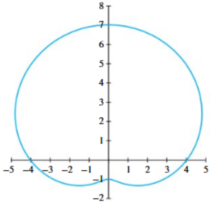


Write About It

79. Graph the limaçons given by the equation $r = a + b\cos\theta$ for $a = 1, 2, 3, 4, 5$, and 6, and $b = 3$. Comment on the effect of $\frac{a}{b}$ on the graph.

80. Graph the limaçons given by the equation $r = a + b\cos\theta$ for $a = 1, 2$, and 3 and $b = 4$. Comment on the effect of $\frac{a}{b}$ on the graph.

Expanding Your Skills

81. Find the values of θ on the interval $[0, 2\pi)$ for which the graphs of $r = 1 + \sin\theta$ and $r = 1 - \sin\theta$ intersect.

82. Find the values of θ on the interval $[0, 2\pi)$ for which the graphs of $r = \cos\theta$ and $r = 1 - \cos\theta$ intersect.

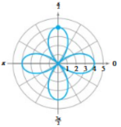
83. Given $r = 6\cos\frac{\theta}{2}$,

- **a.** Replace (r, θ) by $(r, \pi - \theta)$ in the equation. Does this show that the graph of the equation is symmetric with respect to the line $\theta = \frac{\pi}{2}$?
- **b.** The ordered pair $(-r, 2\pi - \theta)$ is another representation of the point $(r, \pi - \theta)$. Replace (r, θ) by $(-r, 2\pi - \theta)$ in the equation. Does this show that the graph of the equation is symmetric with respect to the line $\theta = \frac{\pi}{2}$?
- **c.** What other type of symmetry (if any) does the graph of the equation have?
- **d.** Use a graphing utility to graph the equation on a graphing utility for $0 \le \theta \le 4\pi$.

84. Given $r = 4\sin\frac{\theta}{4}$,

- **a.** Replace (r, θ) by $(r, -\theta)$ in the equation. Does this show that the graph of the equation is symmetric with respect to the polar axis?
- **b.** The ordered pair $(r, 4\pi - \theta)$ is another representation of the point $(r, -\theta)$. Replace (r, θ) by $(r, 4\pi - \theta)$ in the equation. Does this show that the graph of the equation is symmetric with respect to the polar axis?
- **c.** What other type of symmetry (if any) does the graph of the equation have?
- **d.** Use a graphing utility to graph the equation on a graphing utility for $0 \le \theta \le 8\pi$.

85. The graph of $r = 4\cos 2\theta$ is shown.



a. Is $\left(4, \frac{\pi}{2}\right)$ a solution to the equation?

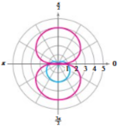
b. The point $\left(4, \frac{\pi}{2}\right)$ appears to lie on the curve.

How do you explain this?

86. Given the equations of two curves in rectangular coordinates, we can find the points of intersection by solving the equations simultaneously. In polar coordinates, solving systems of equations simultaneously may reveal some intersection points without finding others. For example, the graphs of $r = 1 - \sin\theta$ (shown in blue) and $r^2 = 16\sin\theta$ (shown in red) on the interval $0 \le \theta < 2\pi$ intersect in four points.

$r = 1 - \sin\theta$

$r^2 = 16\sin\theta$



a. Solve the system simultaneously for r. (*Hint*: From the second equation, $\sin\theta = \frac{r^2}{16}$. Substitute $\frac{r^2}{16}$ for $\sin\theta$ in the first equation and solve for r.)

b. From the graph, are both values of r reasonable?

c. Find the points of intersection corresponding to the reasonable value of r found in part (a). Round r and θ to 3 decimal places.

d. Do the results in part (c) give all points of intersection between the two curves?

Problem Recognition Exercises

Comparing Equations in Polar and Rectangular Form

For Exercises 1–8,

a. Match the polar or rectangular equation on the left with the letter of the rectangular or polar equation on the right.

b. Describe the graph.

c. Determine if the graph is symmetric with respect to the origin (pole), x-axis (polar axis), or y-axis $\left(\text{line } \theta = \frac{\pi}{2}\right)$.

1. $x^2 + y^2 = 144$

2. $x = -3$

3. $r = \frac{4}{2\cos\theta + \sin\theta}$

4. $r = -6\cos\theta$

5. $y = -\sqrt{3}x$

6. $x^2 + (y + 5)^2 = 25$

7. $r = 5\csc\theta$

8. $r = 2\sin\theta + 2\cos\theta$

A. $2x + y = 4$

B. $r = -10\sin\theta$

C. $r = 12$

D. $\theta = \frac{2\pi}{3}$

E. $y = 5$

F. $r = -3\sec\theta$

G. $(x - 1)^2 + (y - 1)^2 = 2$

H. $(x + 3)^2 + y^2 = 9$

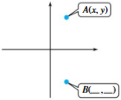
For Exercises 9–10, use a graphing utility to graph the polar equation, and describe the graph in a rectangular coordinate system.

9. $r = \frac{3}{1 - \sin\theta}$

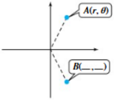
10. $r = \frac{2}{1 + \cos\theta}$

For Exercises 11–13, points *A* and *B* make up a graph that has an indicated type of symmetry for all polar representations of point *B*. Fill in an ordered pair for point *B* that would satisfy the condition of symmetry. Note that for ordered pairs in polar form, there is more than one correct answer.

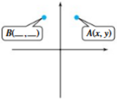
11. a. Symmetry: *x*-axis



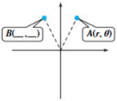
b. Symmetry: polar axis



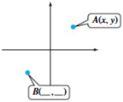
12. a. Symmetry: *y*-axis



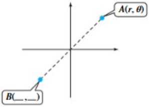
b. Symmetry: Line $\theta = \frac{\pi}{2}$



13. a. Symmetry: Origin



b. Symmetry: Pole



Go online for more practice problems.

Lesson 7-3

Complex Numbers in Polar Form

What Will You Learn?

After completing this lesson, you should be able to:

- Plot Complex Numbers in the Complex Plane
- Write the Polar Form of a Complex Number
- Compute Products and Quotients of Complex Numbers in Polar Form
- Compute Powers of Complex Numbers (De Moivre's Theorem)
- Find the *n*th Roots of a Complex Number

Learn: Plot Complex Numbers in the Complex Plane

A real number can be represented as a point on the real number line. Since a complex number $a + bi$ has two parts, the real part a and the imaginary part b, two axes are required to represent $a + bi$ graphically. The horizontal axis (called the **real axis**) represents the real part of a complex number, and the vertical axis (called the **imaginary axis**) represents the imaginary part. Together, the real and imaginary axes form the **complex plane**. A complex number $z = a + bi$ is written as an ordered pair (a, b) and can then be graphed as such in the complex plane (Figure 7-29).

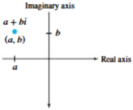


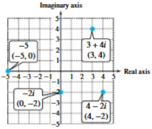
Figure 7-29 Complex Plane

Example 1: Graphing Complex Numbers

Graph the complex numbers in the complex plane.

a. $3 + 4i$ **b.** $4 - 2i$ **c.** -5 **d.** $-2i$

Solution:



a. $3 + 4i$ is represented by the point $(3, 4)$.

b. $4 - 2i$ is represented by the point $(4, -2)$.

c. $-5 = -5 + 0i$ is represented by the point $(-5, 0)$ on the real axis.

d. $-2i = 0 - 2i$ is represented by the point $(0, -2)$ on the imaginary axis.

Insights

Complex numbers are often denoted by the variable z such as $z = 3 + 4i$.

Apply the Skills

1. Graph the complex numbers in the complex plane.

   a. $-1 + 3i$ **b.** $-4 - 5i$ **c.** 3 **d.** $4i$

Geometrically, the absolute value of a real number is its distance from zero on the real number line. Given a complex number $z = a + bi$, the absolute value of z, denoted $|z|$, is the distance between (a, b) and $(0, 0)$ in the complex plane (Figure 7-30). The absolute value of a complex number is more commonly called the *modulus* of the number.

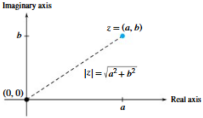


Figure 7-30

Absolute Value of a Complex Number

If $z = a + bi$ is a complex number, then the absolute value of z (also called the **modulus** or **magnitude** of z) is given by

$$|z| = \sqrt{a^2 + b^2}.$$

Insights

Notice that for a complex number of the form $a + 0i$, the absolute value is $\sqrt{a^2 + 0^2} = |a|$. This is consistent with the absolute value of a real number a.

Example 2: Finding the Modulus (Absolute Value) of a Complex Number

Find the modulus.

a. $z = 3 - 4i$ **b.** $z = -8 + 2i$

Solution:

a. $|z| = |3 - 4i| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

b. $|z| = |-8 + 2i| = \sqrt{(-8)^2 + (2)^2} = \sqrt{64 + 4} = \sqrt{68} = 2\sqrt{17}$

Apply the Skills

2. Find the modulus.

a. $z = -5 - 12i$ **b.** $z = 4 + 6i$

Learn: Write the Polar Form of a Complex Number

The complex number $z = a + bi$ is said to be in rectangular (or Cartesian) form because a and b are the coordinates of the point (a, b) in the rectangular coordinate system formed by the complex plane. For a and b not both zero, let r represent the length of the line segment between (a, b) and $(0, 0)$. Therefore, $r = |z| = \sqrt{a^2 + b^2} \geq 0$ (Figure 7-31). Taking angle θ to be in standard position with terminal side through (a, b), we have $a = r\cos\theta$ and $b = r\sin\theta$.

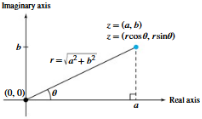


Figure 7-31

$$
\begin{aligned}
\text{Therefore, } z &= a + bi \\
&= r\cos\theta + i \cdot r\sin\theta \\
&= r(\cos\theta + i\sin\theta)
\end{aligned}
$$

This is called the polar form (or trigonometric form) of a complex number.

Polar Form of a Complex Number

The **polar form** (or **trigonometric form**) of a complex number $z = a + bi$ is

$$z = r(\cos\theta + i\sin\theta)$$

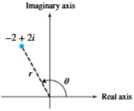
where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = \frac{b}{a}$ $(a \neq 0)$. The value of r is the modulus of z, and we say that θ is an argument of z. There are infinitely many choices for θ, but normally, θ is taken on the interval $0 \leq \theta < 2\pi$ or $0° \leq \theta < 360°$.

Example 3: Convert a Complex Number from Rectangular Form to Polar Form

Write the complex number in polar form.

a. $z = -2 + 2i$ **b.** $z = 8 - 15i$

Solution:



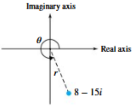
a. The point $z = -2 + 2i$ is in Quadrant II.

$r = \sqrt{a^2 + b^2} = \sqrt{(-2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}$

$\tan\theta = \frac{b}{a} = \frac{2}{-2} = -1$

Taking θ in Quadrant II on the interval $0 \le \theta < 2\pi$, we have $\theta = \frac{3\pi}{4}$.

Polar form: $z = \overset{\text{Modulus}}{2\sqrt{2}}\left(\cos\overset{\text{Argument}}{\frac{3\pi}{4}} + i\sin\frac{3\pi}{4}\right)$



b. The point $z = 8 - 15i$ is in Quadrant IV.

$r = \sqrt{a^2 + b^2} = \sqrt{(8)^2 + (15)^2} = \sqrt{289} = 17$

$\tan\theta = \frac{-15}{8} \Rightarrow \theta = \tan^{-1}\left(-\frac{15}{8}\right)$

Taking θ in Quadrant IV on the interval $0 \le \theta \le 2\pi$, we have $\theta = 2\pi + \tan^{-1}\left(-\frac{15}{8}\right)$.

Polar form: $z = 17\left[\cos\left(2\pi + \tan^{-1}\left(-\frac{15}{8}\right)\right) + i\sin\left(2\pi + \tan^{-1}\left(-\frac{15}{8}\right)\right)\right]$

$z \approx 17(\cos 5.2 + i\sin 5.2)$

Apply the Skills

3. Write the complex number in polar form.

a. $z = 1 - i$

b. $z = 3 + 4i$

Point of Interest

The complex number $z = r(\cos\theta + i\sin\theta)$ is sometimes shortened to $z = r\,\text{cis}\,\theta$, where “cis” stands for “cosine *i* sine.” For example,

$$z = 5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \text{ would be written as } z = 5\,\text{cis}\,\frac{\pi}{4}.$$

Another form in which a complex number can be written makes use of Euler’s formula. Swiss mathematician Leonhard Euler (pronounced “Oy-ler”) (1707–1783), discovered a relationship between the exponential function of a complex number and the trigonometric functions. For a real number θ, Euler’s formula states that $e^{i\theta} = \cos\theta + i\sin\theta$. So, the complex number

$$z = 5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \text{ can be written as } z = 5e^{\frac{\pi}{4}i}.$$

In Example 4, we perform the simpler task of writing a complex number in rectangular form.

Example 4: Convert a Complex Number from Polar Form to Rectangular Form

Write $z = 4(\cos 120° + i \sin 120°)$ in rectangular form, $a + bi$.

Solution:

$$z = 4(\cos 120° + i \sin 120°) = 4\left(-\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}\right) = -2 + 2\sqrt{3}i$$

Apply the Skills

4. Write $z = 6(\cos 150° + i \sin 150°)$ in rectangular form.

Learn: Compute Products and Quotients of Complex Numbers in Polar Form

In Chapter R, we learned how to add, subtract, multiply, and divide complex numbers in rectangular form. We now present the procedure to multiply and divide complex numbers in polar form.

Products and Quotients of Complex Numbers in Polar Form

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad (1)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]\,(z_2 \neq 0) \qquad (2)$$

In words, to multiply two complex numbers in polar form, multiply their moduli and add their arguments. To divide two complex numbers in polar form, divide their moduli and subtract their arguments. *Note*: The word "moduli" is the plural of the word "modulus."

To prove (1), we have

Recall that $i^2 = -1$.

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

Apply the sum formulas for sine and cosine.

The proof of (2) is similar and left as an exercise (Exercise 85).

Example 5: Multiplying Complex Numbers in Polar Form

Given $z_1 = 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ and $z_2 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, find z_1z_2 and write the product in polar form.

Solution:

$$z_1z_2 = 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) \cdot 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= 4 \cdot 2\left[\cos\left(\frac{4\pi}{3} + \frac{\pi}{2}\right) + i\sin\left(\frac{4\pi}{2} + \frac{\pi}{2}\right)\right]$$

Multiply the moduli and add the arguments. Note that $\frac{4\pi}{3} + \frac{\pi}{2} = \frac{8\pi}{6} + \frac{3\pi}{6} = \frac{11\pi}{6}$

$$= 8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) \text{ (polar form)}$$

Apply the Skills

5. Given $z_1 = 3\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$ and $z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, find z_1z_2 and write the product in polar form.

Insights

In Example 5, the product is left in polar form. The corresponding rectangular form is:

$$8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$
$$= 8\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$
$$= 4\sqrt{3} - 4i$$

In Exercise 83, we verify the result of Example 5 by converting the complex numbers to rectangular form and then multiplying.

Example 6: Dividing Complex Numbers in Polar Form

Given $z_1 = 4(\cos 15^\circ + i\sin 15^\circ)$ and $z_2 = 12(\cos 285^\circ + i\sin 285^\circ)$, find $\frac{z_1}{z_2}$ and write the quotient in polar form.

Solution:

$$\frac{z_1}{z_2} = \frac{4(\cos 15^\circ + i\sin 15^\circ)}{12(\cos 285^\circ + i\sin 285^\circ)}$$

$$= \frac{1}{3}[\cos(15^\circ - 285^\circ) + i\sin(15^\circ - 285^\circ)]$$

Divide the moduli and subtract the arguments.

$$= \frac{1}{3}[\cos(-270^\circ) + i\sin(-270^\circ)]$$

$$= \frac{1}{3}[\cos(90^\circ) + i\sin(90^\circ)]$$

90° is coterminal to -270° and on the interval $0^\circ \le \theta < 360^\circ$.

Apply the Skills

6. Given $z_1 = 18(\cos 75° + i\sin 75°)$ and $z_2 = 2(\cos 255° + i\sin 255°)$, find $\frac{z_1}{z_2}$ and write the quotient in polar form.

Learn: Compute Powers of Complex Numbers (De Moivre's Theorem)

Given $z = a + bi$, the value of z^n, where n is a positive integer, can be computed by using repeated multiplication of the base z in its polar form.

Let $z = r(\cos\theta + i\sin\theta)$. Then,

$$z^2 = r(\cos\theta + i\sin\theta) \cdot r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta)$$

$$z^3 = r^2(\cos 2\theta + i\sin 2\theta) \cdot r(\cos\theta + i\sin\theta) = r^3(\cos 3\theta + i\sin 3\theta)$$

$$z^4 = r^3(\cos 3\theta + i\sin 3\theta) \cdot r(\cos\theta + i\sin\theta) = r^4(\cos 4\theta + i\sin 4\theta)$$

$$\vdots$$

$$z^n = r^{n-1}[\cos(n-1)\theta + i\sin(n-1)\theta] \cdot r(\cos\theta + i\sin\theta) = r^n(\cos n\theta + i\sin n\theta).$$

This pattern involving integer powers of a complex number is known as De Moivre's theorem named after the French mathematician Abraham de Moivre (pronounced "duh mwah-vruh") (1667–1754).

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

In words, to raise z to a positive integer power n, raise the modulus to the nth power and multiply the argument by n.

Example 7: Raising a Complex Number to a Power

Compute $[5(\cos 75° + i\sin 75°)]^4$ and write the result in rectangular form $a + bi$.

Solution:

$$[5(\cos 75° + i\sin 75°)]^4 = 5^4[\cos(4 \cdot 75°) + i\sin(4 \cdot 75°)]$$

$= 625\cos 300° + i\sin 300°$ Polar form

$= 625\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$ Evaluate $\cos 300°$ and $\sin 300°$ to convert to rectangular form.

$= \frac{625}{2} - \frac{625\sqrt{3}}{2}i$ Rectangular form

Good Practices

Recall that $a + (-bi)$ is usually written as $a - bi$. Likewise, a real number or pure imaginary number is usually simplified. For example, $a + 0i$ is usually written as a, and $0 + bi$ is usually written as bi.

Apply the Skills

7. Compute $[4(\cos 70° + i\sin 70°)]^3$ and write the result in rectangular form $a + bi$.

Example 8: Raising a Complex Number to a Power

Compute $(-3 + 3i)^6$ and write the result in rectangular form $a + bi$.

Solution:

Computing the product in rectangular form would be very tedious to carry out:

$$(-3 + 3i)^6 = (-3 + 3i)(-3 + 3i)(-3 + 3i)(-3 + 3i)(-3 + 3i)(-3 + 3i)$$

Instead, we will write $-3 + 3i$ in polar form and then apply De Moivre's theorem.

$z = -3 + 3i = (-3, 3)$ is in Quadrant II.

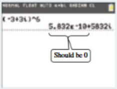
$r = \sqrt{(-3)^2 + (3)^2} = \sqrt{18} = 3\sqrt{2}$

$\tan\theta = \frac{3}{-3} = -1 \Rightarrow \theta = \frac{3\pi}{4}$ for θ in Quadrant II.

$$\begin{aligned} \text{So, } (-3 + 3i)^6 &= \left[(3\sqrt{2})\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^6 \\ &= (3\sqrt{2})^6\left[\cos\left(6 \cdot \frac{3\pi}{4}\right) + i\sin\left(6 \cdot \frac{3\pi}{4}\right)\right] \\ &= 3^6 \cdot 2^3\left(\cos\frac{9\pi}{2} + i\sin\frac{9\pi}{2}\right) \\ &= 5832(0 + i) \\ &= 5832i \end{aligned}$$

TECHNOLOGY CONNECTIONS

The calculation from Example 8 performed on a graphing utility results in a small round-off error. The real part is displayed as 5.832×10^{-10} when in fact it should be zero.



```
NORMAL FLOAT AUTO a+bi RADIAN CL
(-3+3i)^6
                 5.832E-10+5832i
```

Should be 0

Apply the Skills

8. Compute $(-2 - 2i)^4$ and write the result in rectangular form $a + bi$.

Learn: Find the *n*th Roots of a Complex Number

For a positive integer $n \geq 2$, an *n*th root of a complex number z is a complex number w such that $w^n = z$. In Example 7, we found that

$$[5(\cos 75° + i\sin 75°)]^4 = 625(\cos 300° + i\sin 300°)$$

This means that $5(\cos 75° + i\sin 75°)$ is a fourth root of $625(\cos 300° + i\sin 300°)$. Likewise, in Example 8, we found that $(-3 + 3i)^6 = 5832i$. This means that $-3 + 3i$ is a sixth root of $5832i$. In general, there are n distinct *n*th roots of a complex number.

*n*th Roots of a Complex Number

If $z = r(\cos\theta + i\sin\theta)$, then for an integer $n \geq 2$, z has n distinct *n*th roots of the form

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \text{ for } k = 0, 1, 2, \dots, n-1 \qquad (3)$$

or

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right] \text{ for } k = 0, 1, 2, \dots, n-1$$

Equation (3) generates n *n*th roots of a complex number z, and the argument of each root corresponds to a value of k between 0 and $n - 1$, inclusive. Values of $k \geq n$ do not need to be considered because they generate arguments that are coterminal to those generated by $k = 0, 1, 2, \dots, n - 1$ and would therefore be redundant.

While we will not take up a formal proof of equation (3), we can use De Moivre's theorem to show that the complex numbers listed in equation (3) are indeed roots of $z = r(\cos\theta + i\sin\theta)$. We require that $w^n = z$.

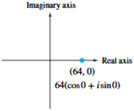
$(w_k)^n = \left(\sqrt[n]{r}\left[\left(\cos\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]\right)^n$	Definition of an *n*th root of a complex number.
$= \left(\sqrt[n]{r}\right)^n\left[\cos\left(n \cdot \frac{\theta + 2\pi k}{n}\right) + i\sin\left(n \cdot \frac{\theta + 2\pi k}{n}\right)\right]$	Apply De Moivre's theorem.
$= r[\cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k)]$	Simplify.
$= r(\cos\theta + i\sin\theta)$	θ and $\theta + 2\pi k$ are coterminal for all integers k.
$= z$	

It is important to note that this is not a complete proof of equation (3) because we have not shown that the n roots given are the only roots that exist.

Example 9: Finding the *n*th Roots of a Complex Number

Find the sixth roots of 64 over the set of complex numbers. Write the results in rectangular form $a + bi$.

Solution:



First write $z = 64 + 0i$ in polar form.

$r = \sqrt{(64)^2 + (0)^2} = 64$ and $\theta = 0$

Therefore, $z = 64(\cos 0 + i\sin 0)$.

The sixth roots of $z = 64(\cos 0 + i\sin 0)$ are

$$w_k = \sqrt[6]{64}\left[\cos\left(\frac{0 + 2\pi k}{6}\right) + i\sin\left(\frac{0 + 2\pi k}{6}\right)\right] \quad \text{for } k = 0, 1, 2, 3, 4, \text{ and } 5$$

$$= 2\left[\cos\left(\frac{\pi}{3}k\right) + i\sin\left(\frac{\pi}{3}k\right)\right] \text{ for } k = 0, 1, 2, 3, 4, \text{ and } 5$$

$$w_0 = 2\left[\cos\left(\frac{\pi}{3}\cdot 0\right) + i\sin\left(\frac{\pi}{3}\cdot 0\right)\right] = 2(\cos 0 + i\sin 0) = 2(1 + 0i) = 2$$

$$w_1 = 2\left[\cos\left(\frac{\pi}{3}\cdot 1\right) + i\sin\left(\frac{\pi}{3}\cdot 1\right)\right] = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}\,i$$

$$w_2 = 2\left[\cos\left(\frac{\pi}{3}\cdot 2\right) + i\sin\left(\frac{\pi}{3}\cdot 2\right)\right] = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}\,i$$

$$w_3 = 2\left[\cos\left(\frac{\pi}{3}\cdot 3\right) + i\sin\left(\frac{\pi}{3}\cdot 3\right)\right] = 2(\cos\pi + i\sin\pi) = 2(-1 + 0i) = -2$$

$$w_4 = 2\left[\cos\left(\frac{\pi}{3}\cdot 4\right) + i\sin\left(\frac{\pi}{3}\cdot 4\right)\right] = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1 - \sqrt{3}\,i$$

$$w_5 = 2\left[\cos\left(\frac{\pi}{3}\cdot 5\right) + i\sin\left(\frac{\pi}{3}\cdot 5\right)\right] = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}\,i$$

Notice that the six roots all have the same magnitude 2. This means that they all lie on a circle centered at the origin of the complex plane with radius 2. Furthermore, the argument of each root differs by a constant angular measure of $\frac{\pi}{3}$ (Figure 7-32). Geometrically, this means that the six 6th roots are all equally spaced around the circle. This is true in general when we find the *n*th roots of a complex number.

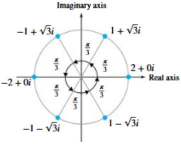


Figure 7-32

Apply the Skills

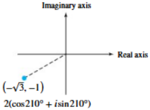
9. Find the cube roots of –64 over the set of complex numbers. Write the results in rectangular form $a + bi$.

From the fundamental theorem of algebra presented in Lesson 2.4, we know that a polynomial equation of degree n has n complex solutions. Therefore, the six 6th roots of 64 can also be found by solving the equation $x^6 = 64$ (Exercise 93).

Example 10: Finding the *n*th Roots of a Complex Number

Find the fourth roots of $-\sqrt{3} - i$ over the set of complex numbers. Write the results in polar form using degree measure for the argument.

Solution:


Imaginary axis
Real axis
$(-\sqrt{3}, -1)$
$2(\cos 210° + i\sin 210°)$

First write $z = -\sqrt{3} - i$ in polar form.

$$r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2$$

$$\tan\theta = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Since $z = -\sqrt{3} - i$ is in Quadrant III, $\theta = 210°$.

So, $z = 2(\cos 210° + i\sin 210°)$

The fourth roots of $z = 2(\cos 210° + i\sin 210°)$ are

$$w_k = \sqrt[4]{2}\left[\cos\left(\frac{210° + 360°k}{4}\right) + i\sin\left(\frac{210° + 360°k}{4}\right)\right] \text{ for } k = 0,\ 1,\ 2, \text{ and } 3$$

$$= \sqrt[4]{2}\,[\cos(52.5° + 90°k) + i\sin(52.5° + 90°k)] \text{ for } k = 0,\ 1,\ 2, \text{ and } 3.$$

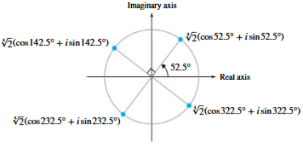
$$w_0 = \sqrt[4]{2}\,[\cos(52.5° + 90° \cdot 0) + i\sin(52.5° + 90° \cdot 0)] = \sqrt[4]{2}(\cos 52.5° + i\sin 52.5°)$$

$$w_1 = \sqrt[4]{2}\,[\cos(52.5° + 90° \cdot 1) + i\sin(52.5° + 90° \cdot 1)] = \sqrt[4]{2}(\cos 142.5° + i\sin 142.5°)$$

$$w_2 = \sqrt[4]{2}\,[\cos(52.5° + 90° \cdot 2) + i\sin(52.5° + 90° \cdot 2)] = \sqrt[4]{2}(\cos 232.5° + i\sin 232.5°)$$

$$w_3 = \sqrt[4]{2}\,[\cos(52.5° + 90° \cdot 3) + i\sin(52.5° + 90° \cdot 3)] = \sqrt[4]{2}(\cos 322.5° + i\sin 322.5°)$$

The fourth roots are equally spaced around a circle of radius $\sqrt[4]{2}$ centered at the origin in the complex plane. The arguments of the polar forms of the roots differ by $\frac{360°}{n} = \frac{360°}{4} = 90°$.


Imaginary axis
$\sqrt[4]{2}(\cos 142.5° + i\sin 142.5°)$
$\sqrt[4]{2}(\cos 52.5° + i\sin 52.5°)$
52.5°
Real axis
$\sqrt[4]{2}(\cos 322.5° + i\sin 322.5°)$
$\sqrt[4]{2}(\cos 232.5° + i\sin 232.5°)$

Apply the Skills

10. Find the fourth roots of $3 + 3\sqrt{3}i$ over the set of complex numbers. Write the results in polar form using degree measure for the argument.

Point of Interest

In addition to his discovery of the roots of a complex number, French mathematician Abraham de Moivre (1667–1754) was noted for his study of probability and statistics. He made numerous publications involving the probabilities associated with games of chance and had investigated mortality statistics that laid the foundation of the theory of annuities. As de Moivre aged, a popular anecdote suggests that he slept 15 min longer each night, and that he correctly predicted the date of his death by linear extrapolation to the day that he would sleep for 24 hr.

Practice 7-3

Practice Exercises

Prerequisite Review

For Exercises R.1–R.5, perform the indicated operations and write the result in standard form $a + bi$.

R.1. $\left(\frac{1}{3}-\frac{1}{2}i\right)-\left(\frac{1}{6}-\frac{7}{12}i\right)$

R.2. $3i(8-5i)$

R.3. $(2+3i)(-3-9i)$

R.4. $(10-3i)^2$

R.5. $\frac{-5+8i}{3-9i}$

Concept Connections

1. The complex plane has two axes. The horizontal axis is called the __________ axis, and the vertical axis is called the __________ axis.

2. Given $z = a + bi$, the value $|z| =$ __________.

3. The rectangular form of a complex number z is $a + bi$ and the polar form (or trigonometric form) is given by $z =$ __________, where $r =$ __________ and $\tan\theta =$ __________ for __________ $\neq 0$.

4. Given $z = r(\cos\theta + i\sin\theta)$, the value r is called the __________ of z and θ is called a(n) __________ of z.

5. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then $z_1 \cdot z_2 =$ ____________________ and $\frac{z_1}{z_2} =$ ____________________ for $z_2 \neq 0$.

6. If $z = r(\cos\theta + i\sin\theta)$, then $z^n =$ ____________________ for a positive integer n. An nth root of z is a complex number w such that __________ $= z$.

Learn: Plot Complex Numbers in the Complex Plane

For Exercises 7–8, graph the complex numbers in the complex plane. (See Example 1, p. 779)

7. **a.** $z = -3 + 4i$ **b.** $z = 2 - 5i$
c. $z = -2i$ **d.** $z = 3$

8. **a.** $z = -2 - i$ **b.** $z = 5 + i$
c. $z = -4$ **d.** $z = 3i$

For Exercises 9–14, find the modulus of each complex number. (See Example 2, p. 780)

9. **a.** $20 - 21i$ **b.** $-11 + 60i$

10. **a.** $5 + 12i$ **b.** $7 - 24i$

11. **a.** $4 - 6i$ **b.** $-5 + 5i$

12. **a.** $-14 + 6i$ **b.** $3 + 9i$

13. **a.** $-\frac{1}{3} - \frac{2}{3}i$ **b.** $\frac{3}{8} + \frac{7}{8}i$

14. **a.** $\frac{\sqrt{2}}{2} - 2i$ **b.** $-1 + \frac{\sqrt{3}}{3}i$

Learn: Write the Polar Form of a Complex Number

For Exercises 15–22, write the complex number in polar form with $0 \leq \theta < 2\pi$. (See Example 3, p. 781)

15. **a.** $3 + 3i$ **b.** $-3 - 3i$

16. **a.** $2 - 2\sqrt{3}i$ **b.** $-2 + 2\sqrt{3}i$

17. **a.** $-4\sqrt{3} + 4i$ **b.** $10\sqrt{3} - 10i$

18. **a.** $3\sqrt{2} - 3\sqrt{2}i$ **b.** $-\sqrt{2} + \sqrt{2}i$

19. **a.** $24 + 7i$ **b.** $-20 + 21i$

20. **a.** $35 - 12i$ **b.** $-28 - 45i$

21. **a.** 17 **b.** $-4i$

22. **a.** $10i$ **b.** -12

For Exercises 23–30, convert the complex number from polar form to rectangular form $a + bi$. (See Example 4, p. 783)

23. $18\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

24. $35(\cos 330^\circ + i\sin 330^\circ)$

25. $15(\cos 315^\circ + i\sin 315^\circ)$

26. $24\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

27. $18.6\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

28. $12.3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

29. $43\pi(\cos 90^\circ + i\sin 90^\circ)$

30. $\sqrt{5}(\cos 180^\circ + i\sin 180^\circ)$

Learn: Compute Products and Quotients of Complex Numbers in Polar Form

For Exercises 31–42, given complex numbers z_1 and z_2,

a. Find z_1z_2 and write the product in polar form.

b. Find $\frac{z_1}{z_2}$ and write the quotient in polar form.

(See Examples 5–6, p. 784)

31. $z_1 = 20(\cos 31° + i\sin 31°)$,
$z_2 = 40(\cos 14° + i\sin 14°)$

32. $z_1 = 27(\cos 67° + i\sin 67°)$,
$z_2 = 9(\cos 53° + i\sin 53°)$

33. $z_1 = 3\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$,
$z_2 = 6\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

34. $z_1 = 10\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$,
$z_2 = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

35. $z_1 = \cos 40° + i\sin 40°$,
$z_2 = \cos 120° + i\sin 120°$

36. $z_1 = \cos 17° + i\sin 17°, z_2 = \cos 73° + i\sin 73°$

37. $z_1 = \frac{3}{4}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$,
$z_2 = \frac{1}{12}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

38. $z_1 = \frac{5}{6}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$,
$z_2 = 30\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

39. $z_1 = 2 - 2i$ and $z_2 = 3 + 3i$

40. $z_1 = -5 - 5i$ and $z_2 = 10 + 10i$

41. $z_1 = -12\sqrt{3} - 12i, z_2 = -6 + 6\sqrt{3}i$

42. $z_1 = 24 + 24\sqrt{3}i, z_2 = 4\sqrt{3} - 4i$

Learn: Compute Powers of Complex Numbers (De Moivre's Theorem)

For Exercises 43–52, use De Moivre's theorem to find the indicated power. Write the result in rectangular form *a* + *bi*. (See Examples 7–8, pp. 785–786)

43. $\left[6\left(\cos\frac{2\pi}{36} + i\sin\frac{\pi}{36}\right)\right]^6$

44. $\left[4\left(\cos\frac{4\pi}{15} + i\sin\frac{4\pi}{15}\right)\right]^5$

45. $\left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^3$

46. $\left[3\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)\right]^4$

47. $[\sqrt{3}(\cos 35° + i\sin 35°)]^9$

48. $[\sqrt{5}(\cos 60° + i\sin 60°)]^5$

49. $[2(\cos 26.25° + i\sin 26.25°)]^8$

50. $[4(\cos 33.75° + i\sin 33.75°)]^4$

51. $(2\sqrt{3} - 2i)^5$

52. $(4 + 4\sqrt{3}i)^3$

Learn: Find the *n*th Roots of a Complex Number

53. When expressing the fourth roots w_k of a complex number generated by $w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$ for $k = 0, 1, 2, \ldots, n - 1$, by how many degrees will consecutive roots differ? In general, given an integer $n \geq 2$, by how many degrees will consecutive *n*th roots differ?

54. When expressing the fifth roots w_k of a complex number generated by $w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$ for $k = 0, 1, 2, \ldots, n - 1$, by how many radians will consecutive roots differ?

For Exercises 55–60, find the indicated complex roots by first writing the number in polar form. Write the results in rectangular form *a* + *bi*. (See Example 9, p. 788)

55. The sixth roots of 729

56. The fourth roots of 625

57. The cube roots of $27i$

58. The cube roots of $-125i$

59. The square roots of $-8 + 8\sqrt{3}i$

60. The square roots of $50 - 50\sqrt{3}i$

For Exercises 61–64, find the indicated complex roots. Write the results in polar form.

61. The square roots of $16\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

62. The square roots of $49\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$

63. The fifth roots of $16\sqrt{3} + 16i$

64. The sixth roots of $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

For Exercises 65–70, find the indicated complex roots. Write the results in polar form using degree measure for the argument. (See Example 10, p. 789)

65. The cube roots of $27(\cos 114° + i\sin 114°)$

66. The fourth roots of $256(\cos 228° + i\sin 228°)$

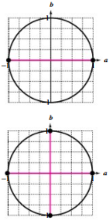
67. The square roots of $\frac{81\sqrt{2}}{2} - \frac{81\sqrt{2}}{2}i$

68. The square roots of $-\frac{3\sqrt{3}}{2} + \frac{3}{2}i$

69. The sixth roots of $-2 - 2\sqrt{3}i$

70. The fifth roots of $5\sqrt{3} - 5i$.

71. The square roots, cube roots, and fourth roots of 1 are shown in the figure from left to right, respectively. Use the pattern to describe how to find the nth roots of 1.



72. Use the results of Exercise 71 to find the fifth roots of 1.

73. Use De Moivre's theorem to show that $2\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)$ is a cube root of $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$.

74. Use De Moivre's theorem to show that $1 - i$ is a cube root of $2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$.

Mixed Exercises

For Exercises 75–78, shade the area in the complex plane.

75. $\{z = a + bi \mid a < 3, b \geq 2\}$

76. $\{z = a + bi \mid a \geq -1, b < 3\}$

77. $\{z = a + bi \mid |z| < 3\}$

78. $\{z = a + bi \mid |z| \geq 2\}$

For Exercises 79–82, convert the complex number from polar form to rectangular form, $a + bi$. (*Hint*: To evaluate the sine or cosine of the given angle, use the sum or difference formulas for sine and cosine.)

79. $4\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$

80. $36\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

81. $\cos 15° + i\sin 15°$

82. $\cos 285° + i\sin 285°$

83. Verify the results of Example 5 by converting z_1 and z_2 to rectangular form and then computing the product.

84. Given $z_1 = 2 + 2i$ and $z_2 = 3i$,

- **a.** Find $\frac{z_1}{z_2}$ by dividing the numbers in rectangular form and then converting the quotient to polar form.
- **b.** Find $\frac{z_1}{z_2}$ by dividing the numbers in polar form.

85. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Prove that $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]\ (z_2 \neq 0)$.

86. Show that $z_1 = r(\cos\theta + i\sin\theta)$ and $z_2 = r[\cos(-\theta) + i\sin(-\theta)]$ are conjugates.

87. Let $z = r(\cos\theta + i\sin\theta)$.

- **a.** Prove that $z^{-1} = r^{-1}[\cos(-\theta) + i\sin(-\theta)]$
- **b.** Prove that $z^{-2} = r^{-2}[\cos(-2\theta) + i\sin(-2\theta)]$

  Repeating this argument, we can show that De Moivre's theorem

  $z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ also holds for negative integers.

88. Use De Moivre's theorem to find the indicated power. Write the result in rectangular form $a + bi$.

- **a.** $(-4\sqrt{2} - 4\sqrt{2}i)^{-2}$
- **b.** $(-1 + \sqrt{3}i)^{-4}$
- **c.** $\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^{-3}$

For Exercises 89–92, simplify and write the solution in rectangular form, $a + bi$. (*Hint*: Convert the complex numbers to polar form before simplifying.)

89. $(1+i)^3(-1-i)^4$

90. $(1+i)^4(-3i)^5$

91. $\dfrac{(-3+\sqrt{3}\,i)^5}{(1-\sqrt{3}\,i)^2}$

92. $\dfrac{(5\sqrt{2}-5\sqrt{2}\,i)^4}{(-\sqrt{3}-i)^3}$

93. Solve $x^6 = 64$ by factoring and applying the zero product property to show that the results are the same as the six 6th roots of 64 as found in Example 9 on page 788.

94. Solve $x^3 = -64$ by factoring and applying the zero product property to show that the results are the same as the three cube roots of −64 as found in the *Apply the Skills* Exercise 9 on page 788.

For Exercises 95–98, find all complex solutions to the equation. Write the solutions in polar form.

95. $z^4 + 8\sqrt{2} = -8\sqrt{2}i$

96. $z^5 = 16\sqrt{2}(1+i)$

97. $2z^3 + \sqrt{3} = i$

98. $z^3 = 8i$

Write About It

99. How is the absolute value of a complex number $z = a + bi$ similar to the absolute value of a real number x?

100. Is the polar representation of a complex number $z = a + bi$ unique? Explain.

Expanding Your Skills

101. Euler's formula states that $e^{i\theta} = \cos\theta + i\sin\theta$.

a. Evaluate $e^{(\pi/4)i}$.

b. Use Euler's formula to show that $e^{\pi i} + 1 = 0$. This equation relates five fundamental numbers used in mathematics (0, 1, e, π, and i).

102. In calculus, we can show that

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2\cdot 1} + \frac{x^3}{3\cdot 2\cdot 1} + \frac{x^4}{4\cdot 3\cdot 2\cdot 1} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2\cdot 1} + \frac{x^4}{4\cdot 3\cdot 2\cdot 1} - \frac{x^6}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + \cdots, \text{ and}$$

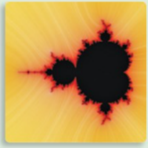
$$\sin x = x - \frac{x^3}{3\cdot 2\cdot 1} + \frac{x^5}{5\cdot 4\cdot 3\cdot 2\cdot 1} - \frac{x^7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + \cdots$$

Write the first eight terms of the expansion of e^{ix} to illustrate that $e^{ix} = \cos x + i\sin x$.

103. Show that $3e^{\pi i/12}$ is a solution to $z^8 - 81z^4 + 6561 = 0$.

104. Show that $e^{7\pi i/12}$ is a solution to $z^8 - z^4 + 1 = 0$.

Point of Interest



The Mandelbrot set is named for the French-American mathematician Benoit Mandelbrot (1924–2010) who discovered it in 1980. The representation of the Mandelbrot set is depicted in numerous places on the internet (and on T-shirts and posters) and shows the members of the set as points on the complex plane. In the image, the Mandelbrot set is depicted in black; all other colors are points outside the set.

The Mandelbrot set is calculated from the equation

$z_{n+1} = (z_n)^2 + c$ where c is a complex number: $c = x + yi$ (c is called the *seed*) and $z_0 = 0$.

The calculation begins with $z_0 = 0$ and proceeds with each previous answer being input into the next step. For example, consider the point on the complex plane $z = 0 + i$. So, we proceed with $c = i$ and $z_0 = 0$:

$$z_1 = (z_0)^2 + c = 0 + i = i$$

$$z_2 = (z_1)^2 + c = i^2 + i = -1 + i$$

$$z_3 = (z_2)^2 + c = (-1 + i)^2 + i = 1 - 2i + i^2 + i = 1 - 2i - 1 + i = -i$$

$$z_4 = (z_3)^2 + c = (-i)^2 + i = i^2 + i = -1 + i$$

Once we observe that the value for z_4 is the same as the value for z_2, we know this list of values will repeat endlessly in the same pattern.

$$0, i, -1 + i, -i, -1 + i, -i, -1 + i, -i, \ldots$$

This list of values is called the *orbit* of z_0 under $z_{n+1} = (z_n)^2 + c$.

We may also consider the modulus of each value:

$$|z_1| = |i| = \sqrt{0^2 + 1^2} = 1$$

$$|z_2| = |-1 + i| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$|z_3| = |-i| = \sqrt{0^2 + (-1)^2} = 1$$

The Mandelbrot set contains all the values c such that the distance from (0, 0) to z_n does *not* become infinitely large. For our example, we have $c = 0 + i$, and as such, we would place a black dot at the point (0, 1) in the complex plane to show it is within the Mandelbrot set.

What about the colors? To answer this question, let's examine another seed, $c = 2i$, and the values of $|z_n|$.

$$z_1 = (z_0)^2 + c = 0 + 2i = 2i \qquad |z_1| = \sqrt{0^2 + 2^2} = 2$$

$$z_2 = (z_1)^2 + c = (2i)^2 + 2i = 4i^2 + 2i = -4 + 2i \qquad |z_2| = \sqrt{(-4)^2 + 2^2} = 2\sqrt{5}$$

$$z_3 = (z_2)^2 + c = (-4 + 2i)^2 + 2i = 12 - 14i \qquad |z_3| = 2\sqrt{85}$$

$$z_4 = (z_3)^2 + c = (12 - 14i)^2 + 2i = -52 - 334i \qquad |z_4| = 2\sqrt{28,565}$$

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The distance from (0, 0) for this list of values appears to grow without bound. We would *not* place a black dot on the point (0, 2) on the complex plane, but we would assign a color to this point.

A color is assigned depending on how many iterations it takes for $|z_n| > 2$. For example, if it takes 2 iterations, we assign red, for 3 iterations, we assign blue, and so on. The color choice is arbitrary—and artistic! Obviously, this detailed graph is not generated or sketched by hand. When generating the graph on a computer, we would also limit the number of iterations before making a color assignment.

There are many zoomable images of the Mandelbrot set available on the Web. Notice that the interesting and deeply detailed portion of the graph is along the boundary. No matter how much you zoom in on a particular boundary region, you will always see a similar kind of crinkly, complex image. The Mandelbrot set is a fractal.

105. Show that the point $c = 0.3 + i$ is *not* in the Mandelbrot set. Consider the point not in the set once $|z_n| > 2$ where $z_{n+1} = (z_n)^2 + c$ and $z_0 = 0$. (For each value of $z = x + yi$, round the values of x and y to two decimal places and the value of $|z|$ to one decimal place.)

106. Examine $z_{n+1} = (z_n)^2 + c$ and $z_0 = 0$ where $c = -0.23 - 0.07i$ through z_5. Do you think this point is in the Mandelbrot set? Justify your answer. (For each value of $z = x + yi$, round the values of x, y, and $|z|$ to two decimal places.)

107. Research *Mandelbrot iteration orbits* on the internet for a visual representation on the complex plane of the list of values created by $z_{n+1} = (z_n)^2 + c$ and $z_0 = 0$. For a journey into the Mandelbrot set, investigate #mandelbrot on YouTube. Also do a search on the phrases *seahorse valley Mandelbrot set* and *elephant valley Mandelbrot set*.

GO ONLINE **Go online for more practice problems.**

Lesson 7-4

Vectors

What Will You Learn?

After completing this lesson, you should be able to:

- Interpret Vectors Geometrically
- Represent Vectors in Component Form
- Perform Operations on Vectors in Component Form
- Represent Vectors in Terms of **i** and **j**
- Relate Component Form and Direction and Magnitude of a Vector
- Use Vectors in Applications

Learn: Interpret Vectors Geometrically

In physics, the mathematical quantities that are used to describe the motion of an object can be divided into two quantities, called scalars and vectors. **Scalars** are quantities that are described by a single value called the **magnitude** (the size or measure of the quantity). **Vectors** are quantities that are described by both magnitude *and* direction. For example, suppose that a bus travels 30 mi. The value 30 mi is a scalar quantity, because it tells us how much distance the bus covered. However, if we say that the bus travels 30 mi east, we know both its magnitude, 30 mi, and direction, east. So this information makes up a vector quantity. This is called a displacement vector because the bus is displaced 30 mi to the east of its starting point.

The **speed** of an object is a scalar quantity that tells us how fast an object travels. By contrast, the **velocity** of an object is a vector quantity, because it tells us how fast the object travels *and* in which direction. For example, suppose that a car travels at a constant speed around a turn at the Daytona International Speedway. Although the speed is constant, the velocity of the car is continuously changing because its direction of travel around the curve changes.

A variety of scalar and vector quantities are used in physical science courses. Some common scalar quantities are distance, speed, area, mass, volume, and temperature. Common vector quantities are displacement, velocity, acceleration, and force.

Vector quantities are so named because they can be represented by a vector in a plane or in space. A **vector** in a plane is a line segment with a specified direction. For example, in Figure 7-33, we show an arrow with endpoints P and Q pointing from P to Q. We denote this vector as $\overrightarrow{PQ}$. We call P the **initial point** of the vector and Q the **terminal point** of the vector. The **magnitude** of $\overrightarrow{PQ}$, denoted $\|\overrightarrow{PQ}\|$, is the length of the line segment.

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Using the context of the **displacement** of an object, the vector $\overrightarrow{PQ}$ refers to the object's overall change in position in moving from point P to point Q.

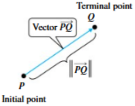


Figure 7-33

Two vectors are equal if they have the same magnitude and direction. The vectors in Figure 7-34 are all equal to $\overrightarrow{AB}$. Notice that the vectors in Figure 7-34 each represent the same displacement left and upward from the initial point to the terminal point.

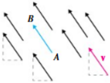


Figure 7-34

To denote vectors, we often use boldface lowercase letters such as **u**, **v**, and **w**. If **v** = $\overrightarrow{AB}$, then **v** represents the set of *all* vectors equal to $\overrightarrow{AB}$.

Good Practices

Because it is difficult to depict boldface when using a pencil and paper, we sometimes use an arrow over a lowercase letter to denote a vector such as *v*.

Example 1: Determining if Two Vectors Are Equal

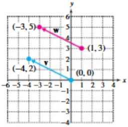
Vector **v** extends from (0, 0) to (–4, 2). Vector **w** extends from (1, 3) to (–3, 5).

- **a.** Find the magnitude of **v** and the magnitude of **w**.
- **b.** Are **v** and **w** equal?

Solution:

a. Find the magnitude (length) of each vector by applying the distance formula.

$$\|\mathbf{v}\| = \sqrt{(-4-0)^2 + (2-0)^2} = \sqrt{16+4}$$
$$= \sqrt{20} = 2\sqrt{5}$$
$$\|\mathbf{w}\| = \sqrt{(-3-1)^2 + (5-3)^2} = \sqrt{16+4}$$
$$= \sqrt{20} = 2\sqrt{5}$$



b. The magnitudes of **v** and **w** are equal, but we must also determine if they have the same direction. Both vectors are directed from the lower right to the upper left. Furthermore, the slopes of the line segments representing **v** and **w** are equal. Therefore, **v** = **w**.

Slope of line through **v**: $m = \frac{2-0}{-4-0} = -\frac{1}{2}$

Slope of line through **w**: $m = \frac{5-3}{-3-1} = -\frac{1}{2}$

Apply the Skills

1. Vector **v** extends from (0, 0) to (3, 4). Vector **w** extends from (–2, 3) to (–5, –1).

   a. Find the magnitude of **v** and magnitude of **w**.

   b. Are **v** and **w** equal?

We can envision the sum or difference of two vectors by regarding their displacements from their initial points to their terminal points. If vector **v** represents the displacement from P to Q and vector **w** represents the vector from Q to R, then the vector **v** + **w** represents the displacement from P to R (Figure 7-35).

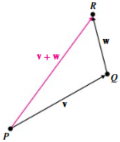


Figure 7-35

Intuitively, the statement $\mathbf{v} + \mathbf{w} = \overrightarrow{PR}$ means that a path directly from P to R represents the same displacement from the starting point as the path from P to Q to R.

Notice that the vector **v** + **w** is formed by placing **v** and **w** "tip to tail" (the initial point of **w** is placed at the terminal point of **v**).

The sum **v** + **w** can also be visualized by placing **v** and **w** at a common initial point ("tail" to "tail"). The sum **v** + **w** is the diagonal of the parallelogram from the initial point of **v** and **w** to the opposite vertex (Figure 7-36).

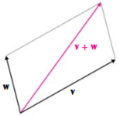


Figure 7-36

If a vector $\mathbf{v}$ is multiplied by a real number k, the magnitude of $k\mathbf{v}$ is $|k|\|\mathbf{v}\|$ (the absolute value of k times the magnitude of $\mathbf{v}$). The direction of $k\mathbf{v}$ is

- in the same direction as $\mathbf{v}$ if $k > 0$ (Figure 7-37).
- in the opposite direction as $\mathbf{v}$ if $k < 0$ (Figure 7-37).

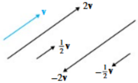
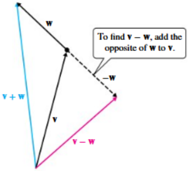


Figure 7-37

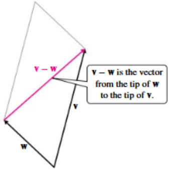
The value k is a scalar, and for this reason, the vector $k\mathbf{v}$ is called a **scalar multiple of v**. Notice that the value of k either stretches or shrinks the length (magnitude) of $\mathbf{v}$ by a factor of $|k|$. If $k = 0$, then $k\mathbf{v}$ is the zero vector, denoted $\mathbf{0}$. The **zero vector** has a magnitude of 0 and is assigned no direction.

The vector $-\mathbf{v}$ is $-1\mathbf{v}$ and represents the opposite of $\mathbf{v}$ (same magnitude, but opposite direction). We use the opposite of a vector to define vector subtraction. The difference of vectors $\mathbf{v}$ and $\mathbf{w}$ is given by $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$. Geometrically, $\mathbf{v} - \mathbf{w}$ is found by adding the opposite of $\mathbf{w}$ to $\mathbf{v}$ (Figure 7-38). With $\mathbf{v}$ and $\mathbf{w}$ placed at a common initial point, the difference $\mathbf{v} - \mathbf{w}$ is visualized as the diagonal of the parallelogram from the "tip" of $\mathbf{w}$ to the "tip" of $\mathbf{v}$ (Figure 7-39).



To find $\mathbf{v} - \mathbf{w}$, add the opposite of $\mathbf{w}$ to $\mathbf{v}$.

Figure 7-38



$\mathbf{v} - \mathbf{w}$ is the vector from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$.

Figure 7-39

Learn: Represent Vectors in Component Form

Working with vectors geometrically can be a cumbersome process. Fortunately, we have a means to represent vectors algebraically. Suppose that for a vector $\mathbf{v} = \overrightarrow{PQ}$, the horizontal displacement moving from P to Q is a units, and the vertical displacement is b units (Figure 7-40). We can represent $\mathbf{v}$ by a pair of numbers of the form $\langle a, b\rangle$. The notation $\langle a, b\rangle$ is called the **component form** of a vector.

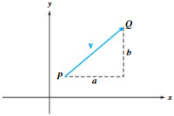
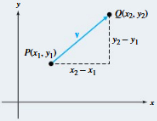


Figure 7-40

Component Form of a Vector

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be points in the plane, and let $\mathbf{v} = \overrightarrow{PQ}$. The component form of **v** is given by

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$$



Sketching a vector with its initial point at the origin of a rectangular coordinate system is often a convenient location to represent a vector. In such a case, the vector is said to be drawn in **standard position.** For a vector $\mathbf{v} = \langle a, b \rangle$ drawn in standard position, the terminal point of **v** is (a, b) (Figure 7-41).

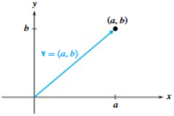


Figure 7-41

Example 2: Finding the Component Form of a Vector

Suppose that vector **v** has initial point (–2, 1) and terminal point (2, –5).

- **a.** Find the component form of **v**.
- **b.** Sketch **v** in standard position.
- **c.** Sketch **v** with its initial point at (2, 3).

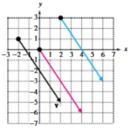
Solution:

a. Label (–2, 1) as (x_1, y_1) and (2, –5) as (x_2, y_2).

Then $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle = \langle 2 - (-2), -5 - 1 \rangle$
$= \langle 4, -6 \rangle$

b. In standard position, the initial point of **v** is (0, 0) and the terminal point is (4, –6), shown in red.

c. If the initial point is (2, 3), then the terminal point is (2 + 4, 3 + (–6)), which is (6, –3), shown in blue.



Apply the Skills

2. Suppose that vector **v** has initial point (–4, 5) and terminal point (–1, 12).

- **a.** Find the component form of **v**.
- **b.** If **v** is placed with initial point at (–2, –4), what is the terminal point of **v**?

Learn: Perform Operations on Vectors in Component Form

With an understanding of the component form of a vector, we have the tools to perform algebraic operations on vectors and to develop important related properties. First, we can state the equality of vectors in terms of their components. Two vectors are equal if and only if their corresponding components are equal.

Equality of Vectors

If $\mathbf{v} = \langle a_1, b_1 \rangle$ and $\mathbf{w} = \langle a_2, b_2 \rangle$, then $\mathbf{v} = \mathbf{w}$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

To add two vectors geometrically, we add their displacements. In component form, this is equivalent to adding their horizontal components and adding their vertical components. Likewise to subtract vectors, we subtract their corresponding components.

Operations on Vectors

Let $\mathbf{v} = \langle a_1, b_1 \rangle$, $\mathbf{w} = \langle a_2, b_2 \rangle$, and c be a real number.

Vector addition: $\mathbf{v} + \mathbf{w} = \langle a_1 + a_2, b_1 + b_2 \rangle$

Vector subtraction: $\mathbf{v} - \mathbf{w} = \langle a_1 - a_2, b_1 - b_2 \rangle$

Multiplication of a vector by a scalar: $c\mathbf{v} = \langle ca_1, cb_1 \rangle$

Example 3: Applying Operations on Vectors Algebraically

Given $\mathbf{v} = \langle 1, 4 \rangle$ and $\mathbf{w} = \langle 3, -2 \rangle$, find

a. $\mathbf{v} + \mathbf{w}$

b. $\mathbf{v} - \mathbf{w}$

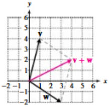
Solution:

a. $\mathbf{v} + \mathbf{w}$

$= \langle 1, 4 \rangle + \langle 3, -2 \rangle$

$= \langle 1 + 3, 4 + (-2) \rangle$

$= \langle 4, 2 \rangle$

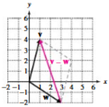


b. $\mathbf{v} - \mathbf{w}$

$= \langle 1, 4 \rangle - \langle 3, -2 \rangle$

$= \langle 1 - 3, 4 - (-2) \rangle$

$= \langle -2, 6 \rangle$



Apply the Skills

3. Given $\mathbf{r} = \langle -3, 10\rangle$ and $\mathbf{s} = \langle 12, -7\rangle$, find

a. $\mathbf{r} + \mathbf{s}$ **b.** $\mathbf{s} - \mathbf{r}$

Example 4: Applying Operations on Vectors Algebraically

Given $\mathbf{r} = \langle -3, 8\rangle$ and $\mathbf{s} = \langle 2, -5\rangle$, find

a. $5\mathbf{r}$ **b.** $2\mathbf{r} - 3\mathbf{s}$

Solution:

a. $5\mathbf{r} = 5\langle -3, 8\rangle = \langle 5(-3), 5(8)\rangle = \langle -15, 40\rangle$

b. $2\mathbf{r} - 3\mathbf{s} = 2\langle -3, 8\rangle - 3\langle 2, -5\rangle$

$= \langle 2(-3), 2(8)\rangle - \langle 3(2), 3(-5)\rangle$

$= \langle -6, 16\rangle - \langle 6, -15\rangle = \langle -6 - 6, 16 - (-15)\rangle = \langle -12, 31\rangle$

Apply the Skills

4. Given $\mathbf{a} = \langle 11, -7\rangle$ and $\mathbf{b} = \langle -4, -6\rangle$, find

a. $6\mathbf{a}$ **b.** $-3\mathbf{a} + 10\mathbf{b}$

Several properties of vectors follow directly from the definition of vector addition, vector subtraction, and multiplication of a vector by a scalar.

Properties of Vectors

If **u**, **v**, and **w** are vectors, and c and d are scalars, then

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
2. $(\mathbf{v} + \mathbf{w}) + \mathbf{u} = \mathbf{v} + (\mathbf{w} + \mathbf{u})$
3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$
4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
5. $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$
6. $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$
7. $(cd)\mathbf{v} = c(d\mathbf{v})$
8. $1(\mathbf{v}) = \mathbf{v}$
9. $0(\mathbf{v}) = \mathbf{0}$
10. $c(\mathbf{0}) = \mathbf{0}$
11. $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$

To show that property 1 is true, let $\mathbf{v} = \langle a_1, b_1\rangle$ and $\mathbf{w} = \langle a_2, b_2\rangle$. Then

$\mathbf{v} + \mathbf{w} = \langle a_1 + a_2, b_1 + b_2\rangle$

$= \langle a_2 + a_1, b_2 + b_1\rangle$ Commutative property of real numbers

$= \mathbf{w} + \mathbf{v}$

Several other properties of vectors are left for verification in the exercises.

Learn: Represent Vectors in Terms of i and j

A vector that has a magnitude of 1 unit is called a **unit vector**. Finding a unit vector in the same direction as a given vector is an important skill in many applications. For example, consider $\mathbf{v} = \langle 3, 4\rangle$ pictured in standard position in Figure 7-42. The magnitude of $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. We want to find a vector $\mathbf{u}$ (pictured in red in Figure 7-42) such that $\mathbf{u}$ is 1 unit in length and in the same direction as $\mathbf{v}$. To do this, we apply a scaling factor of $\frac{1}{\|\mathbf{v}\|}$ to vector $\mathbf{v}$. That is, $\mathbf{u} = \frac{1}{5}\langle 3, 4\rangle = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$. The vectors $\mathbf{u}$ and $\mathbf{v}$ have the same direction because $\mathbf{u}$ is a positive scalar multiple of $\mathbf{v}$. Furthermore, $\mathbf{u}$ is a unit vector because its magnitude is 1.

$$\|\mathbf{u}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$$

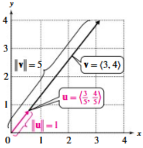
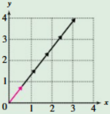


Figure 7-42

Insights

Notice that the length of $\mathbf{u}$ is $\frac{1}{5}$ the length of $\mathbf{v}$ and in the same direction as $\mathbf{v}$. Therefore, $\mathbf{v} = 5\mathbf{u}$.



We generalize this process as follows.

Find a Unit Vector in the Direction of a Given Vector

If $\mathbf{v} = \langle a, b\rangle$, then a unit vector $\mathbf{u}_{\mathbf{v}}$ in the direction of $\mathbf{v}$ is given by

$$\mathbf{u}_{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\|\mathbf{v}\|}\langle a, b\rangle = \left\langle \frac{a}{\|\mathbf{v}\|}, \frac{b}{\|\mathbf{v}\|}\right\rangle$$

Example 5: Finding a Unit Vector

Find a unit vector in the direction of $\mathbf{w} = \langle -3, 2\rangle$.

Solution:

$$\|\mathbf{w}\| = \sqrt{(-3)^2 + (2)^2} = \sqrt{9 + 4} = \sqrt{13}$$

$$\mathbf{u}_{\mathbf{w}} = \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{\sqrt{13}}\langle -3, 2\rangle = \left\langle -\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right\rangle \quad \text{or} \quad \left\langle -\frac{3\sqrt{13}}{13}, \frac{2\sqrt{13}}{13}\right\rangle$$

Check Your Work

We can verify the result from Example 5 by showing that $\|\mathbf{u_w}\| = 1$.

$$\|\mathbf{u_w}\| = \sqrt{\left(-\frac{3}{\sqrt{13}}\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2} = \sqrt{\frac{9}{13} + \frac{4}{13}} = \sqrt{\frac{13}{13}} = 1 \checkmark$$

Apply the Skills

5. Find a unit vector in the direction of $\mathbf{v} = \langle 5, -1 \rangle$.

Two important unit vectors are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ because they are unit vectors in the direction of the positive *x*- and *y*-axes, respectively (Figure 7-43). The importance of vectors **i** and **j** is that any other vector can be expressed as a linear combination of **i** and **j**. Given $\mathbf{v} = \langle a, b \rangle$ we have

$$\begin{aligned} \mathbf{v} &= \langle a, b \rangle \\ &= a\langle 1, 0 \rangle + b\langle 0, 1 \rangle \\ &= a\mathbf{i} + b\mathbf{j} \end{aligned}$$

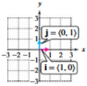


Figure 7-43

Represent Vectors in Terms of i and j

The representation of $\mathbf{v} = \langle a, b \rangle$ in terms of **i** and **j** is $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$.

The values *a* and *b* are called the **scalar horizontal and vertical components** of **v**, respectively.

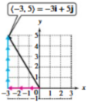
Example 6: Writing Vectors in Terms of i and j

a. Write $\langle -3, 5 \rangle$ in terms of **i** and **j**.

b. Given $\mathbf{w} = 6.2\mathbf{i} - 3.4\mathbf{j}$ and $\mathbf{v} = -1.7\mathbf{i} + 2.2\mathbf{j}$, write $3\mathbf{w} - 5\mathbf{v}$ in terms of **i** and **j**.

Solution:

a.



b.

$$\begin{aligned} 3\mathbf{w} - 5\mathbf{v} &= 3(6.2\mathbf{i} - 3.4\mathbf{j}) - 5(-1.7\mathbf{i} + 2.2\mathbf{j}) \\ &= 18.6\mathbf{i} - 10.2\mathbf{j} + 8.5\mathbf{i} - 11\mathbf{j} \\ &= (18.6 + 8.5)\mathbf{i} + (-10.2 - 11)\mathbf{j} \\ &= 27.1\mathbf{i} - 21.2\mathbf{j} \end{aligned}$$

Apply the Skills

6. a. Write $\langle 12, -1 \rangle$ in terms of **i** and **j**.

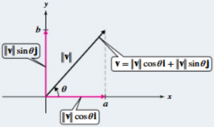
b. Given $\mathbf{a} = -9.4\mathbf{i} + 7.1\mathbf{j}$ and $\mathbf{b} = -0.6\mathbf{i} + 3.2\mathbf{j}$, write $-2\mathbf{a} - 4\mathbf{b}$ in terms of **i** and **j**.

Learn: Relate Component Form and Direction and Magnitude of a Vector

In many applications of vectors we are given the magnitude and direction of a vector rather than its component form. Given a vector in standard position, its **direction angle** θ is measured counterclockwise from the positive x-axis over the interval $0 \leq \theta < 2\pi$ or $0° \leq \theta < 360°$. The relationships among the magnitude, direction, and components of a vector are summarized as follows.

Magnitude, Direction, and Components of a Vector

Let $\mathbf{v} = \langle a, b \rangle$ be a vector in standard position, and let $0° \leq \theta < 360°$ be the direction of $\mathbf{v}$ measured counterclockwise from the positive x-axis.



y
b
$\|\mathbf{v}\| \sin\theta\mathbf{j}$
$\|\mathbf{v}\|$
$\mathbf{v} = \|\mathbf{v}\| \cos\theta\mathbf{i} + \|\mathbf{v}\| \sin\theta\mathbf{j}$
θ
a
x
$\|\mathbf{v}\| \cos\theta\mathbf{i}$

- $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a} (a \neq 0)$ (magnitude and direction of **v**)
- $a = \|\mathbf{v}\| \cos\theta$ and $b = \|\mathbf{v}\| \sin\theta$
- $\mathbf{v} = \langle a, b \rangle = \langle \|\mathbf{v}\| \cos\theta, \|\mathbf{v}\| \sin\theta \rangle$ or $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \|\mathbf{v}\| \cos\theta\mathbf{i} + \|\mathbf{v}\| \sin\theta\mathbf{j}$

In Examples 7–12 and in the exercises, we present vectors in an applied setting with several of the following common vector quantities.

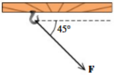
- A vector representing the displacement of an object is called a position or displacement vector.
- A vector representing the speed and direction of an object is called a velocity vector.
- A vector representing the force acting on an object in a given direction is called a force vector. Common units of force are pounds (lb), Newtons (N), and dynes (dyn). Force vectors are often denoted by **F**, by **W** (if the force represents a weight), or by **T** (if the force represents the tension in a rope or wire).

Insights

One Newton is the force required to accelerate a mass of 1 kg at 1 m/sec². So, $1\text{N} = 1\frac{\text{kg} \cdot \text{m}}{\text{sec}^2}$. One dyne is the force required to accelerate a mass of 1 g at 1 cm/sec². So, $1 \text{ dyn} = 1\frac{\text{g} \cdot \text{cm}}{\text{sec}^2}$.

Example 7: Finding the Component Form of a Force Vector

A force of 100 lb is applied to a hook on the ceiling at an angle of 45° with the horizontal. Write the force vector **F** in terms of **i** and **j**.



Solution:

The magnitude of the force is $\|\mathbf{F}\| = 100$ lb. Measuring the direction angle counterclockwise from the positive *x*-axis, we have $\theta = 360° - 45° = 315°$. Then,

$$\begin{aligned}\mathbf{F} &= \|\mathbf{F}\|\cos\theta\,\mathbf{i} + \|\mathbf{F}\|\sin\theta\,\mathbf{j}\\ &= 100\cos 315°\mathbf{i} + 100\sin 315°\mathbf{j}\\ &= 100\cdot\frac{\sqrt{2}}{2}\mathbf{i} + 100\cdot\left(-\frac{\sqrt{2}}{2}\right)\mathbf{j}\\ &= 50\sqrt{2}\,\mathbf{i} - 50\sqrt{2}\,\mathbf{j}\end{aligned}$$

Force **F** is applied in a direction to the right and downward. So, the horizontal component is positive and the vertical component is negative.

Insights

The direction angle θ is usually taken between 0° and 360°, inclusive. However, any angle coterminal to θ results in an equivalent vector. In Example 7,

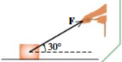
$$F = 100\cos(-45°)\mathbf{i} + 100\sin(-45°)\mathbf{j}$$

also results in the vector

$$50\sqrt{2}\,\mathbf{i} - 50\sqrt{2}\,\mathbf{j}$$

Apply the Skills

7. A person exerts a force of 80 lb on an object by pulling on a rope at an angle of 30° from the horizontal. Write the force vector **F** in terms of **i** and **j**.



In Example 8, we are given a velocity vector in terms of **i** and **j** and asked to find the direction and magnitude of the vector.

Example 8: Finding the Magnitude and Direction of a Vector

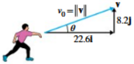
A ball is thrown and at the time of release, the velocity of the ball is given by $\mathbf{v} = 22.6\mathbf{i} + 8.2\mathbf{j}$, where each component is given in m/sec. Find the initial velocity v_0 (magnitude of the velocity vector at the time of release) and the angle above the horizontal (in degrees) at which the ball was thrown. Round to 1 decimal place.

Solution:

The initial velocity is the magnitude of the velocity vector.

$$v_0 = \|\mathbf{v}\| = \sqrt{(22.6)^2 + (8.2)^2} \approx 24.0 \text{ m/sec}$$

To find the direction angle, we have $\tan\theta = \frac{8.2}{22.6}$.



For θ in Quadrant I, $\theta = \tan^{-1}\frac{8.2}{22.6} \approx 19.9°$.

The ball was thrown with an initial velocity of 24.0 m/sec at an angle of 19.9° above the horizontal.

Apply the Skills

8. A ball is thrown from a height of 1 m. At the time of release, the velocity of the ball is given by $\mathbf{v} = 27.2\mathbf{i} + 12.7\mathbf{j}$, where each component is given in m/sec. Find the initial velocity v_0 and the angle θ above the horizontal (in degrees) at which the ball was thrown. Round to 1 decimal place.

Learn: Use Vectors in Applications

In Example 9, a ship travels in open ocean with velocity **v** relative to the water. The ship then encounters a current **c** that alters its course. The resulting velocity **r** of the ship is given by $\mathbf{r} = \mathbf{v} + \mathbf{c}$. As demonstrated in Example 9, finding the component form of vectors **v** and **c** enables us to find the resultant velocity vector.

Example 9: Finding a Resultant Vector in an Application

A ship in open ocean travels 18 mph at a bearing of N64°W and encounters a current of 4 mph acting in the direction of N45°E. Rounding to 1 decimal place,

a. Express the velocity of the ship **v** relative to the water in terms of **i** and **j**.

b. Express the velocity of the current **c** in terms of **i** and **j**.

c. Find the true speed and bearing of the ship.

Solution:

A diagram of the vectors is shown in Figure 7-44.

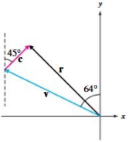


Figure 7-44

a. The direction angle of the ship is $90° + 64° = 154°$, and the magnitude (speed) is 18 mph. Therefore,

$$\begin{aligned}\mathbf{v} &= \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j}\\ &= 18\cos 154°\mathbf{i} + 18\sin 154°\mathbf{j}\\ &\approx -16.2\mathbf{i} + 7.9\mathbf{j}\end{aligned}$$

b. The direction angle of the current is $90° - 45° = 45°$, and the magnitude is 4 mph. Therefore,

$$\begin{aligned}\mathbf{c} &= \|\mathbf{c}\|\cos\theta\,\mathbf{i} + \|\mathbf{c}\|\sin\theta\,\mathbf{j}\\ &= 4\cos 45°\mathbf{i} + 4\sin 45°\mathbf{j}\\ &= 4\cdot\frac{\sqrt{2}}{2}\mathbf{i} + 4\cdot\frac{\sqrt{2}}{2}\mathbf{j} = 2\sqrt{2}\,\mathbf{i} + 2\sqrt{2}\,\mathbf{j} \approx 2.8\mathbf{i} + 2.8\mathbf{j}\end{aligned}$$

c. The true velocity of the ship is given by the sum of vectors **v** and **c**. We call this the resultant vector **r**.

$\mathbf{r} = \mathbf{v} + \mathbf{c} \approx (-16.2\mathbf{i} + 7.9\mathbf{j}) + (2.8\mathbf{i} + 2.8\mathbf{j}) \approx -13.4\mathbf{i} + 10.7\mathbf{j}$

The true speed of the ship is the magnitude of **r**.

$\|\mathbf{r}\| \approx \sqrt{(-13.4)^2 + (10.7)^2} \approx 17.1$ mph

The true direction of the ship is the direction angle of **r**.

$\tan\theta \approx \left(\frac{10.7}{-13.4}\right)$

For θ in Quadrant II, we have $\theta \approx \tan^{-1}\left(-\frac{10.7}{13.4}\right) + 180° \approx 141.4°$.

The ship travels approximately 17.1 mph at a bearing of N51.4°W.

Apply the Skills

9. A plane travels with an air speed of 400 knots at S20°E and encounters a wind of 50 knots acting in the direction of S30°W. Rounding to the nearest whole unit,

a. Express the velocity of the plane **p** relative to the air in terms of **i** and **j**.

b. Express the velocity of the wind **w** in terms of **i** and **j**.

c. Find the true speed (ground speed) and bearing of the plane.

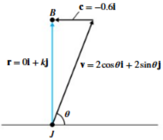
In Example 10, we use vectors to find an unknown direction angle.

Example 10: Calculate a Direction Angle

José enters a triathlon and must swim to a buoy offshore and then back again. A current of 0.6 mph flows due west.

a. If José starts at the closest point J on the shoreline to the buoy B, in what direction θ should he swim in order to arrive at the buoy? Assume that José's speed swimming in still water is 2 mph. Round to the nearest tenth of a degree.

b. What is José's true speed? Round to 1 decimal place.


B
$\mathbf{c} = -0.6\mathbf{i}$
$\mathbf{r} = 0\mathbf{i} + k\mathbf{j}$
$\mathbf{v} = 2\cos\theta\mathbf{i} + 2\sin\theta\mathbf{j}$
θ
J

Solution:

From the diagram, the sum of José's velocity **v** in still water and the velocity **c** of the current must equal the resulting velocity **r**. The magnitude of **r** is José's actual speed k, and the direction of **r** is due north. Therefore, vector **r** is of the form $\mathbf{r} = 0\mathbf{i} + k\mathbf{j}$.

a. $\mathbf{v} + \mathbf{c} = \mathbf{r}$

$$(2\cos\theta\mathbf{i} + 2\sin\theta\mathbf{j}) + (-0.6\mathbf{i}) = 0\mathbf{i} + k\mathbf{j}$$

$$(2\cos\theta - 0.6)\mathbf{i} = 0\mathbf{i}$$

$$2\cos\theta - 0.6 = 0$$

$$\cos\theta = \frac{0.6}{2} \Rightarrow \theta = \cos^{-1}\frac{0.6}{2}$$

Equating the horizontal components results in an equation with one unknown and enables us to solve for θ.

In Quadrant I, we have $\theta \approx 72.5°$.

b. $2\sin\theta\mathbf{j} = k\mathbf{j}$

$$2\sin 72.5° = k$$

$$k \approx 1.9 \text{ mph}$$

José's true speed is the vertical component k of the resultant vector. To solve for k, equate the vertical components.

Apply the Skills

10. Repeat Example 10 with José's speed swimming in still water as 2.5 mph, and a current of 0.5 mph due west.

In many applications of physics, we encounter systems with numerous forces acting on an object. If forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots, \mathbf{F}_n$ act on an object, the sum of the forces is called the resultant force **R**.

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots + \mathbf{F}_n$$

In Example 11, we find the resultant force of three forces acting on an object.

Example 11: Finding the Resultant Force

Three forces, $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, act on an object in the directions shown in Figure 7-45. If the magnitude of $\mathbf{F}_1$ is 60 lb, the magnitude of $\mathbf{F}_2$ is 100 lb, and the magnitude of $\mathbf{F}_3$ is 85 lb, write the resultant force **R** in terms of **i** and **j**. Round the components to the nearest tenth of a pound.

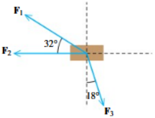


Figure 7-45

Solution:

From Figure 7-45, the direction angle of $\mathbf{F}_1$ is $180° - 32° = 148°$.

The direction angle of $\mathbf{F}_2$ is $180°$.

The direction angle of $\mathbf{F}_3$ is $270° + 18° = 288°$.

$$\begin{aligned}\mathbf{F}_1 = \|\mathbf{F}_1\|\cos\theta\mathbf{i} + \|\mathbf{F}_1\|\sin\theta\mathbf{j} &= 60\cos 148°\mathbf{i} + 60\sin 148°\mathbf{j}\\ &\approx -50.9\mathbf{i} + 31.8\mathbf{j}\end{aligned}$$

Write $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ in terms of **i** and **j**. Then add the results to get the resultant force.

$$\begin{aligned}\mathbf{F}_2 = \|\mathbf{F}_2\|\cos\theta\mathbf{i} + \|\mathbf{F}_2\|\sin\theta\mathbf{j} &= 100\cos 180°\mathbf{i} + 100\sin 180°\mathbf{j}\\ &\approx -100\mathbf{i} + 0\mathbf{j}\\ &= -100\mathbf{i}\end{aligned}$$

$$\begin{aligned}\mathbf{F}_3 = \|\mathbf{F}_3\|\cos\theta\mathbf{i} + \|\mathbf{F}_3\|\sin\theta\mathbf{j} &= 85\cos 288°\mathbf{i} + 85\sin 288°\mathbf{j}\\ &\approx 26.3\mathbf{i} - 80.8\mathbf{j}\end{aligned}$$

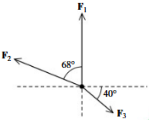
$$\begin{aligned}\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 &\approx (-50.9\mathbf{i} + 31.8\mathbf{j}) + (-100\mathbf{i}) + (26.3\mathbf{i} - 80.8\mathbf{j})\\ &\approx -124.6\mathbf{i} - 49\mathbf{j}\end{aligned}$$

Insights

Force $\mathbf{F}_2$ acts in the negative *x* direction in a coordinate plane. Therefore, there is no vertical component of force, and by inspection, we have that $\mathbf{F}_2 = -100\mathbf{i}$.

Apply the Skills

11. If the magnitude of $\mathbf{F}_1$ is 32 lb, the magnitude of $\mathbf{F}_2$ is 24 lb, and the magnitude of $\mathbf{F}_3$ is 15 lb, write the resultant force **R** in terms of **i** and **j**.



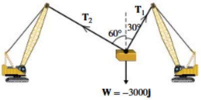
Suppose that forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots, \mathbf{F}_n$ act on an object. The object is said to be in static equilibrium if the resultant force **R** is the zero vector. That is,

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots + \mathbf{F}_n = \mathbf{0}$$

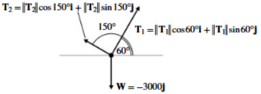
For example, the sum of the three forces in Example 11 is $-124.6\mathbf{i} - 49\mathbf{j}$. If a fourth force $\mathbf{F}_4 = 124.6\mathbf{i} + 49\mathbf{j}$ acted on the object, then the sum of the forces would be $0\mathbf{i} + 0\mathbf{j}$ and the object would be in static equilibrium. Intuitively, this means that the forces counteract one another.

Example 12: Solving for Force Vectors in a System in Static Equilibrium

Two cranes support a 3000-lb crate in static equilibrium, as shown in the figure (not drawn to scale). Find the magnitudes of the tension vectors $\mathbf{T}_1$ and $\mathbf{T}_2$. Round to the nearest pound.



Solution:


$\mathbf{T}_2 = \|\mathbf{T}_2\| \cos 150° \mathbf{i} + \|\mathbf{T}_2\| \sin 150° \mathbf{j}$
150°
$\mathbf{T}_1 = \|\mathbf{T}_1\| \cos 60° \mathbf{i} + \|\mathbf{T}_1\| \sin 60° \mathbf{j}$
60°
$\mathbf{W} = -3000\mathbf{j}$

The weight **W** of the crate is directed straight downward and can be expressed as the vector $\mathbf{W} = -3000\mathbf{j}$.

We need to solve for $\|\mathbf{T}_1\|$ and $\|\mathbf{T}_2\|$. Since the crate is in static equilibrium, we know that the sum of the force vectors is **0**. That is, $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{W} = \mathbf{0}$.

$$(\|\mathbf{T}_1\|\cos 60°\mathbf{i} + \|\mathbf{T}_1\|\sin 60°\mathbf{j}) + (\|\mathbf{T}_2\|\cos 150°\mathbf{i} + \|\mathbf{T}_2\|\sin 150°\mathbf{j}) + (-3000\mathbf{j}) = 0\mathbf{i} + 0\mathbf{j}$$

For two vectors to be equal, their corresponding components must be equal.

$(\|\mathbf{T}_1\|\cos 60° + \|\mathbf{T}_2\|\cos 150°)\mathbf{i} = 0\mathbf{i}$ (1) Equate the horizontal components.

$(\|\mathbf{T}_1\|\sin 60° + \|\mathbf{T}_2\|\sin 150° - 3000)\mathbf{j} = 0\mathbf{j}$ (2) Equate the vertical components.

Simplifying equations (1) and (2), we have

$\frac{1}{2}\|\mathbf{T}_1\| - \frac{\sqrt{3}}{2}\|\mathbf{T}_2\| = 0$ (3)

$\frac{\sqrt{3}}{2}\|\mathbf{T}_1\| + \frac{1}{2}\|\mathbf{T}_2\| - 3000 = 0$ (4)

From (3) we have $\|\mathbf{T}_1\| = \sqrt{3}\|\mathbf{T}_2\|$. Substituting $\sqrt{3}\|\mathbf{T}_2\|$ for $\|\mathbf{T}_1\|$ in equation (4) we have

$$\frac{\sqrt{3}}{2}(\sqrt{3}\|\mathbf{T}_2\|) + \frac{1}{2}\|\mathbf{T}_2\| - 3000 = 0$$
$$\frac{3}{2}\|\mathbf{T}_2\| + \frac{1}{2}\|\mathbf{T}_2\| - 3000 = 0$$
$$2\|\mathbf{T}_2\| = 3000$$
$$\|\mathbf{T}_2\| = 1500 \text{ lb}$$

Using the relationship $\|\mathbf{T}_1\| = \sqrt{3}\|\mathbf{T}_2\|$, we have $\|\mathbf{T}_1\| = \sqrt{3}(1500) \approx 2598$ lb.

The tensions in the cables are 2598 lb and 1500 lb.

Apply the Skills

12. Repeat Example 12, with the two cranes supporting a 1-ton (2000-lb) beam in static equilibrium. The direction angles for $\mathbf{T}_1$ and $\mathbf{T}_2$ are 45° and 120°, respectively. Round to the nearest pound.

Point of Interest

The study of *statics* deals with bodies in equilibrium, whereas the study of *dynamics* focuses on bodies under acceleration. Early principles of statics were known in ancient times. For example, the writings of Archimedes show that the ancient Greeks were familiar with the concept of a lever. The study of dynamics came much later because accurate measurements of time were required to investigate bodies in motion. Major contributions were made by Galileo Galilei (1564–1642) and later by Isaac Newton (1642–1727) who formulated three fundamental laws of motion.

Practice 7-4

Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, use the distance formula to find the distance between the two points.

R.1. (2, 4) and (−2, 2)

R.2. (3, 7) and (−5, 7)

For Exercises R.3–R.4, make a sketch to illustrate the bearing.

R.3. N53°W

R.4. S38°E

Concept Connections

1. A _______ is a quantity that has magnitude only, whereas a vector has both magnitude and _______.

2. Given a vector $\overrightarrow{AB}$, the initial point is _______ and the terminal point is _______. The _______ of $\overrightarrow{AB}$ is the length of the vector and is denoted by _______.

3. Suppose that a race car driver drives around the 2.5-mi oval track at the Daytona International Speedway.

- **a.** What is the distance traveled? Is this a scalar or a vector?
- **b.** What is the displacement? Is this a scalar or a vector?

4. For a real number k, the vector $k\mathbf{v}$ is in the same direction as $\mathbf{v}$ if k is _______ and in the opposite direction as $\mathbf{v}$ if k is _______.

5. Two vectors $\mathbf{r} = \langle a_1, b_1 \rangle$ and $\mathbf{s} = \langle a_2, b_2 \rangle$ are equal if _______.

6. Given a nonzero vector $\mathbf{v} = \langle a, b \rangle$, a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$ is given by $\mathbf{u} =$ _______.

7. In component form, the vector $\mathbf{i} =$ _______ and $\mathbf{j} =$ _______.

8. If a nonzero vector $\mathbf{v} = \langle a, b \rangle$ has magnitude $\|\mathbf{v}\|$ and direction θ, then $a =$ _______ and $b =$ _______.

9. If a nonzero vector $\mathbf{v} = \langle a, b \rangle$, then the magnitude $\|\mathbf{v}\|$ is given by _______, and the direction is given by $\tan\theta =$ _______ for $0 \le \theta < 2\pi$ and $a \neq 0$.

10. Suppose that forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots, \mathbf{F}_n$ are acting on an object in static equilibrium. Then the sum of the forces equals _______.

Learn: Interpret Vectors Geometrically

For Exercises 11–16, vector v has initial point *P* and terminal point *Q*. Vector w has initial point *R* and terminal point *S*. (See Example 1, p. 798)

a. Find the magnitude of **v**.

b. Find the magnitude of **w**.

c. Determine whether $\mathbf{v} = \mathbf{w}$ and explain your reasoning.

11. $P(4, 7)$, $Q(5, 10)$ and $R(-2, 8)$, $S(-1, 11)$

12. $P(-2, -10)$, $Q(-5, -8)$ and $R(9, -3)$, $S(6, -1)$

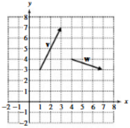
13. $P(4, -1)$, $Q(7, -6)$ and $R(5, 7)$, $S(2, 12)$

14. $P(10, 18)$, $Q(27, -6)$ and $R(9, 12)$, $S(-8, -12)$

15. $P(-12, 10)$, $Q(-16, 7)$ and $R(9, -3)$, $S(1, -9)$

16. $P(-6, 5)$, $Q(10, 35)$ and $R(10, -36)$, $S(34, 9)$

For Exercises 17–20, refer to vectors v and w in the figure.



17. **a.** Sketch $\mathbf{v} + \mathbf{w}$ by placing the initial point of **w** at the terminal point of **v**.

b. Sketch $\mathbf{v} + \mathbf{w}$ by drawing **v** and **w** with the same initial point.

18. **a.** Sketch $\mathbf{v} - \mathbf{w}$ by placing the initial point of $-\mathbf{w}$ at the terminal point of **v**.

b. Sketch $\mathbf{v} - \mathbf{w}$ by drawing **v** and **w** with the same initial point.

19. Sketch $-\mathbf{v}$ and $2\mathbf{w}$ in standard position.

20. Sketch $2\mathbf{v}$ and $-2\mathbf{w}$ in standard position.

Learn: Represent Vectors in Component Form

For Exercises 21–22, given a vector v with initial point *P* and terminal point *Q*,

a. Write the component form of **v**.

b. Sketch **v** from P to Q and sketch **v** in standard position. **(See Example 2, p. 801)**

21. $P(3, -2)$, $Q(5, -5)$

22. $P(-1, 4)$, $Q(-3, 8)$

For Exercises 23–26, write the component form of a vector v with initial point *P* and terminal point *Q*.

23. $P(-8, -3)$, $Q(-8, -6)$

24. $P(10, 10)$, $Q(12, 10)$

25. $P(8.4, -2.3)$, $Q(6.4, -5.8)$

26. $P(2.5, -1.9)$, $Q(10, 3.1)$

27. Given vector **v** with initial point $(16, -30)$ and terminal point $(-22, -10)$,

a. Find the component form of **v**.

b. If **v** is placed with initial point at $(15, 8)$, what is the terminal point of **v**?

28. Given vector **v** with initial point $(17, 80)$ and terminal point $(-72, 53)$,

a. Find the component form of **v**.

b. If **v** is placed with initial point at $(-13, -12)$, what is the terminal point of **v**?

29. Given $\mathbf{v} = \langle -17, 29 \rangle$ with initial point $(4, -10)$, find the terminal point of **v**.

30. Given $\mathbf{v} = \langle 5, -8 \rangle$ with initial point $(-1, -7)$, find the terminal point of **v**.

Learn: Perform Operations on Vectors in Component Form

For Exercises 31–46, perform the indicated operations for the given vectors. (See Examples 3–4, pp. 802–803)

$\mathbf{v} = \langle 8, -10 \rangle$ $\mathbf{w} = \langle -3, 7 \rangle$

$\mathbf{s} = \langle -1, -2 \rangle$ $\mathbf{r} = \langle 2, 10 \rangle$

31. $\mathbf{v} + \mathbf{w}$

32. $\mathbf{s} + \mathbf{w}$

33. $\mathbf{v} + \mathbf{r}$

34. $\mathbf{r} + \mathbf{s}$

35. $\mathbf{w} - \mathbf{v}$

36. $\mathbf{w} - \mathbf{s}$

37. $\mathbf{r} - \mathbf{w}$

38. $\mathbf{v} - \mathbf{r}$

39. $5\mathbf{s}$

40. $-4\mathbf{w}$

41. $-\frac{1}{2}\mathbf{v}$

42. $\frac{2}{3}\mathbf{w}$

43. $\mathbf{v} + \mathbf{s} - \mathbf{r}$

44. $\mathbf{s} - \mathbf{r} + \mathbf{w}$

45. $2\mathbf{r} - (\mathbf{w} + \mathbf{s})$

46. $3(\mathbf{v} + 2\mathbf{w}) - \mathbf{s}$

For Exercises 47–54, let c and d be scalars and let $\mathbf{v} = \langle a_1, b_1 \rangle$, $\mathbf{w} = \langle a_2, b_2 \rangle$, $\mathbf{u} = \langle a_3, b_3 \rangle$, and $\mathbf{0} = \langle 0, 0 \rangle$. Prove the given statement.

47. $(\mathbf{v} + \mathbf{w}) + \mathbf{u} = \mathbf{v} + (\mathbf{w} + \mathbf{u})$

48. $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v}$

49. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

50. $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$

51. $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$

52. $(cd)\mathbf{v} = c(d\mathbf{v})$

53. $0(\mathbf{v}) = \mathbf{0}$

54. $\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$

Learn: Represent Vectors in Terms of i and j

For Exercises 55–60, find a unit vector in the direction of v. (See Example 5, p. 804)

55. $\mathbf{v} = 20\mathbf{i} - 21\mathbf{j}$

56. $\mathbf{w} = -11\mathbf{i} + 60\mathbf{j}$

57. $\mathbf{t} = -0.7\mathbf{i} - 2.4\mathbf{j}$

58. $\mathbf{s} = 3.9\mathbf{i} + 5.2\mathbf{j}$

59. $\mathbf{r} = -10\mathbf{i} + 14\mathbf{j}$

60. $\mathbf{v} = 6\mathbf{i} - 15\mathbf{j}$

For Exercises 61–68, perform the indicated operations for the given vectors. (See Example 6, p. 805)

$\mathbf{v} = -3.3\mathbf{i} + 11\mathbf{j}$ $\quad$ $\mathbf{w} = 4\mathbf{i} + \mathbf{j}$

$\mathbf{s} = 7.2\mathbf{i} - 5.4\mathbf{j}$ $\quad$ $\mathbf{r} = -12\mathbf{i} - 5\mathbf{j}$

61. $\mathbf{v} + \mathbf{s}$

62. $\mathbf{w} + \mathbf{r}$

63. $\mathbf{s} - \mathbf{r}$

64. $\mathbf{v} - \mathbf{w}$

65. $-2\mathbf{r} + 3\mathbf{w}$

66. $6\mathbf{s} - 4\mathbf{v}$

67. $\|\mathbf{r} + \mathbf{w}\|$

68. $\|\mathbf{r} - \mathbf{w}\|$

Learn: Relate Component Form and Direction and Magnitude of a Vector

For Exercises 69–74, write the vector v in the form $a\mathbf{i} + b\mathbf{j}$, where v has the given magnitude and direction angle.

69. $\|\mathbf{v}\| = 12, \theta = 30°$

70. $\|\mathbf{v}\| = 18, \theta = 150°$

71. $\|\mathbf{v}\| = \sqrt{5}, \theta = \frac{3\pi}{4}$

72. $\|\mathbf{v}\| = \sqrt{17}, \theta = \frac{4\pi}{3}$

73. $\|\mathbf{v}\| = \frac{3}{8}, \theta = 300°$

74. $\|\mathbf{v}\| = \frac{6}{7}, \theta = 225°$

For Exercises 75–78, find the magnitude and direction angle ($0° \le \theta < 360°$) for the given vector. Round to 1 decimal place.

75. $\mathbf{v} = 4\mathbf{i} + 6\mathbf{j}$

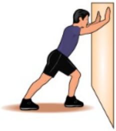
76. $\mathbf{w} = -5\mathbf{i} + 8\mathbf{j}$

77. $\mathbf{s} = -1.8\mathbf{i} - 3.5\mathbf{j}$

78. $\mathbf{r} = \mathbf{i} - 0.8\mathbf{j}$

79. A kick boxer kicks a bag at an angle of 30° from the horizontal with a force of 250 N (Newtons). Write the force vector **F** in terms of **i** and **j**. (**See Example 7, p. 807**)

80. An athlete stretches his Achilles tendon by extending one leg behind the other and pushing against the wall with his hands. If he pushes against the wall with a force of 120 N at an angle of 20° from the horizontal as shown, write the force vector **F** in terms of **i** and **j**. Round the components to the nearest tenth of a Newton.



81. A ship travels 18 mph at a bearing of S30°W. Write the velocity vector **v** in terms of **i** and **j**.

82. A baseball is thrown with a speed of 70 mph at an angle of 14° from the horizontal. Write the velocity vector **v** at the time of release in terms of **i** and **j**. Round the components to the nearest tenth of a mph.

83. A punter kicks a football with an initial velocity given by $\mathbf{v} = 22.2\mathbf{i} + 18.6\mathbf{j}$ m/sec. (**See Example 8, p. 807**)

a. Find the magnitude of the velocity vector at the time the ball leaves the punter's foot. Round to the nearest m/sec.

b. Find the angle from the horizontal at which the ball was kicked. Round to the nearest degree.

84. The velocity of a ship is given by the vector $-6.4\mathbf{i} + 7.7\mathbf{j}$ mph.

a. Find the speed of the ship. Round to the nearest mph.

b. Find the bearing of the ship. Round to the nearest degree.

Learn: Use Vectors in Applications

85. A plane travels N30°W at 450 mph and encounters a wind blowing due west at 30 mph. (**See Example 9, p. 808**)

a. Express the velocity of the plane $\mathbf{v}_p$ relative to the air in terms of **i** and **j**.

b. Express the velocity of the wind $\mathbf{v}_w$ in terms of **i** and **j**.

c. Express the true velocity of the plane $\mathbf{v}_T$ in terms of **i** and **j** and find the true speed of the plane.

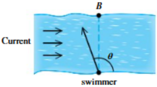
86. A plane flies at a speed of 370 mph on a bearing of S45°E. Relative to the ground, the plane's speed is measured as 345 mph with a true bearing of S40°E. Rounding to the nearest whole unit,

a. Express the velocity of the plane $\mathbf{v}_p$ relative to the air in terms of **i** and **j**.

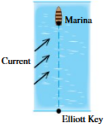
b. Express the true velocity of the plane $\mathbf{v}_T$ in terms of **i** and **j**.

c. Express the velocity of the wind $\mathbf{v}_w$ in terms of **i** and **j** and find the speed of the wind.

87. A swimmer swims 1.5 mph in still water and wants to swim to a point *B* due north from the starting point directly across a river. If the current is 0.5 mph due east, at what angle θ should the swimmer swim to reach point *B*? Round to the nearest tenth of a degree. (**See Example 10, p. 809**)



88. A boat that travels 6 mph in still water leaves a marina in Miami to travel to an island called Elliott Key due south of the marina. The prevailing current is directed northeast (N45°E) at 0.8 mph. At what bearing should the captain of the boat steer the ship?



For Exercises 89–94, the given forces (in units of pounds) act on an object.

a. Find the resultant force, **R**.

b. What additional force **F** is needed for the object to be in static equilibrium? (**See Example 11, p. 810**)

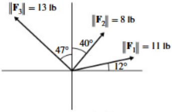
89. $\mathbf{F}_1 = \langle 3, 11 \rangle$ and $\mathbf{F}_2 = \langle -8, 6 \rangle$

90. $\mathbf{F}_1 = \langle -5, 2 \rangle$ and $\mathbf{F}_2 = \langle -7, -3 \rangle$

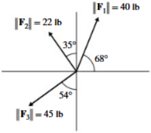
91. $\mathbf{F}_1 = 5\mathbf{i} - \mathbf{j}$, $\mathbf{F}_2 = 4\mathbf{i} - 3\mathbf{j}$, $\mathbf{F}_3 = -9\mathbf{i} + 2\mathbf{j}$

92. $\mathbf{F}_1 = -12\mathbf{i} + 8\mathbf{j}$, $\mathbf{F}_2 = -9\mathbf{i} - 15\mathbf{j}$, $\mathbf{F}_3 = 11\mathbf{i} + 7\mathbf{j}$

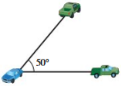
93.


$\|\mathbf{F}_3\| = 13$ lb
$\|\mathbf{F}_2\| = 8$ lb
47°
40°
$\|\mathbf{F}_1\| = 11$ lb
12°

94.


$\|\mathbf{F}_1\| = 40$ lb
$\|\mathbf{F}_2\| = 22$ lb
35°
68°
54°
$\|\mathbf{F}_3\| = 45$ lb

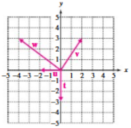
95. Two pickup trucks rescue a car stuck in the sand. The first truck pulls with a force of 5000 lb and the second truck pulls with a force of 4000 lb at an angle of 50° to the first truck. How much force can the combined pull of the two trucks generate? Round to the nearest pound.



96. Two forces act on an object with an angle of 63° between them. If the magnitude of the first force is 48 N and the magnitude of the second force is 70 N, find the magnitude of the resultant force to the nearest Newton.

Mixed Exercises

For Exercises 97–102, refer to vectors t, u, v, and w in the figure and match each vector with an equivalent vector a–f.



a. $-\mathbf{t}$ **b.** $3\mathbf{u}$ **c.** $\mathbf{w}$

d. $\mathbf{u} - 2\mathbf{t}$ **e.** $\mathbf{v}$ **f.** $-3\mathbf{u}$

97. $2\mathbf{u} - \mathbf{t}$

98. $-\mathbf{u} - \mathbf{t}$

99. $\mathbf{w} - \mathbf{v}$

100. $\mathbf{v} + \mathbf{u}$

101. $\mathbf{v} + \mathbf{w}$

102. $\mathbf{v} - \mathbf{w}$

103. A kayaker paddles 3 mph in still water. If she heads due east but encounters a 0.8-mph current flowing south,

a. Find the actual direction of travel. Write the direction as a bearing and round to the nearest degree.

b. How far will she travel in 2 hr? Round to the nearest tenth of a mile.

104. A long-distance swimmer swims 2 mph in still water. If he heads north but encounters a 0.6-mph current flowing northeast,

a. Find the actual direction of travel. Write the direction as a bearing and round to the nearest degree.

b. How far will he swim in 1.5 hr? Round to the nearest tenth of a mile.

For Exercises 105–106, two vectors v and w act on a point in the plane with the indicated force.

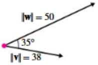
a. Use a parallelogram to sketch the resultant vector $\mathbf{v} + \mathbf{w}$.

b. Use the law of cosines to find the magnitude of the resultant vector. (*Hint:* Adjacent angles in a parallelogram are supplementary.)

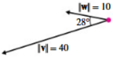
c. Use the law of sines to find the angle that $\mathbf{v} + \mathbf{w}$ makes with $\mathbf{v}$.

Round answers to the nearest whole unit.

105.



106.



107. A 15-lb kettlebell rests on a surface inclined at an angle of 20° to the ground. Find the magnitude of the components of the weight parallel to the surface and perpendicular to the surface. Round to the nearest tenth of a pound.

108. A force of 50 lb is applied at an angle of 20° to the horizontal. Resolve the force into the horizontal and vertical components. Round to the nearest tenth of a pound.

Write About It

109. How is the component form of a vector $\langle a, b\rangle$ related to the graph of the vector in standard position?

110. Explain how the sum $\mathbf{v} + \mathbf{w}$ is found geometrically.

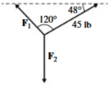
111. Explain the geometric significance of the scalar multiple of a vector, such as $k\mathbf{v}$, where k is a real number.

112. Explain how vectors written in component form are added or subtracted.

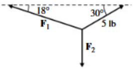
Expanding Your Skills

For Exercises 113–114, assume that the system is in equilibrium with $\mathbf{F}_2$ acting directly downward. Calculate the magnitude of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Round to the nearest tenth of a pound.

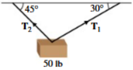
113.



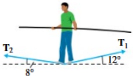
114.



115. A 50-lb box is supported by two ropes attached to the ceiling. Find the magnitudes of the tension vectors $\mathbf{T}_1$ and $\mathbf{T}_2$. Round to the nearest tenth of a pound. (**See Example 12, p. 811**)



116. A tightrope walker and the balance bar together weigh 175 lb. Find the magnitudes of the tension vectors $\mathbf{T}_1$ and $\mathbf{T}_2$. Round to the nearest tenth of a pound.



Go online for more practice problems.

Lesson 7-5

Dot Product

What Will You Learn?

After completing this lesson, you should be able to:

- Compute Dot Product
- Find the Angle Between Two Vectors
- Decompose Vectors into Two Orthogonal Vectors
- Use Vectors to Compute Work

Learn: Compute Dot Product

In this lesson, we present a special type of product of vectors called the dot product. The dot product of two vectors has a variety of applications, including decomposing a vector into components parallel to and orthogonal (perpendicular) to another vector. We will also see in Examples 6–7 that the dot product is helpful in finding the amount of work done by a force in moving an object a specified distance.

Definition of Dot Product

If $\mathbf{v} = \langle a_1, b_1 \rangle$ and $\mathbf{w} = \langle a_2, b_2 \rangle$, the **dot product $\mathbf{v} \cdot \mathbf{w}$** is defined as

$$\mathbf{v} \cdot \mathbf{w} = a_1a_2 + b_1b_2$$

It is important to note that the dot product of two vectors is a scalar.

Example 1: Finding the Dot Product of Two Vectors

Given $\mathbf{v} = \langle 3, -7 \rangle$ and $\mathbf{w} = \langle -2, 5 \rangle$, find

a. $\mathbf{v} \cdot \mathbf{w}$ **b.** $\mathbf{w} \cdot \mathbf{v}$ **c.** $\mathbf{v} \cdot \mathbf{v}$ **d.** $\|\mathbf{v}\|$

Solution:

a. $\mathbf{v} \cdot \mathbf{w} = 3(-2) + (-7)(5) = -6 - 35 = -41$

b. $\mathbf{w} \cdot \mathbf{v} = -2(3) + 5(-7) = -6 - 35 = -41$

c. $\mathbf{v} \cdot \mathbf{v} = 3(3) + (-7)(-7) = 9 + 49 = 58$

d. $\|\mathbf{v}\| = \sqrt{(3)^2 + (-7)^2} = \sqrt{9 + 49} = \sqrt{58}$

Apply the Skills

1. Given $\mathbf{v} = \langle -2, 6 \rangle$ and $\mathbf{w} = \langle -3, -1 \rangle$, find

a. $\mathbf{v} \cdot \mathbf{w}$ **b.** $\mathbf{w} \cdot \mathbf{v}$ **c.** $\mathbf{v} \cdot \mathbf{v}$ **d.** $\|\mathbf{v}\|$

Notice from Examples 1(a) and 1(b) that $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$. We can show that this is true for any two vectors **v** and **w.** That is, the dot product of two vectors is a commutative operation. Given $\mathbf{v} = \langle a_1, b_1 \rangle$ and $\mathbf{w} = \langle a_2, b_2 \rangle$:

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= a_1a_2 + b_1b_2 && \text{Dot product of } \mathbf{v} \text{ and } \mathbf{w} \\ &= a_2a_1 + b_2b_1 && \text{Commutative property of multiplication of real numbers} \\ &= \mathbf{w} \cdot \mathbf{v} && \text{Dot product of } \mathbf{w} \text{ and } \mathbf{v} \end{aligned}$$

Examples 1(c) and 1(d) suggest that the dot product of a vector with itself is the square of the magnitude of the vector. That is, $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$. Given $\mathbf{v} = \langle a_1, b_1 \rangle$:

$$\begin{aligned} \mathbf{v} \cdot \mathbf{v} &= a_1a_1 + b_1b_1 && \text{Dot product of } \mathbf{v} \text{ and } \mathbf{v} \\ &= a_1^2 + b_1^2 && \\ &= \left(\sqrt{a_1^2 + b_1^2}\right)^2 && \text{Recall that } \|\mathbf{v}\| = \sqrt{a_1^2 + b_1^2}. \\ &= \|\mathbf{v}\|^2 && \text{Square of the magnitude of } \mathbf{v} \end{aligned}$$

Several important properties of the dot product are given next. The proofs of the remaining properties in the following box are addressed in Exercises 29–32.

Properties of Dot Product

If **u**, **v**, and **w** are vectors, and c is a scalar, then

1. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
2. $(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w})$
3. $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u}) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{u}$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $\mathbf{0} \cdot \mathbf{v} = 0$
6. $c(\mathbf{v} \cdot \mathbf{w}) = (c\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (c\mathbf{w})$

Learn: Find the Angle Between Two Vectors

Given two vectors **v** and **w** in standard position, the angle θ between **v** and **w** is the angle formed by the two line segments representing **v** and **w**, where $0° \leq \theta \leq 180°$ (Figure 7-46). The vectors **v**, **w**, and $\mathbf{v} - \mathbf{w}$ form a triangle.

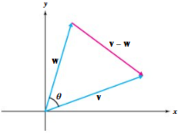


Figure 7-46

Furthermore, by applying the law of cosines, θ can be expressed in terms of the dot product of **v** and **w** and the magnitudes of **v** and **w**.

Angle Between Two Vectors

If θ is the angle between two nonzero vectors **v** and **w**, then

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \text{ and } \theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$$

Proof:

Draw vectors **v** and **w** in standard position with angle θ between **v** and **w** (Figure 7-46). Applying the law of cosines on the triangle formed by **v**, **w**, and $\mathbf{v}-\mathbf{w}$ we have,

$$\begin{aligned}
\|\mathbf{v}-\mathbf{w}\|^2 &= \|\mathbf{v}\|^2+\|\mathbf{w}\|^2-2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta && \text{Apply the law of cosines.}\\
(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) &= \|\mathbf{v}\|^2+\|\mathbf{w}\|^2-2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta && \text{Apply property 4.}\\
\mathbf{v}\cdot(\mathbf{v}-\mathbf{w})-\mathbf{w}\cdot(\mathbf{v}-\mathbf{w}) &= \|\mathbf{v}\|^2+\|\mathbf{w}\|^2-2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta && \text{Apply property 3.}\\
\mathbf{v}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{w}-\mathbf{w}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{w} &= \|\mathbf{v}\|^2+\|\mathbf{w}\|^2-2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta && \text{Apply property 3.}\\
\|\mathbf{v}\|^2-2(\mathbf{v}\cdot\mathbf{w})+\|\mathbf{w}\|^2 &= \|\mathbf{v}\|^2+\|\mathbf{w}\|^2-2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta && \text{Apply properties 1 and 4.}\\
\mathbf{v}\cdot\mathbf{w} &= \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta && \text{Simplify the equation.}\\
\cos\theta &= \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} && \text{Solve for }\cos\theta.
\end{aligned}$$

Example 2: Finding the Angle Between Two Vectors

Find the angle between $\mathbf{v}=\langle -4, 3\rangle$ and $\mathbf{w}=\langle -6, -2\rangle$. Round to the nearest tenth of a degree.

Solution:

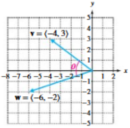
$$\mathbf{v}\cdot\mathbf{w} = -4(-6)+3(-2) = 24-6 = 18$$

$$\|\mathbf{v}\| = \sqrt{(-4)^2+(3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

$$\begin{aligned}
\|\mathbf{w}\| &= \sqrt{(-6)^2+(-2)^2} = \sqrt{36+4}\\
&= \sqrt{40} = 2\sqrt{10}
\end{aligned}$$

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} = \frac{18}{5(2\sqrt{10})} = \frac{9}{5\sqrt{10}}$$

$$\theta = \cos^{-1}\left(\frac{9}{5\sqrt{10}}\right) \approx 55.3^\circ$$



The angle between **v** and **w** is approximately 55.3°.

Apply the Skills

2. Find the angle between $\mathbf{v}=\langle 5, -1\rangle$ and $\mathbf{w}=\langle -5, 12\rangle$. Round to the nearest tenth of a degree.

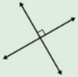
The relationship $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$ can be written as $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$. This gives us an alternative method to compute the dot product of two vectors if their magnitudes and angle between them are known. Furthermore, since $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$ are both positive, $\mathbf{v}\cdot\mathbf{w}$ and $\cos\theta$ must have the same sign. Therefore, the sign of $\mathbf{v}\cdot\mathbf{w}$ gives us information about the measure of θ.

- If $\mathbf{v}\cdot\mathbf{w} > 0$, then $\cos\theta > 0$, which implies that $0° \le \theta < 90°$.
- If $\mathbf{v}\cdot\mathbf{w} < 0$, then $\cos\theta < 0$, which implies that $90° < \theta \le 180°$.
- If $\mathbf{v}\cdot\mathbf{w} = 0$, then $\cos\theta = 0$, which implies that $\theta = 90°$.

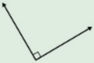
If $\mathbf{v}\cdot\mathbf{w} = 0$, then $\theta = 90°$ and we say that **v** and **w** are **orthogonal.** Orthogonal vectors drawn "tail-to-tail" will form a right angle.

Insights

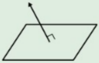
The terms **orthogonal, perpendicular,** and **normal** all refer to objects where the measure of the angle between them is 90°. By convention, we say that two lines in a plane are perpendicular, two vectors in a plane are orthogonal, and a line or vector is normal to a plane.



Perpendicular lines



Orthogonal vectors



Vector normal to a plane

Orthogonal Vectors

Two vectors **v** and **w** are orthogonal if and only if $\mathbf{v}\cdot\mathbf{w} = 0$.

Example 3: Determining if Two Vectors Are Orthogonal

Determine if the given vectors are orthogonal.

a. $\mathbf{v} = 3\mathbf{i} - 6\mathbf{j}$ and $\mathbf{w} = 8\mathbf{i} + 4\mathbf{j}$ **b.** $\mathbf{s} = -2\mathbf{i} - 5\mathbf{j}$ and $\mathbf{r} = -4\mathbf{i} + 8\mathbf{j}$

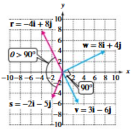
Solution:

a. $\mathbf{v}\cdot\mathbf{w} = 3(8) + (-6)(4) = 24 - 24 = 0$

Since $\mathbf{v}\cdot\mathbf{w} = 0$, **v** and **w** are orthogonal.

b. $\mathbf{s}\cdot\mathbf{r} = (-2)(-4) + (-5)(8) = 8 - 40 = -32$

Since $\mathbf{s}\cdot\mathbf{r} < 0$, we have $90° < \theta \le 180°$ and **s** and **r** are *not* orthogonal.



Apply the Skills

3. Determine if the given vectors are orthogonal.

a. $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 6\mathbf{i} - 10\mathbf{j}$ **b.** $\mathbf{s} = -2\mathbf{i} + \mathbf{j}$ and $\mathbf{t} = 11\mathbf{i} + 22\mathbf{j}$

Recall that two vectors **v** and **w** are parallel if $\mathbf{v} = c\mathbf{w}$ for some real number c. Likewise, if the angle θ between **v** and **w** is either 0° or 180°, then the vectors are parallel. If $\theta = 0°$, the vectors have the same direction, and if $\theta = 180°$ the vectors have opposite directions. Furthermore,

$$\text{If } \theta = 0°,\ \mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos 0° = \|\mathbf{v}\|\|\mathbf{w}\|(1) = \|\mathbf{v}\|\|\mathbf{w}\|$$

$$\text{If } \theta = 180°,\ \mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos 180° = \|\mathbf{v}\|\|\mathbf{w}\|(-1) = -\|\mathbf{v}\|\|\mathbf{w}\|$$

This tells us that if **v** and **w** are parallel, then their dot product equals the product of their magnitudes or the opposite of the product of their magnitudes.

Learn: Decompose Vectors into Two Orthogonal Vectors

In many physical applications, it is important to find the contribution of a vector in a particular direction. For example, consider a car parked on a hill with angle of inclination θ. The force of gravity **F** pulls straight downward. However, **F** can be decomposed into two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting parallel to the hill and orthogonal to the hill, respectively (Figure 7-47).

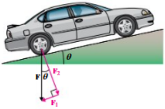


Figure 7-47

$\mathbf{F}_1$ and $\mathbf{F}_2$ are called vector components of **F**, and their sum is the original force vector ($\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}$). The vector components of **F** are important because they determine the opposing forces needed to keep the car in place. The vector $-\mathbf{F}_1$ is the minimum amount of force needed by the brakes to keep the car from rolling down the hill. Force $\mathbf{F}_2$ tells us the amount of force that the tires must withstand against the ground.

To decompose a vector into two orthogonal vectors, we use the idea of a vector projection. Consider two vectors $\mathbf{v}$ and $\mathbf{w}$ drawn with a common initial point $\boldsymbol{A}$ (Figures 7-48 and 7-49). Suppose we want to decompose $\mathbf{v}$ into vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, parallel to $\mathbf{w}$ and orthogonal to $\mathbf{w}$, respectively. We call vector $\mathbf{v}_1$ the projection of $\mathbf{v}$ onto $\mathbf{w}$ and denote this by $\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v}$.

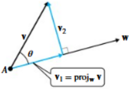


Figure 7-48

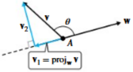


Figure 7-49

We want to find the decomposition of $\mathbf{v}$ into two orthogonal vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v}$, $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$, and $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$ (Figure 7-50).

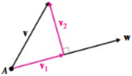


Figure 7-50

We first make two observations.

- $\mathbf{v}_1$ and $\mathbf{w}$ are parallel, so $\mathbf{v}_1 = c\mathbf{w}$ for some scalar c.
- $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$, which means that $\mathbf{v}_2 \cdot \mathbf{w} = 0$.

Then since $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$,

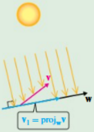
$$\begin{aligned}\mathbf{v} \cdot \mathbf{w} &= (\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w}\\ &= \mathbf{v}_1 \cdot \mathbf{w} + \mathbf{v}_2 \cdot \mathbf{w}\\ &= (c\mathbf{w}) \cdot \mathbf{w} + 0\\ &= c\|\mathbf{w}\|^2.\end{aligned}$$

Solving the equation $\mathbf{v} \cdot \mathbf{w} = c\|\mathbf{w}\|^2$ for c gives $c = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}$.

Therefore, $\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = c\mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}$.

Insights

To visualize the projection of $\mathbf{v}$ onto $\mathbf{w}$, think of sunlight shining onto vector $\mathbf{w}$ at a right angle. The directed shadow cast by $\mathbf{v}$ onto $\mathbf{w}$ is the vector projection of $\mathbf{v}$ onto $\mathbf{w}$.



These results are summarized as follows.

Decomposition of a Vector into Orthogonal Vectors

If **v** and **w** are two nonzero vectors, then

1. The vector projection of **v** onto **w** is given by $\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}$.
2. To decompose **v** into vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ parallel to **w** and orthogonal to **w**, respectively, we have

$$\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} \quad \text{and} \quad \mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$$

Example 4: Decomposing a Vector into Orthogonal Vectors

Given $\mathbf{v} = \langle -5, 4 \rangle$ and $\mathbf{w} = \langle 2, 4 \rangle$,

a. Find $\text{proj}_{\mathbf{w}}\mathbf{v}$.

b. Find vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v}_1$ is parallel to **w**, $\mathbf{v}_2$ is orthogonal to **w**, and $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$.

Solution:

a.

$$\begin{aligned} \text{proj}_{\mathbf{w}}\mathbf{v} &= \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} \\ &= \frac{(-5)(2) + (4)(4)}{(2)^2 + (4)^2}\langle 2, 4 \rangle \\ &= \frac{6}{20}\langle 2, 4 \rangle \\ &= \left\langle \frac{3}{5}, \frac{6}{5} \right\rangle \end{aligned}$$

$\mathbf{v}_1$ is the projection of **v** onto **w**. Recall that for vector $\mathbf{w} = \langle b_1, b_2 \rangle$, $\|\mathbf{w}\|^2 = \left(\sqrt{b_1^2 + b_2^2}\right)^2 = b_1^2 + b_2^2$ or equivalently $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w} = b_1^2 + b_2^2$.

b. $\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = \left\langle \frac{3}{5}, \frac{6}{5} \right\rangle$

$$\begin{aligned} \mathbf{v}_2 &= \mathbf{v} - \mathbf{v}_1 = \langle -5, 4 \rangle - \left\langle \frac{3}{5}, \frac{6}{5} \right\rangle \\ &= \left\langle -\frac{25}{5}, \frac{20}{5} \right\rangle - \left\langle \frac{3}{5}, \frac{6}{5} \right\rangle \\ &= \left\langle -\frac{28}{5}, \frac{14}{5} \right\rangle \end{aligned}$$

Use the relationship $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$ to solve for $\mathbf{v}_2$.

Check Your Work

We can check the solution to Example 4 by showing that vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ meet all the necessary conditions.

1. $\mathbf{v}_1$ is parallel to **w** because $\mathbf{v}_1 = c\mathbf{w}$ for a constant c.

$$\mathbf{v}_1 = \left\langle \frac{3}{5}, \frac{6}{5} \right\rangle = \frac{3}{10}\langle 2, 4 \rangle = \frac{3}{10}\mathbf{w} \checkmark$$

2. $\mathbf{v}_2$ is orthogonal to **w** because $\mathbf{v}_2 \cdot \mathbf{w} = 0$.

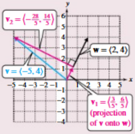
$$\begin{aligned} \mathbf{v}_2 \cdot \mathbf{w} &= \left\langle -\frac{28}{5}, \frac{14}{5} \right\rangle \cdot \langle 2, 4 \rangle \\ &= \left(-\frac{28}{5}\right)(2) + \left(\frac{14}{5}\right)(4) = -\frac{56}{5} + \frac{56}{5} = 0 \checkmark \end{aligned}$$

So, **v** and **w** are orthogonal.

3. $\mathbf{v}_1 + \mathbf{v}_2 \overset{?}{=} \mathbf{v}$

$$\left\langle \frac{3}{5}, \frac{6}{5} \right\rangle + \left\langle -\frac{28}{5}, \frac{14}{5} \right\rangle \overset{?}{=} \langle -5, 4 \rangle$$

$$\left\langle -\frac{25}{5}, \frac{20}{5} \right\rangle \overset{?}{=} \langle -5, 4 \rangle \checkmark$$



Apply the Skills

4. Repeat Example 4 with $\mathbf{v} = \langle 7, -3 \rangle$ and $\mathbf{w} = \langle -2, -6 \rangle$.

Example 5: Finding a Component of Force in a Given Direction

A truck weighing 2000 lb is parked on a hill with a 10° incline.

a. Write the force vector **F** representing the weight of the truck through a single tire. Write **F** in terms of **i** and **j** and assume that the weight of the truck is evenly distributed among all four tires.

b. Find the component vector of **F** parallel to the hill. Round to 1 decimal place.

c. Find the magnitude of the force required by the brakes on each wheel to keep the truck from rolling down the hill.

Solution:

a. The weight of the truck is evenly distributed among all four tires and is a force acting straight downward. The magnitude of **F** is (2000 lb)/4. Therefore, $\mathbf{F} = -500\mathbf{j}$ represents the force of gravity acting straight downward where a given tire meets the road.

b. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the components of **F** parallel and perpendicular to the hill, respectively. Let $\mathbf{u} = \cos 10°\mathbf{i} + \sin 10°\mathbf{j}$ be the unit vector in the direction up the hill. The vector component of **F** parallel to the hill is the projection of **F** onto **u**, and is labeled as $\mathbf{F}_1$ in Figure 7-51.

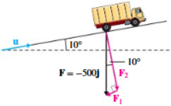


Figure 7-51

$$\mathbf{F}_1 = \text{proj}_{\mathbf{u}}\mathbf{F} = \frac{\mathbf{F} \cdot \mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u} = \frac{(0\mathbf{i} - 500\mathbf{j}) \cdot (\cos 10°\mathbf{i} + \sin 10°\mathbf{j})}{(1)^2}(\cos 10°\mathbf{i} + \sin 10°\mathbf{j})$$

Recall that a unit vector has a magnitude of 1.

$$= \frac{(0)(\cos 10°) + (-500)(\sin 10°)}{1}(\cos 10°\mathbf{i} + \sin 10°\mathbf{j})$$

$$\approx -86.8(\cos 10°\mathbf{i} + \sin 10°\mathbf{j})$$

$$\approx -85.5\mathbf{i} - 15.1\mathbf{j}$$

c. The brakes on each wheel represent the *opposing* force to $\mathbf{F}_1$. That is, the brakes provide a force $\mathbf{B} = -\mathbf{F}_1 = -(-85.5\mathbf{i} - 15.1\mathbf{j}) = 85.5\mathbf{i} + 15.1\mathbf{j}$.

$\|\mathbf{B}\| = \sqrt{(85.5)^2 + (15.1)^2} \approx 86.8 \text{lb}$

The brakes on each wheel must exert a force of approximately 86.8 lb to keep the truck from rolling down the hill.

Insights

As an alternative approach to part (c), use the right triangle formed by $\mathbf{F}$, $\mathbf{F}_1$, and $\mathbf{F}_2$ in Figure 7-51.

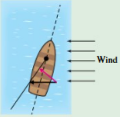
$$\|\mathbf{B}\| = \|\mathbf{F}_1\| = \|\mathbf{F}\|\sin 10° = 500 \sin 10° \approx 86.8 \text{ lb}$$

Apply the Skills

5. Repeat Example 5 with a 1400-lb SUV parked on a hill with an incline of 12°.

Point of Interest

A sailboat is propelled forward by the force of wind on the sail and by the force of water against the keel ("fin" that extends below the boat to prevent sideways "slippage"). The force of wind on the sail can be broken down into two vectors: one acting parallel to the sail and one acting perpendicular to the sail. The component acting parallel to the sail (called the drag component) "slips" by and does not contribute to the forward motion. The component force of the wind acting perpendicular to the sail (shown in red) contributes to the forward motion of the boat.



Learn: Use Vectors to Compute Work

Suppose a person lifts a 30-lb weight straight upward 4 ft. To quantify the amount of effort for this task, we can compute the amount of work done. In physics, the work *W* done by a constant force **F** in moving an object from point *A* to point *B* along the line of motion is given by

$$W = \begin{pmatrix}\text{Magnitude of force in} \\ \text{the direction of motion}\end{pmatrix}\begin{pmatrix}\text{Distance the} \\ \text{object is moved}\end{pmatrix}$$

Since a force of 30 lb is required to lift the weight, the work done raising it 4 ft is

$$W = (30 \text{ lb})(4 \text{ ft}) = 120 \text{ ft·lb}$$

Notice that work is a scalar quantity. In the U.S. customary units of measurement, the common unit for work is the foot-pound (ft·lb). In the metric system, the Newton (N) is a common unit of force, and the Newton·meter (N·m) is the associated unit of work.

It is important to note that if a force vector **F** is *not* directed along the line of motion, then only the component of force in the direction of motion is applied to the work done (Figure 7-52). Therefore, if **D** is the displacement vector of an object moved in a straight line from point *A* to point *B* by a force **F**, the work done is given by

Magnitude of force in the direction of **D**

Distance moved

$$W = \|\text{proj}_{\mathbf{D}}\mathbf{F}\|\|\mathbf{D}\| = \left(\frac{\mathbf{F}\cdot\mathbf{D}}{\|\mathbf{D}\|^2}\|\mathbf{D}\|\right)\|\mathbf{D}\|$$
$$= \mathbf{F}\cdot\mathbf{D}$$

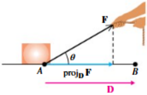


Figure 7-52

Definition of Work

If **D** is the displacement vector of an object in moving the object in a straight line from points *A* to *B* under a constant force **F**, then the work *W* done is computed by

1. $W = \mathbf{F}\cdot\mathbf{D}$.
2. $W = \|\text{proj}_{\mathbf{D}}\mathbf{F}\|\|\mathbf{D}\|$ or $W = \|\mathbf{F}\|\|\mathbf{D}\|\cos\theta$, where θ is the angle between **F** and **D**.

Example 6: Computing Work

Find the work done by a force $\mathbf{F} = 4\mathbf{i} + 9\mathbf{j}$ (in N) in moving an object in a straight line from point *A*(1, 3) to point *B*(4, 8). Assume that the units in the coordinate plane are in meters.

Solution:

The displacement vector for the object is $\mathbf{D} = \overrightarrow{AB} = (4-1)\mathbf{i} + (8-3)\mathbf{j} = 3\mathbf{i} + 5\mathbf{j}$.

The work done is the dot product of the force vector and displacement vector.

$W = \mathbf{F}\cdot\mathbf{D} = (4\mathbf{i} + 9\mathbf{j})\cdot(3\mathbf{i} + 5\mathbf{j}) = (4)(3) + (9)(5) = 12 + 45 = 57$

The amount of work is 57 N·m.

Apply the Skills

6. Find the work done by a force $\mathbf{F} = 2\mathbf{i} + 7\mathbf{j}$ (in N) in moving an object in a straight line from point *A*(–1, 5) to point *B*(9, 11). Assume that the units in the coordinate plane are in meters.

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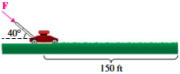
Example 7: Computing Work

A teenager pushes a lawn mower 150 ft across a yard on a horizontal path and exerts a constant force of 30 lb directed downward at an angle of 40° with the horizontal. Find the work done. Round to the nearest ft-lb.



Solution:

Let A and B designate the initial and final positions of the lawn mower. Let $\mathbf{D}$ represent the displacement vector from A to B. In Figure 7-53, we have drawn the force vector $\mathbf{F}$ with initial point at A. The angle θ between the vectors is 40°.

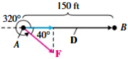


Figure 7-53

Using the projection of **F** onto **D** (shown in blue), we have

$$\begin{aligned} W &= \|\text{proj}_{\mathbf{D}}\mathbf{F}\|\|\mathbf{D}\| \\ &= \mathbf{F}\|\|\mathbf{D}\|\cos\theta \\ &= (30\text{ lb})(150\text{ ft})(\cos 40°) \\ &= 4500\cos 40° \\ &\approx 3447\text{ ft}\cdot\text{lb} \end{aligned}$$

Insights

As an alternative approach, we can write

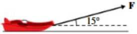
$\mathbf{F} = 30\cos 320°\mathbf{i} + 30\sin 320°\mathbf{j}$ and $\mathbf{D} = 150\mathbf{i} + 0\mathbf{j}$

Then $W = \mathbf{F}\cdot\mathbf{D}$

$$\begin{aligned} &= (30\cos 320°)(150) + (30\sin 320°)(0) \\ &\approx 3447\text{ ft-lb} \end{aligned}$$

Apply the Skills

7. A dog pulls a sled 200 ft across a yard on a horizontal path. The dog exerts a constant force of 25 lb directed upward 15° from the horizontal. Find the work done. Round to the nearest ft-lb.



When the force and displacement vectors are in the same direction, the work done in moving the object is positive. If a force is applied in the opposite direction to a moving object (such as the force applied to brake a skidding car), the work done by the force is negative. In this lesson, we only address situations with positive work.

Practice 7-5

Practice Exercises

Prerequisite Review

R.1. Simplify. $\sqrt{5^2 + 12^2}$

For Exercises R.2–R.3, use a calculator to approximate the degree measure of θ to 1 decimal place.

R.2. $\cos\theta = \frac{5}{6}$ and $0° \le \theta \le 90°$

R.3. $\cos\theta = -\frac{8}{11}$ and $90° \le \theta \le 180°$

Concept Connections

1. The dot product of $\mathbf{v} = \langle a_1, b_1 \rangle$ and $\mathbf{w} = \langle a_2, b_2 \rangle$ equals __________, and the result is a (scalar/vector).
2. If θ is the angle between two nonzero vectors **v** and **w**, then the cosine of θ can be found from the dot product of **v** and **w** as $\cos\theta =$ __________.
3. If θ is the angle between two nonzero vectors **v** and **w**, and $\mathbf{v} \cdot \mathbf{w} = 0$, then $\theta =$ __________°.
4. Two vectors are said to be __________ if and only if they meet at a right angle.
5. The projection of a nonzero vector **v** onto a nonzero vector **w** is given by $\text{proj}_{\mathbf{w}}\mathbf{v} =$ __________.
6. Suppose that **D** is the displacement vector of an object when the object is moved in a straight line from points *A* to *B* under a constant force **F**. Then the work done is given by W = __________.

Learn: Compute Dot Product

For Exercises 7–14, for the given vectors, find

a. $\mathbf{v} \cdot \mathbf{w}$ **b.** $\mathbf{v} \cdot \mathbf{v}$ **c.** $\mathbf{w} \cdot \mathbf{w}$ **(See Example 1, p. 819)**

7. $\mathbf{v} = \langle -3, 5 \rangle$, $\mathbf{w} = \langle 10, -8 \rangle$
8. $\mathbf{v} = \langle 6, -7 \rangle$, $\mathbf{w} = \langle -1, -4 \rangle$
9. $\mathbf{v} = \left\langle \frac{3}{8}, \frac{2}{3} \right\rangle$, $\mathbf{w} = \left\langle -\frac{1}{6}, -\frac{3}{4} \right\rangle$
10. $\mathbf{v} = \left\langle -\frac{1}{3}, -\frac{4}{5} \right\rangle$, $\mathbf{w} = \left\langle \frac{1}{3}, -\frac{5}{8} \right\rangle$
11. $\mathbf{v} = 6\mathbf{i}$, $\mathbf{w} = -3\mathbf{j}$
12. $\mathbf{v} = 2\mathbf{j}$, $\mathbf{w} = 14\mathbf{i}$
13. $\mathbf{v} = 2.1\mathbf{i} - 6.8\mathbf{j}$, $\mathbf{w} = 0.4\mathbf{i} - 0.3\mathbf{j}$
14. $\mathbf{v} = -0.5\mathbf{i} + 1.6\mathbf{j}$, $\mathbf{w} = -0.1\mathbf{i} + 2.3\mathbf{j}$

For Exercises 15–18, use the dot product to find the magnitude of the vector.

15. $\mathbf{v} = 4\mathbf{i} - 8\mathbf{j}$
16. $\mathbf{w} = -18\mathbf{i} + 3\mathbf{j}$
17. $\mathbf{r} = \langle 5, -3 \rangle$
18. $\mathbf{s} = \langle 7, -4 \rangle$

For Exercises 19–28, use vectors $\mathbf{u} = \langle -4, 1 \rangle$, $\mathbf{v} = \langle 5, -2 \rangle$, and $\mathbf{w} = \langle 0, 6 \rangle$ to perform the indicated operation. Then determine whether the result is a scalar or a vector.

19. $4\mathbf{v} \cdot \mathbf{w}$
20. $3(\mathbf{u} \cdot \mathbf{v})$
21. $\mathbf{v} \cdot \mathbf{v}$
22. $\mathbf{w} \cdot \mathbf{w}$
23. $(\mathbf{v} \cdot \mathbf{u})\mathbf{w}$
24. $(\mathbf{u} \cdot \mathbf{w})\mathbf{v}$
25. $\|\mathbf{w}\|\mathbf{u}$
26. $\|\mathbf{u}\|\mathbf{w}$
27. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$
28. $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u})$

For Exercises 29–32, let $\mathbf{v} = \langle a_1, b_1 \rangle$, $\mathbf{w} = \langle a_2, b_2 \rangle$, $\mathbf{u} = \langle a_3, b_3 \rangle$, and c be a real number. Prove the given statement.

29. $(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w})$
30. $\mathbf{0} \cdot \mathbf{v} = 0$
31. $c(\mathbf{v} \cdot \mathbf{w}) = c\mathbf{v} \cdot \mathbf{w}$
32. $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u}) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{u}$

Learn: Find the Angle Between Two Vectors

33. If θ is the angle between two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$, and $\mathbf{v} \cdot \mathbf{w} < 0$, what is the range of θ? Give the answer in degree measure.

34. If θ is the angle between two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$, and $\mathbf{v} \cdot \mathbf{w} > 0$, what is the range of θ? Give the answer in degree measure.

For Exercises 35–40, find the angle θ between v and w. If necessary, round to the nearest tenth of a degree. (See Example 2, p. 821)

35. $\mathbf{v} = -8\mathbf{i} + 6\mathbf{j}$, $\mathbf{w} = 4\mathbf{i} + 18\mathbf{j}$

36. $\mathbf{v} = 12\mathbf{i} - 5\mathbf{j}$, $\mathbf{w} = -3\mathbf{i} - 4\mathbf{j}$

37. $\mathbf{v} = \langle 1, 7\rangle$, $\mathbf{w} = \langle -12, -8\rangle$

38. $\mathbf{v} = \langle 8, 3\rangle$, $\mathbf{w} = \langle 4, -3\rangle$

39. $\mathbf{v} = \cos 45^\circ\mathbf{i} + \sin 45^\circ\mathbf{j}$ and $\mathbf{w} = \cos 60^\circ\mathbf{i} + \sin 60^\circ\mathbf{j}$

40. $\mathbf{v} = \cos 150^\circ\mathbf{i} + \sin 150^\circ\mathbf{j}$ and $\mathbf{w} = \cos 225^\circ\mathbf{i} + \sin 225^\circ\mathbf{j}$

41. If $\|\mathbf{v}\| = 10$ and $\|\mathbf{w}\| = 6$ and the angle between $\mathbf{v}$ and $\mathbf{w}$ is 30°, find $\mathbf{v} \cdot \mathbf{w}$.

42. If $\|\mathbf{v}\| = 2.6$ and $\|\mathbf{w}\| = 8$ and the angle between $\mathbf{v}$ and $\mathbf{w}$ is 60°, find $\mathbf{v} \cdot \mathbf{w}$.

For Exercises 43–50, determine if the given vectors are orthogonal, parallel, or neither. (See Example 3, p. 822)

43. $\mathbf{r} = \langle -12, 6\rangle$, $\mathbf{s} = \langle -8, -16\rangle$

44. $\mathbf{t} = \langle -6, 18\rangle$, $\mathbf{s} = \langle 9, 3\rangle$

45. $\mathbf{w} = \frac{1}{5}\mathbf{i} - \frac{1}{4}\mathbf{j}$, $\mathbf{v} = \frac{15}{4}\mathbf{i} + 3\mathbf{j}$

46. $\mathbf{b} = \frac{5}{2}\mathbf{i} + \frac{25}{6}\mathbf{j}$, $\mathbf{a} = -\frac{10}{9}\mathbf{i} + \frac{2}{3}\mathbf{j}$

47. $\mathbf{p} = \langle 2, 6\rangle$, $\mathbf{q} = \langle -1, -3\rangle$

48. $\mathbf{k} = \langle -\frac{3}{2}, 2\rangle$, $\mathbf{s} = \langle 6, -8\rangle$

49. $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{w} = \mathbf{i} - 3\mathbf{j}$

50. $\mathbf{z} = 14\mathbf{i} + 7\mathbf{j}$, $\mathbf{t} = -7\mathbf{i} - 14\mathbf{j}$

Learn: Decompose Vectors into Two Orthogonal Vectors

51. Given $\mathbf{v} = \langle -3, 6\rangle$ and $\mathbf{w} = \langle 1, 3\rangle$,

- **a.** Find $\text{proj}_{\mathbf{w}}\mathbf{v}$. **(See Example 4, p. 825)**
- **b.** Find vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v}_1$ is parallel to $\mathbf{w}$, $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$, and $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$.
- **c.** Using the results from part (b), show that $\mathbf{v}_1$ is parallel to $\mathbf{w}$ by finding a constant c such that $\mathbf{v}_1 = c\mathbf{w}$.
- **d.** Show that $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$.
- **e.** Show that $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$.

52. Given $\mathbf{v} = \langle 4, -3\rangle$ and $\mathbf{w} = \langle 2, -2\rangle$,

- **a.** Find $\text{proj}_{\mathbf{w}}\mathbf{v}$.
- **b.** Find vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v}_1$ is parallel to $\mathbf{w}$, $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$, and $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$.
- **c.** Using the results from part (b) show that $\mathbf{v}_1$ is parallel to $\mathbf{w}$ by finding a constant c such that $\mathbf{v}_1 = c\mathbf{w}$.
- **d.** Show that $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$.
- **e.** Show that $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$.

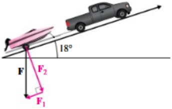
53. Given $\mathbf{v} = 4\mathbf{i} + 9\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} - \mathbf{j}$,

- **a.** Find $\text{proj}_{\mathbf{w}}\mathbf{v}$.
- **b.** Find vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v}_1$ is parallel to $\mathbf{w}$, $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$, and $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$.

54. Given $\mathbf{v} = -8\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = -6\mathbf{i} + 4\mathbf{j}$,

- **a.** Find $\text{proj}_{\mathbf{w}}\mathbf{v}$.
- **b.** Find vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v}_1$ is parallel to $\mathbf{w}$, $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$, and $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$.

55. A boat and trailer weighing a total of 450 lb are parked on a boat ramp with an 18° angle of inclination. Assume that the weight of the boat and trailer is evenly distributed between two wheels. (**See Example 5, p. 826**)

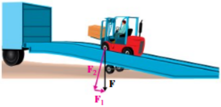


a. Write the force vector **F** in terms of **i** and **j** representing the weight of the boat and trailer for a single tire.

b. Find the component vector of **F** parallel to the ramp. Round values to 1 decimal place.

c. Find the magnitude of the force needed to keep the trailer from moving down the ramp. Round to the nearest pound.

56. A forklift is used to offload freight from a delivery truck. Together the forklift and its contents weigh 800 lb, and the weight is evenly distributed among four wheels. The ramp is inclined 14° from the horizontal.



a. Write the force vector **F** in terms of **i** and **j** representing the weight against a single tire.

b. Find the component vector of **F** parallel to the ramp. Round values to 1 decimal place.

c. Find the magnitude of the force needed to keep the forklift from rolling down the ramp. Round to the nearest pound.

57. Refer to Exercise 55.

a. Find the component vector of **F** perpendicular to the ramp. Round values to 1 decimal place.

b. Find the magnitude of the force that the ramp makes against each tire. Round to the nearest pound.

58. Refer to Exercise 56.

a. Find the component vector of **F** perpendicular to the ramp. Round values to 1 decimal place.

b. Find the magnitude of the force that the ramp makes against each tire. Round to the nearest pound.

Learn: Use Vectors to Compute Work

59. a. Find the amount of work done in lifting a 50-lb weight upward 5 ft.

b. If 200 N of force is applied in the direction of motion in moving an object 10 m, how much work is done?

60. a. Find the amount of work done pulling an object horizontally 10 ft using a force of 40 lb in the direction of motion.

b. If 2250 N of force is applied in the direction of motion in moving an object 50 m, how much work is done?

For Exercises 61–64, find the work done by the given force F acting in the direction from point *A* to point *B*. Assume that the units in the coordinate plane are in meters.

61. $A(3, -1)$, $B(8, 11)$; $\|\mathbf{F}\| = 400\,\text{N}$

62. $A(-3, -2)$, $B(5, 4)$; $\|\mathbf{F}\| = 250\,\text{N}$

63. $A(12, 5)$, $B(-3, 10)$; $\|\mathbf{F}\| = 38\,\text{N}$

64. $A(17, -7)$, $B(27, -1)$; $\|\mathbf{F}\| = 18\,\text{N}$

For Exercises 65–68, find the work *W* done by a force F in moving an object in a straight line given by the displacement vector D. (See Example 6, p. 828)

65. $\mathbf{F} = (40\mathbf{i} - 15\mathbf{j})$ lb; $\mathbf{D} = (30\mathbf{i} + 10\mathbf{j})$ ft

66. $\mathbf{F} = (220\mathbf{i} + 350\mathbf{j})$ lb; $\mathbf{D} = (-12\mathbf{i} + 16\mathbf{j})$ ft

67. $\mathbf{F} = (-26\mathbf{i} + 32\mathbf{j})$ N; $\mathbf{D} = (100\mathbf{i} + 120\mathbf{j})$ m

68. $\mathbf{F} = (6\mathbf{i} + 3\mathbf{j})$ N; $\mathbf{D} = (14\mathbf{i} - 8\mathbf{j})$ m

For Exercises 69–72, find the work done by a force F (in lb) in moving an object in a straight line from point *A* to point *B*. Assume that the units in the coordinate plane are in feet. (See Example 6, p. 828)

69. $A(0, 0)$, $B(2, 6)$; $\mathbf{F} = 3\mathbf{i} + 7\mathbf{j}$

70. $A(0, 0)$, $B(-5, 4)$; $\mathbf{F} = -2\mathbf{i} + 10\mathbf{j}$

71. $A(20, 35)$, $B(32, 57)$; $\mathbf{F} = 150\mathbf{i} + 200\mathbf{j}$

72. $A(-14, -20)$, $B(8, -35)$; $\mathbf{F} = 40\mathbf{i} + 26\mathbf{j}$

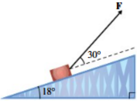
73. Find the amount of work done pulling an object horizontally 50 ft if a force of 60 lb is directed 30° above the horizontal.

74. Find the amount of work done if an object is pushed horizontally 70 m by a force of 25 N directed 60° above the horizontal.

75. A tree removal service hooks a chain around the stump of a tree and attaches the other end of the chain to a pickup truck. The truck drags the stump horizontally 40 m to a wood chipper. The chain is directed upward 32° from the horizontal, and the tension in the chain is 15,000 N. Find the amount of work done. Round to the nearest N·m. **(See Example 7, p. 829)**

76. A child exerts a force of 28 N on the handle of a small wagon. If the handle of the wagon is directed upward 18° from the horizontal, find the amount of work done in moving the wagon 20 m horizontally. Round to the nearest N·m.

77. A force of 16 lb is used to pull a block up a ramp. If the ramp is inclined 18° above the horizontal, and the force is directed 30° above the ramp, find the work done moving the block 15 ft along the ramp.



78. A force of 28 lb is used to pull a block up a ramp. If the ramp is inclined 10° above the horizontal, and the force is directed 45° above the ramp, find the work done moving the block 20 ft along the ramp.

Mixed Exercises

79. Given $\mathbf{v} = 22\mathbf{i} - 55\mathbf{j}$ and $\mathbf{w} = -4\mathbf{i} + 10\mathbf{j}$,

- **a.** Find the angle between $\mathbf{v}$ and $\mathbf{w}$.
- **b.** Are the vectors parallel? If yes, find a real number c such that $\mathbf{v} = c\mathbf{w}$.

80. Given $\mathbf{v} = -21\mathbf{i} + 33.6\mathbf{j}$ and $\mathbf{w} = 5\mathbf{i} - 8\mathbf{j}$,

- **a.** Find the angle between $\mathbf{v}$ and $\mathbf{w}$.
- **b.** Are the vectors parallel? If yes, find a real number c such that $\mathbf{v} = c\mathbf{w}$.

81. Given $r = \langle -3, 4 \rangle$ and $s = \langle 9, -12 \rangle$,

- **a.** Find the product $\|\mathbf{r}\|\|\mathbf{s}\|$.
- **b.** Find $\mathbf{r} \cdot \mathbf{s}$.
- **c.** Based on the results of parts (a) and (b), what do you know about the two vectors?

82. Given $c = \langle 5, -12 \rangle$ and $d = \langle 10, -24 \rangle$,

- **a.** Find the product $\|\mathbf{c}\|\|\mathbf{d}\|$.
- **b.** Find $\mathbf{c} \cdot \mathbf{d}$.
- **c.** Based on the results of parts (a) and (b), what do you know about the two vectors?

83. If $\text{proj}_{\mathbf{w}}\mathbf{v} = \mathbf{v}$, what do you know about $\mathbf{v}$ and $\mathbf{w}$?

84. If $\text{proj}_{\mathbf{w}}\mathbf{v} = 0$, what do you know about $\mathbf{v}$ and $\mathbf{w}$?

85. Given $\mathbf{v} = 4\mathbf{i} - 10\mathbf{j}$,

- **a.** Find two vectors parallel to $\mathbf{v}$, one in the same direction as $\mathbf{v}$ and one in the opposite direction as $\mathbf{v}$. Answers will vary.
- **b.** Find two vectors orthogonal to $\mathbf{v}$. Answers will vary.

86. Given $\mathbf{v} = -3\mathbf{i} - 9\mathbf{j}$,

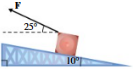
- **a.** Find two vectors parallel to $\mathbf{v}$, one in the same direction as $\mathbf{v}$ and one in the opposite direction as $\mathbf{v}$. Answers will vary.
- **b.** Find two vectors orthogonal to $\mathbf{v}$. Answers will vary.

The dot product of vectors can be used in business applications. For Exercises 87–88, find the dot product and interpret the results.

87. The components of $\mathbf{n} = \langle 500, 330\rangle$ represent the number of T-shirts and hats, respectively, in the inventory of a surf shop. The components of $\mathbf{p} = \langle 15, 9\rangle$ represent the price (in $) per T-shirt and hat, respectively. Find $\mathbf{n}\cdot\mathbf{p}$ and interpret the result.

88. The components of $\mathbf{n} = \langle 4, 2\rangle$ represent the number of tacos and drinks, respectively, that a restaurant patron had for lunch. The components of $\mathbf{c} = \langle 120, 90\rangle$ represent the number of calories per taco and number of calories per drink, respectively. Find $\mathbf{n}\cdot\mathbf{c}$ and interpret the result.

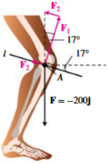
89. A force of 40 lb is used to pull a block up a ramp. If the ramp is inclined 10° above the horizontal, and the force is directed 25° from the horizontal, find the work done moving the block 25 ft along the ramp. Round to the nearest unit.



90. A force of 65 lb is used to pull a block up a ramp. If the ramp is inclined 18° above the horizontal, and the force is directed 38° from the horizontal, find the work done moving the block 12 ft along the ramp. Round to the nearest unit.

91. Suppose that the contact point between the femur (large bone in the thigh) and tibia (supporting bone from lower leg) meet at point *A*. The planar surface *l* between the two bones is directed 17° downward from the horizontal, and the individual's body weight is 200 lb directed downward. Rounding to the nearest tenth of a pound,

- **a.** Find the magnitude of the force $\mathbf{F}_1$ that the tibia must exert normal to *l* to oppose the weight.
- **b.** Find the magnitude of force $\mathbf{F}_2$ within the knee joint that prevents the bones from slipping.



92. The individual in Exercise 91 is directed to lose weight by a physician. After 6 months of exercise and dietary changes, the individual's new weight is 175 lb. Recalculate parts (a) and (b) with the new body weight.

Write About It

93. Explain how the dot product can be used to determine whether the angle between two nonzero vectors is less than 90° or greater than 90°.

94. Explain what is meant by decomposing a nonzero vector **v** into orthogonal vectors.

Expanding Your Skills

95. Prove that
$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\mathbf{v} \cdot \mathbf{w}$.

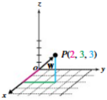
96. Prove that
$\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 = 4(\mathbf{v} \cdot \mathbf{w})$.

97. Prove that
$\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)$.

98. If **v** and **w** are vectors with the same magnitude, show that **v** + **w** and **v** − **w** are orthogonal.

A vector v in a plane is a line segment with a specified direction, where the component form is given by two coordinates $\langle a, b\rangle$. Similarly, we may define a vector w in three-dimensional space as a line segment in space with a specified direction where the component form is given by three coordinates $\langle a, b, c\rangle$.

For example, a vector w from the origin to a point $P(2, 3, 3)$ is given in component form as $\mathbf{w} = \langle 2, 3, 3\rangle$ or, in terms of the unit vectors $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$, as $\mathbf{w} = 2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$. Use this convention for Exercises 99–100.



99. If $\mathbf{v} = \langle a_1, b_1, c_1\rangle$ and $\mathbf{w} = \langle a_2, b_2, c_2\rangle$, the **dot product $\mathbf{v} \cdot \mathbf{w}$** is defined as $\mathbf{v} \cdot \mathbf{w} = a_1a_2 + b_1b_2 + c_1c_2$. Evaluate $\mathbf{v} \cdot \mathbf{w}$ for

a. $\mathbf{v} = \langle -2, 1, 4\rangle$, $\mathbf{w} = \langle 3, -1, 5\rangle$

b. $\mathbf{v} = -6\mathbf{j} + 3\mathbf{k}$, $\mathbf{w} = 10\mathbf{i} + 12\mathbf{k}$

100. The magnitude of $\mathbf{v} = \langle a, b, c\rangle$ is given by $\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}$. Find the magnitude of each vector.

a. $\mathbf{v} = \langle -8, 2, 12\rangle$

b. $\mathbf{w} = 3\mathbf{i} - 9\mathbf{j} + 12\mathbf{k}$

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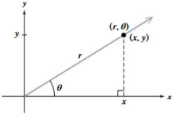
Chapter 7 Review

Chapter 7 Key Concepts

Lesson 7-1 Polar Coordinates

Key Concepts	Reference
A **polar coordinate system** consists of a fixed point O called the **pole** (or origin), and a ray called the **polar axis,** with endpoint at the pole. Each point P in the plane is defined by an ordered pair of the form (r, θ), where r is the directed distance from the pole to P, and θ is a directed angle from the polar axis to line $\overleftrightarrow{OP}$.	p. 744
A point in a plane can be described by an ordered pair (x, y) in rectangular coordinates or by an ordered pair (r, θ) in polar coordinates. $x = r\cos\theta$ $r^2 = x^2 + y^2$ $y = r\sin\theta$ $\tan\theta = \frac{y}{x}$ for $x \neq 0$  	p. 747
A **polar equation** is an equation described in a polar coordinate system with variables r and θ. Convert between rectangular coordinates and polar coordinates by using the relationships $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, and $\tan\theta = \frac{y}{x}$ for $x \neq 0$.	p. 750

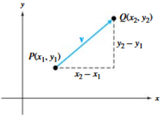
Lesson 7-2 Graphs of Polar Equations

Key Concepts	Reference
A **solution** to a polar equation is an ordered pair (r, θ) that satisfies the equation. The graph of a polar equation is the graph of all solutions to the equation.	p. 759
The graphs of polar equations may be symmetric with respect to the polar axis, the line $\theta = \frac{\pi}{2}$, or the pole. For an equation in polar coordinates, if the given substitution produces an equivalent equation, then the graph has the indicated type of symmetry. (1) Symmetry with respect to the polar axis: Replace (r, θ) by $(r, -\theta)$. (2) Symmetry with respect to the line $\theta = \frac{\pi}{2}$: Replace (r, θ) by $(r, \pi - \theta)$. (3) Symmetry with respect to the pole: Replace (r, θ) by $(-r, \theta)$.	p. 761
Polar equations and their graphs may be members of specific categories of polar equations representing lines, circles, spirals, limaçons, roses, or lemniscates.	pp. 767–768

Lesson 7-3 Complex Numbers in Polar Form

Key Concepts	Reference
In the **complex plane,** the horizontal axis (called the **real axis**) represents the real part of a complex number, and the vertical axis (called the **imaginary axis**) represents the imaginary part. The complex number $z = a + bi$ is written as an ordered pair (a, b) and can then be graphed as such in the complex plane.	p. 779
The absolute value or **modulus** of a complex number is its distance from the origin in the complex plane. If $z = a + bi$ is a complex number, then $\{z\} = \sqrt{a^2 + b^2}$.	p. 780
The **polar form** (or **trigonometric form**) of a complex number $z = a + bi$ is $z = r(\cos\theta + i\sin\theta)$, where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = \frac{b}{a}(a \neq 0)$. There are infinitely many choices for θ, but normally θ is taken on the interval $0 \leq \theta < 2\pi$.	p. 781
The product and quotient of complex numbers can be found using their polar forms. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$ and $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$ for$(z_2 \neq 0)$	p. 783
De Moivre's theorem enables us to raise a complex number to an integer power and to find its nth roots. Let $z = r(\cos\theta + i\sin\theta)$. If n is a positive integer, then $z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ For an integer $n \geq 2$, z has n distinct nth roots of the form $w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$ for $k = 0, 1, 2, \ldots, n - 1$	pp. 785–787

Lesson 7-4 Vectors

Key Concepts	**Reference**
A **scalar** is a quantity that has magnitude only (such as speed). A **vector quantity** (such as velocity) has both magnitude *and* direction. A **vector** in the plane is a line segment with a specified direction. We denote a vector pointing from point P to point Q as $\overrightarrow{PQ}$. Point P is called the **initial point** of the vector and Q is called the **terminal point** of the vector. The **magnitude** of $\overrightarrow{PQ}$, denoted $\lVert\overrightarrow{PQ}\rVert$, is the length of the line segment. Two vectors are equal if they have the same magnitude and direction. A vector drawn in **standard position** has its initial point at the origin.	p. 797
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be points in the plane, and let $\mathbf{v} = \overrightarrow{PQ}$. The component form of $\mathbf{v}$ is given by $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$  y x $Q(x_2, y_2)$ v $y_2 - y_1$ $P(x_1, y_1)$ $x_2 - x_1$	p. 801
Two vectors may be added or subtracted geometrically, or by adding or subtracting the corresponding components. Multiplying a vector $\mathbf{v}$ by a scalar c results in a parallel vector $c\mathbf{v}$, and the length of $c\mathbf{v}$ is the length of $\mathbf{v}$ scaled by a factor of $\lvert c\rvert$.	p. 802
Several properties of vectors follow directly from the definition of vector addition, vector subtraction, and multiplication of a vector by a scalar.	p. 803
A unit vector has a length of 1 unit. If $\mathbf{v} = \langle a, b \rangle$, then a unit vector $\mathbf{u_v}$ in the direction of $\mathbf{v}$ is given by $\mathbf{u_v} = \frac{1}{\lVert\mathbf{v}\rVert}\mathbf{v} = \frac{1}{\lVert\mathbf{v}\rVert}\langle a, b \rangle = \left\langle \frac{a}{\lVert\mathbf{v}\rVert}, \frac{b}{\lVert\mathbf{v}\rVert} \right\rangle$	p. 804
Two important unit vectors are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ because they are unit vectors in the direction of the positive x- and y-axes, respectively. The representation of $\mathbf{v} = \langle a, b \rangle$ in terms of $\mathbf{i}$ and $\mathbf{j}$ is $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$. The values a and b are called the **scalar horizontal and vertical components** of $\mathbf{v}$, respectively.	p. 805
It is often important to find the component form of a vector $\mathbf{v}$ given its magnitude $\lVert\mathbf{v}\rVert$ and direction θ: $\mathbf{v} = \langle \lVert\mathbf{v}\rVert\cos\theta, \lVert\mathbf{v}\rVert\sin\theta \rangle$. Likewise, given the component form $\mathbf{v} = \langle a, b \rangle$, the magnitude and direction of a vector are given by $\lVert\mathbf{v}\rVert = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}(a \neq 0)$.	p. 806
Vectors have many useful applications in physics, navigation, and engineering such as finding the true direction and speed of a boat or ship under the influence of a current or wind.	p. 808
Suppose that forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots, \mathbf{F}_n$ act on an object. The sum of these forces is called the **resultant force.** The object is said to be in **static equilibrium** if the resultant force $\mathbf{R}$ is the zero vector, that is, if $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots + \mathbf{F}_n = \mathbf{0}$.	p. 811

Lesson 7-5 Dot Product

Key Concepts	**Reference**
If $\mathbf{v} = \langle a_1, b_1 \rangle$ and $\mathbf{w} = \langle a_2, b_2 \rangle$, the **dot product** $\mathbf{v} \cdot \mathbf{w}$ is defined as the scalar quantity $\mathbf{v} \cdot \mathbf{w} = a_1a_2 + b_1b_2$.	p. 819
The following are important properties of the dot product. If **u, v,** and **w** are vectors, and c is a scalar, then **1.** $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ **4.** $\mathbf{v} \cdot \mathbf{v} = \lVert \mathbf{v} \rVert^2$ **2.** $(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w})$ **5.** $\mathbf{0} \cdot \mathbf{v} = 0$ **3.** $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u}) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{u}$ **6.** $c(\mathbf{v} \cdot \mathbf{w}) = (c\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (c\mathbf{w})$	p. 820
If θ is the angle between two nonzero vectors **v** and **w,** then $\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\lVert \mathbf{v} \rVert \lVert \mathbf{w} \rVert}$ and $\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\lVert \mathbf{v} \rVert \lVert \mathbf{w} \rVert}\right)$.	p. 821
Two vectors **v** and **w** are orthogonal if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.	p. 822
In many applications it is important to decompose a vector into orthogonal vectors. If **v** and **w** are two nonzero vectors, then 1. The vector projection of **v** onto **w** is given by $\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\lVert \mathbf{w} \rVert^2}\mathbf{w}$. 2. To decompose **v** into vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ parallel to **w** and orthogonal to **w**, respectively, we have $\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\lVert \mathbf{w} \rVert^2}\mathbf{w}$ and $\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$.	p. 825
In physics, the work W done by a constant force **F** in moving an object from point A to point B along the line of motion is given by $W = \left(\begin{array}{c}\text{Magnitude of force in} \\ \text{the direction of motion}\end{array}\right)\left(\begin{array}{c}\text{Distance the} \\ \text{object is moved}\end{array}\right)$	p. 827
If a force vector **F** is *not* directed along the line of motion, then only the component of force in the direction of motion is applied to the work done. Therefore, if **D** is the displacement vector of an object moved in a straight line from point A to point B by a force **F**, the work done is given by 1. $W = \lVert \text{proj}_{\mathbf{D}}\mathbf{F} \rVert \, \lVert \mathbf{D} \rVert$ 2. $W = \lVert \mathbf{F} \rVert \, \lVert \mathbf{D} \rVert \cos\theta$, where θ is the angle between **F** and **D** 3. $W = \mathbf{F} \cdot \mathbf{D}$	p. 828

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Chapter 7 Test

1. Convert $\left(-5, \frac{5\pi}{4}\right)$ in polar coordinates to rectangular coordinates.

2. Convert $(-6\sqrt{3}, -6)$ in rectangular coordinates to polar coordinates with $r > 0$ and $0 \leq \theta < 2\pi$.

For Exercises 3–4, write an equivalent equation using polar coordinates.

3. $(x+2)^2 + y^2 = 4$

4. $y = x$

For Exercises 5–6, write an equivalent equation using rectangular coordinates.

5. $r = 8$

6. $r = 5\sec\theta$

For Exercises 7–10, graph the polar equation and identify the type of curve.

7. $r = 2\cos 3\theta$

8. $r = 2 - 6\cos\theta$

9. $r = 5 + 5\sin\theta$

10. $r = -5\sin 2\theta$

For Exercises 11–12, determine whether the tests for symmetry in Table 7-2 detect symmetry with respect to

a. The polar axis. Replace (r, θ) by $(r, -\theta)$.

b. The line $\theta = \frac{\pi}{2}$. Replace (r, θ) by $(r, \pi - \theta)$.

c. The pole. Replace (r, θ) by $(-r, \theta)$.

Otherwise, indicate that the test is inconclusive.

11. $r = 5\cos 3\theta$

12. $r = -7\sin\theta$

13. Find the modulus of $5 - 5i$ and write the number in polar form with $0 \leq \theta < 2\pi$.

14. Convert $24\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ from polar form to rectangular form, $a + bi$.

15. Find $[12(\cos 143^\circ + i\sin 143^\circ)][8(\cos 220^\circ + i\sin 220^\circ)]$ and write the product in polar form.

16. Find $\frac{18(\cos 100^\circ + i\sin 100^\circ)}{16(\cos 55^\circ + i\sin 55^\circ)}$ and write the quotient in polar form.

17. Use De Moivre's theorem to find $[3(\cos 210^\circ + i\sin 210^\circ)]^4$ and write the result in rectangular form $a + bi$.

18. Find the cube roots of $8\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$.

For Exercises 19–24, perform the indicated operations for $\mathbf{v} = \langle -8, 0 \rangle$ and $\mathbf{w} = \langle 5, -3 \rangle$.

19. $\mathbf{v} + 3\mathbf{w}$

20. $\|\mathbf{v} + \mathbf{w}\|$

21. $-\frac{6}{5}\mathbf{w}$

22. $8\mathbf{v} - 10\mathbf{w}$

23. $\mathbf{v} \cdot \mathbf{w}$

24. $(\mathbf{v} \cdot \mathbf{v})\mathbf{w}$

25. Given $\mathbf{v} = \langle 3, -10 \rangle$,

- **a.** Find $\|\mathbf{v}\|^2$.
- **b.** Find $\mathbf{v} \cdot \mathbf{v}$.
- **c.** How are $\|\mathbf{v}\|^2$ and $\mathbf{v} \cdot \mathbf{v}$ related?

26. Find a unit vector in the direction of $\mathbf{v} = -8\mathbf{i} + 6\mathbf{j}$.

27. Write the vector $\mathbf{v}$ in the form $a\mathbf{i} + b\mathbf{j}$, where $\|\mathbf{v}\| = 12$ and $\mathbf{v}$ has direction angle $\theta = 225^\circ$.

28. Find the magnitude and direction angle $(0^\circ \leq \theta < 360^\circ)$ for the vector $\mathbf{v} = -2\mathbf{i} + 7\mathbf{j}$. Round to 1 decimal place.

29. Find the angle θ between $\mathbf{v} = -12\mathbf{i}$ and $\mathbf{w} = 4\mathbf{i} + 6\mathbf{j}$. Round to the nearest tenth of a degree.

30. The forces $\mathbf{F}_1 = \langle -8, 1 \rangle$ and $\mathbf{F}_2 = \langle 5, 6 \rangle$ act on an object. Find the resultant force, **R**.

31. The forces $\mathbf{F}_1 = 6\mathbf{i} - \mathbf{j}$, $\mathbf{F}_2 = -7\mathbf{i} - 2\mathbf{j}$, and $\mathbf{F}_3 = 9\mathbf{i} + 3\mathbf{j}$ act on an object. What additional force **F** is needed for the object to be in static equilibrium?

32. Given $\mathbf{v} = \langle 7, -5 \rangle$ and $\mathbf{w} = \langle 10, 12 \rangle$, find $\text{proj}_{\mathbf{w}}\mathbf{v}$.

33. A plane heading S21°E with an an air speed of 375 mph encounters a wind blowing due east at 28 mph.

a. Express the velocity of the plane $\mathbf{v}_p$ relative to the air and the velocity of the wind $\mathbf{v}_w$ in terms of **i** and **j**. Round components to 1 decimal place.

b. Find the velocity of the plane relative to the ground $\mathbf{v}_g$ and the speed of the plane relative to the ground. Round the speed to the nearest mph.

c. Find the bearing of the plane relative to the ground. Round to the nearest tenth of a degree.

34. Find the magnitude of the force needed to keep a 250-lb crate from sliding down a ramp inclined 28° from the horizontal. Round to the nearest pound.

35. During a field goal kick, a football is kicked at an angle of 45° from the horizontal with a speed of 86 mph. Write the velocity vector **v** at the instant of the kick in terms of **i** and **j**.

36. Find the work done by a force $\mathbf{F} = -7\mathbf{i} + 3\mathbf{j}$ (in N) in moving an object in a straight line from point $A(6, 1)$ to point $B(-5, 4)$. Assume that the units in the coordinate plane are in meters.

37. Find the amount of work done if an object is pushed horizontally 120 m by a force of 15 N directed 40° above the horizontal. Round to the nearest N·m.

Go online for more practice problems.

Chapter 7 Cumulative Review Exercises

For Exercises 1–7, solve for y. Write solutions to inequalities in interval notation.

1. $A = \frac{x}{2}(x + y)$

2. $z = 10 - \sqrt{x^2 + y^2}$

3. $sy^2 + 2ty = r$

4. $A = A_0 e^{-ky}$

5. $R = 10 \log\left(\frac{y}{y_0}\right)$

6. $\frac{y^2 - 3y - 4}{y + 2} \leq 0$

7. $3 - |y - 10| \leq 0$

8. Write an equation of a circle in rectangular coordinates and in standard form with center at (−8, 5) and radius 8.

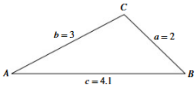
9. Write an equation of a circle in polar coordinates with center at (−8, 0) and radius 8.

10. Write an equation of a line in slope-intercept form in rectangular coordinates that passes through the points (−8, 5) and (5, −8).

11. Write an equation of a line in polar coordinates that passes through the points (−8, 8) and (8, −8).

For Exercises 12–14, solve each triangle. Round to the nearest tenth.

12.



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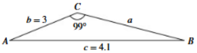

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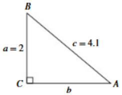


15. Evaluate the difference quotient $\frac{f(x + h) - f(x)}{h}$ for $f(x) = \frac{3}{x}$.

16. Give the domain and range of $f(x) = \frac{3}{x}$.

17. Find the period and phase shift of each function.

- **a.** $y = \tan\left(2x + \frac{\pi}{3}\right)$
- **b.** $y = \sin\left(2x + \frac{\pi}{3}\right)$

18. The height h (in feet) of a swing above ground level at time t (in sec) is modeled by

$$h(t) = -2\cos\left(\frac{2\pi}{3}t\right) + 3$$

- **a.** Find the maximum and minimum height of the swing.
- **b.** When is the first time after $t = 0$ that the swing is at a height of 4 ft?
- **c.** What is the period of the swing?
- **d.** What is the frequency of the swing?

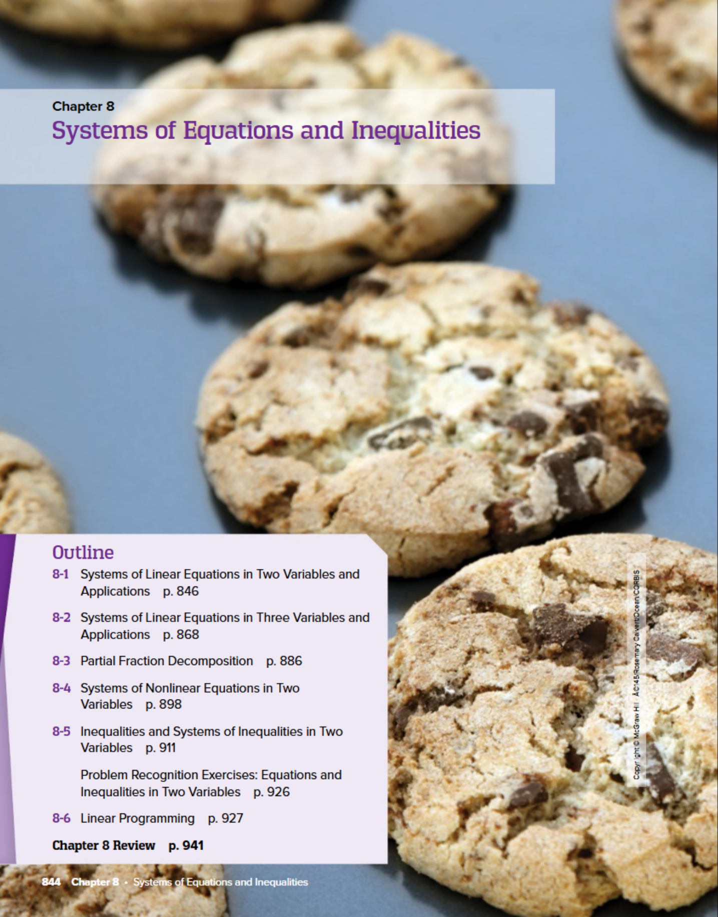
19. Given $\mathbf{v} = \langle 8, -3 \rangle$ and $\mathbf{w} = \langle 12, -4 \rangle$, find the vector projection of **v** onto **w**.

20. Find the amount of work done pulling an object horizontally 25 ft if a force of 120 lb is directed 38° above the horizontal. Round to the nearest ft · lb.

Go online for more practice problems.

Chapter 8

Systems of Equations and Inequalities



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Launch Activity: Mathematical Modeling

Making Cookies

The school cafeteria bakes enough cookies each day for every student in the school to have 2 cookies at lunch. At least 17 pounds of chocolate chips or chocolate chunks are needed to make cookies each day. At most, the school can spend $425 per week on chocolate chips and chocolate chunks. The cost for one 16-ounce bag of chocolate chips is $2.50, while the cost for one 32-ounce bag of chocolate chunks is $13.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer these questions: Which type of tool will you use to create a graph to represent this situation? After graphing, how could you identify and interpret a solution that shows how many chocolate chunks and chocolate chips are needed to make cookies for one week?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

GO ONLINE to answer these questions.

Lesson 8-1

Systems of Linear Equations in Two Variables and Applications

What Will You Learn?

After completing this lesson, you should be able to:

- Identify Solutions to Systems of Linear Equations in Two Variables
- Solve Systems of Linear Equations in Two Variables
- Use Systems of Linear Equations in Applications

Learn: Identify Solutions to Systems of Linear Equations in Two Variables

The point of intersection between two curves is important in the analysis of supply and demand. For example, suppose that a small theater company wants to set an optimal price for tickets to a show. If the theater sells tickets for $1 each, there would be a high demand and many people would buy tickets. However, the revenue brought in would not cover the expense of the show. Therefore, the theater is not willing to offer (supply) tickets at this low price. If the theater sells tickets for $100 each, chances are that few people would buy tickets (demand would be low).

In an open market, the price *y* of an item is dependent on the number of items *x* supplied by the producer and demanded by the consumers. Competition between buyers and sellers steers the price toward an equilibrium price. This is the point where the supply curve and demand curve intersect (Figure 8-1).

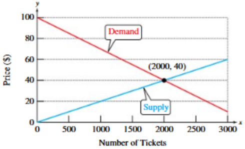


Figure 8-1

Suppose that the following linear equations represent the supply and demand curves for theater tickets. Taken together these equations form a *system* of linear equations.

$$\left.\begin{aligned}\text{Supply: } y &= 0.02x\\ \text{Demand: } y &= -0.03x + 100\end{aligned}\right\}\text{ System of linear equations}$$

Two or more linear equations make up a **system of linear equations.** A **solution** to a system of equations in two variables is an ordered pair that is a solution to each individual equation. Graphically, this is a point of intersection of the graphs of the equations. For the system given here, the solution is (2000, 40). This means that 2000 tickets are bought and sold at the equilibrium price of $40. The **solution set** to a system of equations is the set of all solutions to the system. In this case, the solution set is {(2000, 40)}.

Good Practices

A solution to a system of linear equations in two variables is an ordered pair. To express the solution set, we enclose the solution(s) within set brackets.

Example 1: Determining if an Ordered Pair Is a Solution to a System of Equations

Determine if the ordered pair is a solution to the system.

$$6x + y = -2$$
$$4x - 3y = 17$$

a. $\left(\frac{1}{2}, -5\right)$ **b.** $(0, -2)$

Solution:

a.

First equation	Second equation	
$6x + y = -2$	$4x - 3y = 17$	Test the ordered pair $\left(\frac{1}{2}, -5\right)$ in each equation.
$6\left(\frac{1}{2}\right) + (-5) \stackrel{?}{=} -2$	$4\left(\frac{1}{2}\right) - 3(-5) \stackrel{?}{=} 17$	The ordered pair is a solution to both equations.
$3 + (-5) \stackrel{?}{=} -2$ ✓ true	$2 + 15 \stackrel{?}{=} 17$ ✓ true	

The ordered pair $\left(\frac{1}{2}, -5\right)$ is a solution to the system.

b.

First equation	Second equation	
$6x + y = -2$	$4x - 3y = 17$	Test the ordered pair $(0, -2)$ in each equation.
$6(0) + (-2) \stackrel{?}{=} -2$	$4(0) - 3(-2) \stackrel{?}{=} 17$	
$0 + (-2) \stackrel{?}{=} -2$	$0 + 6 \stackrel{?}{=} 17$	The ordered pair is *not* a solution to the second equation.
$-2 \stackrel{?}{=} -2$ ✓ true	$2 + 15 \stackrel{?}{=} 17$ false	

The ordered pair $(0, -2)$ is *not* a solution to the system.

Check Your Work

For an ordered pair to be a solution to a system of equations, it must satisfy *both* equations. If it fails in either equation, it is *not* a solution.

The lines from Example 1 are shown in Figure 8-2. From the graph, we can verify that $\left(\frac{1}{2}, -5\right)$ is the only solution to the system of equations.

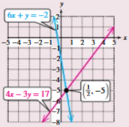


Figure 8-2

Apply the Skills

1. Determine if the ordered pair is a solution to the system.

$$3x - y = 10$$
$$x + \frac{1}{4}y = 1$$

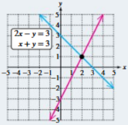
a. $(2, -4)$

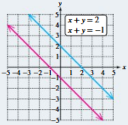
b. $\left(\frac{1}{3}, -9\right)$

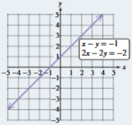
There are three different possibilities regarding the number of solutions to a system of linear equations.

Solutions to Systems of Linear Equations in Two Variables

- **One Unique Solution:** If a system of linear equations represents intersecting lines, then it has exactly one solution.
- **No Solution:** If a system of linear equations represents parallel lines, then the lines do not intersect, and the system has no solution. In such a case, we say that the system is **inconsistent.**
- **Infinitely Many Solutions:** If a system of linear equations represents the same line, then all points on the common line satisfy each equation. Therefore, the system has infinitely many solutions. In such a case, we say that the equations are **dependent.**







Learn: Solve Systems of Linear Equations in Two Variables

Graphing a system of equations is one method to find the solution(s) to the system. However, sometimes it is difficult to determine the solution(s) using this method because of limitations in the accuracy of the graph. Instead we often use algebraic methods to solve a system of equations. The first method we present is called the **substitution method.**

Solving a System of Equations by Using the Substitution Method

Step 1 Isolate one of the variables from one equation.

Step 2 Substitute the expression found in step 1 into the *other* equation.

Step 3 Solve the resulting equation.

Step 4 Substitute the value found in step 3 back into the equation in step 1 to find the value of the remaining variable.

Step 5 Check the ordered pair in each equation and write the solution as an ordered pair in set notation.

Example 2: Solving a System of Equations by the Substitution Method

Solve the system by using the substitution method.

$$-5x - 4y = 2$$
$$4x + y = 5$$

Solution:

$$-5x - 4y = 2$$
$$4x + y = 5 \longrightarrow y = \underbrace{-4x + 5}$$

Step 1: Isolate one of the variables from one of the equations. A variable with coefficient 1 or −1 is easily isolated.

$$-5x - 4(-4x + 5) = 2$$

Step 2: Substitute the expression from step 1 into the other equation.

$$-5x + 16x - 20 = 2$$
$$11x = 22$$
$$x = 2$$

Step 3: Solve for the remaining variable.

$$y = -4x + 5$$
$$y = -4(2) + 5$$
$$y = -3$$

Step 4: Substitute the known value of x into the equation where y is isolated. From step 1, this is $y = -4x + 5$.

Step 5: Check the ordered pair in each original equation.

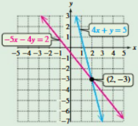
Check (2, −3).

$$-5x - 4y = 2 \qquad\qquad 4x + y = 5$$
$$-5(2) - 4(-3) \overset{?}{=} 2 \qquad\qquad 4(2) + (-3) \overset{?}{=} 5$$
$$-10 + 12 \overset{?}{=} 2 \checkmark \text{ true} \qquad\qquad 8 - 3 \overset{?}{=} 5 \checkmark \text{ true}$$

The solution set is {(2, −3)}.

Insights

The lines from Example 2 are shown here. The graph shows the point of intersection at (2, –3).



Apply the Skills

2. Solve the system by using the substitution method.

$$\begin{aligned} 3x + 4y &= 5 \\ x - 3y &= 6 \end{aligned}$$

Now consider the following system of equations.

$$\begin{aligned} 5x - 4y &= 6 \\ -3x + 7y &= 1 \end{aligned}$$

None of the variable terms has a coefficient of 1 or –1. Therefore, if we isolate x or y from either equation, the resulting equation will have one or more terms with fractional coefficients. To avoid this scenario, we can use another method called the **addition method** (also called the elimination method).

Insights

The addition method is sometimes called the *elimination method*. However, since both the substitution method and addition method eliminate a variable, we use the name addition method to emphasize the technique used.

Solving a System of Equations by Using the Addition Method

Step 1 Write both equations in standard form: $Ax + By = C$.

Step 2 Clear fractions or decimals (optional).

Step 3 Multiply one or both equations by nonzero constants to create opposite coefficients for one of the variables.

Step 4 Add the equations from step 3 to eliminate one variable.

Step 5 Solve for the remaining variable.

Step 6 Substitute the known value found in step 5 into one of the original equations to solve for the other variable.

Step 7 Check the ordered pair in each equation and write the solution set.

Example 3: Solving a System of Equations by the Addition Method

Solve the system by using the addition method.

$$5x = 4y + 6$$
$$-3x + 7y = 1$$

Solution:

$$5x = 4y + 6 \xrightarrow{\text{Subtract } 4y.} 5x - 4y = 6$$
$$-3x + 7y = 1 \qquad\qquad -3x + 7y = 1$$

Step 1: Write each equation in standard form: $Ax + By = C$.

Step 2: There are no decimals or fractions.

$$5x - 4y = 6 \xrightarrow{\text{Multiply by 3.}} 15x - 12y = 18$$
$$-3x + 7y = 1 \xrightarrow[\text{Multiply by 5.}]{} \underline{-15x + 35y = 5}$$
$$23y = 23$$

Step 3: Multiply the first equation by 3. Multiply the second equation by 5.

Step 4: Add the equations to eliminate x

$$y = 1$$

Step 5: Solve for y.

$$5x = 4y + 6$$
$$5x = 4(1) + 6$$
$$5x = 10$$
$$x = 2$$

Step 6: Substitute $y = 1$ into one of the original equations to solve for x.

Check (2, 1). **Step 7:** Check the ordered pair in each original equation.

$$5x = 4y + 6 \qquad -3x + 7y = 1$$
$$5(2) \stackrel{?}{=} 4(1) + 6 \qquad -3(2) + 7(1) \stackrel{?}{=} 1$$
$$10 \stackrel{?}{=} 4 + 6 \checkmark \text{ true} \qquad -6 + 7 \stackrel{?}{=} 1 \checkmark \text{ true}$$

The solution set is {(2, 1)}.

Insights

In Example 3, the variable x was eliminated.

Alternatively, we could have eliminated y by multiplying the first equation by 7 and the second equation by 4. This would create new equations with coefficients of –28 and 28 on the y terms.

Apply the Skills

3. Solve the system by using the addition method.

$$2x - 9y = 1$$
$$3x = 17 - 2y$$

TECHNOLOGY CONNECTIONS

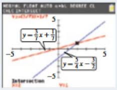
Solving a System of Linear Equations in Two Variables Using Intersect

The solution to a system of linear equations can be checked on a graphing calculator by first writing the equations in slope-intercept form. From Example 3 we have

$$5x = 4y + 6 \longrightarrow y = \frac{5}{4}x - \frac{3}{2}$$
$$-3x + 7y = 1 \longrightarrow y = \frac{3}{7}x + \frac{1}{7}$$

Then graph the equations and use the Intersect feature to approximate the point of intersection.

The Intersect feature gives the solution (2, 1).



NORMAL FLOAT AUTO a+bi DEGREE CL
CALC INTERSECT
5
$y = \frac{3}{7}x + \frac{1}{7}$
−5
5
$y = \frac{5}{4}x - \frac{3}{2}$
−5
Intersection

It is important to write the individual equations in a system of equations in standard form so that the variables line up. Also consider clearing decimals or fractions within an equation to make integer coefficients.

Example 4: Solving a System of Equations by the Addition Method

Solve the system by using the addition method.

$$\frac{2}{5}x - y = \frac{19}{10}$$
$$5(x + y) = -7y - 41$$

Solution:

$$\frac{2}{5}x - y = \frac{19}{10} \xrightarrow{\text{Multiply by 10.}} 4x - 10y = 19$$
$$5(x + y) = -7y - 41 \xrightarrow{\text{Simplify.}} 5x + 12y = -41$$

Clear the fractions in the first equation. Write the second equation in standard form.

$$4x - 10y = 19 \xrightarrow{\text{Multiply by } -5.} -20x + 50y = -95$$
$$5x + 12y = -41 \xrightarrow[\text{Multiply by 4.}]{} \underline{20x + 48y = -164}$$
$$98y = -259$$

The LCM of the *x* coefficients, 4 and 5, is 20. Create opposite coefficients on *x* of 20 and −20.

$$y = \frac{-259}{98}$$

Solve for *y*.

$$y = -\frac{37}{14}$$

Simplify to lowest terms.

Substituting $y = -\frac{37}{14}$ back into one of the original equations to solve for x would be cumbersome. Alternatively, we can solve for x by repeating the addition method. This time we will eliminate y by creating opposite coefficients on the y terms and then solving for x.

$$\begin{array}{lcr}
4x - 10y = 19 & \xrightarrow{\text{Multiply by 6.}} & 24x - 60y = 114 \\
5x + 12y = -41 & \xrightarrow[\text{Multiply by 5.}]{} & \underline{25x + 60y = -205} \\
 & & 49x = -91
\end{array}$$

The LCM of 10 and 12 is 60. Create opposite coefficients of –60 and 60 on the y terms.

$$x = \frac{-91}{49}$$

Solve for x.

$$x = -\frac{13}{7}$$

Simplify.

The ordered pair $\left(-\frac{13}{7}, -\frac{37}{14}\right)$ checks in both original equations.

The solution set is $\left\{\left(-\frac{13}{7}, -\frac{37}{14}\right)\right\}$.

Apply the Skills

4. Solve the system by using the addition method.

$$\begin{aligned} 2(x - 2y) &= y + 14 \\ \frac{1}{2}x + \frac{7}{6}y &= -\frac{13}{3} \end{aligned}$$

Point of Interest

The study of systems of linear equations is a topic in a broader branch of mathematics called linear algebra.

The systems in Examples 1–4 each have one unique solution. That is, the lines represented by the two equations intersect in exactly one point. In Examples 5 and 6, we investigate systems with no solution or infinitely many solutions.

Example 5: Identifying a System of Equations with No Solution

Solve the system. $2x + y = 4$

$6x + 3y = 6$

Solution:

$$\begin{array}{lcr}
2x + y = 4 & \xrightarrow{\text{Multiply by } -3.} & -6x - 3y = -12 \\
6x + 3y = 6 & \xrightarrow{\hspace{3em}} & \underline{6x + 3y = 6} \\
 & & 0 = -6
\end{array}$$

We can eliminate either the x terms or y terms by multiplying the first equation by –3.

Both the x and y terms are eliminated, leading to the contradiction $0 = -6$.

The system of equations reduces to a contradiction. This indicates that there is no solution and the system is inconsistent. The equations represent parallel lines, and two parallel lines do not intersect.

The solution set is { }.

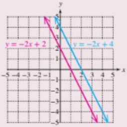
Check Your Work

To verify that the system in Example 5 has no solution, write the equations in slope-intercept form.

$$2x + y = 4 \Rightarrow y = -2x + 4$$

$$6x + 3y = 6 \Rightarrow y = -2x + 2$$

The equations have the same slope, but different y-intercepts. Therefore, the lines are parallel and do not intersect.



Apply the Skills

5. Solve the system.

$$3x - y = 2$$

$$-9x + 3y = 4$$

Example 6: Solving a System of Dependent Equations

Solve the system.

$$y = 2x - 1$$

$$8x - 4y = 4$$

Solution:

$y = \overbrace{2x - 1}$

$8x - 4y = 4 \qquad 8x - 4(2x - 1) = 4$ With y already isolated in the first equation, apply the substitution method.

$8x - 8x + 4 = 4$ Solve the resulting equation.

$4 = 4$ Identity.

Notice that both the variable terms and the constant terms were eliminated. The system of equations is reduced to the identity $4 = 4$. Therefore, the two original equations are dependent and represent the same line. The solution set consists of an infinite number of ordered pairs (x, y) that fall on the common line of intersection, $y = 2x - 1$. Therefore, the solutions are ordered pairs of the form $(x, 2x - 1)$.

The solution set is $\{(x, 2x - 1) \mid x \text{ is any real number}\}$.

Check Your Work

To verify the solution to Example 6, write the two equations in slope-intercept form.

$$y = 2x - 1 \Rightarrow y = 2x - 1$$
$$8x - 4y = 4 \Rightarrow y = 2x - 1$$

The equations have the same slope-intercept form and define the same line.

Apply the Skills

6. Solve the system.

$$x = 5 - 3y$$
$$2x + 6y = 10$$

Insights

Sometimes the solution to a system of dependent equations is written with an arbitrary variable called a **parameter.** For example, letting t represent any real number, then the solution set to Example 6 can be written as $\{(t, 2t - 1) \mid t \text{ is a real number}\}$.

The solution set $\{(x, 2x - 1) \mid x$ is any real number)$\}$ is called the **general solution** to the system in Example 6. By varying the value of x, we can produce any number of specific solutions to the system of equations. For example, if we randomly select the values of $x = 1, 2, 3$, and -1, we can calculate the following solutions.

x	$(x, 2x - 1)$	Solution
1	$(1, 2(1) - 1)$	$(1, 1)$
2	$(2, 2(2) - 1)$	$(2, 3)$
3	$(3, 2(3) - 1)$	$(3, 5)$
-1	$(-1, 2(-1) - 1)$	$(-1, -3)$

Notice that the solutions fall on the common line of intersection as expected (Figure 8-3).

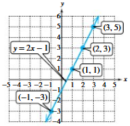


Figure 8-3

We should also note that the general solution to a system of dependent equations can be written in a variety of forms. For instance, the solutions to the system in Example 6 fall on the line $y = 2x - 1$. By solving the equation for x, we have $x = \frac{y+1}{2}$. So, the solution set can also be written as $\left\{\left(\frac{y+1}{2}, y\right) \middle| y \text{ is any real number}\right\}$.

Insights

The equations $y = 2x - 1$ and $8x - 4y = 4$ represent the same relationship between x and y. Therefore, we have only one unique equation, but two variables. As a result, y *depends* on the choice of x and vice versa.

Learn: Use Systems of Linear Equations in Applications

When solving an application involving two unknowns, sometimes it is convenient to use a system of two independent equations as demonstrated in Examples 7–9.

Example 7: Solving an Application Involving Mixtures

A hospital uses a 15% bleach solution to disinfect a quarantine area. How much 6% bleach solution must be mixed with an 18% bleach solution to make 50L of a 15% bleach solution?

Solution:

Let x represent the amount of 6% bleach solution.

Let y represent the amount of 18% bleach solution.

The amount of pure bleach in each mixture is found by multiplying the amount of solution by the concentration rate. This information can be organized in a table.

	6% Solution	18% Solution	15% Solution
Amount of mixture	x	y	50
Amount of pure bleach	$0.06x$	$0.18y$	0.15(50)

There are two unknown quantities. We will set up a system of two independent equations, first in words and then using *x* and *y*.

$$\begin{pmatrix}\text{Amount of}\\ \text{6\% mixture}\end{pmatrix} + \begin{pmatrix}\text{Amount of}\\ \text{18\% mixture}\end{pmatrix} = \begin{pmatrix}\text{Amount of}\\ \text{15\% mixture}\end{pmatrix}$$

$$\begin{pmatrix}\text{Amount of}\\ \text{pure bleach in}\\ \text{6\% mixture}\end{pmatrix} + \begin{pmatrix}\text{Amount of}\\ \text{pure bleach in}\\ \text{18\% mixture}\end{pmatrix} = \begin{pmatrix}\text{Amount of}\\ \text{pure bleach in}\\ \text{15\% mixture}\end{pmatrix}$$

$$\begin{array}{ccccc} x & + & y & = & 50 \\ 0.06x & + & 0.18y & = & 0.15(50) \end{array}$$

$$\begin{array}{rcrcr} x + y = 50 & & x + y = 50 & \longrightarrow & x + y = 50 \\ 0.06x + 0.18y = 7.5 & \xrightarrow[\text{Multiply by 100.}]{} & 6x + 18y = 750 & \xrightarrow[\text{Divide by } -6.]{} & \underline{-x - 3y = -125} \\ & & & & -2y = -75 \\ & & & & y = 37.5 \end{array}$$

Substitute $y = 37.5$ back into the equation $x + y = 50$.

$$\begin{aligned} x + y &= 50 \\ x + 37.5 &= 50 \\ x &= 12.5 \end{aligned}$$

Therefore, 12.5 L of the 6% bleach solution should be mixed with 37.5 L of the 18% bleach solution to make 50 L of 15% bleach solution.

Check Your Work

Check that the answer is reasonable. The total amount of the resulting solution is 12.5 L + 37.5 L, which is 50 L.

Amount of pure bleach in the 50 L solution:
$0.06(12.5\text{ L}) + 0.18(37.5\text{ L}) = 0.75\text{ L} + 6.75\text{ L} = 7.5\text{ L}$

$0.15(50\text{ L}) = 7.5\text{ L}$ ✓

Apply the Skills

7. How many ounces of 20% and 35% acid solution should be mixed to produce 15 oz of 30% acid solution?

Example 8: Solving an Application Involving Uniform Motion

A riverboat traveling upstream against the current on the Mississippi River takes 3 hr to travel 24 mi. The return trip downstream with the current takes only 2 hr. Find the speed of the boat in still water and the speed of the current.

Solution:

Let b represent the speed of the boat in still water.

Let c represent the speed of the current.

The given information can be organized in a table.

	Distance (mi)	Rate (mph)	Time (hr)
Upstream	24	$b - c$	3
Downstream	24	$b + c$	2

Use the relationship $d = rt$; that is, distance = (rate)(time).

$$\begin{pmatrix}\text{Distance}\\ \text{upstream}\end{pmatrix} = \begin{pmatrix}\text{Rate}\\ \text{upstream}\end{pmatrix}\begin{pmatrix}\text{Time}\\ \text{upstream}\end{pmatrix} \longrightarrow 24 = (b - c) \cdot 3 = 3b - 3c$$

$$\begin{pmatrix}\text{Distance}\\ \text{downstream}\end{pmatrix} = \begin{pmatrix}\text{Rate}\\ \text{downstream}\end{pmatrix}\begin{pmatrix}\text{Time}\\ \text{downstream}\end{pmatrix} \longrightarrow 24 = (b + c) \cdot 2 = 2b + 2c$$

$$\begin{aligned}
24 = 3b - 3c &\xrightarrow{\text{Divide by 3.}} \quad 8 = b - c\\
24 = 2b + 2c &\xrightarrow[\text{Divide by 2.}]{} \quad \underline{12 = b + c}\\
& \qquad\qquad\quad 20 = 2b\\
& \qquad\qquad\quad 10 = b
\end{aligned}$$

Substitute $b = 10$ into the equation $12 = b + c$, which gives $c = 2$.

The boat's speed in still water is 10 mph and the speed of the current is 2 mph.

Check Your Work

Check that the answer is reasonable.

Speed upstream

(10 − 2) mph is 8 mph

Speed downstream

(10 + 2) mph is 12 mph

Distance upstream

24 mi = (8 mph)(3 hr) ✓

Distance downstream

24 mi = (12 mph)(2 hr) ✓

Apply the Skills

8. A boat takes 3 hr to go 24 mi upstream against the current. It can go downstream with the current a distance of 48 mi in the same amount of time. Determine the speed of the boat in still water and the speed of the current.

Example 9: Applying a System of Equations in Business

A lawn service company has fixed monthly costs of \$500 and variable costs (labor, gasoline, and depreciation) of \$40 per lawn. If the service charges \$60 per lawn,

a. Write a cost function representing the cost $C(x)$ to the company to service x lawns per month.

b. Write a revenue function representing the revenue $R(x)$ to service x lawns per month.

c. Determine the number of lawns that must be serviced in a month for the company to break even.

Solution:

Let x represent the number of lawns serviced for a given month.

a. $\begin{pmatrix}\text{Monthly}\\\text{cost}\end{pmatrix} = \begin{pmatrix}\text{Fixed}\\\text{cost}\end{pmatrix} + \begin{pmatrix}\text{Variable}\\\text{cost}\end{pmatrix} \xrightarrow{\qquad} C(x) = 500 + 40x$

b. $\begin{pmatrix}\text{Monthly}\\\text{Revenue}\end{pmatrix} = \begin{pmatrix}\text{Revenue}\\\text{per lawn}\end{pmatrix}\begin{pmatrix}\text{Number}\\\text{of lawns}\end{pmatrix} \xrightarrow{\qquad} R(x) = 60x$

c. To break even, the cost must equal revenue, $C(x) = R(x)$.

$$500 + 40x = 60x$$
$$500 = 20x$$
$$x = 25$$

To break even, the company must service 25 lawns.

Insights

Notice that the result of Example 9(c) is found by solving the system of equations found in parts (a) and (b).

$$y = 500 + 40x$$
$$y = 60x$$

Check Your Work

Check that the answer is reasonable.

$$C(25) = 500 + 40(25)$$
$$= \$1500$$
$$R(25) = 60(25)$$
$$= \$1500$$

Cost and revenue are the same for 25 lawns serviced.

Apply the Skills

9. A storage company rents its units for $120 per month. The company has fixed monthly costs of $2100 and variable costs (air-conditioning and service) of $50 per unit.

a. Write a cost function representing the monthly cost $C(x)$ to the company for x units.

b. Write a revenue function representing the revenue $R(x)$ when x units per month are rented.

c. Determine the number of units that must be rented in a month for the company to break even.

Practice 8-1

Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, solve the equation.

R.1. $4 = 3 + 3(3x + 1)$

R.2. $0.65(x - 6) + 0.35(x + 4) = 0.5$

R.3. $3(7 - 4n) + 2 = -2n - 3 - 10n$

R.4. $-4 + 3x = 3(x - 3) + 5$

R.5. Graph the equation and identify the x- and y-intercepts. $4x + y = 4$

R.6. Graph the equation and identify the slope and y-intercept. $-x + 5y = -5$

Concept Connections

1. Two or more linear equations taken together make up a __________ of linear equations.

2. A __________ to a system of equations in two variables is an ordered pair that is a solution to each individual equation in the system.

3. Two algebraic methods to solve a system of linear equations in two variables are the __________ method and the __________ method.

4. A system of linear equations in two variables may have no solution. In such a case, the equations represent __________ lines.

5. A system of equations that has no solution is called an __________ system.

6. A system of linear equations in two variables may have infinitely many solutions. In such a case, the equations are said to be __________ .

Learn: Identify Solutions to Systems of Linear Equations in Two Variables

For Exercises 7–10, determine if the ordered pair is a solution to the system of equations. (See Example 1, p. 847)

7. $3x - 5y = -7$
$x - 4y = -7$

- **a.** $(1, 2)$
- **b.** $\left(-\frac{2}{3}, 1\right)$

8. $-11x + 6y = -4$
$7x + 3y = 23$

- **a.** $\left(1, \frac{7}{6}\right)$
- **b.** $(2, 3)$

9. $y = \frac{3}{2}x - 5$
$6x - 4y = 20$

- **a.** $(2, -2)$
- **b.** $(-4, -11)$

10. $y = -\frac{1}{5}x + 2$
$2x + 10y = 10$

- **a.** $(5, 1)$
- **b.** $(-10, 4)$

For Exercises 11–14, a system of equations is given in which each equation is written in slope-intercept form. Determine the number of solutions. If the system does not have one unique solution, state whether the system is inconsistent or whether the equations are dependent.

11. $y = \frac{2}{5}x - 7$
$y = \frac{1}{4}x + 7$

12. $y = 6x - \frac{2}{3}$
$y = 6x + 4$

13. $y = 8x - \frac{1}{2}$
$y = 8x - \frac{1}{2}$

14. $y = \frac{1}{2}x + 3$
$y = 2x + \frac{1}{3}$

Learn: Solve Systems of Linear Equations in Two Variables

For Exercises 15–20, solve the system of equations by using the substitution method. (See Example 2, p. 849)

15. $x + 3y = 5$
$3x - 2y = -18$

16. $2x + y = 2$
$5x + 3y = 9$

17. $2x + 7y = 1$
$3y - 7 = 2$

18. $3x = 2y - 11$
$6 + 5x = 1$

19. $2(x + y) = 2 - y$
$4x - 1 = 2 - 5y$

20. $5(x + y) = 9 - 2y$
$6y - 2 = 10 - 7x$

For Exercises 21–28, solve the system of equations by using the addition method. (See Examples 3–4, pp. 851–852)

21. $3x - 7y = 1$
$6x + 5y = -17$

22. $5x - 2y = -2$
$3x + 4y = 30$

23. $11x = -5 - 4y$
$2(x - 2y) = 22 + y$

24. $-3(x - y) = y - 14$
$2x + 2 = 7y$

25. $0.6x + 0.1y = 0.4$
$2x - 0.7y = 0.3$

26. $0.25x - 0.04y = 0.24$
$0.15x - 0.12y = 0.12$

27. $2x + 11y = 4$
$3x - 6y = 5$

28. $3x - 4y = 9$
$2x + 9y = 2$

For Exercises 29–34, solve the system by using any method. If a system does not have one unique solution, state whether the system is inconsistent or whether the equations are dependent. (See Examples 5–6, pp. 853–854)

29. $3x - 4y = 6$
$9x = 12y + 4$

30. $-4x - 8y = 2$
$2x = 8 - 4y$

31. $3x + y = 6$
$x + \frac{1}{3}y = 2$

32. $2x - y = 8$
$x - \frac{1}{2}y = 4$

33. $2x + 4 = 4 - 5y$
$2 + 4(x + y) = 7y + 2$

34. $3(x - 3y) = 2y$
$2x + 5 = 5 - 7y$

For Exercises 35–36.

a. Write the general solution.

b. Find three individual solutions. Answers will vary.

35. $-5x - y = 6$
$10x = -2(y + 6)$

36. $2y = 6 - 4x$
$8x = 12 - 4y$

Mixed Exercises

For Exercises 37–50, solve the system using any method.

37. $3x - 10y = 1900$
$5y + 800 = x$

38. $2x - 7y = 2400$
$-4x + 1800 = y$

39. $5(2x + y) = y - x - 8$
$x - \frac{3}{2}y = \frac{5}{2}$

40. $3(2x - y) = 2 - x$
$x + \frac{5}{4}y = \frac{3}{2}$

41. $y = \frac{2}{3}x - 1$
$y = \frac{1}{6}x + 2$

42. $y = -\frac{1}{4}x + 7$
$y = -\frac{3}{2}x + 17$

43. $4(x - 2) = 6y + 3$
$\frac{1}{4}x - \frac{3}{8}y = -\frac{1}{2}$

44. $\frac{1}{14}x - \frac{1}{7}y = \frac{1}{2}$
$2(x - 2y) + 3 = 20$

45. $2x = \frac{y}{2} + 1$
$0.04x - 0.01y = 0.02$

46. $0.05x + 0.01y = 0.03$
$x + \frac{y}{5} = \frac{3}{5}$

47. $y = \quad 2.4y - 1.54$
$y = -3.5x + 7.9$

48. $y = -0.18x + 0.129$
$y = -0.15x + 0.1275$

49. $\frac{x-2}{8} + \frac{y+1}{2} = -6$
$\frac{x-2}{2} - \frac{y+1}{4} = 12$

50. $\frac{x+1}{2} - \frac{y-2}{10} = -1$
$\frac{x+1}{6} + \frac{y-2}{2} = 21$

Learn: Use Systems of Linear Equations in Applications

51. One antifreeze solution is 36% alcohol and another is 20% alcohol. How much of each mixture should be added to make 40 L of a solution that is 30% alcohol? (**See Example 7, p. 856**)

52. A pharmacist wants to mix a 30% saline solution with a 10% saline solution to get 200 mL of a 12% saline solution. How much of each solution should be used?

53. A radiator has 16 L of a 36% antifreeze solution. How much must be drained and replaced by pure antifreeze to bring the concentration level up to 50%?

54. A student performed an experiment for a science fair project and learned that rinsing lettuce in vinegar kills more bacteria than rinsing with water or with a popular commercial product. As a follow-up to the project, the student wants to determine the percentage of bacteria killed by rinsing with a diluted solution of vinegar.

a. How much water and how much vinegar should be mixed to produce 10 cups of a mixture that is 40% vinegar?

b. How much pure vinegar and how much 40% vinegar solution should be mixed to produce 10 cups of a mixture that is 60% vinegar?

55. Michelle borrows a total of $5000 in student loans from two lenders. One charges 4.6% simple interest and the other charges 6.2% simple interest. She is not required to pay off the principal or interest for 3 yr. However, at the end of 3 yr, she will owe a total of $762 for the interest from both loans. How much did she borrow from each lender?

56. Juan borrows $100,000 to pay for medical school. Part of the money is borrowed at 4.5% simple interest. The rest is borrowed through a government loan at 6% interest. In both cases Juan is not required to pay off the principal or interest while in medical school. However, at the end of 4 yr, Juan will owe a total of $19,200 for the interest from both loans. How much was borrowed from each source?

57. Hakeem pays back two student loans over a 4-yr period. One loan charges the equivalent of 3% simple interest and the other charges the equivalent of 5.5% simple interest. If the total amount borrowed was $24,000 and the total amount of interest paid after 4 yr is $3280, find the amount borrowed from each loan.

58. A total of \$6000 is invested for 5 yr with a total return of \$1080. Part of the money is invested in a fund that returns the equivalent of 2% simple interest. The rest of the money is invested at 4% simple interest. Determine the amount invested in each account

59. Monique and Tara each make an ice cream sundae. Monique gets 2 scoops of Cherry ice cream and 1 scoop of Mint Chocolate Chunk ice cream for a total of 43 g of fat. Tara has 1 scoop of Cherry and 2 scoops of Mint Chocolate Chunk for a total of 47 g of fat. How many grams of fat does 1 scoop of each type of ice cream have?

60. Bryan and Jadyn had barbeque potato chips and soda at a football party. Bryan ate 3 oz of chips and drank 2 cups of soda for a total of 700 mg of sodium. Jadyn ate 1 oz of chips and drank 3 cups of soda for a total of 350 mg of sodium. How much sodium is in 1 oz of chips and how much is in 1 cup of soda?

61. The average weekly salary of two employees is \$1350. One makes \$300 more than the other. Find their salaries.

62. The average of an electrician's hourly wage and a plumber's hourly wage is \$33. One day a contractor hires the electrician for 8 hr of work and the plumber for 5 hr of work and pays a total of \$438 in wages. Find the hourly wage for the electrician and for the plumber.

63. A moving sidewalk in an airport moves people between gates. It takes Jason's 9-year-old daughter Josie 40 sec to travel 200 ft walking with the sidewalk. It takes 30 sec to walk 90 ft against the moving sidewalk (in the opposite direction). Find the speed of the sidewalk and find Josie's speed walking on non-moving ground. **(See Example 8, p. 858)**

64. A fishing boat travels along the east coast of the United States and encounters the Gulf Stream current. It travels 44 mi north with the current in 2 hr. It travels 56 mi south against the current in 4 hr. Find the speed of the current and the speed of the boat in still water.

65. Two runners begin at the same point on a 390-m circular track and run at different speeds. If they run in opposite directions, they pass each other in 30 sec. If they run in the same direction, they meet each other in 130 sec. Find the speed of each runner.

66. Two particles begin at the same point and move at different speeds along a circular path of circumference 280 ft. Moving in opposite directions, they pass in 10 sec. Moving in the same direction, they pass in 70 sec. Find the speed of each particle.

67. A cleaning company charges \$100 for each office it cleans. The fixed monthly cost of \$480 for the company includes telephone service and the depreciation on cleaning equipment and a van. The variable cost is \$52 per office and includes labor, gasoline, and cleaning supplies. **(See Example 9, p. 859)**

a. Write a linear cost function representing the cost $C(x)$ (in \$) to the company to clean x offices per month.

b. Write a linear revenue function representing the revenue $R(x)$ (in \$) for cleaning x offices per month.

c. Determine the number of offices to be cleaned per month for the company to break even.

d. If 28 offices are cleaned, will the company make money or lose money?

68. A vendor at a carnival sells cotton candy and caramel apples for \$2.00 each. The vendor is charged \$100 to set up the booth. Furthermore, the vendor's average cost for each product produced is approximately \$0.75.

a. Write a linear cost function representing the cost $C(x)$ (in \$) to the vendor to produce x products.

b. Write a linear revenue function representing the revenue $R(x)$ (in \$) for selling x products.

c. Determine the number of products to be produced and sold for the vendor to break even.

d. If 60 products are sold, will the vendor make money or lose money?

For Exercises 69–70, refer to Figure 8-1 and the narrative on page 846.

69. Suppose that the price p (in \$) of theater tickets is influenced by the number of tickets x offered by the theater and demanded by consumers.

Supply: $p = 0.025x$

Demand: $p = -0.04x + 104$

a. Solve the system of equations defined by the supply and demand models.

b. What is the equilibrium price?

c. What is the equilibrium quantity?

70. The price p (in \$) of a cookbook is determined by the number of cookbooks x demanded by consumers and supplied by the publisher.

Supply: $p = 0.002x$

Demand: $p = -0.005x + 70$

a. Solve the system of equations defined by the supply and demand models.

b. What is the equilibrium price?

c. What is the equilibrium quantity?

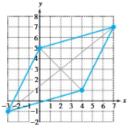
71. a. Sketch the lines defined by $y = 2x$ and $y = -\frac{1}{2}x + 5$.

b. Find the area of the triangle bounded by the lines in part (a) and the x-axis.

72. a. Sketch the lines defined by $y = x + 2$ and $y = -\frac{1}{2}x + 2$.

b. Find the area of the triangle bounded by the lines in part (a) and the x-axis.

73. The **centroid** of a region is the geometric center. For the region shown, the centroid is the point of intersection of the diagonals of the parallelogram.

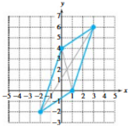


a. Find an equation of the line through the points (−3, −1) and (7, 7).

b. Find an equation of the line through the points (0, 5) and (4, 1).

c. Find the centroid of the region.

74. The centroid of the region shown is the point of intersection of the diagonals of the parallelogram.



a. Find an equation of the line through the points (−2, −2) and (3, 6).

b. Find an equation of the line through the points (1, 0) and (0, 4).

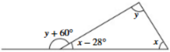
c. Find the centroid of the region.

75. Two angles are complementary. The measure of one angle is 6° less than twice the measure of the other angle. Find the measure of each angle.

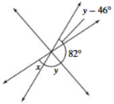
76. Two angles are supplementary. The measure of one angle is 12° more than 5 times the measure of the other angle. Find the measure of each angle.

For Exercises 77–78, find the measure of angles x and y.

77.



78.



Mixed Exercises

79. Write a system of linear equations with solution set $\{(-3, 5)\}$.

80. Write a system of linear equations with solution set $\{(4, -3)\}$.

81. Find C and D so that the solution set to the system is $\{(4, 1)\}$.

$$\begin{aligned} Cx + 5y &= 13 \\ -2x + Dy &= -5 \end{aligned}$$

82. Find A and B so that the solution set to the system is $\{(-5, 2)\}$.

$$\begin{aligned} 3x + Ay &= -3 \\ Bx - \ \ y &= -12 \end{aligned}$$

83. Given $f(x) = mx + b$, find m and b if $f(3) = -3$ and $f(-12) = -8$.

84. Given $g(x) = mx + b$, find m and b if $g(2) = 1$ and $g(-4) = 10$.

For Exercises 85–86, find an equation of the form $y = ax^2 + b$ for a parabola that passes through the given points.

85. $(1, -1)$ and $(2, -7)$

86. $(3, 9)$ and $\left(1, \frac{11}{3}\right)$

For Exercises 87–88, use the substitution $u = \frac{1}{x}$ and $v = \frac{1}{y}$ to rewrite the equations in the system in terms of the variables u and v. Solve the system in terms of u and v. Then back substitute to determine the solution set to the original system in terms of x and y.

87. $\frac{1}{x} + \frac{2}{y} = 1$
$-\frac{1}{x} + \frac{4}{y} = -7$

88. $-\frac{3}{x} + \frac{4}{y} = 11$
$\frac{1}{x} - \frac{2}{y} = -5$

89. During a race, Marta bicycled 12 mi and ran 4 mi in a total of 1 hr 20 min $\left(\frac{4}{3}\text{ hr.}\right)$ In another race, she bicycled 21 mi and ran 3 mi in 1 hr 40 min $\left(\frac{5}{3}\text{ hr}\right)$. Determine the speed at which she bicycles and the speed at which she runs. Assume that her bicycling speed was the same in each race and that her running speed was the same in each race.

90. Shelia swam 1 mi and ran 6 mi in a total of 1 hr 15 min $\left(\frac{5}{3}\text{ hr}\right)$. In another training session she swam 2 mi and ran 8 mi in a total of 2 hr. Determine the speed at which she swims and the speed at which she runs. Assume that her swimming speed was the same each day and that her running speed was the same each day.

91. A certain pickup truck gets 16 mpg in the city and 22 mpg on the highway. If a driver drives 254 mi on 14 gal of gas, determine the number of city miles and highway miles that the truck was driven.

92. A sedan gets 12 mpg in the city and 18 mpg on the highway. If a driver drives a total of 420 mi on 26 gal of gas, how many miles in the city and how many miles on the highway did he drive?

Write About It

93. A system of linear equations in x and y can represent two intersecting lines, two parallel lines, or a single line. Describe the solution set to the system in each case.

94. When solving a system of linear equations in two variables using the substitution or addition method, explain how you can detect whether the equations are dependent.

95. When solving a system of linear equations in two variables using the substitution or addition method, explain how you can detect whether the system is inconsistent.

96. Consider a system of linear equations in two variables in which the solution set is $\{(x, x+2) \mid x \text{ is any real number}\}$. Why do we say that the equations in the system are dependent?

Expanding Your Skills

For Exercises 97–102, solve for *x* and *y*. Assume that *a* and *b* are constant real numbers.

97. $$\frac{2}{x+y} + \frac{3}{x-y} = 16a \qquad \text{where } a \neq 0$$
$$\frac{5}{x+y} + \frac{1}{x-y} = 23a$$

98. $3(x+y) + 4(x-y) = 8a + 7b - 1$

$5(x-y) - 4(x+y) = 10a + b - 9$

99. $3x - 4y = -a - 7b$

$\frac{x-y}{2} = -b$

100. $5x + 6y = ab + 11$

$\frac{x+y}{2} - \frac{x}{2} = \frac{1+ab}{2}$

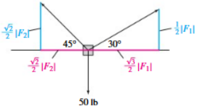
101. $2(x-y) + 3(x+y) = 5a + 2ab$

$ax - \frac{b}{2a}y = a^2 - b^2$ where $a \neq 0$ and $2a^2 + 5b \neq 0$

102. $6(x+y) + 3(x-y) = 18a$

$x - 2y = 16a$

103. A 50-lb weight is supported from two cables and the system is in equilibrium. The magnitudes of the forces on the cables are denoted by $|F_1|$ and $|F_2|$, respectively. An engineering student knows that the horizontal components of the two forces (shown in red) must be equal in magnitude. Furthermore, the sum of the magnitudes of the vertical components of the forces (shown in blue) must be equal to 50 lb to offset the downward force of the weight. Find the values of $|F_1|$ and $|F_2|$. Write the answers in exact form with no radical in the denominator. Also give approximations to 1 decimal place.


$\frac{\sqrt{2}}{2}|F_2|$
$\frac{1}{2}|F_1|$
45°
30°
$\frac{\sqrt{2}}{2}|F_2|$
$\frac{\sqrt{3}}{2}|F_1|$
50 lb

Technology Connections

For Exercises 104–107, use a graphing utility to approximate the solution to the system of equations. Round the *x* and *y* values to 3 decimal places.

104. $y = -3.729x + 6.958$

$y = 2.615x - 8.713$

105. $y = -0.041x + 0.068$

$y = 0.019x - 0.053$

106. $-0.25x + 0.04y = -0.42$

$6.775x + 2.5y = -38.1$

107. $0.36x - 0.075y = -0.813$

$0.066x + 0.008y = 0.194$

Go online for more practice problems.

Lesson 8-2

Systems of Linear Equations in Three Variables and Applications

What Will You Learn?

After completing this lesson, you should be able to:

- Identify Solutions to a System of Linear Equations in Three Variables
- Solve Systems of Linear Equations in Three Variables
- Use Systems of Linear Equations in Applications
- Modeling with Linear Equations in Three Variables

Learn: Identify Solutions to a System of Linear Equations in Three Variables

In Lesson 8.1 we solved systems of linear equations in two variables. In this lesson, we expand the discussion to solving systems involving three variables. A **linear equation in three variables** is an equation that can be written in the form

$$Ax + By + Cz = D, \text{ where } A, B, \text{ and } C \text{ are not all zero.}$$

For example, $x + 2y + z = 4$ is a linear equation in three variables. A solution to a linear equation in three variables is an ordered triple (x, y, z) that satisfies the equation. For example, several solutions to $x + 2y + z = 4$ are given here.

Solution	**Check:** $x + 2y + z = 4$
(1, 1, 1)	$(1) + 2(1) + (1) \stackrel{?}{=} 4$ ✓ true
(4, 0, 0)	$(4) + 2(0) + (0) \stackrel{?}{=} 4$ ✓ true
(0, 2, 0)	$(0) + 2(2) + (0) \stackrel{?}{=} 4$ ✓ true
(0, 0, 4)	$(0) + 2(0) + (4) \stackrel{?}{=} 4$ ✓ true

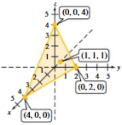


Figure 8-4

There are infinitely many solutions to the equation $x + 2y + z = 4$. The set of all solutions to a linear equation in three variables can be represented graphically by a plane in space. Figure 8-4 shows a portion of the plane defined by $x + 2y + z = 4$.

In many applications, we are interested in determining the point or points of intersection of two or more planes. This is given by the solutions to a system of linear equations in three variables. For example:

$$\begin{aligned} 2x + y - 3z &= -2 \\ x - 4y + z &= 24 \\ -3x - y + 4z &= 0 \end{aligned}$$

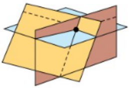


Figure 8-5

A solution to a system of linear equations in three variables is an **ordered triple** (x, y, z) that satisfies each equation in the system. Geometrically, a solution is a point of intersection of the planes represented by the equations in the system (Figure 8-5).

Point of Interest

Beautiful planar surfaces are seen in both art and architecture. For example, the great pyramid of Khufu outside Cairo, Egypt, has four triangular sides. The planes representing the sides form an angle of 52° with the ground.

Example 1: Determining if an Ordered Triple Is a Solution to a System of Equations

Determine if the ordered triple is a solution to the system.

$$\begin{aligned} 2x + y - 3z &= -2 \\ x - 4y + z &= 24 \\ -3x - y + 4z &= 0 \end{aligned}$$

a. (3, –5, 1) **b.** (2, –3, 1)

Solution:

Test the ordered triple in each equation.

a.

First equation	Second equation	Third equation
$2x + y - 3z = -2$	$x - 4y + z = 24$	$-3x - y + 4z = 0$
$2(3) + (-5) - 3(1) \stackrel{?}{=} -2$	$(3) - 4(-5) + (1) \stackrel{?}{=} 24$	$-3(3) - (-5) + 4(1) \stackrel{?}{=} 0$
$-2 \stackrel{?}{=} -2$ ✓ true	$24 \stackrel{?}{=} 24$ ✓ true	$0 \stackrel{?}{=} 0$ ✓ true

The ordered triple (3, –5, 1) is a solution to the system of equations.

b.

First equation	Second equation	Third equation
$2x + y - 3z = -2$	$x - 4y + z = 24$	$-3x - y + 4z = 0$
$2(2) + (-3) - 3(1) \stackrel{?}{=} -2$	$(2) - 4(-3) + (1) \stackrel{?}{=} 24$	$-3(2) - (-3) + 4(1) \stackrel{?}{=} 0$
$-2 \stackrel{?}{=} -2$ ✓ true	$15 \stackrel{?}{=} 24$ false	$1 \stackrel{?}{=} 0$ false

If an ordered triple fails to be a solution to any of the equations in the system, then it is not a solution to the system. The ordered triple (2, –3, 1) is *not* a solution to the second or third equation. Therefore, (2, –3, 1) is *not* a solution to the system of equations.

Apply the Skills

1. Determine if the ordered triple is a solution to the system.

$$\begin{aligned} 5x - y + 3z &= -7 \\ 3x + 4y - z &= 5 \\ 9x + 5y + 7z &= 1 \end{aligned}$$

a. (–2, –6, 3) **b.** (–1, 2, 0)

Learn: Solve Systems of Linear Equations in Three Variables

To solve a system of three linear equations in three variables, we first eliminate one variable. The system is then reduced to a two-variable system that can be solved by the techniques learned in Lesson 8-1.

Solving a System of Three Linear Equations in Three Variables

Step 1 Write each equation in standard form $Ax + By + Cz = D$.

Step 2 Choose a pair of equations and eliminate one of the variables by using the addition method.

Step 3 Choose a different pair of equations and eliminate the *same* variable.

Step 4 Once steps 2 and 3 are complete, the resulting system has two equations in two variables. Solve this system by using the substitution or addition method.

Step 5 Substitute the values of the variables found in step 4 into any of the three original equations that contain the third variable. Solve for the third variable.

Step 6 Check the ordered triple in each original equation. Then write the solution as an ordered triple in set notation.

Example 2: Solving a System of Equations in Three Variables

Solve the system.

$$\begin{aligned} 3x - 2y + z &= 2 \\ 5x + y - 2z &= 1 \\ 4x - 3y + 3z &= 7 \end{aligned}$$

Solution:

$$\begin{array}{ll} \boxed{A} & 3x - 2y + z = 2 \\ \boxed{B} & 5x + y - 2z = 1 \\ \boxed{C} & 4x - 3y + 3z = 7 \end{array}$$

Step 1: The equations are already in standard form.

It is helpful to label the equations $\boxed{A}$, $\boxed{B}$, and $\boxed{C}$.

The *z* variable can easily be eliminated from equations $\boxed{A}$ and $\boxed{B}$ and from equations $\boxed{A}$ and $\boxed{C}$. This is accomplished by creating opposite coefficients for the *z* terms and then adding the equations.

Step 2: Eliminate *z* from equations $\boxed{A}$ and $\boxed{B}$.

$$\begin{array}{llcl} \boxed{A} & 3x - 2y + z = 2 & \xrightarrow{\text{Multiply by 2.}} & 6x - 4y + 2z = 4 \\ \boxed{B} & 5x + y - 2z = 1 & & \underline{5x + y - 2z = 1} \\ & & & 11x - 3y \phantom{{}+2z} = 5 \quad \boxed{D} \end{array}$$

Step 3: Eliminate z from equations $\boxed{A}$ and $\boxed{C}$.

$$\begin{array}{llcr}
\boxed{A} & 3x - 2y + z = 2 & \xrightarrow{\text{Multiply by } -3.} & -9x + 6y - 3z = -6 \\
\boxed{B} & 4x - 3y + 3z = 7 & & \underline{4x - 3y + 3z = 7} \\
 & & & -5x + 3y \qquad = 1 \quad \boxed{E}
\end{array}$$

Step 4: Solve the system of equations $\boxed{D}$ and $\boxed{E}$.

$$\begin{array}{lrcl}
\boxed{D} & 11x - 3y = 5 & \longrightarrow & \boxed{D}\ 11(1) - 3y = 5 \\
\boxed{E} & \underline{-5x + 3y = 1} & & 11 - 3y = 5 \\
 & 6x \qquad = 6 & & -3y = -6 \\
 & x = 1 & & y = 2
\end{array}$$

Step 5: Substitute the values of the known variables x and y back into one of the original equations. We have chosen equation $\boxed{A}$.

$$\begin{aligned}
\boxed{A} \qquad 3x - 2y + z &= 2 \\
3(1) - 2(2) + z &= 2 \\
3 - 4 + z &= 2 \\
-1 + z &= 2 \\
z &= 3
\end{aligned}$$

Step 6: Check the ordered triple (1, 2, 3) in the three original equations.

$$\begin{array}{lll}
\boxed{A}\ 3x - 2y + z = 2 & \boxed{B}\ 5x + y - 2z = 1 & \boxed{C}\ 4x - 3y + 3z = 7 \\
3(1) - 2(2) + (3) \overset{?}{=} 2 & 5(1) + (2) - 2(3) \overset{?}{=} 1 & 4(1) - 3(2) + 3(3) \overset{?}{=} 7 \\
2 \overset{?}{=} 2 \checkmark \text{ true} & 1 \overset{?}{=} 1 \checkmark \text{ true} & 7 \overset{?}{=} 7 \checkmark \text{ true}
\end{array}$$

The solution set is {(1, 2, 3)}.

Insights

In Example 2, the y terms can also be eliminated easily because the coefficient on y in equation $\boxed{B}$ is 1. Therefore, y can be eliminated from equations $\boxed{A}$ and $\boxed{B}$ by multiplying equation $\boxed{B}$ by 2. Likewise y can be eliminated from equations $\boxed{B}$ and $\boxed{C}$ by multiplying equation $\boxed{B}$ by 3.

Apply the Skills

2. Solve the system.

$$\begin{aligned}
2x - y + 5z &= -7 \\
x + 4y - 2z &= 1 \\
3x + 2y + z &= -7
\end{aligned}$$

In Example 3, we solve a system of linear equations in which one or more equations has a missing term.

Example 3: Solving a System of Equations in Three Variables

Solve the system.

$$\begin{aligned} 2x + y &= -2 \\ 3y &= 5z - 12 \\ 5(x + z) &= 2z + 5 \end{aligned}$$

Solution:

Step 1: Write the equations in standard form.

$$\begin{array}{lll} \boxed{A}\ 2x + y = -2 & \longrightarrow & 2x + y = -2 \\ \boxed{B}\ 3y = 5z - 12 & \longrightarrow & 3y - 5z = -12 \\ \boxed{C}\ 5(x + z) = 2z + 5 & \longrightarrow & 5x + 3z = 5 \end{array}$$

Notice that the equations already have missing variable terms.

Steps 2 and 3: This system of equations has several missing terms. For example, equation $\boxed{C}$ is missing the variable y. If we eliminate y from equations $\boxed{A}$ and $\boxed{B}$, then we will have a second equation with variable y missing.

$$\begin{array}{lll} \boxed{A}\ 2x + y = -2 & \xrightarrow{\text{Multiply by } -3.} & -6x - 3y = 6 \\ \boxed{B}\ 3y - 5z = -12 & & \underline{ \quad 3y - 5z = -12} \\ & & -6x - 5z = -6 \quad \boxed{D} \end{array}$$

Step 4: Pair up equations $\boxed{C}$ and $\boxed{D}$. These equations form a system of linear equations in two variables. To solve the system with equations $\boxed{C}$ and $\boxed{D}$ we have chosen to eliminate the z variable.

$$\begin{array}{lll} \boxed{C}\ \ 5x + 3z = 5 & \xrightarrow{\text{Multiply by } 5.} & 25x + 15z = 25 \\ \boxed{D}\ -6x - 5z = -6 & \xrightarrow[\text{Multiply by } 3.]{} & \underline{-18x - 15z = -18} \\ & & 7x = 7 \\ & & x = 1 \end{array}$$

$$\begin{aligned} \boxed{C}\quad 5x + 3z &= 5 \\ 5(1) + 3z &= 5 \\ z &= 0 \end{aligned}$$

Step 5: Substitute the values of the known variables x and z back into one of the original equations containing y. We have chosen equation $\boxed{B}$.

$$\begin{aligned} \boxed{B}\quad 3y &= 5z - 12 \\ 3y &= 5(0) - 12 \\ 3y &= -12 \\ y &= -4 \end{aligned}$$

Step 6: The ordered triple $(1, -4, 0)$ checks in each original equation.

The solution set is $\{(1, -4, 0)\}$.

Apply the Skills

3. Solve the system.

$$\begin{aligned} a \qquad + 3c &= 4 \\ b + 2c &= -1 \\ 2a - 4b \qquad &= 14 \end{aligned}$$

A system of linear equations in three variables may have no solution. This occurs if the equations represent planes that do not all intersect (Figure 8-6). In such a case, we say that the system is **inconsistent.**

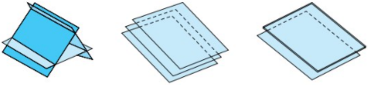


Figure 8-6

A system of linear equations in three variables may also have infinitely many solutions. This occurs if the equations represent planes that intersect in a common line or common plane (Figure 8-7). In such a case, we say that the equations are **dependent.**

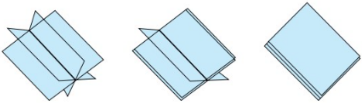


Figure 8-7

Example 4: Determining the Number of Solutions to a System

a. Determine the number of solutions to the system.

b. State whether the system is inconsistent or the equations are dependent.

c. Write the solution set.

$$\begin{aligned} -x + 6y - 3z &= -8 \\ x - 2y + 2z &= 3 \\ 3x + 2y + 4z &= -6 \end{aligned}$$

Solution:

Begin by eliminating a variable from two different pairs of equations. We will eliminate *x* from $\boxed{A}$ and $\boxed{B}$ and from $\boxed{A}$ and $\boxed{C}$.

$$\begin{array}{lrcl} \boxed{A} & -x + 6y - 3z = -8 & \longrightarrow & -x + 6y - 3z = -8 \\ \boxed{B} & x - 2y + 2z = 3 & \longrightarrow & \underline{x - 2y + 2z = 3} \\ \boxed{C} & 3x + 2y + 4z = -6 & & 4y - z = -5 \quad \boxed{D} \end{array}$$

Add equations $\boxed{A}$ and $\boxed{B}$ to eliminate *x*.

$$\begin{array}{lll} \boxed{A}\ -x+6y-3z=-8 & \xrightarrow{\text{Multiply by 3.}} & -3x+18y-9z=-24 \\ \boxed{C}\ \ 3x+2y+4z=-6 & \xrightarrow{\qquad\qquad} & \underline{\ \ 3x+\ 2y+4z=-6} \\ & & \qquad 20y-5z=-30\ \ \boxed{E} \end{array}$$

Multiply equation $\boxed{A}$ by 3 and add the result to $\boxed{C}$.

$$\begin{array}{lll} \boxed{D}\ \ 4y-\ z=-5 & \xrightarrow{\text{Multiply by }-5.} & -20y+5z=25 \\ \boxed{E}\ \ 20y-5z=-30 & & \underline{\ 20y-5z=-30} \\ & & \qquad 0=-5 \end{array}$$

Solving the system of equations $\boxed{D}$ and $\boxed{E}$ results in a contradiction.

The system of equations reduces to a contradiction.

- **a.** There is no solution.
- **b.** The system is inconsistent.
- **c.** The solution set is { }.

Apply the Skills

4. Repeat Example 4 with the given system.

$$\begin{aligned} x+y+4z&=-1 \\ 3x+y-4z&=3 \\ -4x-y+8z&=-2 \end{aligned}$$

In Example 5, we investigate the case in which a system of equations has infinitely many solutions.

Example 5: Determining the Number of Solutions to a System

- **a.** Determine the number of solutions to the system.
- **b.** State whether the system is inconsistent, or the equations are dependent.
- **c.** Write the solution set.

$$\begin{aligned} 2x+\ y\qquad\ &=-3 \\ 2y+16z&=-10 \\ -7x-3y+\ 4z&=8 \end{aligned}$$

Solution:

Eliminate variable z from equations $\boxed{B}$ and $\boxed{C}$.

$$\begin{array}{lll} \boxed{A}\ \ 2x+\ y\qquad\ =-3 & & \\ \boxed{B}\qquad\ 2y+16z=-10 & \xrightarrow{\qquad\qquad} & \qquad\ \ 2y+16z=-10 \\ \boxed{C}\ -7x-3y+\ 4z=8 & \xrightarrow[\text{Multiply by }-4.]{} & \underline{28x+12y-16z=-32} \\ & & 28x+14y\qquad\ =-42\ \ \boxed{D} \end{array}$$

Pair up equations $\boxed{A}$ and $\boxed{D}$ to solve for x and y.

$$\begin{array}{lll} \boxed{A}\ \ 2x+\ y=-3 & \xrightarrow{\qquad\qquad} & 2x+y=-3 \\ \boxed{D}\ \ 28x+14y=-42 & \xrightarrow[\text{Divide by }-14.]{} & \underline{-2x-y=3} \\ & & \qquad 0=0 \end{array}$$

The system reduces to the identity $0 = 0$. This implies that

a. There are infinitely many solutions.

b. The equations are dependent.

c. To find the general solution, we need to express the dependency among the variables as an ordered triple. Note that from equation $\boxed{A}$, we can solve for x in terms of y, and from equation $\boxed{B}$, we can solve for z in terms of y. Therefore,

Equation $\boxed{A}$: $2x = -y - 3 \Rightarrow x = \frac{-y-3}{2} \Rightarrow x = -\frac{y+3}{2}$

Equation $\boxed{B}$: $16z = -2y - 10 \Rightarrow z = \frac{-2y-10}{16} \Rightarrow z = -\frac{y+5}{8}$

Therefore, the solution set is $\left\{\left(-\frac{y+3}{2}, y, -\frac{y+5}{8}\right) \middle| y \text{ is any real number}\right\}$.

Apply the Skills

5. Repeat Example 5 with the given system.

$$\begin{aligned} 5y + z &= 0 \\ -x \qquad + 4z &= 0 \\ -x + 5y + 5z &= 0 \end{aligned}$$

The general solution to the system in Example 5 can be written in a number of forms.

Solve for x and z in terms of y: $\left\{\left(-\frac{y+3}{2}, y, -\frac{y+5}{8}\right) \middle| y \text{ is any real number}\right\}$

Solve for y and z in terms of x: $\left\{\left(x, -2x-3, \frac{x-1}{4}\right) \middle| x \text{ is any real number}\right\}$

Solve for x and y in terms of z: $\{(4z+1, -8z-5, z) | z \text{ is any real number}\}$

Check Your Work

Any form of the general solution can be checked by substitution in the original three equations.

Check: $\left\{\left(-\frac{y+3}{2}, y, -\frac{y+5}{8}\right) \middle| y \text{ is any real number}\right\}$

$\boxed{A}$

$$\begin{aligned} 2x + y &= -3 \\ 2\left(-\frac{y+3}{2}\right) + y &\stackrel{?}{=} -3 \\ -y - 3 + y &= -3 \checkmark \end{aligned}$$

$\boxed{B}$

$$\begin{aligned} 2y + 16z &= -10 \\ 2y + 16\left(-\frac{y+5}{8}\right) &\stackrel{?}{=} -10 \\ 2y - 2y - 10 &= -10 \checkmark \end{aligned}$$

$\boxed{C}$

$$\begin{aligned} -7x - 3y + 4z &= 8 \\ -7\left(-\frac{y+3}{2}\right) - 3y + 4\left(-\frac{y+5}{8}\right) &\stackrel{?}{=} 8 \\ \frac{7}{2}y + \frac{21}{2} - 3y - \frac{1}{2}y - \frac{5}{2} &\stackrel{?}{=} 8 \\ \frac{6}{2}y - 3y + \frac{16}{2} &= 8 \checkmark \end{aligned}$$

The topic of dependent equations will be discussed in more detail when we learn matrix methods to solve systems of linear equations. With additional tools available, we can investigate whether the dependent equations represent planes that intersect in a line or whether the equations all represent the same plane (Figure 8-7 on page 873).

Learn: Use Systems of Linear Equations in Applications

When solving an application involving three unknowns, sometimes it is convenient to use a system of three independent equations, as demonstrated in Examples 6 and 7.

Example 6: Solving an Application Involving Finance

Janette invested a total of \$18,000 in three different mutual funds. A bond fund returned 4% the first year. An aggressive growth fund lost 8% for the year, and an international fund returned 2%. Janette invested \$2000 more in the growth fund than in the other two funds combined. If the net loss was \$540 for the year, how much did Janette invest in each fund?

Solution:

Let x represent the amount invested in the bond fund.

Let y represent the amount invested in the growth fund.

Let z represent the amount invested in the international fund.

Label the variables.

With three unknowns, we need three independent equations.

$x + y + z = 18{,}000$ The total amount invested was \$18,000.

$y = (x + z) + 2000$ The amount invested in the growth fund was \$2000 more than the combined amount in the other two funds.

$0.04x - 0.08y + 0.02z = -540$ The sum of the gain and loss from each fund equals −\$540.

$$\begin{aligned}
&\boxed{A}\ x + y + z = 18{,}000 & &\boxed{A}\quad x + y + z = 18{,}000\\
&\boxed{B}\ y = (x + z) + 2000 \xrightarrow{\text{Standard form}} & &\boxed{B}\ -x + y - z = 2000\\
&\boxed{C}\ 0.04x - 0.08y + 0.02z = -540 \xrightarrow{\text{Multiply by 100.}} & &\boxed{C}\ 4x - 8y + 2z = -54{,}000
\end{aligned}$$

Eliminate x from equations $\boxed{A}$ and $\boxed{B}$ and equations $\boxed{B}$ and $\boxed{C}$.

$$\begin{aligned}
\boxed{A}\quad & x + y + z = 18{,}000 & 4\cdot\boxed{B}\quad & -4x + 4y - 4z = 8000\\
\boxed{B}\quad & \underline{-x + y - z = 2000} & \boxed{C}\quad & \underline{4x - 8y + 2z = -54{,}000}\\
& 2y = 20{,}000 & & -4y - 2z = -46{,}000 \quad \boxed{D}\\
& y = 10{,}000 & &
\end{aligned}$$

Back substitute.

$$\begin{aligned}
\boxed{D}\quad -4y - 2z &= -46{,}000 & \boxed{A}\quad x + y + z &= 18{,}000\\
-4(10{,}000) - 2z &= -46{,}000 & x + (10{,}000) + (3000) &= 18{,}000\\
-2z &= -6000 & x + 13{,}000 &= 18{,}000\\
z &= 3000 & x &= 5000
\end{aligned}$$

Janette invested \$5000 in the bond fund, \$10,000 in the aggressive growth fund, and \$3000 in the international fund.

Check Your Work

For Example 6, verify that the conditions of the problem have been met.

Principal: \$5000 + \$10,000 + \$3000 = \$18,000 ✓

Return: $0.04(\$5000) - 0.08(\$10{,}000) + 0.02(\$3000) = -\$540$ ✓

More invested in growth: (\$5000 + \$3000) + \$2000 = \$10,000 ✓

Apply the Skills

6. Javier mixes three solutions of acid with concentrations of 10%, 15%, and 5%. Javier wants to make 30 L of a mixture that is 12% acid and uses four times as much of the 15% solution as the 5% solution. How much of each of the three solutions must be used?

Learn: Modeling with Linear Equations in Three Variables

In Lesson 2.1, we used regression to find a quadratic model $y = ax^2 + bx + c$ from observed data points. We will now learn how to find a quadratic model by using a system of linear equations in three variables. The premise is that any three noncollinear points (points that do not all fall on the same line) define a unique parabola given by $y = ax^2 + bx + c$ $(a \neq 0)$.

Example 7: Using a System of Linear Equations to Create a Quadratic Model

Given the noncollinear points (4, 2), (1, –1), and (–1, 7), find an equation of the form $y = ax^2 + bx + c$ that defines the parabola through the points.

Solution:

$$y = ax^2 + bx + c$$

Substitute (4, 2): $2 = a(4)^2 + b(4) + c \longrightarrow \boxed{A} \quad 16a + 4b + c = 2$

Substitute (1, –1): $-1 = a(1)^2 + b(1) + c \longrightarrow \boxed{B} \quad a + b + c = -1$

Substitute (–1, 7): $7 = a(-1)^2 + b(-1) + c \longrightarrow \boxed{C} \quad a - b + c = 7$

Eliminate variable b from equations $\boxed{A}$ and $\boxed{C}$ and from equations $\boxed{B}$ and $\boxed{C}$.

$$\begin{array}{rlcrl} \boxed{A} & 16a + 4b + c = 2 & \qquad & \boxed{B} & a + b + c = -1 \\ 4\cdot\boxed{C} & \underline{4a - 4b + 4c = 28} & & \boxed{C} & \underline{a - b + c = 7} \\ & 20a \quad + 5c = 30 \quad \boxed{D} & & & 2a \quad + 2c = 6 \quad \boxed{E} \end{array}$$

$$\begin{array}{ll} \boxed{D}\ \ 20a+5c=30 \xrightarrow{\text{Divide by 5.}} & 4a+c=6 \\ \boxed{E}\ \ 2a+2c=6 \xrightarrow[\text{Divide by }-2.]{} & \underline{-a-c=-3} \\ & 3a \qquad\ \ =3 \\ & \qquad\ \ a=1 \end{array}$$

$$\text{Back substitute.}\left\{\begin{array}{r} \boxed{E}\ \ 2a+2c=6 \\ 2(1)+2c=6 \\ c=2 \\ \\ \boxed{B}\ \ a+b+c=-1 \\ (1)+b+(2)=-1 \\ b=-4 \end{array}\right.$$

Substituting $a=1$, $b=-4$, and $c=2$ into the equation $y=ax^2+bx+c$ gives

$$y=x^2-4x+2.$$

The graph of $y=x^2-4x+2$ passes through the points (4, 2), (1, –1), and (–1, 7) as shown in Figure 8-8.

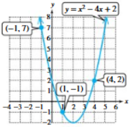
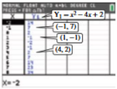


Figure 8-8

The results can also be verified by using the Table feature of a graphing utility.

Enter $Y_1 = x^2 - 4x + 2$.



Apply the Skills

7. Given the noncollinear points (–3, 2), (–4, 1), and (–6, –7), find an equation of the form $y=ax^2+bx+c$ that defines the parabola through the points.

Insights

We can verify that the three points given in Example 7 are *not* collinear by showing that the slopes between two pairs of points are different.

For (4, 2) and (1, –1),

$$m=\frac{-1-2}{1-4}=1$$

For (4, 2) and (–1, 7),

$$m=\frac{7-2}{-1-4}=-1$$

Practice 8-2

Practice Exercises

Prerequisite Review

R.1. How much interest will a student have to pay if the student borrows $2000 for 2 yr at a simple interest rate of 3%?

R.2. Fatima needs to have a toilet repaired. The cost of the new plumbing fixtures is $110 and labor is $70/hr. Write a model that represents the cost of the repair C (in $) in terms of the number of hours of labor x.

R.3. Solve the equation. $\frac{v-4}{5}+\frac{v}{8}=\frac{v-2}{5}-7$

Concept Connections

1. The graph of a linear equation in two variables is a line in a two-dimensional coordinate system. The graph of a linear equation in three variables is a ________________ in a three-dimensional coordinate system.

2. A solution to a system of linear equations in three variables is an ordered ________________ that satisfies each equation in the system. Graphically, this is a point of ________________ of three planes.

Learn: Identify Solutions to a System of Linear Equations in Three Variables

For Exercises 3–4, find three ordered triples that are solutions to the linear equation in three variables.

3. $2x+4y-6z=12$

4. $3x-5y+z=15$

For Exercises 5–8, determine if the ordered triple is a solution to the system of equations. (See Example 1, p. 869)

5.
$$\begin{aligned} -x+3y-7z&=7\\ 2x+4y+z&=16\\ 3x-5y+6z&=-9 \end{aligned}$$

a. (2, 3, 0) **b.** (−2, 4, 1)

6.
$$\begin{aligned} 2x-3y+z&=-12\\ x+y-2z&=9\\ -3x+2y-z&=7 \end{aligned}$$

a. (2, 5, −1)

b. (1, 4, −2)

7.
$$\begin{aligned} x+y+z&=2\\ x+2y-z&=2\\ 3x+5y-z&=6 \end{aligned}$$

a. (2, 0, 0)

b. (−1, 2, 1)

8.
$$\begin{aligned} -x-y+z&=3\\ 3x+4y-z&=1\\ 5x+7y-z&=-1 \end{aligned}$$

a. (1, 2, 6)

b. (3, −1, 5)

Learn: Solve Systems of Linear Equations in Three Variables

For Exercises 9–32, solve the system. If a system has one unique solution, write the solution set. Otherwise, determine the number of solutions to the system, and determine whether the system is inconsistent, or the equations are dependent. (See Examples 2–5, pp. 870–874)

9.
$$\begin{aligned} x-2y+z&=-9\\ 3x+4y+5z&=9\\ -2x+3y-z&=12 \end{aligned}$$

10.
$$\begin{aligned} 2x-y+z&=6\\ -x+5y-z&=10\\ 3x+y-3z&=12 \end{aligned}$$

11.
$$\begin{aligned} 4x&=3y-2z-5\\ 2(x+y)&=y+z-6\\ 6(x-y)+z&=x-5y-8 \end{aligned}$$

12. $3x = 5y - z + 13$

$-(x - y) - z = x - 3$

$5(x + y) = 3y - 3z - 4$

13. $2x + 5z = 2$

$3y - 7z = 9$

$-5x + 9y = 22$

14. $3x - 2y = -8$

$5y + 6z = 2$

$7x + 11z = -33$

15. $-4x - 3y = 0$

$3y + z = -1$

$4x - z = 12$

16. $4x - y + 2z = 1$

$3x + 5y - z = -2$

$-9x - 15y + 3z = 0$

17. $2x = 3y - 6z - 1$

$6y = 12z - 10x + 9$

$3z = 6y - 3x - 1$

18. $5x = 2y - 3z - 3$

$4y = -1 - 10x - 5z$

$2z = 5x - 6y$

19. $x + 2y + 4z = 3$

$y + 3z = 5$

$x - 2z = -7$

20. $3x + 2y + 5z = 6$

$3y - z = 4$

$3x + 17y = 26$

21. $0.2x = 0.1y - 0.6z$

$0.004x + 0.005y - 0.001z = 0$

$30x = 50z - 20y$

22. $0.3x = 0.5y - 1.2z$

$0.05x + 0.1y = 0.04z$

$100x = 300y - 700z$

23. $\frac{1}{12}x + \frac{1}{4}y + \frac{1}{3}z = \frac{7}{12}$

$-\frac{1}{10}x + \frac{1}{2}y - \frac{1}{5}z = -\frac{17}{10}$

$\frac{1}{2}x + \frac{1}{4}y + z = 3$

24. $x + \frac{7}{2}y + \frac{1}{2}z = 4$

$\frac{3}{4}x + y + \frac{1}{2}z = -1$

$\frac{1}{10}x - \frac{2}{5}y - \frac{3}{10}z = 1$

25. $3x + 2y + 5z = 12$

$3y + 8z = -8$

$10z = 20$

26. $-4x + 6y + z = -4$

$-2y - 4z = -24$

$6z = 48$

27. $\frac{x+2}{3} + \frac{y-4}{2} + \frac{z+1}{6} = 8$

$-\frac{x+2}{3} + \frac{z+1}{2} = 8$

$\frac{y-4}{4} - \frac{z+1}{6} = -1$

28. $\frac{x-1}{7} + \frac{y-2}{3} + \frac{z+2}{4} = 13$

$\frac{y-2}{9} - \frac{z+2}{8} = 3$

$\frac{x-1}{7} + \frac{z+2}{2} = 3$

29. $3(x + y) = 6 - 4z + y$

$4 = 6y + 5z$

$-3x + 4y + z = 0$

30. $4(x - y) = 8 - z - y$

$3 = 3x + 4z$

$-x + 3y + 3z = 1$

31. $-3x + 4y - z = -4$

$x + 2y + z = 4$

$-12x + 16y - 4z = -16$

32. $x + 2y - 3z = 5$

$-2x - 5y + 4z = -6$

$3x + 6y - 9z = 15$

For Exercises 33–36, solve the system from the indicated exercise and write the general solution. (See Example 5, p. 874)

33. Exercise 19

34. Exercise 20

35. Exercise 31

36. Exercise 32

For Exercises 37–38, the general solution is given for a system of linear equations. Find three individual solutions to the system.

37.
$$\begin{aligned} -x + 4y + 2z &= 4 \\ x - 3y - z &= -2 \\ -x + y - z &= -2 \end{aligned}$$

Solution: $\{(-2z + 4, -z + 2, z) \;\; z \text{ is any real number}\}$

38.
$$\begin{aligned} 2x - 3y + z &= 1 \\ x + 4y - z &= 3 \\ 5x - 2y + z &= 5 \end{aligned}$$

Solution: $\{(x, -3x + 4, -11x + 13) \;\; x \text{ is any real number}\}$

Learn: Use Systems of Linear Equations in Applications

39. Devon invested $8000 in three different mutual funds. A fund containing large cap stocks made 6.2% return in 1 yr. A real estate fund lost 13.5% in 1 yr, and a bond fund made 4.4% in 1 yr. The amount invested in the large cap stock fund was twice the amount invested in the real estate fund. If Devon had a net return of $66 across all investments, how much was invested in each fund? (**See Example 6, p. 876**)

40. Melek inherited $120,000 from his uncle and decided to invest the money. Part of the money was put in a money market account that earns 2.2% simple interest. The remaining money was invested in a stock that returned 6% in the first year and a mutual fund that lost 2% in the first year. Melek invested $10,000 more in the stock than in the mutual fund, and the net gain for 1 yr was $2820. Determine the amount invested in each account

41. A basketball player scored 26 points in one game. In basketball, some baskets are worth 3 points, some are worth 2 points, and free-throws are worth 1 point. The player scored four more 2-point baskets than 3-point baskets. The number of free-throws equaled the sum of the number of 2-point and 3-point shots made. How many free-throws, 2-point shots, and 3-point shots did the player make?

42. A sawmill cuts boards for a lumber supplier. When saws A, B, and C all work for 6 hr, they cut 7200 linear board-ft of lumber. It would take saws A and B working together 9.6 hr to cut 7200 ft of lumber. Saws B and C can cut 7200 ft of lumber in 9 hr. Find the rate (in ft/hr) that each saw can cut lumber.

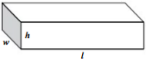
43. Plant fertilizers are categorized by the percentage of nitrogen (N), phosphorus (P), and potassium (K) they contain, by weight. For Example, a fertilizer that has N-P-K numbers of 8-5-5 has 8% nitrogen, 5% phosphorus, and 5% potassium by weight. Suppose that a fertilizer has twice as much potassium by weight as phosphorus. The percentage of nitrogen equals the sum of the percentages of phosphorus and potassium. If nitrogen, phosphorus, and potassium make up 42% of the fertilizer, determine the proper N-P-K label on the fertilizer.

a. 14-7-14
b. 21-7-14
c. 14-7-21
d. 14-21-21

44. A theater charges $50 per ticket for seats in Section A, $30 per ticket for seats in Section B, and $20 per ticket for seats in Section C. For one play, 4000 tickets were sold for a total of $120,000 in revenue. If 1000 more tickets in Section B were sold than the other two sections combined, how many tickets in each section were sold?

45. The perimeter of a triangle is 55 in. The shortest side is 7 in. less than the longest side. The middle side is 19 in. less than the combined lengths of the shortest and longest sides. Find the lengths of the three sides.

46. A package in the shape of a rectangular solid is to be mailed. The combination of the girth (perimeter of a cross section defined by *w* and *h*) and the length of the package is 48 in. The width is 2 in. greater than the height, and the length is 12 in. greater than the width. Find the dimensions of the package.



47. The measure of the largest angle in a triangle is 100° larger than the sum of the measures of the other two angles. The measure of the smallest angle is two-thirds the measure of the middle angle. Find the measure of each angle.

48. The measure of the largest angle in a triangle is 18° more than the sum of the measures of the other two angles. The measure of the smallest angle is one-half the measure of the middle angle. Find the measure of each angle.

Learn: Modeling with Linear Equations in Three Variables

49. a. Show that the points (1, 0), (3, 10), and (−2, 15) are not collinear by finding the slope between (1, 0) and (3, 10), and the slope between (3, 10) and (−2, 15). (**See Example 7, p. 877**)

b. Find an equation of the form $y = ax^2 + bx + c$ that defines the parabola through the points.

c. Use a graphing utility to verify that the graph of the equation in part (b) passes through the given points.

50. a. Show that the points (2, 9), (−1, −6), and (−4, −3) are not collinear by finding the slope between (2, 9) and (−1, −6), and the slope between (2, 9) and (−4, −3).

b. Find an equation of the form $y = ax^2 + bx + c$ that defines the parabola through the points.

c. Use a graphing utility to verify that the graph of the equation in part (b) passes through the given points.

For Exercises 51–52, find an equation of the form $y = ax^2 + bx + c$ that defines the parabola through the three noncollinear points given.

51. (0, 6), (2, −6), (−1, 9)

52. (0, −4), (2, −6), (−3, −31)

The motion of an object traveling along a straight path is given by $s(t) = \frac{1}{2}at^2 + v_0t + s_0$, where $s(t)$ is the position relative to the origin at time t. For Exercises 53–54, three observed data points are given. Find the values of a, v_0, and s_0.

53. $s(1) = 30$, $s(2) = 54$, $s(3) = 82$

54. $s(1) = -7$, $s(2) = 12$, $s(3) = 37$

Many statistics courses cover a topic called *multiple regression*. This provides a means to predict the value of a dependent variable *y* based on two or more independent variables $x_1, x_2, \ldots, xn$. The model $y = ax_1 + bx_2 + c$ is a linear model that predicts *y* based on two independent variables x_1 and x_2. While statistical techniques may be used to find the values of *a*, *b*, and *c* based on a large number of data points, we can form a crude model given three data values (x_1, x_2, y). Use the information given in Exercises 55–56 to form a system of three equations and three variables to solve for *a*, *b*, and *c*.

55. The selling price of a home *y* (in \$1000) is given based on the living area x_1 (in 100 ft^2) and on the lot size x_2 (in acres).

Living Area (100 ft^2) x_1	Lot Size (acres) x_2	Selling Price (\$1000) y
28	0.5	225
25	0.8	207
18	0.4	154

a. Use the data to create a model of the form $y = ax_1 + bx_2 + c$.

b. Use the model from part (a) to predict the selling price of a home that is 2000 ft^2 on a 0.4-acre lot.

56. The gas mileage y (in mpg) for city driving is given based on the weight of the vehicle x_1 (in lb) and on the number of cylinders.

Weight (lb) x_1	Cylinders x_2	Mileage (mpg) y
3500	6	20
3200	4	26
4100	8	18

a. Use the data to create a model of the form $y = ax_1 + bx_2 + c$.

b. Use the model from part (a) to predict the gas mileage of a vehicle that is 3800 lb and has 6 cylinders.

Write About It

57. Give a geometric description of the solution set to a linear equation in three variables.

58. If a system of linear equations in three variables has no solution, then what can be said about the three planes represented by the equations in the system?

59. Explain the procedure presented in this lesson to solve a system of linear equations in three variables.

60. Explain how to check a solution to a system of linear equations in three variables.

Expanding Your Skills

For Exercises 61–62, find all solutions of the form (a, b, c, d).

61.
$$\begin{aligned} 2a + b - c + d &= 7 \\ 3b + 2c - 2d &= -11 \\ a + 3c + 3d &= 14 \\ 4a + 2b - 5c &= 6 \end{aligned}$$

62.
$$\begin{aligned} 3a - 4b + 2c + d &= 8 \\ 2a + 3b + 2d &= 7 \\ 5b - 3c + 4d &= -4 \\ -a + b - 2c &= -7 \end{aligned}$$

For Exercises 63–64, find all solutions of the form (u, v, w).

63.
$$\begin{aligned} \frac{u-3}{4} + \frac{v+1}{3} + \frac{w-2}{8} &= 1 \\ \frac{u-3}{2} + \frac{v+1}{2} + \frac{w-2}{4} &= 0 \\ \frac{u-3}{4} - \frac{v+1}{2} + \frac{w-2}{2} &= -6 \end{aligned}$$

64.
$$\begin{aligned} \frac{u+1}{6} + \frac{v-1}{6} + \frac{w+3}{4} &= 11 \\ \frac{u+1}{3} - \frac{v-1}{2} + \frac{w+3}{4} &= 7 \\ \frac{u+1}{2} - \frac{v-1}{6} + \frac{w+3}{2} &= 20 \end{aligned}$$

For Exercises 65–66, solve for x, y, and z. Assume that a, b, and c are constant real numbers.

65.
$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = a$$
$$\frac{a}{x} + \frac{b}{y} - \frac{c}{z} = b$$
where $a \neq -c$, $b \neq c$, $a \neq b$, $abc \neq 0$
$$\frac{a}{x} - \frac{b}{y} - \frac{c}{z} = c$$

66.
$$\frac{a}{x} + \frac{b}{y} + \frac{2c}{z} = 4$$
$$\frac{a}{x} + \frac{2b}{y} - \frac{c}{z} = 2 \quad \text{where } abc \neq 0$$
$$\frac{2a}{x} - \frac{b}{y} - \frac{c}{z} = 0$$

Recall that an equation of a circle can be written in the form $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius. Expanding terms, the equation can also be written in the form $x^2 + y^2 + Ax + By + C = 0$. For Exercises 67–68,

a. Find an equation of the form $x^2 + y^2 + Ax + By + C = 0$ that represents the circle that passes through the given points.

b. Find the center and radius of the circle.

67. (2, 2), (6, 0), (7, –3)

68. (–1, 12), (5, 10), (9, 2)

For Exercises 69–74, find constants A, B, and C, so that the expressions on the left and right side of the equation are equal. (*Hint*: Create a system of linear equations by equating the constant terms and by equating the coefficients on the variables.)

69. $4x + 5 = (A + B)x + (2A - B)$

70. $5x + 13 = Ax + Bx + 2A + B$

71. $9x^2 - 3x - 9 = (A + B + C)x^2 + (2A - B + C)x + (A + B - 2C)$

72. $x^2 + 4x + 10 = (2A + 3B - C)x^2 + (4A - 6B + 3C)x + (3B + 4C)$

73. $11x^2 + 26x - 5 = 2Ax^2 + 5Ax + 3A + Bx^2 - 2Bx - 8B + 2Cx^2 - 7Cx - 4C$

74. $3x^2 + 37x - 82 = Ax^2 + Ax - 12A + 3Bx^2 - 10Bx + 3B + 3Cx^2 + 11Cx - 4C$

Point of Interest

Dr. Valerie L. Thomas (1943–present) is a physicist, mathematician, and inventor who worked on numerous projects for NASA between 1964 and 1995. In the 1970s, Dr. Thomas worked on the Landsat satellite program in the area of image processing to enable satellites to transmit images from space. But Dr. Thomas is perhaps best known for her invention of the *illusion transmitter*, a device that enables three-dimensional images to be sent from one location to another. Dr. Thomas recognized that an object placed in front of a flat mirror produces an image that appears to be *behind* the mirror, but a concave mirror creates an image that appears to be in *front*. The illusion transmitter uses one parabolic mirror to create an optical illusion of a three-dimensional object. A camera then records the image and sends the video signal to a receiver at a different location, which in turn projects the image onto a second parabolic mirror, where it can be observed.

This technology behind the illusion transmitter has been adapted for television and video production, surgery, and magnetic resonance imaging (MRI). Furthermore, assigning a three-dimensional coordinate system to a three-dimensional object, such as a human heart, enables the object to be mapped and studied mathematically. Key points on the surface are defined by ordered triples that, when taken collectively, create the “form” or “map” of the object.

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75. a. Plot the ordered triples on a three-dimensional coordinate system. You can do this by hand on paper or by using an online graphing utility that accommodates 3-D. If you use a graphing utility, you can view the points from different directions by holding down the left mouse key and dragging the cursor over the image. Move the coordinate system so that the positive *x*-axis is oriented toward the front of the monitor, the positive *y*-axis is to the right, and the positive *z*-axis is upward.

$A(1, 2, 4)$, $B(-1, 2, 4)$, $C(1, -2, 4)$, $D(-1, -2, 4)$,
$E(1, 2, -4)$, $F(-1, 2, -4)$, $G(1, -2, -4)$, $H(-1, -2, -4)$

b. Draw the line segments on the graph.

$\overline{AB}$, $\overline{BD}$, $\overline{DC}$, $\overline{CA}$, $\overline{EF}$, $\overline{FH}$, $\overline{HG}$, $\overline{GE}$, $\overline{BF}$, $\overline{AE}$, $\overline{DH}$, and $\overline{CG}$.

c. As drawn in part (b), what type of shape do the eight points define?

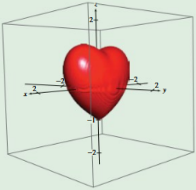
d. Shapes of more complexity can be formed by using a large number of points to define an object such as a character in a computer game. Then the object can be moved (shifted), stretched, and rotated in much the same way we perform transformations on two-dimensional graphs. Explain how a three-dimensional computerized rendering of an object such as a human heart might be helpful to a medical doctor.

e. Three-dimensional surfaces can also be made from equations using three variables in much the same way that curves in a plane can be rendered from equations in two variables. For example, the equation

$$\left(x^2 + \frac{9}{4}y^2 + z^2 - 1\right)^3 - x^2z^3 - \frac{9}{80}y^2z^3 = 0$$

represents a "Valentine's heart."

$$\left(x^2 + \frac{9}{4}y^2 + z^2 - 1\right)^3 - x^2z^3 - \frac{9}{80}y^2z^3 = 0$$



Use an online three-dimensional graphing utility to determine the shape of the graph formed by the equations

$z = \sqrt{4 - x^2 - y^2}$ and $z = -\sqrt{4 - x^2 - y^2}$.

Further Reading

Kooser, Amanda. "Astounding 3D-Printed Human Heart Model Cuts and Sutures Like the Real Thing." 18 November 2020. https://www.cnet.com/news/first-3d-printed-human-heart-model-cuts-and-sutures-like-the-real-thing/

GO ONLINE Go online for more practice problems.

Lesson 8-3

Partial Fraction Decomposition

What Will You Learn?

After completing this lesson, you should be able to:

- Set Up a Partial Fraction Decomposition
- Decompose $\frac{f(x)}{g(x)}$, Where $g(x)$ Is a Product of Linear Factors
- Decompose $\frac{f(x)}{g(x)}$, Where $g(x)$ Has Irreducible Quadratic Factors

Learn: Set Up a Partial Fraction Decomposition

In Lesson R.4 we learned how to add and subtract rational expressions. For example:

$$\begin{aligned}\frac{5}{x+2}+\frac{3}{x-5}&=\frac{5(x-5)}{(x+2)(x-5)}+\frac{3(x+2)}{(x-5)(x+2)}\\&=\frac{5(x-5)+3(x+2)}{(x+2)(x-5)}\\&=\frac{8x-19}{(x+2)(x-5)}\end{aligned}$$

The fraction $\frac{8x-19}{(x+2)(x-5)}$ is the result of adding two simpler fractions, $\frac{5}{x+2}$ and $\frac{3}{x-5}$. The sum $\frac{5}{x+2}+\frac{3}{x-5}$ is called the **partial fraction decomposition** of $\frac{8x-19}{(x+2)(x-5)}$. In some applications in higher mathematics, it is more convenient to work with the partial fraction decomposition than the more complicated single fraction. In this lesson, we will learn the technique of partial fraction decomposition to write a rational expression as a sum of simpler fractions. That is, we will reverse the process of adding two or more fractions. There are two parts to this process.

I. First we set up the "form" or "structure" for the partial fraction decomposition into simpler fractions. For example, the denominator of $\frac{8x-19}{(x+2)(x-5)}$ consists of the distinct linear factors $(x+2)$ and $(x-5)$. From the preceding discussion, the partial fraction decomposition must be of the form:

$$\frac{8x-19}{(x+2)(x-5)}=\frac{A}{x+2}+\frac{B}{x-5}$$

The expression on the right is the "form" or "structure" for the partial fraction decomposition of $\frac{8x-19}{(x+2)(x-5)}$.

II. Next, we solve for the constants A and B. To do so, multiply both sides of the equation by the LCD, and set up a system of linear equations.

$$(x+2)(x-5)\cdot\left[\frac{8x-19}{(x+2)(x-5)}\right]=(x+2)(x-5)\cdot\left[\frac{A}{x+2}+\frac{B}{x-5}\right]$$

Multiply by the LCD to clear fractions.

$8x-19=A(x-5)+B(x+2)$

$8x-19=Ax-5A+Bx+2B$ Apply the distributive property.

$8x-19=(A+B)x+(-5A+2B)$ Simplify and combine like terms.

x coefficients are equal.

$$8x - 19 = \overbrace{(A + B)}x + \underbrace{(-5A + 2B)}$$

Constants are equal.

Two polynomials are equal if and only if the coefficients on like terms are equal.

$$A + B = 8$$

Equate the coefficients on x.

$$-5A + 2B = -19$$

Equate the constant terms.

$$\boxed{1}\quad A + B = 8 \xrightarrow{\text{Multiply by 5.}} 5A + 5B = 40$$

Solve the system of linear equations.

$$\boxed{2}\quad -5A + 2B = -19 \qquad \underline{-5A + 2B = -19}$$

$$7B = 21$$

$$B = 3$$

$$\boxed{1}\quad A + B = 8$$

$$A + 3 = 8$$

$$A = 5$$

$$A = 5 \text{ and } B = 3$$

$$\frac{8x - 19}{(x + 2)(x - 5)} = \frac{A}{x + 2} + \frac{B}{x - 5} = \frac{5}{x + 2} + \frac{3}{x - 5}$$

Substitute the values of A and B into the partial fraction decomposition.

Insights

To illustrate why two polynomials are equal if and only if their coefficients are equal, consider the statement:

$$2x^2 + 5x = Ax^2 + Bx.$$

By inspection, we can see that the two polynomials will be equal only for $A = 2$ and $B = 5$.

We begin partial fraction decomposition by factoring the denominator into linear factors $(ax + b)$ and quadratic factors $(ax^2 + bx + c)$ that are irreducible over the integers. A quadratic factor that is irreducible over the integers cannot be factored as a product of binomials with integer coefficients. From the factorization of the denominator, we then determine the proper form for the partial fraction decomposition using the following guidelines.

Decomposition of $\frac{f(x)}{g(x)}$ into Partial Fractions

Consider a rational expression $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials with real coefficients, $g(x) \neq 0$, and the degree of $f(x)$ is less than the degree of $g(x)$.

PART I:

Step 1 Factor the denominator $g(x)$ completely into linear factors of the form $(ax + b)^m$ and quadratic factors of the form $(ax^2 + bx + c)^n$ that are not further factorible over the integers (irreducible over the integers).

Step 2 Set up the form for the decomposition. That is, write the original rational expression $\frac{f(x)}{g(x)}$ as a sum of simpler fractions using these guidelines.

Note that $A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_n$, and $C_1, C_2, \dots, C_n$ are constants.

- **Linear factors of $g(x)$:** For each linear factor of $g(x)$, the partial fraction decomposition must include the sum:

$$\frac{A_1}{(ax+b)^1} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

- **Quadratic factors of $g(x)$:** For each quadratic factor of $g(x)$, the partial fraction decomposition must include the sum:

$$\frac{B_1x + C_1}{(ax^2+bx+c)^1} + \frac{B_2x + C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2+bx+c)^n}$$

PART II:

Step 3 With the form of the partial fraction decomposition set up, multiply both sides of the equation by the LCD to clear fractions.

Step 4 Using the equation from step 3, set up a system of linear equations by equating the constant terms and equating the coefficients of like powers of x.

Step 5 Solve the system of equations from step 4 and substitute the solutions to the system into the partial fraction decomposition.

Check Your Work

It is important to note that to perform partial fraction decomposition on a rational expression, the degree of the numerator must be less than the degree of the denominator.

Insights

The constants $A_1, A_2, \dots, A_m$ are often replaced by A, B, C, and so on to avoid confusion with subscript notation.

The constants $B_1, B_2, \dots, B_n$ and $C_1, C_2, \dots, C_n$ are often replaced by A, B, C, D, E, and so on to avoid confusion with subscript notation.

In Examples 1 and 2, we focus on setting up the proper form for a partial fraction decomposition (Part I). In each example, note that the factors of the denominator will fall into one of the following categories:

Linear factors: $ax + b$

Repeated linear factors: $(ax + b)^m$ ($m \geq 2$, an integer)

Quadratic factors (irreducible over the integers): $ax^2 + bx + c$

Repeated quadratic factors (irreducible over the integers): $(ax^2 + bx + c)^n$ ($n \geq 2$, an integer)

Example 1: Setting Up the Form for a Partial Fraction Decomposition

Set up the form for the partial fraction decomposition for the given rational expressions.

a. $\dfrac{4x - 15}{(2x + 3)(x - 2)}$

b. $\dfrac{4x^2 + 10x + 9}{x^3 + 6x^2 + 9x}$

Solution:

a. $\dfrac{4x - 15}{(2x + 3)^1(x - 2)^1} = \dfrac{A}{(2x + 3)^1} + \dfrac{B}{(x - 2)^1}$

The denominator has distinct linear factors of the form $(2x + 3)^1$ and $(x - 2)^1$. Since each linear factor is raised to the first power, only one fraction is needed for each factor.

b. $\dfrac{4x^2 + 10x + 9}{x^3 + 6x^2 + 9x} = \dfrac{4x^2 + 10x + 9}{x(x^2 + 6x + 9)}$

Factor the denominator completely. The denominator has a linear factor of x^1 and a *repeated* linear factor $(x + 3)^2$.

$$= \dfrac{4x^2 + 10x + 9}{x^1(x + 3)^2} = \dfrac{A}{x^1} + \dfrac{B}{(x + 3)^1} + \dfrac{C}{(x + 3)^2}$$

Include one fraction with $(x + 3)$ raised to each positive integer up to and including 2.

For a repeated factor that occurs m times, one fraction must be given for each power less than or equal to m.

Apply the Skills

1. Set up the form for the partial fraction decomposition for the given rational expressions.

a. $\dfrac{-x + 18}{(3x + 1)(x + 4)}$

b. $\dfrac{-x^2 + 3x + 8}{x^3 + 4x^2 + 4x}$

In Example 2, we practice setting up the form for the partial decomposition of a rational expression that contains irreducible quadratic factors in the denominator.

Example 2: Setting Up the Form for a Partial Fraction Decomposition

Set up the form for the partial fraction decomposition for the given rational expressions.

a. $\frac{2x^2 - 3x + 4}{x^3 + 4x}$

b. $\frac{3x^2 + 8x + 14}{(x^2 + 2x + 5)^2}$

Solution:

a. $\frac{2x^2 - 3x + 4}{x^3 + 4x} = \frac{2x^2 - 3x + 4}{x(x^2 + 4)}$

Factor the denominator completely. The denominator has one linear factor x^1 and one irreducible quadratic factor $(x^2 + 4)^1$.

$= \frac{2x^2 - 3x + 4}{x^1(x^2 + 4)^1} = \frac{A}{x^1} + \frac{Bx + C}{(x^2 + 4)^1}$

Since each factor is raised to the first power, only one fraction is needed for each factor.

$= \frac{2x^2 - 3x + 4}{x(x^2 + 4)^1} = \frac{A}{x} + \frac{Bx + C}{(x^2 + 4)}$

Insights

For a first-degree (linear) denominator, the numerator is constant (degree 0). For a second-degree (quadratic) denominator, the numerator is linear (degree 1).

b. $\frac{3x^2 + 8x + 14}{(x^2 + 2x + 5)^2}$

The quadratic factor $x^2 + 2x + 5$ does not factor further over the integers.

$= \frac{Ax + B}{(x^2 + 2x + 5)^1} = \frac{Cx + D}{(x^2 + 2x + 5)^2}$

The factor $(x^2 + 2x + 5)$ appears to the *second* power in the denominator. Therefore, in the partial fraction composition, one fraction must have $(x^2 + 2x + 5)^1$ in the denominator, and one fraction must have $(x^2 + 2x + 5)^2$ in the denominator.

Apply the Skills

2. Set up the form for the partial fraction decomposition for the given rational expressions.

a. $\frac{7x^2 + 2x + 12}{x^3 + 3x}$

b. $\frac{-3x^2 - 5x - 19}{(x^2 + 3x + 6)^2}$

Learn: Decompose $\frac{f(x)}{g(x)}$, Where $g(x)$ Is a Product of Linear Factors

In Example 3, we find the partial fraction decomposition of a rational expression in which the denominator is a product of distinct linear factors.

Example 3: Decomposing $\frac{f(x)}{g(x)}$, Where $g(x)$ Has Distinct Linear Factors

Find the partial fraction decomposition. $\frac{4x-15}{(2x+3)(x-2)}$

Solution:

$\frac{4x-15}{(2x+3)(x-2)} = \frac{A}{2x+3} + \frac{B}{x-2}$

From Example 1(a), we have the form for the partial fraction decomposition.

$(2x+3)(x-2)\left[\frac{4x-15}{(2x+3)(x-2)}\right]$

$= (2x+3)(x-2)\left[\frac{A}{2x+3} + \frac{B}{x-2}\right]$

To solve for *A* and *B*, first multiply both sides by the LCD to clear fractions.

$4x - 15 = A(x-2) + B(2x+3)$ Apply the distributive property.

$4x - 15 = Ax - 2A + 2Bx + 3B$

$4x - 15 = (A + 2B)x + (-2A + 3B)$ Combine like terms.

$\boxed{1}\quad 4 = A + 2B$ Equate the *x* term coefficients.

$\boxed{2}\quad -15 = -2A + 3B$ Equate the constant terms.

Solve the system of linear equations by using the substitution method or addition method.

$\boxed{1}\quad 4 = A + 2B \xrightarrow{\text{Multiply by 2.}} 2A + 4B = 8$

$\boxed{2}\quad -15 = -2A + 3B \qquad \underline{-2A + 3B = -15}$

$7B = -7$

$B = -1$

$\boxed{1}\quad 4 = A + 2B$

$4 = A + 2(-1)$

$A = 6$

$A = 6$ and $B = -1$.

$\frac{4x-15}{(2x+3)(x-2)} = \frac{A}{2x+3} + \frac{B}{x-2}$ Substitute $A = 6$ and $B = -1$ into the partial fraction decomposition

$\frac{4x-15}{(2x+3)(x-2)} = \frac{6}{2x+3} + \frac{-1}{x-2}$ or equivalently $\frac{6}{2x+3} - \frac{1}{x-2}$

Apply the Skills

3. Find the partial fraction decomposition. $\frac{-x+18}{(3x+1)(x+4)}$

Check Your Work

Always remember that the result of a partial fraction decomposition can be checked by adding the partial fractions and verifying that the sum equals the original rational expression.

To verify the result of Example 3, we can add the rational expressions.

$$\begin{aligned}\frac{6}{2x+3}+\frac{-1}{x-2}&=\frac{6(x-2)}{(2x+3)(x-2)}+\frac{-1(2x+3)}{(x-2)(2x+3)}\\&=\frac{6(x-2)-1(2x+3)}{(2x+3)(x-2)}\\&=\frac{4x-15}{(2x+3)(x-2)}\ \checkmark\end{aligned}$$

In Example 4, we perform partial fraction decomposition with a rational expression that has repeated linear factors in the denominator.

Example 4: Decomposing $\frac{f(x)}{g(x)}$, Where $g(x)$ Has Repeated Linear Factors

Find the partial fraction decomposition. $\frac{4x^2+10x+9}{x^3+6x^2+9x}$

Solution:

$$\frac{4x^2+10x+9}{x(x+3)^2}=\frac{A}{x}+\frac{B}{(x+3)^1}+\frac{C}{(x+3)^2}$$

From Example 1(b), we have the form for the partial fraction decomposition.

$$x(x+3)^2\left[\frac{4x^2+10x+9}{x(x+3)^2}\right]=x(x+3)^2\left[\frac{A}{x}+\frac{B}{(x+3)^1}+\frac{C}{(x+3)^2}\right]$$

$$4x^2+10x+9=A(x+3)^2+Bx(x+3)+Cx$$

$$4x^2+10x+9=A(x^2+6x+9)+Bx^2+3Bx+Cx$$

$$4x^2+10x+9=Ax^2+6Ax+9A+Bx^2+3Bx+Cx$$

$$4x^2+10x+9=(A+B)x^2+(6A+3B+C)x+9A$$

$4=A+B$ Equate the x^2 term coefficients.

$10=6A+3B+C$ Equate the x term coefficients.

$9=9A$ Equate the constant terms.

Solve the system of linear equations.

From the third equation $A=1$.

Substituting 1 for A in the first equation gives us $4=1+B$, so $B=3$.

Substituting 1 for A and 3 for B in the second equation gives us

$$\begin{aligned}10&=6(1)+3(3)+C\\10&=6+9+C\\-5&=C\end{aligned}$$

$A=1$, $B=3$, and $C=-5$

$$\frac{4x^2+10x+9}{x(x+3)^2} = \frac{A}{x} + \frac{B}{(x+3)^1} + \frac{C}{(x+3)^2}$$

Substitute $A = 1$, $B = 3$, and $C = -5$ into the partial fraction decomposition.

$$\frac{4x^2+10x+9}{x(x+3)^2} = \frac{1}{x} + \frac{3}{(x+3)^1} + \frac{-5}{(x+3)^2}$$

$$= \frac{1}{x} + \frac{3}{x+3} = \frac{5}{(x+3)^2}$$

Apply the Skills

4. Find the partial fraction decomposition. $\dfrac{-x^2+3x+8}{x^3+4x^2+4x}$

Learn: Decompose $\frac{f(x)}{g(x)}$, Where $g(x)$ Has Irreducible Quadratic Factors

We now turn our attention to performing partial fraction decomposition where the denominator of a rational expression contains quadratic factors irreducible over the integers. In Example 5, we also address the situation in which the given rational expression is an **improper rational expression;** that is, the degree of the numerator is greater than or equal to the degree of the denominator. In such a case, we use long division to write the expression in the form:

(polynomial) + (proper rational expression)

where a **proper rational expression** is one in which the degree of the numerator is less than the degree of the denominator.

Example 5: Decomposing $\frac{f(x)}{g(x)}$, Where $g(x)$ Has an Irreducible Quadratic Factor

Find the partial fraction decomposition. $\dfrac{x^4+3x^3+6x^2+9x+4}{x^3+4x}$

Solution:

First note that the degree of the numerator is not less than the degree of the denominator. Therefore, perform long division first.

$$\frac{x^4+3x^3+6x^2+9x+4}{x^3+4x} \xrightarrow{\text{Long division}} \begin{array}{r} x+3 \\ x^3+4x\overline{)x^4+3x^3+6x^2+9x+4} \\ -(x^4 \qquad +4x^2) \\ \hline 3x^3+2x^2+9x \\ -(3x^3 \qquad +12x) \\ \hline 2x^2-3x+4 \end{array}$$

$$= x+3+\frac{2x^2-3x+4}{x^3+4x} \xleftarrow{\text{Equivalent form}}$$

$$= \overbrace{x+3}^{\text{polynomial}} + \overbrace{\frac{2x^2-3x+4}{x(x^2+4)}}^{\text{proper rational expression}} \qquad \text{Factor the denominator.}$$

$$\frac{2x^2-3x+4}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{(x^2+4)}$$

Perform partial fraction decomposition on the proper fraction. From Example 2(a), we have the form for the partial fraction decomposition.

$$x(x^2+4)\left[\frac{2x^2-3x+4}{x(x^2+4)}\right]=x(x^2+4)\left[\frac{A}{x}+\frac{Bx+C}{(x^2+4)}\right]$$

To solve for A, B, and C, multiply both sides by the LCD to clear fractions.

$$2x^2-3x+4=A(x^2+4)+(Bx+C)x$$

Apply the distributive property.

$$2x^2-3x+4=Ax^2+4A+Bx^2+Cx$$

$$2x^2-3x+4=(A+B)x^2+Cx+4A$$

Combine like terms.

$2=A+B$ Equate the x^2 term coefficients.

$-3=C$ Equate the x term coefficients.

$4=4A$ Equate the constant terms.

From the second equation, $C=-3$. Solve the system of linear equations.

From the third equation, $A=1$.

Substituting 1 for A in the first equation gives us $2=1+B$, so $B=1$.

$$\frac{2x^2-3x+4}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{(x^2+4)}$$

Substitute $A=1$, $B=1$, and $C=-3$.

$$\frac{2x^2-3x+4}{x(x^2+4)}=\frac{1}{x}+\frac{1x+(-3)}{x^2+4}\quad\text{or}\quad\frac{1}{x}+\frac{x-3}{x^2+4}$$

Therefore, $\dfrac{x^4+3x^3+6x^2+9x+4}{x^3+4x}=x+3+\dfrac{1}{x}+\dfrac{x-3}{x^2+4}$.

Apply the Skills

5. Find the partial fraction decomposition. $\dfrac{x^4+2x^3+10x^2+8x+12}{x^3+3x}$

In Example 6, we demonstrate the case in which a rational expression contains a repeated quadratic factor.

Example 6: Decomposing $\frac{f(x)}{g(x)}$, Where $g(x)$ Has a Repeated Irreducible Quadratic Factor

Find the partial fraction decomposition. $\dfrac{3x^2+8x+14}{(x^2+2x+5)^2}$

Solution:

$$\frac{3x^2+8x+14}{(x^2+2x+5)^2}=\frac{Ax+B}{(x^2+2x+5)^1}+\frac{Cx+D}{(x^2+2x+5)^2}$$

From Example 2(b), we have the form for the partial fraction decomposition.

To solve for *A*, *B*, *C*, and *D*, multiply both sides by the LCD to clear fractions.

$$(x^2+2x+5)^2\left[\frac{3x^2+8x+14}{(x^2+2x+5)^2}\right] = (x^2+2x+5)^2\left[\frac{Ax+B}{x^2+2x+5}+\frac{Cx+D}{(x^2+2x+5)^2}\right]$$

$$3x^2+8x+14 = (Ax+B)(x^2+2x+5)+(Cx+D)$$

$$3x^2+8x+14 = Ax^3+2Ax^2+5Ax+Bx^2+2Bx+5B+Cx+D$$

$3x^2+8x+14 = Ax^3+(2A+B)x^2+(5A+2B+C)x+5B+D$ Combine like terms.

$0 = A$ Equate the x^3 term coefficients.

$3 = 2A+B$ Equate the x^2 term coefficients.

$8 = 5A+2B+C$ Equate the x term coefficients.

$14 = 5B+D$ Equate the constant terms.

From the first equation, $A = 0$, and substituting 0 for *A* in the second equation gives us $B = 3$. Solve the system of linear equations.

Substituting 0 for *A* and 3 for *B* in the third equation gives us

$$8 = 5(0)+2(3)+C$$

$$8 = 6+C$$

$$C = 2.$$

Substituting 3 for *B* in the fourth equation gives us

$$14 = 5(3)+D$$

$$14 = 15 + D$$

$$D = -1.$$

$A = 0$, $B = 3$, $C = 2$, and $D = -1$

$$\frac{3x^2+8x+14}{(x^2+2x+5)^2} = \frac{Ax+B}{x^2+2x+5}+\frac{Cx+D}{(x^2+2x+5)^2}$$

Substitute $A = 0$, $B = 3$, $C = 2$, and $D = -1$ into the partial fraction decomposition.

$$\frac{3x^2+8x+14}{(x^2+2x+5)^2} = \frac{(0)x+(3)}{x^2+2x+5}+\frac{(2)x+(-1)}{(x^2+2x+5)^2} \quad \text{or} \quad \frac{3}{x^2+2x+5}+\frac{2x-1}{(x^2+2x+5)^2}$$

Apply the Skills

6. Find the partial fraction decomposition. $\dfrac{-3x^2-5x-19}{(x^2+3x+6)^2}$

Practice 8-3

Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, factor completely.

R.1. $12y^3 - 16y - 9y^2 + 12$

R.2. $64u^2 + 80u + 25$

R.3. Add. $\frac{4}{4a+3} + \frac{4}{a-3}$

R.4. Use long division to divide.

$$\frac{25x^4 - 5x^3 + 15x^2 + 4x - 2}{5x^2 - x + 4}$$

R.5. Solve. $\frac{16}{x} = \frac{16}{x-2} = \frac{4}{x}$

Concept Connections

1. The process of decomposing a rational expression into two or more simpler fractions is called partial ________.

2. When setting up a partial fraction decomposition, if a fraction has a linear denominator, then the numerator should be (constant/linear). That is, should the numerator be set up as A or $Ax + B$?

3. When setting up a partial fraction decomposition, if the denominator of a fraction is a quadratic polynomial irreducible over the integers, then the numerator should be (constant/linear). That is, should the numerator be set up as A or $Ax + B$?

4. In what situation should long division be used before attempting to decompose a rational expression into partial fractions?

Learn: Set Up a Partial Fraction Decomposition

For Exercises 5–20, set up the form for the partial fraction decomposition. Do not solve for *A*, *B*, *C*, and so on. (See Examples 1–2, pp. 889–890)

5. $\frac{-x - 37}{(x+4)(2x-3)}$

6. $\frac{20x - 4}{(x-5)(3x+1)}$

7. $\frac{8x - 10}{x^2 - 2x}$

8. $\frac{y - 12}{y^2 + 3y}$

9. $\frac{6w - 7}{w^2 + w - 6}$

10. $\frac{-10t - 11}{t^2 + 5t - 6}$

11. $\frac{x^2 + 26x + 100}{x^3 + 10x^2 + 25x}$

12. $\frac{-3x^2 + 2x + 8}{x^3 + 4x^2 + 4x}$

13. $\frac{13x^2 + 2x + 45}{2x^3 + 18x}$

14. $\frac{17x^2 - 7x + 18}{7x^3 + 42x}$

15. $\frac{2x^3 - x^2 + 13x - 5}{x^4 + 10x^2 + 25}$

16. $\frac{3x^3 - 4x^2 + 11x - 12}{x^4 + 6x^2 + 9}$

17. $\frac{5x^2 - 4x + 8}{(x-4)(x^2 + x + 4)}$

18. $\frac{x^2 + 15x - 6}{(x+6)(x^2 + 2x + 6)}$

19. $\frac{2x^5 + 3x^3 + 4x^2 + 5}{x(x+2)^3(x^2 + 2x + 7)^2}$

20. $\frac{6x^4 - 5x^3 + 2x^2 - 5}{(x-3)(2x+9)^2(x^2+1)^2}$

Learn: Decompose $\frac{f(x)}{g(x)}$ into Partial Fractions

For Exercises 21–42, find the partial fraction decomposition. (See Examples 3–6, pp. 891–894)

21. $\frac{-x - 37}{(x+4)(2x-3)}$

22. $\frac{20x - 4}{(x-5)(3x+1)}$

23. $\frac{8x - 10}{x^2 - 2x}$

24. $\frac{y - 12}{y^2 + 3y}$

25. $\frac{6w - 7}{w^2 + w - 6}$

26. $\frac{-10t - 11}{t^2 + 5t - 6}$

27. $\frac{x^2 + 26x + 100}{x^3 + 10x^2 + 25x}$

28. $\frac{-3x^2 + 2x + 8}{x^3 + 4x^2 + 4x}$

29. $\frac{13x^2 + 2x + 45}{2x^3 + 18x}$

30. $\frac{17x^2 - 7x + 18}{7x^3 + 42x}$

31. $\frac{x^4 - 3x^3 + 13x^2 - 28x + 28}{x^3 + 7x}$

32. $\frac{x^4 - 4x^3 + 11x^2 - 13x + 12}{x^3 + 2x}$

33. $\frac{2x^3 - x^2 + 13x - 5}{x^4 + 10x^2 + 25}$

34. $\frac{3x^3 - 4x^2 + 11x - 12}{x^4 + 6x^2 + 9}$

35. $\frac{5x^2 - 4x + 8}{(x - 4)(x^2 + x + 4)}$

36. $\frac{x^2 + 15x - 6}{(x + 6)(x^2 + 2x + 6)}$

37. $\frac{4x^3 - 4x^2 + 11x - 7}{x^4 + 5x^2 + 6}$

38. $\frac{3x^3 - 4x^2 + 6x - 7}{x^4 + 5x^2 + 4}$

39. $\frac{2x^3 - 11x^2 - 4x + 24}{x^2 - 3x - 10}$

40. $\frac{3x^3 + 11x^2 + x + 10}{x^2 + 3x - 4}$

41. $\frac{3x^3 + 2x^2 - x - 5}{x^2 + 2x + 1}$

42. $\frac{2x^3 - 17x^2 + 54x - 68}{x^2 - 6x + 9}$

43. **a.** Factor. $x^3 - x^2 - 21x + 45$ (*Hint*: Use the rational zero theorem.)

b. Find the partial fraction decomposition for $\frac{-3x^2 + 35x - 70}{x^3 - x^2 - 21x + 45}$.

44. **a.** Factor. $x^3 + 2x^2 - 7x + 4$

b. Find the partial fraction decomposition for $\frac{10x^2 + 17x - 17}{x^3 + 2x^2 - 7x + 4}$.

45. **a.** Factor. $x^3 + 6x^2 + 12x + 8$

b. Find the partial fraction decomposition for $\frac{3x^2 + 8x + 5}{x^3 + 6x^2 + 12x + 8}$.

46. **a.** Factor. $x^3 - 9x^2 + 27x - 27$

b. Find the partial fraction decomposition for $\frac{2x^2 - 17x + 37}{x^3 - 9x^2 + 27x - 27}$.

Write About It

47. Write an informal explanation of partial fraction decomposition.

48. Suppose that a proper rational expression has a single repeated linear factor $(ax + b)^3$ in the denominator. Explain how to set up the partial fraction decomposition.

49. What is meant by a *proper* rational expression?

50. Given an improper rational expression, what must be done first before the technique of partial fraction decomposition may be performed?

Expanding Your Skills

51. **a.** Determine the partial fraction decomposition for $\frac{2}{n(n + 2)}$.

b. Use the partial fraction decomposition for $\frac{2}{n(n + 2)}$ to rewrite the infinite sum

$$\frac{2}{1(3)} + \frac{2}{2(4)} + \frac{2}{3(5)} + \frac{2}{4(6)} + \frac{2}{5(7)} \cdots$$

c. Determine the value of $\frac{1}{n + 2}$ as $n \to \infty$.

d. Find the value of the sum from part (b).

52. **a.** Determine the partial fraction decomposition for $\frac{3}{n(n + 3)}$.

b. Use the partial fraction decomposition for $\frac{3}{n(n + 3)}$ to rewrite the infinite sum

$$\frac{3}{1(4)} + \frac{3}{2(5)} + \frac{3}{3(6)} + \frac{3}{4(7)} + \frac{3}{5(8)} \cdots$$

c. Determine the value of as $n \to \infty$.

d. Find the value of the sum from part (b).

For Exercises 53–54, find the partial fraction decomposition. Assume that *a* and *b* are nonzero constants.

53. $\frac{1}{x(a + bx)}$

54. $\frac{1}{a^2 - x^2}$

For Exercises 55–56, find the partial fraction decomposition for the given expression. [*Hint*: Use the substitution $u = e^x$ and recall that $e^{2x} = (e^x)^2$.]

55. $\frac{5e^x + 7}{e^{2x} + 3e^x + 2}$

56. $\frac{-3e^x - 22}{e^{2x} + 3e^x - 4}$

Go online for more practice problems.

Lesson 8-4

Systems of Nonlinear Equations in Two Variables

What Will You Learn?

After completing this lesson, you should be able to:

- Solve Nonlinear Systems of Equations by the Substitution Method
- Solve Nonlinear Systems of Equations by the Addition Method
- Use Nonlinear Systems of Equations to Solve Applications

Learn: Solve Nonlinear Systems of Equations by the Substitution Method

The attending physician in an emergency room treats an unconscious patient. The physician needs to know the concentration of a medication in the bloodstream at the time it was taken. The patient's family does not know the original amount of medication taken, but believes that the it was taken by injection 3 hr before arriving at the hospital. Blood work at the time of arrival ($t = 3$ hr after the patient had taken the medication) showed that the concentration in the bloodstream was 0.69 μg/dL. One hour later ($t = 4$ hr), the level had dropped to 0.655 μg/dL.

The physician can solve the following system of nonlinear equations to determine the concentration of the medication in the bloodstream at the time of injection. The value A_0 represents the initial concentration, and the value k is related to the rate at which the kidneys can remove the medication.

$$0.69 = A_0 e^{-3k}$$

$$0.655 = A_0 e^{-4k}$$

The solution to this problem is discussed in Exercise 59.

A **nonlinear system of equations** is a system in which one or more equations is nonlinear. For example:

$$\begin{aligned} -x + 7y &= 50 \\ x^2 + y^2 &= 100 \end{aligned}$$

Second equation nonlinear

$$\left.\begin{aligned} 2x^2 + y^2 &= 17 \\ x^2 + 2y^2 &= 22 \end{aligned}\right\}$$

Both equations nonlinear

A **solution** to a nonlinear system of equations in two variables is an ordered pair with real-valued coordinates that satisfies each equation in the system. Graphically, these are the points of intersection of the graphs of the equations. A nonlinear system of equations may have no solution or one or more solutions. See Figure 8-9 through Figure 8-11.

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Two solutions

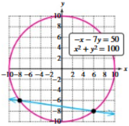


Figure 8-9

Four solutions

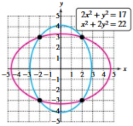


Figure 8-10

No solution

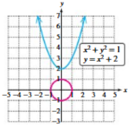
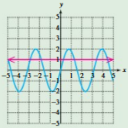


Figure 8-11

Insights

A nonlinear system of equations may also have infinitely many solutions. In the graph shown, the "wave" pattern extends infinitely far in both directions.



We will solve nonlinear systems of equations by the substitution method and by the addition method. In Example 1, we begin with the substitution method.

Example 1: Solving a System of Nonlinear Equations by Using the Substitution Method

Solve the system by using the substitution method.

$$-x - 7y = 50$$

$$x^2 + y^2 = 100$$

Solution:

$\boxed{A}$ $-x - 7y = 50$ Equation $\boxed{A}$ is a line and can be written in slope-intercept form as $y = -\frac{1}{7}x - \frac{50}{7}$.

$\boxed{B}$ $x^2 + y^2 = 100$ Equation $\boxed{B}$ represents a circle centered at (0, 0) with radius 10. A sketch of the two equations suggests that the curves intersect at (−8, −6) and (6, −8). See Figure 8-12.

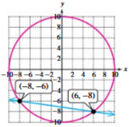


Figure 8-12

$\boxed{A}$ $-x - 7y = 50 \longrightarrow x = -7y - 50$

$\boxed{B}$ $x^2 + y^2 = 100$

To solve the system algebraically by the substitution method, first solve for x or y from either equation.

$\boxed{B}$ $(-7y - 50)^2 + y^2 = 100$

Substitute $x = -7y - 50$ from equation $\boxed{A}$ into equation $\boxed{B}$.

$$49y^2 + 700y + 2500 + y^2 = 100$$

Solve the resulting equation for y.

$$50y^2 + 700y + 2400 = 0$$

$$50(y^2 + 14y + 48) = 0$$

Factor out the GCF of 50.

$$50(y + 6)(y + 8) = 0$$

$$y = -6 \quad \text{or} \quad y = -8$$

For each value of y, find the corresponding x value by substituting y into the equation in which x is isolated: $x = -7y - 50$

$y = -6$: $x = -7(-6) - 50 = -8$ The solution is $(-8, -6)$.

$y = -8$: $x = -7(-8) - 50 = 6$ The solution is $(6, -8)$.

Check Your Work

Check: $(-8, -6)$

$\boxed{A}$ $-(-8) - 7(-6) = 50$ ✓

$\boxed{B}$ $(-8)^2 + (-6)^2 = 100$ ✓

Check: $(6, -8)$

$\boxed{A}$ $-(6) - 7(-8) = 50$ ✓

$\boxed{B}$ $(6)^2 + (-8)^2 = 100$ ✓

The solutions both check in each equation.

The solution set is $\{(-8, -6), (6, -8)\}$.

Apply the Skills

1. Solve the system by using the substitution method.

$$2x + y = 5$$
$$x^2 + y^2 = 50$$

As we solve systems of equations, we will consider only solutions with real coordinates. In Example 2, we have a system of equations in which one equation is $y = 3\sqrt{x - 8}$. The expression $\sqrt{x - 8}$ is a real number for values of x on the interval $[8, \infty)$. Therefore, any ordered pair with an x-coordinate less than 8 must be rejected as a potential solution.

Example 2: Solving a System of Nonlinear Equations by Using the Substitution Method

Solve the system by using the substitution method. $(x - 5)^2 + y^2 = 25$

$$y = 3\sqrt{x - 8}$$

Solution:

$\boxed{A}\quad (x-5)^2 + y^2 = 25$

$\boxed{B}\quad y = 3\sqrt{x-8}$

Label the equations. The graphs of the equations are shown in Figure 8-13. The graph suggests that there is only one solution: (9, 3).

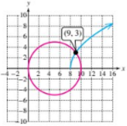


Figure 8-13

$\boxed{A}\quad (x-5)^2 + (3\sqrt{x-8})^2 = 25$

Substitute $3\sqrt{x-8}$ for y in equation A.

$x^2 - 10x + 25 + 9(x-8) = 25$

Square each term.

$$x^2 - x - 72 = 0$$

$$(x-9)(x+8) = 0$$

$$x = 9 \quad \text{or} \quad \cancel{x=-8}$$

Reject $x = -8$ because $3\sqrt{x-8}$ is not a real number for $x = -8$.

Given $x = 9$, solve for y:

$\boxed{B}\quad y = 3\sqrt{x-8}$

$y = 3\sqrt{9-8} = 3$

The solution is (9, 3) and checks in each original equation.

The solution set is $\{(9, 3)\}$.

Apply the Skills

2. Solve the system by using the substitution method.

$$x^2 + y^2 = 90$$

$$y = \sqrt{x}$$

Learn: Solve Nonlinear Systems of Equations by the Addition Method

The substitution method is used most often to solve a system of nonlinear equations. In some situations, however, the addition method is an efficient way to find a solution. Examples 3 and 4 demonstrate that we can eliminate a variable from both equations in a system provided the terms containing the corresponding variables are like terms.

Example 3: Solving a System of Nonlinear Equations by Using the Addition Method

Solve the system by using the addition method.

$$\begin{aligned} 2x^2 + y^2 &= 17 \\ x^2 + 2y^2 &= 20 \end{aligned}$$

Solution:

Using the addition method, the goal is to create opposite coefficients on either the x^2 terms or the y^2 terms. In this case, we have chosen to eliminate the x^2 terms.

$$\boxed{A}\quad 2x^2 + y^2 = 17 \qquad\qquad 2x^2 + y^2 = 17$$

$$\boxed{B}\quad x^2 + 2y^2 = 22 \xrightarrow[\text{Multiply by } -2.]{} \underline{-2x^2 - 4y^2 = -44}$$

$$-3y^2 = -27$$

$$y^2 = 9$$

$$y = \pm 3$$

$y = 3$: $\boxed{B}$ $x^2 + 2(3)^2 = 22$

$$x^2 = 4$$

$$x = \pm 2$$

Substitute $y = \pm 3$ into either equation $\boxed{A}$ or $\boxed{B}$ to solve for the corresponding values of x.

The solutions are (2, 3), (−2, 3).

$y = -3$: $\boxed{B}$ $x^2 + 2(-3)^2 = 22$

$$x^2 = 4$$

$$x = \pm 2$$

The solutions are (2, −3), (−2, −3).

The solution set is {(2, 3), (−2, 3), (2, −3), (−2, −3)}.

The solutions all check in the original equations.

Apply the Skills

3. Solve the system by using the addition method.

$$\begin{aligned} x^2 + y^2 &= 17 \\ x^2 - 2y^2 &= -31 \end{aligned}$$

TECHNOLOGY CONNECTIONS

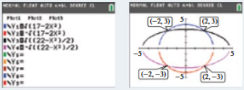
Solving a System of Nonlinear Equations Using Intersect

The equations in Example 3 each represent a curve called an ellipse. We do not yet know how to graph an ellipse; however, we can graph the curves on a graphing calculator. First solve each equation for *y*.

$$2x^2 + y^2 = 17 \xrightarrow{\text{Solve for } y.} y = \pm\sqrt{17 - 2x^2}$$

$$x^2 + 2y^2 = 22 \longrightarrow y = \pm\sqrt{\frac{22 - x^2}{2}}$$

Enter the resulting functions in the calculator and use the Intersect feature to approximate the points of intersection (Figure 8-14).



```
Plot1  Plot2  Plot3
\Y1■√(17-2X²)
\Y2■-√(17-2X²)
\Y3■√((22-X²)/2)
\Y4■-√((22-X²)/2)
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
```

Figure 8-14

Insights

The image produced by a graphing calculator in Figure 8-14 does not show the curves touching the *x*-axis, when indeed they do.

Example 4 illustrates that a nonlinear system of equations may have no solution.

Example 4: Solving an Inconsistent System by Using the Addition Method

Solve the system by using the addition method.

$$\begin{aligned} x^2 + 4y^2 &= 4 \\ x^2 - y^2 &= 9 \end{aligned}$$

Solution:

$$x^2 + 4y^2 = 4 \qquad\qquad x^2 + 4y^2 = 4$$

$$x^2 - y^2 = 9 \xrightarrow[\text{Multiply by } -1.]{} \underline{-x^2 + y^2 = -9}$$

$$5y^2 = -5$$

$$y^2 = -1$$

$$y = \pm i$$

We use the addition method because like terms are aligned vertically.

The values for *y* are not real numbers. Therefore, there is no solution to the system of equations over the set of real numbers.

The solution set is { }.

Insights

Solutions to a system of equations are limited to ordered pairs with real-valued coordinates because we are interested in the points of intersection of the graphs of the equations.

Apply the Skills

4. Solve the system by using the addition method.

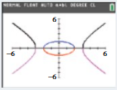
$$\begin{aligned} x^2 + y^2 &= 16 \\ 4x^2 + 9y^2 &= 36 \end{aligned}$$

TECHNOLOGY CONNECTIONS

Solving a Nonlinear System of Equations

The equations in Example 4 represent two curves that have not yet been studied: an ellipse and a hyperbola. However, we can graph the curves on a graphing calculator by solving for *y* and entering the functions into the calculator.

$$\begin{aligned} x^2 + 4y^2 = 4 &\xrightarrow[\text{Solve for } y]{} y = \pm\sqrt{\frac{4 - x^2}{4}} \\ x^2 - y^2 = 9 &\xrightarrow{\hspace{2.5cm}} y = \pm\sqrt{x^2 - 9} \end{aligned}$$



From the graph, we see that the curves do not intersect.

Learn: Use Nonlinear Systems of Equations to Solve Applications

In Example 5, we set up a system of nonlinear equations to model an application involving two independent relationships between two variables.

Example 5: Solving an Application of a Nonlinear System

The perimeter of a television screen is 140 in. The area is 1200 in.^{2}.

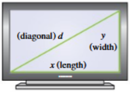
a. Find the length and width of the screen.

b. Find the length of the diagonal.

Solution:

Let *x* represent the length of the screen.

Let *y* represent the width of the screen.



The statement of the problem gives two independent relationships between the length and width of the screen.

The perimeter of a television screen is 140 in. $2x + 2y = 140$

The area is 1200 in.². $xy = 1200$

a. Solve the nonlinear system of equations for x and y.

$$\boxed{A}\quad 2x + 2y = 140$$

$$\boxed{B}\quad xy = 1200 \xrightarrow{\text{Solve for } y.} y = \frac{1200}{x}$$

Using the substitution method, solve for x or y in either equation.

$$\boxed{A}\quad 2x + 2\left(\frac{1200}{x}\right) = 140$$

$$2x + \frac{2400}{x} = 140$$

Substitute $y = \frac{1200}{x}$ from equation $\boxed{B}$ into equation $\boxed{A}$.

$$x \cdot \left(2x + \frac{2400}{x}\right) = x \cdot (140)$$

Multiply both sides by the LCD to clear fractions.

$$2x^2 + 2400 = 140x$$

$$2x^2 - 140x + 2400 = 0$$

The resulting equation is quadratic.

$$2(x^2 - 70x + 1200) = 0$$

$$2(x - 40)(x - 30) = 0$$

Factor the left side.

$$x = 40 \quad \text{or} \quad x = 30$$

There are two possible values for the length x.

Substitute $x = 40$ and $x = 30$ into the equation $y = \frac{1200}{x}$.

$x = 40$: $y = \frac{1200}{40} = 30$ If the length x is 40 in., then the width y is 30 in.

$x = 30$: $y = \frac{1200}{30} = 40$ If the length x is 30 in., then the width y is 40 in.

Taking the length to be the longer side, we have that the length is 40 in. and the width is 30 in.

b. $x^2 + y^2 = d^2$

$$(40)^2 + (30)^2 = d^2$$

$$1600 + 900 = d^2$$

$$2500 = d^2$$

$$d = \pm 50$$

Use the Pythagorean theorem to determine the measure of the diagonal of the screen.

Excluding the negative solution for d, the diagonal is 50 in.

Check Your Work

Check the solution to Example 5(a) by computing the perimeter and area.

Perimeter:

2(40 in.) + 2(30 in.) = 140 in. ✓

Area:

(40 in.)(30 in.) = 1200 in.² ✓

Apply the Skills

5. The perimeter of a rectangular rug is 40 ft and the area is 96 ft². Find the dimensions of the rug.

Practice 8-4

Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, graph the equation.

R.1. $y = x^2 + 4x + 3$

R.2. $(x + 4)^2 + (y - 1)^2 = 4$

R.3. $y = \sqrt{x + 2}$

For Exercises R.4–R.5, solve the equation.

R.4. $\ln x = 17$ **R.5.** $3x = 81$

Concept Connections

1. A ___________ system of equations in two variables is a system in which one or more equations in the system is nonlinear.
2. A solution to a nonlinear system of equations in two variables is an ___________ pair with real-valued coordinates that satisfies each equation in the system. Graphically, a solution is a point of ___________ of the graphs of the equations.

Learn: Solve Nonlinear Systems of Equations by the Substitution Method

For Exercises 3–14,

a. Graph the equations in the system.

b. Solve the system by using the substitution method. (**See Examples 1–2, pp. 899–900**)

3. $y = x^2 - 2$
$2x - y = 2$

4. $y = -x^2 + 3$
$y - 2x = 0$

5. $x^2 + y^2 = 25$
$x + y = 1$

6. $x^2 + y^2 = 25$
$3y = 4x$

7. $y = \sqrt{x}$
$x^2 + y^2 = 20$

8. $x^2 + y^2 = 10$
$y = \sqrt{x - 2}$

9. $(x + 2)^2 + y^2 = 9$
$y = 2x - 4$

10. $x^2 + (y - 3)^2 = 4$
$y = -x - 4$

11. $y = x^3$
$y = x$

12. $y = \sqrt[3]{x}$
$y = x$

13. $y = -(x - 2)^2 + 5$
$y = 2x + 1$

14. $y = (x + 3)^2 - 1$
$y = 2x + 5$

Learn: Solve Nonlinear Systems of Equations by the Addition Method

For Exercises 15–22, solve the system by using the addition method. (See Examples 3–4, pp. 902–903)

15. $2x^2 + 3y^2 = 11$
$x^2 + 4y^2 = 8$

16. $3x^2 + y^2 = 21$
$4x^2 - 2y^2 = -2$

17. $x^2 - xy = 20$
$-2x^2 + 3xy = -44$

18. $4xy + 3y^2 = -9$
$2xy + y^2 = -5$

19. $5x^2 - 2y^2 = 1$
$2x^2 - 3y^2 = -4$

20. $6x^2 + 5y^2 = 38$
$7x^2 - 3y^2 = 9$

21. $x^2 = 1 - y^2$
$9x^2 - 4y^2 = 36$

22. $4x^2 = 4 - y^2$
$16y^2 = 144 + 9x^2$

Mixed Exercises

For Exercises 23–34, solve the system by using any method.

23. $x^2 - 4xy + 4y^2 = 1$
$x + y = 4$

24. $x^2 - 6xy + 9y^2 = 0$
$x - y = 2$

25. $y = x^2 + 4x + 5$
$y = 4x + 5$

26. $y = x^2 - 6x + 9$
$y = -2x + 5$

27. $y = x^2$
$y = \frac{1}{x}$

28. $y = \frac{1}{x}$
$y = \sqrt{x}$

29. $x^2 + (y - 4)^2 = 25$
$y = -x^2 + 9$

30. $(x - 10)^2 + y^2 = 100$
$x = y^2$

31. $y = -x^2 + 6x - 7$
$y = x^2 - 10x + 23$

32. $y = -x^2 + 6x - 9$
$y = x^2 - 2x - 3$

33. $\frac{x^2}{4} + \frac{y^2}{16} = 1$
$x^2 + y = 4$

34. $\frac{x^2}{4} + y^2 = 1$
$x = -2y^2 + 2$

For Exercises 35–36, use the substitutions $u = \frac{1}{x^2}$ and $v = \frac{1}{y^2}$ to solve the system of equations.

35. $\frac{4}{x^2} - \frac{3}{y^2} = -23$
$\frac{5}{x^2} + \frac{1}{y^2} = 14$

36. $-\frac{3}{x^2} + \frac{1}{y^2} = 13$
$\frac{5}{x^2} - \frac{1}{y^2} = -5$

Learn: Use Nonlinear Systems of Equations to Solve Applications

37. Find two numbers whose sum is 12 and whose product is 35.

38. Find two numbers whose sum is 9 and whose product is −36.

39. The sum of the squares of two positive numbers is 29 and the difference of the squares of the numbers is 21. Find the numbers.

40. The sum of the squares of two negative numbers is 145 and the difference of the squares of the numbers is 17. Find the numbers.

41. The difference of two positive numbers is 2 and the difference of their squares is 44. Find the numbers.

42. The sum of two numbers is 4 and the difference of their squares is 64. Find the numbers.

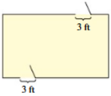
43. The ratio of two numbers is 3 to 4 and the sum of their squares is 225. Find the numbers.

44. The ratio of two numbers is 5 to 12 and the sum of their squares is 676. Find the numbers.

45. Find the dimensions of a rectangle whose perimeter is 36 m and whose area is 80 m².

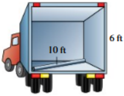
46. Find the dimensions of a rectangle whose perimeter is 56 cm and whose area is 192 cm².

47. The floor of a rectangular bedroom requires 240 ft² of carpeting. Molding is placed around the base of the floor except at two 3-ft doorways. If 58 ft of molding is required around the base of the floor, determine the dimensions of the floor. (**See Example 5, p. 904**)



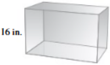
48. An electronic sign for a grocery store is in the shape of a rectangle. The perimeter of the sign is 72 ft and the area is 320 ft². Find the length and width of the sign.

49. A rental truck has a cargo capacity of 288 ft³. A 10-ft pipe just fits resting diagonally on the floor of the truck. If the cargo space is 6 ft high, find the dimensions of the truck.



50. A rectangular window has a 15-yd diagonal and an area of 108 yd². Find the dimensions of the window.

51. An aquarium is 16 in. high with volume of 4608 in.³ (approximately 20 gal). If the amount of glass used for the bottom and four sides is 1440 in.², determine the dimensions of the aquarium.



52. A closed box is in the shape of a rectangular solid with height 3 m. Its surface area is 268 m². If the volume is 240 m³, find the dimensions of the box.

53. The hypotenuse of a right triangle is $\sqrt{65}$ ft. The sum of the lengths of the legs is 11 ft. Find the lengths of the legs.

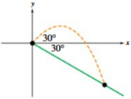
54. The hypotenuse of a right triangle is $\sqrt{73}$ in. The sum of the lengths of the legs is 11 in. Find the lengths of the legs.

55. A ball is kicked off the side of a hill at an angle of elevation of 30°. The hill slopes downward 30° from the horizontal. Consider a coordinate system in which the origin is the point on the edge of the hill from which the ball is kicked. The path of the ball and the line of declination of the hill can be approximated by

$y = -\frac{x^2}{192} + \frac{\sqrt{3}}{3}x$ Path of the ball

$y = -\frac{\sqrt{3}}{3}x$ Line of declination of the hill

Solve the system to determine where the ball will hit the ground.



56. A child kicks a rock off the side of a hill at an angle of elevation of 60°. The hill slopes downward 30° from the horizontal. Consider a coordinate system in which the origin is the point on the edge of the hill from which the rock is kicked. The path of the rock and the line of declination of the hill can be approximated by

$y = -\frac{x^2}{36} + \sqrt{3}x$ Path of the rock

$y = -\frac{\sqrt{3}}{3}x$ Line of declination of the hill

Solve the system to determine where the rock will hit the ground.

Write About It

57. What is the difference between a system of linear equations and a system of nonlinear equations?

58. Describe a situation in which the addition method is an efficient technique to solve a system of nonlinear equations.

Expanding Your Skills

59. The initial concentration A_0 of a certain medication in the bloodstream is unknown. However, the physician knows that after 3 hr, the medication concentration in the blood is 0.69 μg/dL and after 4 hr, the concentration is 0.655 μg/dL. The model $A(t) = A_0e^{-kt}$ represents the concentration $A(t)$ (in μg/dL) in the bloodstream t hours after injection of the medication. The value of k is a constant related to the rate at which the medication is removed by the body.

- **a.** Substitute 0.69 for $A(t)$ and 3 for t in the model and write the resulting equation.
- **b.** Substitute 0.655 for $A(t)$ and 4 for t in the model and write the resulting equation.
- **c.** Use the system of equations from parts (a) and (b) to solve for k. Round to 3 decimal places.
- **d.** Use the system of equations from parts (a) and (b) to approximate the initial concentration A_0 (in μg/dL) at the time of injection. Round to 2 decimal places.
- **e.** Determine the concentration of the medication after 12 hr. Round to 2 decimal places.

60. A patient undergoing a heart scan is given a sample of fluorine-18 (^{18}F). After 4 hr, the radioactivity level in the patient is 44.1 MBq (megabecquerel). After 5 hr, the radioactivity level drops to 30.2 MBq. The radioactivity level $Q(t)$ can be approximated by $Q(t) = Q_0e^{-kt}$, where t is the time in hours after the initial dose Q_0 is administered.

- **a.** Determine the value of k. Round to 4 decimal places.
- **b.** Determine the initial dose, Q_0. Round to the nearest whole unit.
- **c.** Determine the radioactivity level after 12 hr. Round to 1 decimal place.

61. The population $P(t)$ of a culture of bacteria grows exponentially for the first 72 hr according to the model $P(t) = P_0e^{kt}$. The variable t is the time in hours since the culture is started. The population of bacteria is 60,000 after 7 hr. The population grows to 80,000 after 12 hr.

- **a.** Determine the constant k to 3 decimal places.
- **b.** Determine the original population P_0. Round to the nearest thousand.
- **c.** Determine the time required for the population to reach 300,000. Round to the nearest hour.

62. An investment grows exponentially under continuous compounding. After 2 yr, the amount in the account is \$7328.70. After 5 yr, the amount in the account is \$8774.10. Use the model $A(t) = Pe^{rt}$ to

- **a.** Find the interest rate r. Round to the nearest percent.
- **b.** Find the original principal P. Round to the nearest dollar.
- **c.** Determine the amount of time required for the account to reach a value of \$15,000. Round to the nearest year.

For Exercises 63–66, find the number of solutions by graphing the system of equations. (*Hint*: Rewrite the system of equations into familiar forms to graph.)

63. $\log_2 y - \log_2 x = 2$
$x^2 - 6x + y^2 - 4y = -9$

64. $\log(y + 4) - 2\log x = \log 3$
$x^2 + y^2 = 9$

65. $\ln 4 + \ln y = 2\ln x$
$x^2 + y^2 - 6y - 7 = 0$

66. $\ln 4 = 2\ln x - \ln y$
$x^2 + y^2 - 6y + 5 = 0$

For Exercises 67–68, determine the number of solutions to the system of equations.

67. $y = 2^{x+1}$

$-1 + \log_2 y = x$

68. $x^2 - y^2 = 0$

$|x| = |y|$

For Exercises 69–74, solve the system.

69. $\log x + 2\log y = 5$

$2\log x - \log y = 0$

70. $\log_2 x + 3\log_2 y = 6$

$\log_2 x - \log_2 y = 2$

71. $2^x + 2^y = 6$

$4^x - 2^y = 14$

72. $3^x - 9^y = 18$

$3^x + 3^y = 30$

73. $(x - 1)^2 + (y + 1)^2 = 5$

$x^2 + (y + 4)^2 = 29$

74. $(x + 3)^2 + (y - 2)^2 = 4$

$(x - 1)^2 + y^2 = 8$

For Exercises 75–76, use substitution to solve the system for the set of ordered triples (x, y, λ) that satisfy the system.

75. $2 = 2\lambda x$

$6 = 2\lambda y$

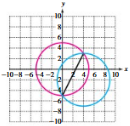
$x^2 + y^2 = 10$

76. $8 = 4\lambda x$

$2 = 2\lambda y$

$2x^2 + y^2 = 9$

77. Two circles intersect as shown.



a. Find the points of intersection.

$x^2 + y^2 = 25$

$(x - 4)^2 + (y + 2)^2 = 25$

b. Find an equation of the chord common to both circles (shown in black). (*Hint*: A chord is a line segment on the interior of a circle with both endpoints on the circle.)

78. The minimum and maximum distances from a point P to a circle are found using the line determined by the given point and the center of the circle. Given the circle defined by $x^2 + y^2 = 9$ and the point $P(4, 5)$,

a. Find the point on the circle closest to the point $(4, 5)$.

b. Find the point on the circle farthest from the point $(4, 5)$.

Technology Connections

For Exercises 79–84, use a graphing utility to approximate the solution(s) to the system of equations. Round the coordinates to 3 decimal places.

79. $y = -0.6x + 7$

$y = e^x - 5$

80. $y = -0.7x + 4$

$y = \ln x$

81. $x^2 + y^2 = 40$

$y = -x^2 + 8.5$

82. $x^2 + y^2 = 32$

$y = 0.8x^2 - 9.2$

83. $y = x^2 - 8x + 20$

$y = 4\log x$

84. $y = 0.2e^x$

$y = -0.6x^2 - 2x - 3$

Go online for more practice problems.

Lesson 8-5

Inequalities and Systems of Inequalities in Two Variables

What Will You Learn?

After completing this lesson, you should be able to:

- Solve Linear Inequalities in Two Variables
- Solve Nonlinear Inequalities in Two Variables
- Solve Systems of Inequalities in Two Variables

Learn: Solve Linear Inequalities in Two Variables

Adriana has 12 hr of available study time before taking tests in algebra and biology in back-to-back classes. Suppose that x represents the time spent studying algebra and y represents the time spent studying biology. Then the inequality $x + y \leq 12$, where $x \geq 0$ and $y \geq 0$ represents the distribution of time that can be allocated for studying for each subject.

An inequality of the form $Ax + By < C$, where A and B are not both zero, is called a **linear inequality in two variables.** (Note that the symbols $>$, $\leq$, and $\geq$ can be used in place of $<$ in the definition.) A **solution** to an inequality in two variables is an ordered pair that satisfies the inequality. The set of all such ordered pairs is called the **solution set** to the inequality. Graphically, the solution set is a region in the xy-plane.

To graph the solution set to a linear inequality in two variables, follow these guidelines.

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Graphing a Linear Inequality in Two Variables

Step 1 Graph the related equation. That is, replace the inequality sign with an $=$ sign and graph the line represented by the equation.

- If the inequality is strict (stated with the symbols $<$ or $>$), then draw the line as a dashed line to indicate that the line *is not* part of the solution set.
- If the inequality is stated with the symbols $\leq$ or $\geq$, then draw the line as a solid line to indicate that the line *is* part of the solution set.

Step 2 Choose a test point from either side of the line (not a point on the line itself) and substitute the ordered pair into the inequality.

- If a true statement results, then shade the region (half-plane) from which the test point was taken.
- If a false statement results, then shade the region (half-plane) on the opposite side of the line from which the test point was taken.

Insights

To graph the line in step 1, we can

- Use the slope-intercept form of the equation of the line.
- Graph the *x*- and *y*-intercepts.
- Create a table of points.

Example 1: Graphing a Linear Inequality in Two Variables

Graph the solution set. $3x - 2y < 6$

Solution:

$$3x - 2y < 6 \xrightarrow[\text{equation}]{\text{related}} 3x - 2y = 6$$

Step 1: Graph the related equation $3x - 2y = 6$ using any technique for graphing. In this case, we have chosen to find the *x*- and *y*-intercepts.

x-intercept:

$3x - 2(0) = 6$

$x = 2$

y-intercept:

$3(0) - 2y = 6$

$y = -3$

The *x*- and *y*-intercepts are (2, 0) and (0, −3).

Graph the line through the intercepts (Figure 8-15). Because the inequality is strict, draw the line as a dashed line.

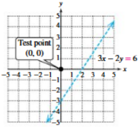


Figure 8-15

Test (0, 0):

$3x - 2y < 6$

$3(0) - 2(0) \overset{?}{<} 6$

$0 \overset{?}{<} 6$ ✓ true

Step 2: Select a test point either above or below the line and test the ordered pair in the original inequality. In Figure 8-16, we have chosen (0, 0) as a test point.

The test point (0, 0) is a representative point above the line. Since (0, 0) satisfies the original inequality, then it and all other points above the line are solutions.

The solution set is the set of ordered pairs in the region (half-plane) above the line (Figure 8-16).

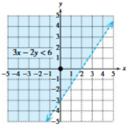


Figure 8-16

Insights

We can also graph the inequality from Example 1 by solving the inequality for *y*.

$$3x - 2y < 6$$
$$-2y < -3x + 6$$
$$y > \frac{3}{2}x - 3$$

Then shade the half-plane *above* the bounding line because this region contains points with *y*-coordinates greater than those on the bounding line.

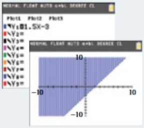
Apply the Skills

1. Graph the solution set. $4x - y > 3$

TECHNOLOGY CONNECTIONS

Graphing a System of Inequalities in Two Variables

A graphing utility can be used to graph an inequality in two variables. In most cases, we solve for *y* first and enter the related equation in the graphing editor. Place the cursor to the left of Y_1 and press ENTER two times. This will set the graph style to shade the region above the line ◥. Notice that the calculator image does not differentiate between a solid and dashed bounding line (Figure 8-17).



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NORMAL FLOAT AUTO a+bi DEGREE CL
 Plot1  Plot2  Plot3
◥Y1=1.5X-3
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
```

Figure 8-17

Note: With the cursor placed to the left of Y_1,

- Select the upper right triangle ◥ for inequalities of the form $Y_1 > f(x)$ and $Y_1 \geq f(x)$.
- Select the lower left triangle ◣ for inequalities of the form $Y_1 < f(x)$ and $Y_1 \leq f(x)$.

In Example 2, we graph the solutions to an inequality in which the bounding line passes through the origin.

Example 2: Graphing a Linear Inequality

Graph the solution set. $4y \leq 3x$

Solution:

$$4y \leq 3x \xrightarrow[\text{equation}]{\text{related}} 4y = 3x$$

$$y = \frac{3}{4}x + 0$$

Step 1: Graph the related equation $4y = 3x$. In this case, we have chosen to find the slope-intercept form of the equation and then graph the line using the slope and y-intercept.

Graph the line having y-intercept $(0, 0)$ and slope $\frac{3}{4}$ (Figure 8-18). Because the inequality symbol $\leq$ allows for equality, draw the line as a solid line.

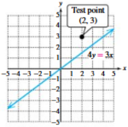


Figure 8-18

Test (2, 3):

$$4y \leq 3x$$

$$4(3) \overset{?}{\leq} 3(2)$$

$$12 \overset{?}{\leq} 6 \text{ false}$$

Step 2: Select a test point. We have chosen (2, 3).

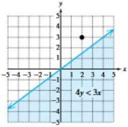


Figure 8-19

The test point (2, 3) is a representative point above the line. Since (2, 3) does *not* satisfy the original inequality, then points on the other side of the line are solutions. Shade below the line.

The solution set is the set of ordered pairs on and below the line (Figure 8-19).

Check Your Work

As a check, we can select a test point *below* the line, such as (1, −1), and verify that it does indeed satisfy the original inequality.

Test (1, −1):

$$4(-1) \overset{?}{\leq} 3(1)$$

$$-4 \overset{?}{\leq} 3 \checkmark \text{ true}$$

Apply the Skills

2. Graph the solution set. $2y \geq 5x$

Recall that for a constant k, the equation $x = k$ represents a vertical line in the xy-plane. The inequalities $x < k$ and $x > k$ represent half-planes to the left or right of the vertical line $x = k$ (Figure 8-20). Likewise, $y = k$ represents a horizontal line in the xy-plane. The inequalities $y < k$ and $y > k$ represent half-planes below or above the line $y = k$ (Figure 8-21).

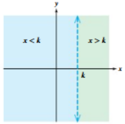


Figure 8-20

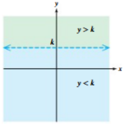


Figure 8-21

Example 3: Graphing Linear Inequalities with a Horizontal or Vertical Bounding Line

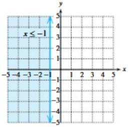
Graph the solution set.

a. $x \le -1$ **b.** $3y > 5$

Solution:

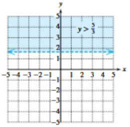
a. $x \le -1$

- The related equation $x = -1$ is a vertical line.
- The inequality $x \le -1$ represents all points to the *left* of or on the line $x = -1$.



b. $3y > 5$

- The inequality is equivalent to $y > \frac{5}{3}$. The related equation $y = \frac{5}{3}$ is a horizontal line.
- The inequality $y > \frac{5}{3}$ represents all points strictly above the line $y = \frac{5}{3}$.



Apply the Skills

3. Graph the solution set.

a. $2y < 6$ **b.** $x \ge \frac{9}{4}$

Learn: Solve Nonlinear Inequalities in Two Variables

The same approach used to graph a linear inequality in two variables is used to graph a *nonlinear* inequality in two variables. This is demonstrated in Example 4.

Example 4: Graphing a Nonlinear Inequality

Graph the solution set. $(x-2)^2+y^2>9$

$$(x-2)^2+y^2>9 \xrightarrow[\text{equation}]{\text{related}} (x-2)^2+y^2=9$$

Step 1: Graph the related equation. The equation represents a circle centered at (2, 0) with radius 3. Because the inequality is strict, draw the circle as a dashed curve (Figure 8-22).

Center: (2, 0)

Radius: 3

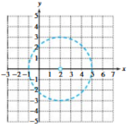


Figure 8-22

Test (2, 0):

$$(x-2)^2+y^2>9$$

$$(2-2)^2+(0)^2 \overset{?}{>} 9$$

$$0 \overset{?}{>} 9 \text{ false}$$

Step 2: Select a test point. We have chosen the center (2, 0).

The test point inside the circle does not satisfy the original inequality. Therefore, the solution set consists of the points strictly outside the circle (Figure 8-23).

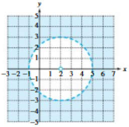


Figure 8-23

Apply the Skills

4. Graph the solution set. $x^2+(y+1)^2<16$

Learn: Solve Systems of Inequalities in Two Variables

Two or more inequalities in two variables make up a system of inequalities in two variables. The solution set is the set of ordered pairs that satisfy each inequality in the system. To graph the solution set to a system of inequalities, graph the solution sets to the individual inequalities first. The solution to the system of inequalities is the *intersection* of the graphs. This is demonstrated in Example 5.

Example 5: Graphing the Solution Set to a System of Linear Inequalities

Graph the solution set to the system of inequalities.

$$y \le \frac{1}{2}x + 2$$
$$3x - y < 3$$

Solution:

$$y \le \frac{1}{2}x + 2$$
$$3x - y < 3$$

First graph the solutions to the individual inequalities (Figures 8-24 and 8-25). Next, find the intersection (area of overlap) of the solution sets shown in purple in Figure 8-26.

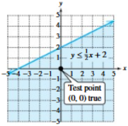


Figure 8-24

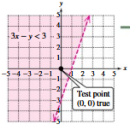


Figure 8-25

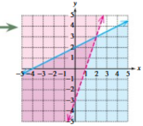


Figure 8-26

Notice that the point of intersection between the two bounding lines is graphed as an open dot (Figure 8-27). This indicates that it is *not* part of the solution set. The reason is that it is not a solution to the strict inequality $3x - y < 3$.

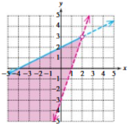


Figure 8-27

Check Your Work

For points on the line $y = \frac{1}{2}x + 2$ *not* adjacent to the shaded region, the line is drawn as a dashed line. This indicates that points on the line outside the shaded region are *not* part of the solution set.

The point of intersection can be found by solving the system of related equations. We have used the substitution method.

$$\begin{aligned}
&y = \tfrac{1}{2}x + 2 \\
&3x - y = 3 \qquad 3x - \left(\tfrac{1}{2}x + 2\right) = 3 \\
&\qquad\qquad\qquad 3x - \tfrac{1}{2}x - 2 = 3 \\
&\qquad\qquad\qquad\qquad \tfrac{5}{2}x = 5 \\
&\qquad\qquad\qquad\qquad x = 2 \longrightarrow y = \tfrac{1}{2}(2) + 2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad y = 3
\end{aligned}$$

The point of intersection is (2, 3) and is excluded from the solution set.

Apply the Skills

5. Graph the solution set to the system of inequalities.

$$\begin{aligned} y &< -\tfrac{1}{3}x + 1 \\ -2x + y &\le 1 \end{aligned}$$

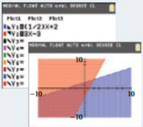
TECHNOLOGY CONNECTIONS

Graphing a System of Inequalities in Two Variables

To graph the system of inequalities from Example 5 on a graphing calculator, solve each inequality for y.

$$y \le \tfrac{1}{2}x + 2$$

$$3x - y < 3 \Rightarrow -y < -3x + 3 \Rightarrow y > 3x - 3$$


NORMAL FLOAT AUTO a+bi DEGREE CL
Plot1 Plot2 Plot3
NORMAL FLOAT AUTO a+bi DEGREE CL
10
−10
10
−10

```
◣Y1■(1/2)X+2
◥Y2■3X−3
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
```

Then enter the related equations in the graphing editor. Choose the appropriate graphing style ◥ or ◣. For $Y_1 \le \frac{1}{2}x + 2$, choose ◣. For $Y_2 > 3x - 3$, choose ◥.

In Example 6, we graph a system of nonlinear inequalities in two variables. This is a system of inequalities in which one or more of the individual inequalities is nonlinear.

Example 6: Solving a System of Nonlinear Inequalities

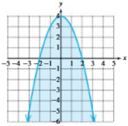
Graph the solution set to the system of inequalities.

$$y \le -x^2 + 4$$
$$x - y \ge -2$$
$$y > -5$$

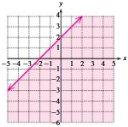
Solution:

To graph the solution set to the given system, first graph each individual inequality.

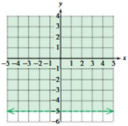
$y \le -x^2 + 4$



$x - y \ge -2$



$y > -5$



The solution set to the system is the intersection of the three shaded regions (Figure 8-28).

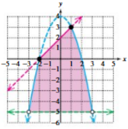


Figure 8-28

The points of intersection are found by pairing up the related equations and solving the system of equations. The inequalities $y \le -x^2 + 4$ and $x - y \ge -2$ include equality. Therefore, the intersection points between the parabola and slanted line are solutions to the system. These points are plotted as closed dots.

On the other hand, because the inequality $y > -5$ is strict, the intersection points between the parabola and horizontal line are *not* solutions to the system. These points are plotted as open dots.

To find the points of intersection between the parabola $y = -x^2 + 4$ and the line $x - y = -2$, solve the system:

$y = -x^2 + 4$

$x - y = -2$ The solutions are (−2, 0) and (1, 3).

To find the points of intersection between the parabola $y = -x^2 + 4$ and the line $y = -5$, solve the system:

$y = -x^2 + 4$

$y = -5$ The solutions are $(-3, -5)$ and $(3, -5)$.

Apply the Skills

6. Graph the solution set to the system of inequalities.

$$\begin{aligned} x^2 + y^2 &\leq 25 \\ -x + y &< 1 \\ y &\geq -4 \end{aligned}$$

In Example 7, we refer to the problem addressed at the beginning of this lesson and set up a system of linear inequalities to model the allocation of time studying algebra and biology.

Example 7: Solving a System of Inequalities in an Application

Adriana has 12 available study hours for algebra and biology.

- Let x represent the number of hours Adriana spends studying algebra.
- Let y represent the number of hours that Adriana studies biology.

  a. Set up an inequality that indicates that the number of hours spent studying algebra cannot be negative.

  b. Set up an inequality that indicates that the number of hours spent studying biology cannot be negative.

  c. Set up an inequality that indicates that the combined number of hours spent studying for these two classes is at most 12 hr.

  d. Graph the solution set to the system of inequalities from parts (a)–(c).

Solution:

a.	$x \geq 0$	The number of hours spent studying algebra is 0 or more.
b.	$y \geq 0$	The number of hours spent studying biology is 0 or more.
c.	$x + y \leq 12$	The sum of time spent studying algebra and the time spent studying biology cannot exceed the maximum number of study hours available. Therefore, the sum is less than or equal to 12 hr.

d. $x \geq 0$
$y \geq 0$
$x + y \leq 12$

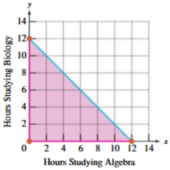


Figure 8-29

The inequalities $x \geq 0$ and $y \geq 0$ together represent the set of points in the first quadrant, including the bounding axes.

To graph the inequality $x + y \leq 12$, we graph the related equation $x + y = 12$. A test point (0, 0) taken below the line results in a true statement. Therefore, the inequality $x + y \leq 12$ represents the set of points on and below the line $x + y = 12$.

The solution to the system of inequalities is shown in Figure 8-29.

Insights

The points of intersection of the bounding lines are closed dots because they are part of the solution set.

Apply the Skills

7. A family plans to spend two 8-hr days at Disney World and will split time between the Magic Kingdom and Epcot Center. Let x represent the number of hours spent at the Magic Kingdom and let y represent the number of hours spent at Epcot Center.

a. Set up two inequalities that indicate that the number of hours spent at the Magic Kingdom and the number of hours spent at Epcot cannot be negative.

b. Set up an inequality that indicates that the combined number of hours spent at the two parks is at most 16 hr.

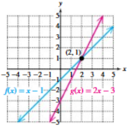
c. Graph the solution set to the system of inequalities.

Practice 8-5

Practice Exercises

Prerequisite Review

R.1. Use the graph to solve the equation and inequalities. Write the solutions to the inequalities in interval notation.



a. $x - 1 = 2x - 3$

b. $x - 1 < 2x - 3$

c. $x - 1 \geq 2x - 3$

Concept Connections

1. An inequality that can be written in the form $Ax + By < C$ (where A and B are not both zero) is called a _______________ inequality in two variables.

2. For a constant real number k, the inequality $x < k$ represents the half-plane to the (left/right) of the (horizontal/vertical) line $x = k$.

3. For a constant real number k, the inequality $y > k$ represents the half-plane (above/below) the (horizontal/vertical) line $y = k$.

4. Given the inequality $y \leq 2x + 1$, the bounding line $y = 2x + 1$ is drawn as a (dashed/solid) line.

5. The solution set to the system of inequalities $x < 0, y > 0$ represents the points in quadrant (I, II, III, IV).

6. The equation $x^2 + y^2 = 4$ is a circle centered at _______________ with radius _______________. The solution set to the inequality $x^2 + y^2 < 4$ represents the set of points (inside/outside) the circle $x^2 + y^2 = 4$.

Learn: Solve Linear Inequalities in Two Variables

For Exercises 7–10, determine whether the ordered pair is a solution to the inequality.

7. $3x + 4y < 12$

- **a.** $(-1, 3)$
- **b.** $(5, 1)$
- **c.** $(4, 0)$

8. $2x + 3y > 6$

- **a.** $(-3, 3)$
- **b.** $(5, -1)$
- **c.** $(0, 2)$

9. $y \geq (x - 3)^2$

- **a.** $(-3, 30)$
- **b.** $(1, 4)$
- **c.** $(5, 5)$

10. $y \leq x^3 - 1$

- **a.** $(-1, -2)$
- **b.** $(2, 6)$
- **c.** $(-4, -50)$

11. **a.** Graph the solution set. $4x - 5y \leq 20$ (**See Example 1, p. 912**)

- **b.** Explain how the graph would differ for the inequality $4x - 5y < 20$.
- **c.** Explain how the graph would differ for the inequality $4x - 5y > 20$.

12. **a.** Graph the solution set. $2x + 5y > 10$

- **b.** Explain how the graph would differ for the inequality $2x + 5y \geq 10$.
- **c.** Explain how the graph would differ for the inequality $2x + 5y < 10$.

For Exercises 13–24, graph the solution set. (See Examples 1–3, pp. 912–915)

13. $2x + 5y > 5$

14. $-5x + 4y \leq 8$

15. $-30x \geq 20y + 600$

16. $-400x < 100y + 8000$

17. $5x \leq 6y$

18. $3x > 2y$

19. $3 + 2(x + y) > y + 3$

20. $-4 - 3(x - y) < 2y - 4$

21. $x < 6$

22. $y \leq 5$

23. $-\frac{1}{2}y + 4 \leq 5$

24. $-\frac{1}{3}x + 2 < 4$

Learn: Solve Nonlinear Inequalities in Two Variables

25. a. Graph the solution set. $x^2 + y^2 < 4$ **(See Example 4, p. 916)**

b. Explain how the graph would differ for the inequality $x^2 + y^2 > 4$.

c. Explain how the graph would differ for the inequality $x^2 + y^2 \geq 4$.

26. a. Graph the solution set. $y \geq x^2 - 1$

b. Explain how the graph would differ for the inequality $y \leq x^2 - 1$.

c. Explain how the graph would differ for the inequality $y > x^2 - 1$.

For Exercises 27–36, graph the solution set. (See Example 4, p. 916)

27. $y < -x^2$

28. $x^2 + y^2 \geq 16$

29. $y \leq (x - 2)^2 + 1$

30. $y \geq -(x + 1)^2 - 2$

31. $|x| \leq 3$

32. $|y| \leq 2$

33. $2|y| > 2$

34. $|x| + 1 > 3$

35. $y \geq \sqrt{x}$

36. $y < \sqrt{x - 1}$

Learn: Solve Systems of Inequalities in Two Variables

37. a. Is the point (2, 1) a solution to the inequality $y < 2x + 3$?

b. Is the point (2, 1) a solution to the inequality $x + y \leq 1$?

c. Is the point (2, 1) a solution to the system of inequalities?

$$\begin{aligned} y &< 2x + 3 \\ x + y &\leq 1 \end{aligned}$$

38. a. Is the point (3, 2) a solution to the inequality $y < -x + 5$?

b. Is the point (3, 2) a solution to the inequality $3x + y \geq 11$?

c. Is the point (3, 2) a solution to the system of inequalities?

$$\begin{aligned} y &< -x + 5 \\ 3x + y &\geq 11 \end{aligned}$$

For Exercises 39–40, determine whether the ordered pair is a solution to the system of inequalities.

39. $x + y < 4$
$y \leq 2x + 1$

a. (0, 1) **b.** (3, 1) **c.** (2, 0) **d.** (1, 4)

40. $y < -x^2 + 3$
$x + 2y \leq 2$

a. (–2, –1) **b.** (0, –2) **c.** (0, 1) **d.** (3, –6)

For Exercises 41–58, graph the solution set. If there is no solution, indicate that the solution set is the empty set. (See Examples 5–6, pp. 917–919)

41. $y < \frac{1}{2}x - 4$
$y > -2x + 1$

42. $y \geq \frac{1}{3}x - 2$
$y \leq x - 4$

43. $2x + 5y \leq 5$
$-3x + 4y \geq 4$

44. $4x - 3y > 3$
$x + 4y < -4$

45. $x^2 + y^2 \geq 9$
$x^2 + y^2 \leq 16$

46. $x^2 + y^2 \geq 1$
$x^2 + y^2 < 25$

47. $y \geq 3x + 3$
$-3x + y < 1$

48. $y < 2x - 4$
$-2x + y \geq 2$

49. $|x| < 3$
$|y| < 3$

50. $|x| \geq 2$
$|y| \geq 2$

51. $y \geq x^2 - 2$
$y > x$
$y \leq 4$

52. $y \leq -x^2 + 7$
$y \leq -x + 5$
$y > 1$

53. $x^2 + y^2 \leq 100$
$y < \frac{4}{3}x$
$x \leq 8$

54. $x^2 + y^2 < 100$
$y \geq x$
$y \geq 1$

55. $y < e^x$
$y > 1$
$x < 2$

56. $y \leq \frac{2}{x}$
$y > 0$
$y < x$

57. $(x + 2)^2 + (y - 3)^2 \leq 9$
$x - y > 2$

58. $(x - 4)^2 + (y + 1)^2 < 25$
$2x - y < -4$

Mixed Exercises

For Exercises 59–64, write an inequality to represent the statement.

59. x is at most 6.

60. y is no more than 7.

61. y is at least -2.

62. x is no less than $\frac{1}{2}$.

63. The sum of x and y does not exceed 18.

64. The difference of x and y is not less than 4.

65. Let x represent the number of hours that Trenton spends studying algebra, and let y represent the number of hours he spends studying history. For parts (a)–(e), write an inequality to represent the given statement. **(See Example 7, p. 920)**

a. Trenton has a total of at most 9 hr to study for both algebra and history combined.

b. Trenton will spend at least 3 hr studying algebra.

c. Trenton will spend no more than 4 hr studying history.

d. The number of hours spent studying algebra cannot be negative.

e. The number of hours spent studying history cannot be negative.

f. Graph the solution set to the system of inequalities from parts (a)–(e).

66. Let x represent the number of country songs that Sierra puts on a playlist. Let y represent the number of rock songs on the same playlist. For parts (a)–(e), write an inequality to represent the given statement.

a. Sierra will put at least 6 country songs on the playlist.

b. Sierra will put no more than 10 rock songs on the playlist.

c. Sierra wants to limit the length of the playlist to at most 20 songs.

d. The number of country songs cannot be negative.

e. The number of rock songs cannot be negative.

f. Graph the solution set to the system of inequalities from parts (a)–(e).

67. A couple has \$60,000 to invest for retirement. They plan to put x dollars in stocks and y dollars in bonds. For parts (a)–(d), write an inequality to represent the given statement.

- **a.** The total amount invested is at most \$60,000.
- **b.** The couple considers stocks a riskier investment, so they want to invest at least twice as much in bonds as in stocks.
- **c.** The amount invested in stocks cannot be negative.
- **d.** The amount invested in bonds cannot be negative.
- **e.** Graph the solution set to the system of inequalities from parts (a)–(d).

68. A college theater has a seating capacity of 2000. It reserves x tickets for students and y tickets for general admission. For parts (a)–(d) write an inequality to represent the given statement.

- **a.** The total number of seats available is at most 2000.
- **b.** The college wants to reserve at least 3 times as many student tickets as general admission tickets.
- **c.** The number of student tickets cannot be negative.
- **d.** The number of general admission tickets cannot be negative.
- **e.** Graph the solution set to the system of inequalities from parts (a)–(d).

69. A weak earthquake occurred roughly 9 km south and 12 km west of the center of Hawthorne, Nevada. The quake could be felt 16 km away. Suppose that the origin of a map is placed at the center of Hawthorne with the positive x-axis pointing east and the positive y-axis pointing north.

- **a.** Find an inequality that describes the points on the map for which the earthquake could be felt.
- **b.** Could the earthquake be felt at the center of Hawthorne?

70. A coordinate system is placed at the center of a town with the positive x-axis pointing east, and the positive y-axis pointing north. A cell tower is located 4 mi west and 5 mi north of the origin.

- **a.** If the tower has an 8-mi range, write an inequality that represents the points on the map serviced by this tower.
- **b.** Can a resident 5 mi east of the center of town get a signal from this tower?

71. Write a system of inequalities that represents the points in the first quadrant less than 3 units from the origin.

72. Write a system of inequalities that represents the points in the second quadrant more than 4 units from the origin.

73. Write a system of inequalities that represents the points inside the triangle with vertices $(-3, -4)$, $(3, 2)$, and $(-5, 4)$.

74. Write a system of inequalities that represents the points inside the triangle with vertices $(-4, -4)$, $(1, 1)$, and $(5, -1)$.

Write About It

75. Under what circumstances should a dashed line or curve be used when graphing the solution set to an inequality in two variables?

76. Explain how test points are used to determine the region of the plane that represents the solution to an inequality in two variables.

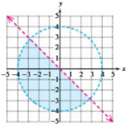
77. Explain how to find the solution set to a system of inequalities in two variables.

78. Describe the solution set to the system of inequalities. $x \geq 0, y \geq 0, x \leq 1, y \leq 1$

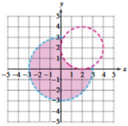
Expanding Your Skills

For Exercises 79–82, write a system of inequalities to represent the shaded region.

79.



80.



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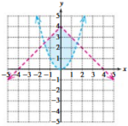

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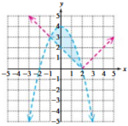


For Exercises 83–84, graph the solution set.

83. $|x| \geq |y|$

84. $|x| + |y| \leq 1$

Technology Connections

For Exercises 85–86, use a graphing utility to graph the solution set to the system of inequalities.

85. $y \geq 0.4e^x$

$y \leq 0.25x^3 - 4x$

86. $y < \dfrac{4}{x^2+1}$

$y > \dfrac{-2}{x^2+0.5}$

Problem Recognition Exercises

Equations and Inequalities in Two Variables

For Exercises 1–2, for parts (a) and (b), graph the equation. For part (c), solve the system of equations. For parts (d) and (e) graph the solution set to the system of inequalities. If there is no solution, indicate that the solution set is the empty set.

1. **a.** $y = -3x + 5$

b. $-2x + y = 0$

c. $y = -3x + 5$
$-2x + y = 0$

d. $y > -3x + 5$
$-2x + y < 0$

e. $y < -3x + 5$
$-2x + y > 0$

2. **a.** $y = 2x - 3$

b. $4x - 2y = -2$

c. $y = 2x - 3$
$4x - 2y = -2$

d. $y \geq 2x - 3$
$4x - 2y \geq -2$

e. $y \leq 2x - 3$
$4x - 2y \leq -2$

For Exercises 3–4, for part (a), graph the equations in the system and determine the solution set. For parts (b) and (c), graph the solution set to the inequality.

3. **a.** $y = x^2$
$y = \frac{1}{2}x^2$

b. $y \leq x^2$
$y \geq \frac{1}{2}x^2$

c. $y \geq x^2$
$y \leq \frac{1}{2}x^2$

4. **a.** $x - y = 1$
$y = (x - 3)^2$

b. $x - y \geq 1$
$y \geq (x - 3)^2$

c. $x - y \leq 1$
$y \leq (x - 3)^2$

Go online for more practice problems.

Lesson 8-6

Linear Programming

What Will You Learn?

After completing this lesson, you should be able to:

- Write an Objective Function
- Solve Linear Programming Applications

Learn: Write an Objective Function

When a company manufactures a product, the goal is to obtain maximum profit at minimum cost. However, the production process is often limited by certain constraints such as the amount of labor available, the capacity of machinery, and the amount of money available for the company to invest in the process. In this lesson, we will study a process called **linear programming** that enables us to maximize or minimize a function under specified constraints.

The function to be optimized (maximized or minimized) in a linear programming application is called the **objective function.** The objective function often has two independent variables. For Example, given $z = f(x, y)$, the variable z is dependent on the two independent variables x and y.

Good Practices The notation $z = f(x, y)$ is read as "z is a function of x and y."

Example 1: Writing an Objective Function

Suppose that a college wants to rent several buses to transport students to a championship college football game. A large bus costs $1200 to rent and a small bus costs $800 to rent.

Let x represent the number of large buses.

Let y represent the number of small buses.

Write an objective function that represents the total cost z (in $) to rent x large buses and y small buses.

Solution:

$$\begin{array}{rccccc} & \begin{pmatrix}\text{Total}\\ \text{cost}\end{pmatrix} & = & \begin{pmatrix}\text{Cost to rent}\\ x\text{ large buses}\end{pmatrix} & + & \begin{pmatrix}\text{Cost to rent}\\ y\text{ small buses}\end{pmatrix} \\ \text{Cost:} & z & = & 1200x & + & 800y \end{array}$$

The objective function is defined by $z = 1200x + 800y$.

Insights

An objective function represents a quantity that is to be maximized or minimized. In Example 1, the school will want to minimize the cost to transport the students.

Apply the Skills

1. An office manager needs to staff the office. Full-time employees are hired at \$36 per hour and part-time employees at \$24 per hour. Write an objective function that represents the total cost (in \$) to staff the office with x full-time employees and y part-time employees for 1 hr.

Learn: Solve Linear Programming Applications

In Example 2, we identify constraints imposed on the resources that affect the number of buses that the college can rent.

Example 2: Writing a System of Constraints

Refer to the scenario from Example 1. We now set several constraints that affect the number of buses that can be rented.

Let x represent the number of large buses.

Let y represent the number of small buses.

For parts (a)–(d), write an inequality that represents the given statement.

a. The number of large buses cannot be negative.

b. The number of small buses cannot be negative.

c. Large buses can carry 60 people and small buses can carry 45 people. The college must transport at least 3600 students.

d. The number of available bus drivers is at most 75.

Solution:

a. $x \geq 0$ The number of large buses cannot be negative.

b. $y \geq 0$ The number of small buses cannot be negative.

c.

$$\begin{pmatrix}\text{Number of students}\\ \text{carried by large buses}\end{pmatrix} + \begin{pmatrix}\text{Number of students}\\ \text{carried by small buses}\end{pmatrix} \overset{\text{is at least}}{\geq} 3600$$

$$60x \quad + \quad 45y \quad \geq 3600$$

Large buses hold 60 people and small buses hold 45 people.

d.

$$\begin{pmatrix}\text{Number of}\\ \text{large buses}\end{pmatrix} + \begin{pmatrix}\text{Number of}\\ \text{small buses}\end{pmatrix} \overset{\text{is at most}}{\leq} 75$$

$$x \quad + \quad y \quad \leq 75$$

Because each bus has only one driver, the total number of buses is the same as the total number of drivers.

Apply the Skills

2. Refer to Apply the Skills 1. Suppose that the office manager needs at least 20 employees, but not more than 24 full-time employees. Furthermore, to make the office run smoothly, the manager knows that the number of full-time employees must always be greater than or equal to the number of part-time employees. Write a system of inequalities that represents the constraints on the number of full-time employees x and the number of part-time employees y.

The constraints in Example 2 make up a system of linear inequalities.

The number of large buses is nonnegative.	$x \geq 0$
The number of small buses is nonnegative.	$y \geq 0$
The school must transport at least 3600 students.	$60x + 45y \geq 3600$
The number of drivers (and therefore buses) is at most 75.	$x + y \leq 75$

The region in the plane that represents the solution set to the system of constraints is called the **feasible region** (Figure 8-30). The points of intersection of the bounding lines in the feasible region are called the **vertices** of the feasible region.

The vertices in Figure 8-30 are (60, 0), (75, 0), and (15, 60).

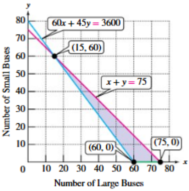


Figure 8-30

Insights

The point (15, 60) is the point of intersection of the lines $60x + 45y = 3600$ and $x + y = 75$.

The point (60, 0) is the point of intersection of the line $60x + 45y = 3600$ and $y = 0$.

The point (75, 0) is the point of intersection of the line $x + y = 75$ and $y = 0$.

Points within the shaded region meet all the constraints in the problem. For Example, the ordered pair (65, 5) represents 65 large buses and 5 small buses.

65 large buses and 5 small buses $\longrightarrow$ 70 buses ($\le$ 75 drivers) ✓

65 large buses and 5 small buses $\longrightarrow$ transports 4125 students ($\ge$ 3600) ✓

However, points such as (65, 15) and (40, 20) are *outside* the feasible region and do not satisfy all constraints.

65 large buses and 15 small buses $\longrightarrow$ 80 buses (exceeds the number of drivers)

40 large buses and 20 small buses $\longrightarrow$ transports only 3300 students

The goal of a linear programming application is to find the maximum or minimum value of the objective function $z = f(x, y)$ when x and y are restricted to the ordered pairs in the feasible region. Fortunately, it has been proven mathematically that if a maximum or minimum value of a function exists, it occurs at one or more of the vertices of the feasible region. This is the basis for the following procedure to solve a linear programming application.

Solving an Application Involving Linear Programming

Step 1 Write an objective function $z = f(x, y)$.

Step 2 Write a system of inequalities defining the constraints on x and y.

Step 3 Graph the feasible region and identify the vertices.

Step 4 Evaluate the objective function at each vertex of the feasible region. Use the results to identify the values of x and y that optimize the objective function. Identify the optimal value of z.

In Example 3, we optimize the objective function found in Example 1 subject to the constraints defined in Example 2.

Example 3: Solving a Linear Programming Application

A college wants to rent buses to transport at least 3600 students to a championship college football game. Large buses hold 60 people and small buses hold 45 people. Furthermore, the number of available bus drivers is at most 75. Each large bus costs $1200 to rent and each small bus costs $800 to rent. Find the optimal number of large and small buses that will minimize cost.

Solution:

Let x represent the number of large buses. Define the relevant variables.

Let y represent the number of small buses.

Step 1: Write an objective function. Since cost is to be minimized, write a cost function. (See Example 1.)

Cost: $z = 1200x + 800y$

Step 2: Write a system of constraints on the relevant variables. (See Example 2.)

Constraints:

$x \geq 0$
$y \geq 0$
$60x + 45y \geq 3600$
$x + y \leq 75$

Step 3: Graph the feasible region and identify the vertices.

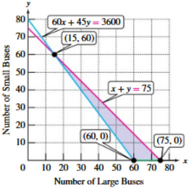
The vertices are found by identifying the points of intersection between the bounding lines.

Bounding lines:

$60x + 45y = 3600$
$x + y = 75$ Point of intersection (15, 60)

$60x + 45y = 3600$
$y = 0$ Point of intersection (60, 0)

$x + y = 75$
$y = 0$ Point of intersection (75, 0)



Step 4: Evaluate the objective function $z = 1200x + 800y$ at each vertex.

Cost function: $z = 1200x + 800y$

at (15, 60) $z = 1200(15) + 800(60) = \boxed{\$66{,}000}$

at (60, 0) $z = 1200(60) + 800(0) = \$72{,}000$

at (75, 0) $z = 1200(75) + 800(0) = \$90{,}000$

The cost would be minimized if the college rents 15 large buses and 60 small buses. The minimum cost is $66,000.

Apply the Skills

3. Refer to Apply the Skills 1 and 2. The office manager needs at least 20 employees, but no more than 24 full-time employees. Furthermore, to make the office run smoothly, the manager knows that the number of full-time employees must always be greater than or equal to the number of part-time employees. If full-time employees are paid \$36 per hour and part-time employees \$24 per hour, determine the number of full-time and part-time employees the manager should hire to minimize total labor cost per hour.

In Example 4, we investigate a situation in which we maximize profit.

Example 4: Solving a Linear Programming Application

A baker produces whole wheat bread and cheese bread to sell at the farmer's market. The whole wheat bread is denser and requires more baking time, whereas the cheese bread requires more labor. The baking times and average amount of labor per loaf are given in the table along with the profit for each loaf.

	Time to Bake	Labor	Profit
Wheat bread	1.5 hr	$\frac{1}{3}$ hr	\$1.20
Cheese bread	1 hr	$\frac{1}{2}$ hr	\$1.00

The oven space restricts the baker from baking more than 120 loaves. Furthermore, the amount of oven time for baking is no more than 165 hr and the amount of available labor is at most 55 hr. Determine the number of loaves of each type of bread that the baker should bake to maximize profit. Assume that all loaves of bread produced are sold.

Solution:

Let x represent the number of loaves of wheat bread.

Let y represent the number of loaves of cheese bread.

Step 1: Write an objective function. In this Example, we need to maximize profit.

$$\text{Profit} = \begin{pmatrix}\text{Profit from}\\ \text{wheat bread}\end{pmatrix} + \begin{pmatrix}\text{Profit from}\\ \text{cheese bread}\end{pmatrix}$$

$$z = 1.20x + 1.00y$$

Step 2: Write a system of constraints on the independent variables.

$x \geq 0$	Number of loaves of wheat bread cannot be negative.
$y \geq 0$	Number of loaves of cheese bread cannot be negative.
$x + y \leq 120$	Total number of loaves is no more than 120.
$1.5x + y \leq 165$	The total amount of baking time is no more than 165 hr.
$\frac{1}{3}x + \frac{1}{2}y \leq 55$	The total amount of available labor is at most 55 hr.

Step 3: Graph the feasible region and identify the vertices. Find the points of intersection between pairs of bounding lines:

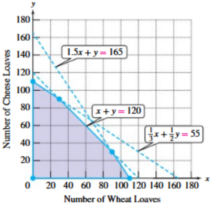
$x + y = 120$
$\frac{1}{3}x + \frac{1}{2}y = 55$ Point of intersection (30, 90)

$x + y = 120$
$1.5x + y = 165$ Point of intersection (90, 30)

$1.5x + y = 165$
$y = 0$ Point of intersection (110, 0)

$\frac{1}{3}x + \frac{1}{2}y = 55$
$x = 0$ Point of intersection (0, 110)

$x = 0$
$y = 0$ Point of intersection (0, 0)



Step 4: Evaluate the objective function at each vertex.

Profit: $z = 1.2x + y$

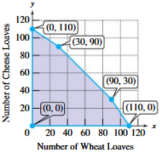
at (0, 0) $z = 1.2(0) + (0) = \$0$

at (0, 110) $z = 1.2(0) + (110) = \$110$

at (30, 90) $z = 1.2(30) + (90) = \$126$

at (90, 30) $z = 1.2(90) + (30) = \boxed{\$138}$

at (110, 0) $z = 1.2(110) + (0) = \$132$



The profit will be maximized if the baker bakes 90 whole wheat loaves and 30 cheese loaves. The maximum profit is $138.

Apply the Skills

4. A manufacturer produces two sizes of leather handbags. It takes longer to cut and dye the leather for the smaller bag, but it takes more time sewing the larger bag. The production constraints and profit for each type of bag are given in the table.

	Cutting and Dying	Sewing	Profit
Large bag	0.6 hr	2 hr	\$30
Small bag	1 hr	1.5 hr	\$25

The machinery limits the number of bags produced to at most 1000 per week. If the company has 900 hr per week available for cutting and dying and 1800 hr available per week for sewing, determine the number of each type of bag that should be produced weekly to maximize profit. Assume that all bags produced are also sold.

To find the maximum or minimum value of an objective function, we evaluate the function at the vertices of the feasible region. It seems reasonable that the profit would be maximized at a point on the upper edge of the feasible region. These are the points in the feasible region where the combined values of x and y are the greatest.

The goal of Example 4 was to find the values of x and y that maximized the profit function $z = 1.2x + y$. To see why the profit z was maximized at a vertex of the feasible region, write the equation in slope-intercept form:

$$y = -1.2x + z$$

In this form, the objective function represents a family of parallel lines with slope −1.2 and y-intercept $(0, z)$. To maximize z, we want the line with the greatest y-intercept that still remains in contact with the feasible region. In Figure 8-31, we see that this occurs for the line passing through the point (90, 30) as expected.

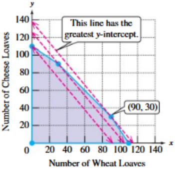


Figure 8-31

Point of Interest

Linear programming was first introduced by Russian mathematician Leonid Kantorovich. The technique was used in World War II to minimize costs to the army. Later in 1975, Kantorovich won the Nobel Prize in economics for his contributions to the theory of optimum allocation of resources.

Practice 8-6

Practice Exercises

Prerequisite Review

R.1. A salesperson makes a base salary of \$500 per week plus 11% commission on sales.

- **a.** Write a linear function to model the salesperson's weekly salary $S(x)$ for x dollars in sales.
- **b.** Evaluate $S(8000)$ and interpret the meaning in the context of this problem.

R.2. A luxury car rental company charges \$123 per day in addition to a flat fee of \$75 for insurance.

- **a.** Write an equation that represents the cost y (in \$) to rent the car for x days.
- **b.** What is the y-intercept and what does it mean in the context of this problem?

Concept Connections

1. The process that maximizes or minimizes a function subject to linear constraints is called __________ programming.

2. The function to be optimized in a linear programming application is called the __________ function.

3. The region in the plane that represents the solution set to a system of constraints is called the __________ region.

4. The points of intersection of a feasible region are called the __________ of the region.

Learn: Write an Objective Function

5. A diner makes a profit of \$0.80 for a cup of coffee and \$1.10 for a cup of tea. Write an objective function $z = f(x, y)$ that represents the total profit for selling x cups of coffee and y cups of tea. (**See Example 1, p. 927**)

6. An athlete burns 10 calories per minute running and 8 calories per minute lifting weights. Write an objective function $z = f(x, y)$ that represents the total number of calories burned by running for x minutes and lifting weights for y minutes.

7. A courier company makes deliveries with two different trucks. Truck A costs \$0.62/mi to operate and truck B costs \$0.50/mi to operate. Write an objective function $z = f(x, y)$ that represents the total cost for driving truck A for x miles and driving truck B for y miles.

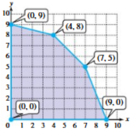
8. The cost for an animal shelter to spay a female cat is \$82 and the cost to neuter a male cat is \$55. Write an objective function $z = f(x, y)$ that represents the total cost for spaying x female cats and neutering y male cats.

Learn: Solve Linear Programming Applications

For Exercises 9–12,

- **a.** Determine the values of x and y that produce the maximum or minimum value of the objective function on the given feasible region.
- **b.** Determine the maximum or minimum value of the objective function on the given feasible region.

9. Maximize: $z = 3x + 2y$



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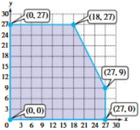
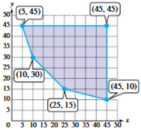
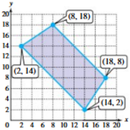
10. Maximize: $z = 1.8x + 2.2y$



11. Minimize: $z = 1000x + 900y$



12. Minimize: $z = 6x + 9y$



For Exercises 13–18,

a. For the given constraints, graph the feasible region and identify the vertices.

b. Determine the values of *x* and *y* that produce the maximum or minimum value of the objective function on the feasible region.

c. Determine the maximum or minimum value of the objective function on the feasible region.

13. $x \geq 0, y \geq 0$
$x + y \leq 60$
$y \leq 2x$
Maximize: $z = 250x + 150y$

14. $x \geq 0, y \geq 0$
$2x + y \leq 40$
$x + 2y \leq 50$
Maximize: $z = 9.2x + 8.1y$

15. $x \geq 0, y \geq 0$
$3x + y \geq 50$
$2x + y \geq 40$
Minimize: $z = 3x + 2y$

16. $x \geq 0, y \geq 0$
$4x + 3y \geq 60$
$2x + 3y \geq 36$
Minimize: $z = 4.5x + 6y$

17. $x \geq 0, y \geq 0$
$x \leq 36$
$y \leq 40$
$x + y \leq 48$
Maximize: $z = 150x + 90y$

18. $x \geq 0, y \geq 0$
$x \leq 10$
$y \leq 8$
$x + y \leq 12$
Maximize: $z = 50x + 70y$

For Exercises 19–20, use the given constraints to find the maximum value of the objective function and the ordered pair (*x*, *y*) that produces the maximum value.

19. $x \geq 0, y \geq 0$
$3x + 4y \leq 48$
$2x + y \leq 22$
$y \leq 9$

a. Maximize: $z = 100x + 120y$

b. Maximize: $z = 100x + 140y$

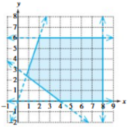
20. $x \geq 0, y \geq 0$
$x + y \leq 20$
$x + 2y \leq 36$
$x \leq 14$

a. Maximize: $z = 12x + 15y$

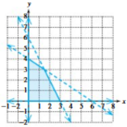
b. Maximize: $z = 15x + 12y$

For Exercises 21–22, write a system of inequalities with the given solution set.

21.



22.



23. A furniture manufacturer builds tables. The cost for materials and labor to build a kitchen table is \$240 and the profit is \$160. The cost to build a dining room table is \$320 and the profit is \$240. **(See Examples 2–3, pp. 928–930)**

Let x represent the number of kitchen tables produced per month. Let y represent the number of dining room tables produced per month.

a. Write an objective function representing the monthly profit for producing and selling x kitchen tables and y dining room tables.

b. The manufacturing process is subject to the following constraints. Write a system of inequalities representing the constraints.

- The number of each type of table cannot be negative.
- Due to labor and equipment restrictions, the company can build at most 120 kitchen tables.
- The company can build at most 90 dining room tables.
- The company does not want to exceed a monthly cost of \$48,000.

c. Graph the system of inequalities represented by the constraints.

d. Find the vertices of the feasible region.

e. Test the objective function at each vertex.

f. How many kitchen tables and how many dining room tables should be produced to maximize profit? (Assume that all tables produced will be sold.)

g. What is the maximum profit?

24. Youssef makes \$24/hr tutoring chemistry and \$20/hr tutoring math. Let x represent the number of hours per week that Youssef spends tutoring chemistry. Let y represent the number of hours per week that Youssef spends tutoring math.

a. Write an objective function representing Youssef's weekly income for tutoring x hours of chemistry and y hours of math.

b. The time that Youssef devotes to tutoring is limited by the following constraints. Write a system of inequalities representing the constraints.

- The number of hours spent tutoring each subject cannot be negative.
- Due to academic demands, Youssef tutors at most 18 hr per week.
- The tutoring center requires that Youssef tutor math at least 4 hr per week.
- The demand for math tutors is greater than the demand for chemistry tutors. Therefore, the number of hours Youssef spends tutoring math must be at least twice the number of hours spent tutoring chemistry.

c. Graph the system of inequalities represented by the constraints.

d. Find the vertices of the feasible region.

e. Test the objective function at each vertex.

f. How many hours tutoring math and how many hours tutoring chemistry should Youssef work to maximize his income?

g. What is the maximum income?

h. Explain why the maximum income is found at a point on the line $x + y = 18$.

25. A plant nursery sells two sizes of oak trees to landscapers. Large trees cost the nursery \$120 from the grower. Small trees cost the nursery \$80. The profit for each large tree sold is \$35 and the profit for each small tree sold is \$30. The monthly demand is at most 400 oak trees. Furthermore, the nursery does not want to allocate more than \$43,200 each month on inventory for oak trees.

a. Determine the number of large oak trees and the number of small oak trees that the nursery should have in its inventory each month to maximize profit. (Assume that all trees in inventory are sold.)

b. What is the maximum profit?

c. If the profit on large trees were \$50, and the profit on small trees remained the same, then how many of each should the nursery have to maximize profit?

26. A sporting goods store sells two types of exercise bikes. The deluxe model costs the store \$400 from the manufacturer and the standard model costs the store \$300 from the manufacturer. The profit that the store makes on the deluxe model is \$180 and the profit on the standard model is \$120. The monthly demand for exercise bikes is at most 30. Furthermore, the store manager does not want to spend more than \$9600 on inventory for exercise bikes.

a. Determine the number of deluxe models and the number of standard models that the store should have in its inventory each month to maximize profit. (Assume that all exercise bikes in inventory are sold.)

b. What is the maximum profit?

c. If the profit on the deluxe bikes were \$150 and the profit on the standard bikes remained the same, how many of each should the store have to maximize profit?

27. A paving company delivers gravel for a road construction project. The company has a large truck and a small truck. The large truck has a greater capacity, but costs more for fuel to operate. The load capacity and cost to operate each truck per load are given in the table.

	Load Capacity	Cost per Load
Small truck	18 yd^3	\$120
Large truck	24 yd^3	\$150

The company must deliver at least 288 yd^3 of gravel to stay on schedule. Furthermore, the large truck takes longer to load and cannot make as many trips as the small truck. As a result, the number of trips made by the large truck is at most $\frac{3}{4}$ times the number of trips made by the small truck.

a. Determine the number of trips that should be made by the large truck and the number of trips that should be made by the small truck to minimize cost.

b. What is the minimum cost to deliver gravel under these constraints?

28. A large department store needs at least 3600 labor hours covered per week. It employs full-time staff 40 hr/wk and part-time staff 25 hr/wk. The cost to employ a full-time staff member is more because the company pays benefits such as health care and life insurance.

	Hours per Week	Cost per Hour
Full time	40 hr	\$25
Part time	25 hr	\$18

The store manager also knows that to make the store run efficiently, the number of full-time employees must be at least 1.25 times the number of part-time employees.

a. Determine the number of full-time employees and the number of part-time employees that should be used to minimize the weekly labor cost.

b. What is the minimum weekly cost to staff the store under these constraints?

29. A manufacturer produces two models of a gas grill. Grill A requires 1 hr for assembly and 0.4 hr for packaging. Grill B requires 1.2 hr for assembly and 0.6 hr for packaging. The production information and profit for each grill are given in the table. **(See Example 4, p. 932)**

	Assembly	Packaging	Profit
Grill A	1 hr	0.4 hr	\$90
Grill B	1.2 hr	0.6 hr	\$120

The manufacturer has 1200 hr of labor available for assembly and 540 hr of labor available for packaging.

a. Determine the number of grill A units and the number of grill B units that should be produced to maximize profit assuming that all grills will be sold.

b. What is the maximum profit under these constraints?

c. If the profit on grill A units is \$110 and the profit on grill B units is unchanged, how many of each type of grill unit should the manufacturer produce to maximize profit?

30. A manufacturer produces two models of patio furniture. Model A requires 2 hr for assembly and 1.2 hr for painting. Model B requires 3 hr for assembly and 1.5 hr for painting. The production information and profit for selling each model are given in the table.

	Assembly	Painting	Profit
Model A	2 hr	1.2 hr	\$150
Model B	3 hr	1.5 hr	\$200

The manufacturer has 1200 hr of labor available for assembly and 660 hr of labor available for painting.

a. Determine the number of model A units and the number of model B units that should be produced to maximize profit assuming that all furniture will be sold.

b. What is the maximum profit under these constraints?

c. If the profit on model A units is \$180 and the profit on model B units remains the same, how many of each type should the manufacturer produce to maximize profit?

31. A farmer has 1200 acres of land and plans to plant corn and soybeans. The input cost (cost of seed, fertilizer, herbicide, and insecticide) for 1 acre for each crop is given in the table along with the cost of machinery and labor. The profit for 1 acre of each crop is given in the last column.

	Input Cost per Acre	Labor/ Machinery Cost per Acre	Profit per Acre
Corn	\$180	\$80	\$120
Soybeans	\$120	\$100	\$100

Suppose the farmer has budgeted a maximum of \$198,000 for input costs and a maximum of \$110,000 for labor and machinery.

a. Determine the number of acres of each crop that the farmer should plant to maximize profit. (Assume that all crops will be sold.)

b. What is the maximum profit?

c. If the profit per acre were reversed between the two crops (that is, \$100 per acre for corn and \$120 per acre for soybeans), how many acres of each crop should be planted to maximize profit?

32. To protect soil from erosion, some farmers plant winter cover crops such as winter wheat and rye. In addition to conserving soil, cover crops often increase crop yields in the row crops that follow in spring and summer. Suppose that a farmer has 800 acres of land and plans to plant winter wheat and rye. The input cost for 1 acre for each crop is given in the table along with the cost for machinery and labor. The profit for 1 acre of each crop is given in the last column.

	Input Cost per Acre	Labor/ Machinery Cost per Acre	Profit per Acre
Wheat	\$90	\$50	\$42
Rye	\$120	\$40	\$35

Suppose the farmer has budgeted a maximum of \$90,000 for input costs and a maximum of \$36,000 for labor and machinery.

a. Determine the number of acres of each crop that the farmer should plant to maximize profit. (Assume that all crops will be sold.)

b. What is the maximum profit?

c. If the profit per acre for wheat were \$40 and the profit per acre for rye were \$45, how many acres of each crop should be planted to maximize profit?

Write About It

33. What is the purpose of linear programming?

34. What is an objective function?

35. How is the feasible region determined?

36. If an optimal value exists for an objective function, it exists at one of the vertices of the feasible region. Explain how to find the vertices.

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Chapter 8 Review

Chapter 8 Key Concepts

Lesson 8-1 Systems of Linear Equations in Two Variables and Applications

Key Concepts	Reference
Two or more linear equations taken together form a **system of linear equations.** A **solution** to a system of equations in two variables is an ordered pair that is a solution to each individual equation. Graphically, this is a point of intersection of the graphs of the equations.	p. 846
The substitution method and the addition method are often used to solve a system of linear equations in two variables.	pp. 849–850
A system of linear equations in two variables will have no solution if the equations in the system represent parallel lines. In such a case, we say that the system is **inconsistent.**	p. 848
A system of linear equations in two variables will have infinitely many solutions if the equations represent the same line. In such a case, we say that the equations are **dependent.**	p. 848

Lesson 8-2 Systems of Linear Equations in Three Variables and Applications

Key Concepts	Reference
A **linear equation in three variables** is an equation that can be written in the form $Ax + By + Cz = D$ where A, B, and C are not all zero.	p. 868
A solution to a system of linear equations in three variables is an **ordered triple** (x, y, z) that satisfies each equation in the system. Geometrically, a solution is a point of intersection of the planes represented by the equations in the system.	p. 868

Lesson 8-3 Partial Fraction Decomposition

Key Concepts	Reference
Partial fraction decomposition is used to write a rational expression as a sum of simpler fractions.	p. 886
There are two basic parts to find the partial fraction decomposition of a rational expression. I. Factor the denominator of the expression into linear factors and quadratic factors that are not further factorable over the integers. Then set up the "form" or "structure" for the partial fraction decomposition into simpler fractions. II. Next, multiply both sides of the equation by the LCD. Then set up a system of linear equations to find the coefficients of the terms in the numerator of each fraction. *Note*: The numerator of the original rational expression must be of lesser degree than the denominator. If this is not the case, first use long division.	p. 886

Lesson 8-4 Systems of Nonlinear Equations in Two Variables

Key Concepts	Reference
A **nonlinear system of equations** is a system in which one or more equations is nonlinear.	p. 898
The substitution method is often used to solve a nonlinear system of equations.	p. 899
In some cases, the addition method can be used provided that the terms containing the corresponding variables are like terms.	p. 901

Lesson 8-5 Inequalities and Systems of Inequalities in Two Variables

Key Concepts	Reference
An inequality of the form $Ax + By < C$, where A and B are not both zero, is called a **linear inequality in two variables.** (The symbols $>$, $\leq$, and $\geq$ can be used in place of $<$ in the definition.)	p. 911
The basic steps to solve a linear inequality in two variables are as follows: 1. Graph the related equation. The resulting line is drawn as a dashed line if the inequality is strict, and is otherwise drawn as a solid line. 2. Select a test point from either side of the line. If the ordered pair makes the original inequality true, then shade the half-plane from which the point was taken. Otherwise, shade the other half-plane.	p. 911
A nonlinear inequality in two variables is solved using the same basic procedure.	p. 916
Two or more inequalities in two variables make up a system of inequalities in two variables. The solution set to the system is the region of overlap (intersection) of the solution sets of the individual inequalities.	p. 917

Lesson 8-6 Linear Programming

Key Concepts	Reference
A process called **linear programming** enables us to maximize or minimize a function under specified constraints. The function to be maximized or minimized is called the **objective function.**	p. 927
The steps to solve a linear programming application are outlined here. **Step 1** Write an objective function, $z = f(x, y)$. **Step 2** Write a system of inequalities defining the constraints on x and y. **Step 3** Graph the feasible region and identify the vertices. **Step 4** Evaluate the objective function at each vertex of the feasible region. Use the results to identify the values of x and y that optimize the objective function and identify the optimal value of z.	p. 930

Go online for a detailed chapter summary.

Chapter 8 Test

For Exercises 1–3, determine if the ordered pair or ordered triple is a solution to the system.

1. $x - 5y = -3$

$y = 2x - 12$

a. $(7, 2)$

b. $(-3, 0)$

2. $2x - 3y + z = -5$

$5x + y - 3z = -18$

$-x + 2y + 5z = 8$

a. $(0, 1, -2)$

b. $(-3, 0, 1)$

3. $2x - 4y < 9$

$-3x + y \geq 4$

a. $(-6, 1)$

b. $(1, 4)$

For Exercises 4–14, solve the system. If the system does not have one unique solution, also state whether the system is inconsistent or whether the equations are dependent.

4. $x = 5 - 4y$

$-3x + 7y = 4$

5. $0.2x = 0.35y - 2.5$

$0.16x + 0.5y = 5.8$

6. $x - \frac{2}{3}y = \frac{3}{10}$

$5x = 2y + \frac{3}{2}$

7. $7(x - y) = 3 - 5y$

$4(3x - y) = -2x$

8. $a + 6b + 3c = -14$

$2a + b - 2c = -8$

$-3a + 2b + c = -8$

9. $x + 4z = 10$

$3y - 2z = 9$

$2x + 5y = 21$

10. $2x - y + z = -3$

$x - 3y = 2$

$x + 2y + z = -7$

11. $(x - 4)^2 + y^2 = 25$

$x - y = 3$

12. $5x^2 + y^2 = 14$

$x^2 - 2y^2 = -17$

13. $2xy - y^2 = -24$

$-3xy + 2y^2 = 38$

14. $\frac{1}{x+3} - \frac{2}{y-1} = -7$

$\frac{3}{x+3} + \frac{1}{y-1} = 7$

15. Solve the system and write the general solution.

$x - 2z = 6$

$y + 3z = 2$

$x + y + z = 8$

16. At a candy and nut shop, the manager wants to make a nut mixture that is 56% peanuts. How many pounds of peanuts must be added to an existing mixture of 45% peanuts to make 20 lb of a mixture that is 56% peanuts?

17. Two runners begin at the same point on a 400-m track. If they run in opposite directions they pass each other in 40 sec. If they run in the same direction, they will meet again in 200 sec. Find the speed of each runner.

18. Dylan invests $15,000 in three different stocks. One stock is very risky and after 1 yr loses 8%. The second stock returns 3.2%, and a third stock returns 5.8%. At the end of 1 yr, the total return is $274. If Dylan invested $2000 more in the second stock than in the third stock, determine the amount Dylan invested in each stock.

19. The difference of two positive numbers is 3 and the difference of their squares is 33. Find the numbers.

20. A rectangular television screen has a perimeter of 154 in. and an area of 1452 in.². Find the dimensions of the screen.

21. Use a system of linear equations in three variables to find an equation of the form $y = ax^2 + bx + c$ that defines the parabola through the points $(1, -1)$, $(2, 1)$, and $(-1, 7)$.

For Exercises 22–23, set up the form for the partial fraction decomposition. Do not solve for *A*, *B*, *C*, and so on.

22. $\frac{-15x+15}{3x^2+x-2}$ **23.** $\frac{5x^6+3x^5-4x^3+x-3}{x^3(x-3)(x^2+5x+1)^2}$

For Exercises 24–28, perform the partial fraction decomposition.

24. $\frac{-12x-29}{2x^2+11x+15}$ **25.** $\frac{6x+8}{x^2+4x+4}$

26. $\frac{x^4-6x^3+4x^2+20x-32}{x^3-4x^2}$

27. $\frac{x^2-2x-21}{x^3+7x}$

28. $\frac{7x^3+4x^2+63x+15}{x^4+11x^2+18}$

For Exercises 29–33, graph the solution set.

29. $2(x+y) > 6-y$

30. $(x+3)^2+y^2 \geq 9$

31. $|x| < 4$

32. $x+y \leq 4$

$2x-y > -2$

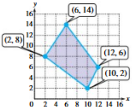
33. $y \leq -x^2+5$

$y > 1$

$x+y \leq 3$

34. A donut shop makes a profit of $2.40 on a dozen donuts and $0.55 per muffin. Write an objective function $z = f(x, y)$ that represents the total profit for selling *x* dozen donuts and *y* muffins.

35. For the feasible region given and the objective function $z = 4x + 5y$,



a. Determine the values of *x* and *y* that produce the minimum value of the objective function on the feasible region.

b. Determine the minimum value of the objective function on the feasible region.

36. For the given constraints and objective function, $z = 600x + 850y$,

a. Graph the feasible region and identify the vertices.

$x \geq 0,\ y \geq 0$

$x + y \leq 48$

$y \leq 3x$

b. Determine the values of *x* and *y* that produce the maximum value of the objective function on the feasible region.

c. Determine the maximum value of the objective function on the feasible region.

37. A weight lifter wants to mix two types of protein powder. One is a whey protein and one is a soy protein. The fat, carbohydrate, and protein content (in grams) for 1 scoop of each powder is given in the table.

	Fat	Carbohydrates	Protein
Whey	3 g	3 g	20 g
Soy	2 g	4 g	18 g

Suppose that the weight lifter wants to make at most 60 scoops of a protein powder mixture, while limiting the total fat content to at most 150 g and the total carbohydrate content to at most 216 g.

a. Determine the number of scoops of each type of powder that will maximize the total protein content under these constraints.

b. What is the maximum total protein content?

c. If the protein content were reversed between the two brands (that is, 18 g for the whey protein and 20 g for the soy protein), then how much of each type of protein powder should be used to maximize the amount of protein?

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Chapter 8 Cumulative Review Exercises

For Exercises 1–5, solve the equation.

1. $2x(x + 2) = 5x + 7$

2. $\sqrt{2t + 8} - t = 4$

3. $(x^2 - 4)^2 - 7(x^2 - 4) - 60 = 0$

4. $\log_2(x - 4) = \log_2(x + 1) + 3$

5. $50e^{2x+1} = 2000$

For Exercises 6–7, solve the inequality. Write the solution set in interval notation.

6. $\frac{x + 4}{x - 2} \le 1$

7. $3|x + 2| - 1 > 8$

8. Find the partial fraction decomposition. $\frac{-5x + 17}{x^2 + 6x + 9}$

9. Given $f(x) = 2x^2 - 3x$ and $g(x) = 5x + 1$,

- **a.** Find $(f \circ g)(x)$.
- **b.** Find $(g \circ f)(x)$.

10. Given $f(x) = \sqrt[3]{x - 2}$, write an equation for $f^{-1}(x)$.

11. Use a calculator to approximate the value of $\log_5 256$. Round to 4 decimal places.

12. Given $f(x) = -\frac{1}{2}x^3 + 4x^2 + 2$, find the average rate of change on the interval [1, 3].

13. Write an equation of the line perpendicular to the line $x + 3y = 6$ and passing through the point $(2, -1)$.

14. Find all zeros of $f(x) = x^4 - 2x^3 + 10x^2 - 18x + 9$ and state the multiplicity of each zero.

For Exercises 15–16,

- **a.** Write the domain in interval notation.
- **b.** Write the range in interval notation.

15. $f(x) = 2|x - 1| - 3$

16. $f(x) = \ln(x - 3)$

For Exercises 17–19, solve the system.

17.
$$\begin{aligned} 3x &= 5y + 1 \\ y &= \tfrac{3}{5}x + 4 \end{aligned}$$

18.
$$\begin{aligned} 5a - 2b + 3c &= 10 \\ -3a + b - 2c &= -7 \\ a + 4b - 4c &= -3 \end{aligned}$$

19.
$$\begin{aligned} -2x^2 + 3y^2 &= 10 \\ 5x^2 + 2y^2 &= 13 \end{aligned}$$

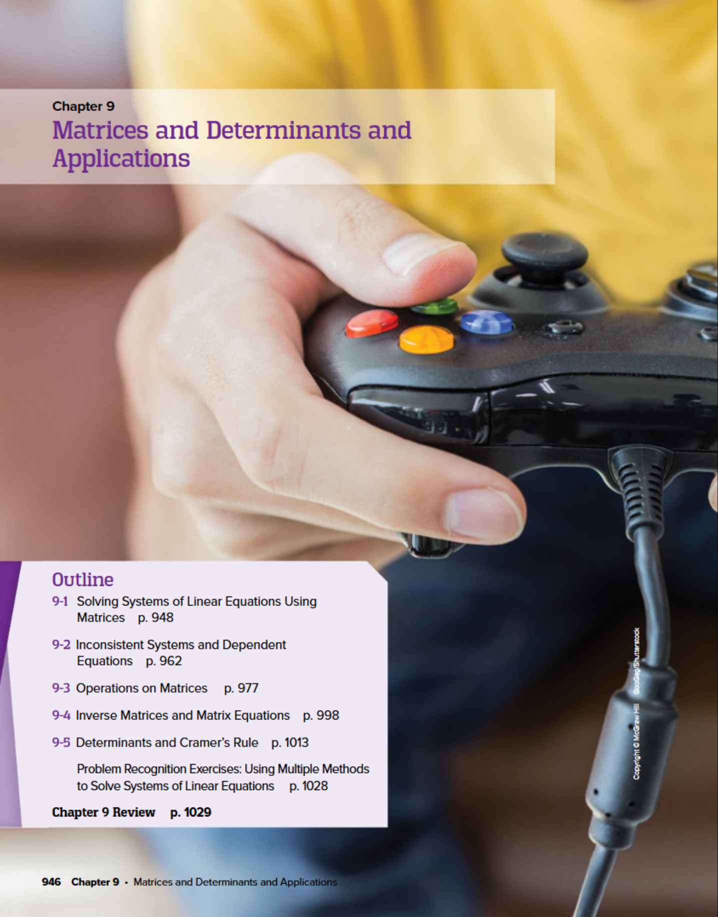
20. Given $f(x) = -x^2 + 5x + 1$, find the difference quotient

21. Shen invested \$8000. After 5 yr with interest compounded continuously, the account is worth \$10,907.40.

- **a.** Write a model of the form $A(t) = Pe^{rt}$, where $A(t)$ represents the amount (in \$) in the account if P dollars in principal is invested at interest rate r for t years.
- **b.** How long will it take for the investment to double? Round to the nearest tenth of a year.

22. The variable y varies jointly as x and the square of z. If y is 36 when x is 10 and z is 3, find the value of y when $x = 12$ and z is 4.

GO ONLINE **Go online for more practice problems.**



Chapter 9

Matrices and Determinants and Applications

Outline

9-1 Solving Systems of Linear Equations Using Matrices p. 948

9-2 Inconsistent Systems and Dependent Equations p. 962

9-3 Operations on Matrices p. 977

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Problem Recognition Exercises: Using Multiple Methods to Solve Systems of Linear Equations p. 1028

Chapter 9 Review p. 1029

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Launch Activity: Abstract Reasoning

Making Purchases

José wanted to buy some new games and a new game controller so that he could play games with his friends and younger sister. After 6 months of working at the library, he saved enough money to make his purchases. He purchased one game controller and five games to play with his friends which cost him $295. Recently, José went back to the store to buy more games and controllers for his sister's birthday. This time, José spent $295.50 on three games and 2 controllers.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer this question: What are three different methods you could use to determine the individual costs of a game and a controller? Discuss, choose a method and solve.

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

GO ONLINE to answer these questions.

Lesson 9-1

Solving Systems of Linear Equations Using Matrices

What Will You Learn?

After completing this lesson, you should be able to:

- Write an Augmented Matrix
- Use Elementary Row Operations
- Use Gaussian Elimination and Gauss-Jordan Elimination

Learn: Write an Augmented Matrix

The data in Table 9-1 represent the body mass index (BMI) for people of selected heights and weights. The National Institutes of Health suggests that a BMI should ideally be between 18.5 and 24.9. A table of data values such as Table 9-1 can be represented as a rectangular array of elements called a **matrix.**

TABLE 9-1 Body Mass Index

Height	Weight of 140 lb	Weight of 160 lb	Weight of 180 lb	Weight of 200 lb
5′7″	21.9	25.1	28.2	31.3
5′10″	20.1	23.0	25.8	28.7
6′1″	18.5	21.1	23.7	26.4

(Source: National Institutes of Health, www.nih.gov)

$$\xrightarrow{\text{matrix}} \begin{bmatrix} 21.9 & 25.1 & 28.2 & 31.3 \\ 20.1 & 23.0 & 25.8 & 28.7 \\ 18.5 & 21.1 & 23.7 & 26.4 \end{bmatrix}$$

A matrix can be used to represent a system of linear equations written in standard form. To do so, we extract the coefficients of each term in the equation to form an **augmented matrix.** For example:

System of Equations

$$\begin{aligned} 3x + 2y \phantom{{}+3z} &= 5 \\ x - y + 3z &= 1 \\ 2x + y + z &= 4 \end{aligned}$$

Augmented Matrix

$$\left[\begin{array}{rrr|r} 3 & 2 & 0 & 5 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 1 & 4 \end{array}\right]$$

The vertical bar within the augmented matrix separates the coefficients of the variable terms in the equations from the constant terms.

Example 1: Writing and Interpreting an Augmented Matrix

a. Write the augmented matrix for the system.

$$\begin{aligned} 2x &= 5y + 5 \\ 3(x + y) &= 17 + y \end{aligned}$$

b. Write a system of linear equations represented by the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Solution:

a. Write each equation in standard form.

$$\left.\begin{array}{rcl} 2x=5y+5 & \longrightarrow & 2x-5y=5 \\ 3(x+y)=17+y & \longrightarrow & 3x+2y=17 \end{array}\right\} \xrightarrow{\text{augmented matrix}} \left[\begin{array}{cc|c} 2 & -5 & 5 \\ 3 & 2 & 17 \end{array}\right]$$

b. $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 4 \end{array}\right] \begin{array}{l} \longrightarrow x+0y+0z=6 \\ \longrightarrow 0x+y+0z=-10 \\ \longrightarrow 0x+0y+z=4 \end{array} \quad \text{or simply} \quad \begin{array}{l} x=6 \\ y=-10 \\ z=4 \end{array}$$

Apply the Skills

1. a. Write the augmented matrix.

$$7x=9+2y$$
$$2(x-y)=4$$

b. Write a system of linear equations represented by the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 3 \end{array}\right]$$

Learn: Use Elementary Row Operations

We can use an augmented matrix to solve a system of linear equations in much the same way as we use the addition method. When using the addition method, notice that the following operations produce an equivalent system of equations. This means that the two systems have the same solution set.

- Interchange two equations.

  Example: $$\begin{array}{r} -3x-5y=-13 \\ x+2y=5 \end{array} \xrightarrow{\text{Interchange equations.}} \begin{array}{r} x+2y=5 \\ -3x-5y=-13 \end{array}$$

- Multiply an equation by a nonzero constant.

  Example: $$\begin{array}{r} x+2y=5 \\ -3x-5y=-13 \end{array} \xrightarrow{\text{Multiply by 3.}} \begin{array}{r} 3x+6y=15 \\ -3x-5y=-13 \end{array}$$

- Add a nonzero multiple of one equation to another equation.

  Example: $$\begin{array}{r} x+2y=5 \\ -3x-5y=-13 \end{array} \xrightarrow{\text{Multiply by 3.}} \left.\begin{array}{r} 3x+6y=15 \\ -3x-5y=-13 \\ \hline y=2 \end{array}\right\} \text{Add the equations.}$$

These same operations performed on a matrix are called **elementary row operations.**

Elementary Row Operations

1. Interchange two rows.

Example: $$\left[\begin{array}{rr|r} -3 & -5 & -13 \\ 1 & 2 & 5 \end{array}\right] \xrightarrow[\text{Interchange rows 1 and 2.}]{R_1 \Leftrightarrow R_2} \left[\begin{array}{rr|r} 1 & 2 & 5 \\ -3 & -5 & -13 \end{array}\right]$$

2. Multiply a row by a nonzero constant.

Example: $$\left[\begin{array}{rr|r} 1 & 2 & 5 \\ -3 & -5 & -13 \end{array}\right] \xrightarrow[\text{Multiply row 1 by 3.}]{3R_1 \rightarrow R_1} \left[\begin{array}{rr|r} 3 & 6 & 15 \\ -3 & -5 & -13 \end{array}\right]$$

3. Add a nonzero multiple of one row to another row.

Example: $$\left[\begin{array}{rr|r} 1 & 2 & 5 \\ -3 & -5 & -13 \end{array}\right] \xrightarrow[\text{Add 3 times row 1 to row 2.}]{3R_1 + R_2 \rightarrow R_2} \left[\begin{array}{rr|r} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right]$$

$$\begin{array}{r} 3R_1 \\ +\,R_2 \\ \hline \rightarrow R_2 \end{array} \quad \begin{array}{rr|r} 3 & 6 & 15 \\ -3 & -5 & -13 \\ \hline 0 & 1 & 2 \end{array}$$

Check Your Work

When we add a constant multiple of one row to another row, we do not change the row that was multiplied by the constant.

Two matrices are said to be **row equivalent** if one matrix can be transformed into the other matrix through a series of elementary row operations.

Example 2: Performing Elementary Row Operations

Given $\left[\begin{array}{rrr|r} 4 & 3 & 0 & 5 \\ 1 & 4 & -1 & 9 \\ 2 & 0 & -3 & -8 \end{array}\right]$, perform the following elementary row operations.

a. $R_1 \Leftrightarrow R_2$

b. $\frac{1}{4}R_1 \rightarrow R_1$

c. $-2R_2 + R_3 \rightarrow R_3$

Solution:

a. $$\left[\begin{array}{rrr|r} 4 & 3 & 0 & 5 \\ 1 & 4 & -1 & 9 \\ 2 & 0 & -3 & -8 \end{array}\right] \xrightarrow[\text{Interchange rows 1 and 2.}]{R_1 \Leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & 4 & -1 & 9 \\ 4 & 3 & 0 & 5 \\ 2 & 0 & -3 & -8 \end{array}\right]$$

b. $$\left[\begin{array}{ccc|c} 4 & 3 & 0 & 5 \\ 1 & 4 & -1 & 9 \\ 2 & 0 & -3 & -8 \end{array}\right] \xrightarrow[\substack{\text{Multiply} \\ \text{row 1 by } \frac{1}{4}.}]{\frac{1}{4}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & \frac{3}{4} & 0 & \frac{5}{4} \\ 1 & 4 & -1 & 9 \\ 2 & 0 & -3 & -8 \end{array}\right]$$

Insights

The notation $\frac{1}{4}R_1 \to R_1$ means to multiply row 1 by $\frac{1}{4}$ and then *replace* row 1 by the result.

c. $$\left[\begin{array}{ccc|c} 4 & 3 & 0 & 5 \\ 1 & 4 & -1 & 9 \\ 2 & 0 & -3 & -8 \end{array}\right] \qquad \begin{array}{r} -2R_2 \\ +\quad R_3 \\ \hline \to R_3 \end{array} \qquad \begin{array}{rrr|r} -2 & -8 & 2 & -18 \\ 2 & 0 & -3 & -8 \\ \hline 0 & -8 & -1 & -26 \end{array}$$

This now replaces row 3.

$$\left[\begin{array}{ccc|c} 4 & 3 & 0 & 5 \\ 1 & 4 & -1 & 9 \\ 2 & 0 & -3 & -8 \end{array}\right] \xrightarrow[\substack{\text{Add } -2 \text{ times row 2 to row 3.} \\ \text{The result replaces row 3.}}]{-2R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 4 & 3 & 0 & 5 \\ 1 & 4 & -1 & 9 \\ 0 & -8 & -1 & -26 \end{array}\right]$$

Apply the Skills

2. Use the matrix from Example 2 to perform the given row operations.

a. $R_2 \Leftrightarrow R_3$ **b.** $-\frac{1}{2}R_3$ **c.** $-4R_2 + R_1 \to R_1$

Learn: Use Gaussian Elimination and Gauss-Jordan Elimination

When an elementary row operation is performed on an augmented matrix, a new row-equivalent augmented matrix is obtained that represents an equivalent system of equations. If we perform repeated row operations, we can form an augmented matrix that represents a system of equations that is easier to solve than the original system. In particular, it is easy to solve a system whose augmented matrix is in *row-echelon form* or *reduced row-echelon form*.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix is in **row-echelon form** if it satisfies the following conditions.

1. Any rows consisting entirely of zeros are at the bottom of the matrix.
2. For all other rows, the first nonzero entry is 1. This is called the leading 1.
3. The leading 1 in each nonzero row is to the right of the leading 1 in the row immediately above.

Note: A matrix is in **reduced row-echelon form** if it is in row-echelon form with the added condition that each row with a leading entry of 1 has zeros above the leading 1.

The following matrices illustrate row-echelon form and reduced row-echelon form.

Row-Echelon Form

$$\left[\begin{array}{cc|c} 1 & 5 & 3 \\ 0 & 1 & 6 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -4 & -\frac{1}{2} & 6 \\ 0 & 1 & 9 & -\frac{1}{3} \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 4.1 & 1.2 & 3.1 \\ 0 & 1 & 0.6 & 4.7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced Row-Echelon Form

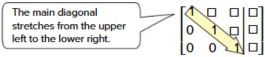
$$\left[\begin{array}{cc|c} 1 & 0 & -27 \\ 0 & 1 & 6 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 150 \\ 0 & 1 & 0 & -85 \\ 0 & 0 & 1 & 12 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 25 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

After writing an augmented matrix in row-echelon form, the corresponding system of linear equations can be solved using back substitution. This method is called **Gaussian elimination.**

Solving a System of Linear Equations Using Gaussian Elimination

1. Write the augmented matrix for the system.
2. Use elementary row operations to write the augmented matrix in row-echelon form.
3. Use back substitution to solve the resulting system of equations.

When writing an augmented matrix in row-echelon form, the goal is to make the elements along the main diagonal 1 and the entries below the main diagonal 0. The **main diagonal** refers to the elements on the diagonal from the upper left to the lower right all to the left of the vertical bar.


The main diagonal stretches from the upper left to the lower right.

To write an augmented matrix in row-echelon form, work one column at a time from left to right. This is demonstrated in Example 3.

Example 3: Solving a System Using Gaussian Elimination

Solve the system by using Gaussian elimination.

$$\begin{aligned} 3x + 7y - 15z &= -12 \\ x + 2y - 4z &= -3 \\ -4x - 6y + 15z &= 16 \end{aligned}$$

Solution:

Need a 1 here

$$\begin{aligned} 3x + 7y - 15z &= -12 \\ x + 2y - 4z &= -3 \\ -4x - 6y + 15z &= 16 \end{aligned} \qquad \left[\begin{array}{ccc|c} 3 & 7 & -15 & -12 \\ 1 & 2 & -4 & -3 \\ -4 & -6 & 15 & 16 \end{array}\right]$$

Set up the augmented matrix. Use elementary row operations to write the augmented matrix in row-echelon form.

$$R_1 \Leftrightarrow R_2 \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & -3 \\ 3 & 7 & -15 & -12 \\ -4 & -6 & 15 & 16 \end{array}\right]$$

Begin working with column 1.

To obtain a leading 1 in the first row, interchange rows 1 and 2.

Insights

To obtain a leading entry of 1 in the first row, we have the option of multiplying the first row by $\frac{1}{3}$. However, this would introduce fractions in the first row. Interchanging rows 1 and 2 is "cleaner."

Now we want zeros below the leading 1 in the first row.

- Multiply row 1 by –3 and add the result to row 2.
- Multiply row 1 by 4 and add the result to row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & -3 \\ 3 & 7 & -15 & -12 \\ -4 & -6 & 15 & 16 \end{array}\right]$$

Need 0's here

$$\begin{array}{r|rrr|r} -3R_1 & -3 & -6 & 12 & 9 \\ +R_2 & 3 & 7 & -15 & -12 \\ \hline \to R_2 & 0 & 1 & -3 & -3 \end{array} \qquad \begin{array}{r|rrr|r} 4R_1 & 4 & 8 & -16 & -12 \\ +R_3 & -4 & -6 & 15 & 16 \\ \hline \to R_3 & 0 & 2 & -1 & 4 \end{array}$$

$$\begin{array}{r} -3R_1 + R_2 \to R_2 \longrightarrow \\ 4R_1 + R_3 \to R_3 \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -4 & -3 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right]$$

Next, work with column 2.

We already have a leading 1 in row 2.

We need 0 below the leading 1 in row 2.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & -3 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right]$$

Need 0 here

- Multiply row 2 by –2 and add the result to row 3.

$$\begin{array}{r|rrr|r} -2R_2 & 0 & -2 & 6 & 6 \\ +R_3 & 0 & 2 & -1 & 4 \\ \hline \to R_3 & 0 & 0 & 5 & 10 \end{array}$$

$$-2R_2 + R_3 \to R_3 \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & -3 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

$$\tfrac{1}{5}R_3 \to R_3 \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & -3 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Now work with column 3. We need a leading 1 in row 3. Multiply row 3 by $\frac{1}{5}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & -3 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{r} \longrightarrow x + 2y - 4z = -3 \\ \longrightarrow y - 3z = -3 \\ \longrightarrow z = 2 \end{array}$$

We now have row-echelon form.

The corresponding system of equations has z isolated.

Use back substitution to find x and y.

$$\begin{aligned} y - 3z &= -3 \\ y - 3(2) &= -3 \\ y &= 3 \end{aligned}$$

Substitute $z = 2$ in the second equation.

$$\begin{aligned} x + 2y - 4z &= -3 \\ x + 2(3) - 4(2) &= -3 \\ x + 6 - 8 &= -3 \\ x &= -1 \end{aligned}$$

Substitute $z = 2$ and $y = 3$ into the first equation.

The solution set is $\{(-1, 3, 2)\}$. The solution $(-1, 3, 2)$ checks in each original equation.

Apply the Skills

3. Solve the system by using Gaussian elimination.

$$\begin{aligned} 2x + 7y + z &= 14 \\ x + 3y - z &= 2 \\ x + 7y + 12z &= 45 \end{aligned}$$

Example 3 illustrates that a system of equations represented by an augmented matrix in row-echelon form is easily solved by using back substitution. If we write an augmented matrix in *reduced* row-echelon form, we can solve the corresponding system of equations by inspection. This is called the Gauss-Jordan elimination method [named after Carl Friedrich Gauss (1777–1855) and German scientist Wilhelm Jordan (1842–1899)].

When writing an augmented matrix in *reduced* row-echelon form, the goal is to make the elements along the main diagonal 1 and the entries above and below the main diagonal 0. This is demonstrated in Examples 4 and 5.

$$\left[\begin{array}{cc|c} 1 & 0 & \square \\ 0 & 1 & \square \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & \square \\ 0 & 1 & 0 & \square \\ 0 & 0 & 1 & \square \end{array}\right]$$

Order of Row Operations to Obtain Reduced Row-Echelon Form

To write an augmented matrix with n rows in reduced row-echelon form, transform the entries in the matrix in the following order.

Column 1: Obtain a leading element of 1 in row 1. Then obtain 0's below this element.

Column 2: Obtain a leading element of 1 in row 2. Then obtain 0's above and below this element.

Column 3: Obtain a leading element of 1 in row 3. Then obtain 0's above and below this element.

⋮

Column n: Obtain a leading element of 1 in row n. Then obtain 0's above this element.

Example 4: Solving a System Using Gauss-Jordan Elimination

Solve the system by using Gauss-Jordan elimination.

$$\begin{aligned} x &= 17 - 2y \\ 3(x + 2y) &= 47 - y \end{aligned}$$

Solution:

$$\begin{aligned} x = 17 - 2y &\longrightarrow x + 2y = 17 \\ 3(x + 2y) = 47 - y &\longrightarrow 3x + 7y = 47 \end{aligned}$$

To use Gaussian elimination or Gauss-Jordan elimination, always begin by writing each equation in standard form.

$$\begin{aligned} x + 2y &= 17 \\ 3x + 7y &= 47 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & 2 & 17 \\ \boxed{3} & 7 & 47 \end{array}\right]$$

Need 0 here

Set up the augmented matrix.
The leading element in row 1 is already 1.
Now we need a zero below the leading 1 in row 1.

Multiply row 1 by −3 and add the result to row 2.

$$\begin{array}{r|cc|c} -3R_1 & -3 & -6 & -51 \\ +R_2 & 3 & 7 & 47 \\ \hline \to R_2 & 0 & 1 & -4 \end{array} \qquad -3R_1 + R_2 \to R_2 \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 17 \\ 0 & 1 & -4 \end{array}\right]$$

Need 0 here

$$\left[\begin{array}{cc|c} 1 & \boxed{2} & 17 \\ 0 & 1 & -4 \end{array}\right]$$

Now work with column 2. The leading element in row 2 is already 1.
Now we need a zero above the leading 1 in row 2.

Multiply row 2 by −2 and add the result to row 1.

$$\begin{array}{r|cc|c} R_1 & 1 & 2 & 17 \\ + -2R_2 & 0 & -2 & 8 \\ \hline \to R_1 & 1 & 0 & 25 \end{array} \qquad -2R_2 + R_1 \to R_1 \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 25 \\ 0 & 1 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 25 \\ 0 & 1 & -4 \end{array}\right] \begin{array}{l} \longrightarrow x = 25 \\ \longrightarrow y = -4 \end{array}$$

From the corresponding system of equations, we can determine the solution (25, −4) by inspection.

The solution set is {(25, −4)}.

The solution checks in each original equation.

Apply the Skills

4. Solve the system by using Gauss-Jordan elimination.

$$\begin{aligned} x - 2y &= -1 \\ 4x - 7y &= 1 \end{aligned}$$

In Example 5, we use the Gauss-Jordan elimination method to solve a three-variable system of linear equations.

Example 5: Solving a System Using Gauss-Jordan Elimination

Solve the system by using Gauss-Jordan elimination.

$$\begin{aligned} 2x - 5y - 21z &= 39 \\ x - 3y - 10z &= 22 \\ x + 3y + 2z &= -8 \end{aligned}$$

Solution:

$$\begin{aligned} 2x - 5y - 21z &= 39 \\ x - 3y - 10z &= 22 \\ x + 3y + 2z &= -8 \end{aligned} \qquad \left[\begin{array}{rrr|r} 2 & -5 & -21 & 39 \\ 1 & -3 & -10 & 22 \\ 1 & 3 & 2 & -8 \end{array}\right]$$

Set up the augmented matrix.

$$R_1 \Leftrightarrow R_2 \longrightarrow \left[\begin{array}{rrr|r} 1 & -3 & -10 & 22 \\ 2 & -5 & -21 & 39 \\ 1 & 3 & 2 & -8 \end{array}\right]$$

Use elementary row operations to write the augmented matrix in reduced row-echelon form.

To obtain a leading 1 in the first row, interchange rows 1 and 2.

$$\begin{aligned} -2R_1 + R_2 &\to R_2 \longrightarrow \\ -R_1 + R_3 &\to R_3 \longrightarrow \end{aligned} \left[\begin{array}{rrr|r} 1 & -3 & -10 & 22 \\ 0 & 1 & -1 & -5 \\ 0 & 6 & 12 & -30 \end{array}\right]$$

Multiply row 1 by –2 and add the result to row 2. Multiply row 1 by –1 and add the result to row 3.

This results in zeros below the leading 1 in the first row.

$$\begin{aligned} 3R_2 + R_1 &\to R_1 \longrightarrow \\ -6R_2 + R_3 &\to R_3 \longrightarrow \end{aligned} \left[\begin{array}{rrr|r} 1 & 0 & -13 & 7 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 18 & 0 \end{array}\right]$$

Row 2 already has a leading 1. Multiply row 2 by 3 and add the result to row 1. Multiply row 2 by –6 and add the result to row 3.

$$\tfrac{1}{18}R_3 \to R_3 \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & -13 & 7 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Multiply row 3 by $\frac{1}{18}$ to obtain a leading 1 in row 3.

$$\begin{aligned} 13R_3 + R_1 &\to R_1 \longrightarrow \\ R_3 + R_2 &\to R_2 \longrightarrow \end{aligned} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Multiply row 3 by 13 and add the result to row 1.

Multiply row 3 by 1 and add the result to row 2.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \longrightarrow x = 7 \\ \longrightarrow y = -5 \\ \longrightarrow z = 0 \end{array}$$

The augmented matrix is in reduced row-echelon form. From the corresponding system of equations, we can determine the solution (7, –5, 0) by inspection.

The solution set is {(7, –5, 0)}.

The solution checks in each original equation.

Apply the Skills

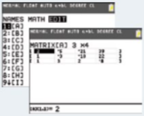
5. Solve the system by using Gauss-Jordan elimination.

$$\begin{aligned} 2x + 7y + 11z &= 11 \\ x + 2y + 8z &= 14 \\ x + 3y + 6z &= 8 \end{aligned}$$

TECHNOLOGY CONNECTIONS

Finding the Row-Echelon Form and Reduced Row-Echelon Form of a Matrix

Many graphing utilities have the capability to enter and manipulate matrices. To enter the augmented matrix from Example 5, select the MATRIX menu and then the EDIT menu. Select a name for the matrix such as A. Then enter the number of rows and columns of the matrix (in this case 3 × 4) followed by the individual elements.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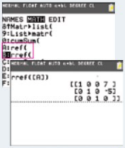
MATRIX[A] 3 ×4
2

To access the commands for row-echelon form and reduced row-echelon form, select the MATH menu from within the MATRIX menu. Select either *ref* for row-echelon form or *rref* for reduced row-echelon form. In the home screen, insert the matrix name within the parentheses for the *ref* and *rref* functions.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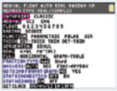
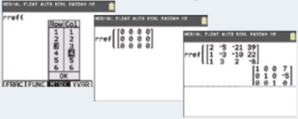
0:cumSum(
A:ref(
B:rref(
C:
D:
E:
F:

```
rref([A])
                    [[1 0 0 7 ]
                     [0 1 0 -5]
                     [0 0 1 0 ]]
```

Alternatively, we can find the row-echelon form or reduced row-echelon form of a matrix if the calculator is in “MATHPRINT” mode. Select MODE, and using the arrow keys to navigate through the menu, highlight “MATHPRINT.”


NORMAL FLOAT AUTO REAL RADIAN MP
MATHPRINT CLASSIC
NORMAL SCI ENG
FLOAT 0123456789
RADIAN DEGREE
FUNCTION PARAMETRIC POLAR SEQ
THICK DOT-THICK THIN DOT-THIN

In the home screen, select the MATRIX menu, followed by MATH. Then select either *ref* or *rref*. Next, press ALPHA the key followed by F3. Enter the dimensions of the matrix (in this case, 3 × 4). Then enter the elements of the matrix using the arrow keys to navigate through the matrix. Hit ENTER to complete the calculation.


NORMAL FLOAT AUTO REAL RADIAN MP
rref(
Row Col
1 1
2 2
3 3
4 4
5 5
6 6
OK
FRAC FUNC MTRX YVAR

$$\text{rref}\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$\text{rref}\left(\begin{bmatrix} 2 & -5 & -21 & 39 \\ 1 & -3 & -10 & 22 \\ 1 & 3 & 2 & -6 \end{bmatrix}\right)$$

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Practice 9-1

Practice Exercises

Prerequisite Review

R.1. Solve the equation and check the solution.

$$3n + 25 = 73$$

R.2. How much interest will Roxanne have to pay if she borrows $1000 for 2 yr at a simple interest rate of 5%?

For Exercises R.3–R.4, set up the form for the partial fraction decomposition. Do not solve for *A*, *B*, *C*, and so on.

R.3 $\dfrac{-12t - 11}{4t^2 - 16t + 15}$ **R.4** $\dfrac{-5x^2 - 10x - 13}{7x^3 + 7x}$

Concept Connections

1. A rectangular array of elements is called a ________________.

2. Identify the elements on the main diagonal.

$$\left[\begin{array}{ccc|c} -4 & -1 & 1 & 8 \\ 2 & 0 & 5 & 11 \\ 0 & 1 & -7 & -6 \end{array}\right]$$

3. Explain the meaning of the notation $R_2 \Leftrightarrow R_3$.

4. Explain the meaning of the notation $-R_2 \rightarrow R_2$.

5. Explain the meaning of the notation $3R_1 \rightarrow R_1$.

6. Explain the meaning of $R_1 \Leftrightarrow R_3$.

7. Explain the meaning of the notation $3R_1 + R_2 \rightarrow R_2$.

8. Explain the meaning of the notation $4R_2 + R_3 \rightarrow R_3$.

Learn: Write an Augmented Matrix

For Exercises 9–14, write the augmented matrix for the given system. (See Example 1)

9. $$\begin{aligned} -3x + 2y - z &= 4 \\ 8x \quad\quad + 4z &= 12 \\ 2y - 5z &= 1 \end{aligned}$$

10. $$\begin{aligned} -4x - y + z &= 8 \\ 2x \quad\quad + 5z &= 11 \\ y - 7z &= -6 \end{aligned}$$

11. $$\begin{aligned} 4(x - 2y) &= 6y + 2 \\ 3x &= 5y + 7 \end{aligned}$$

12. $$\begin{aligned} 2(y - x) &= 4 - 8x \\ 5y &= 6 - x \end{aligned}$$

13. $$\begin{aligned} x &= 2 \\ y &= \tfrac{6}{7} \\ z &= 12 \end{aligned}$$

14. $$\begin{aligned} x &= 4 \\ y &= 2 \\ z &= -\tfrac{1}{2} \end{aligned}$$

For Exercises 15–20, write a system of linear equations represented by the augmented matrix. (See Example 1 on p. 948)

15. $\left[\begin{array}{cc|c} -4 & 6 & 11 \\ -3 & 9 & 1 \end{array}\right]$

16. $\left[\begin{array}{cc|c} -3 & 7 & -2 \\ -1 & 1 & 4 \end{array}\right]$

17. $\left[\begin{array}{ccc|c} 1 & 4 & 3 & 8 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & 1 & 6 \end{array}\right]$

18. $\left[\begin{array}{ccc|c} 1 & 1 & -2 & -4 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right]$

19. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right]$

20. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$

Learn: Use Elementary Row Operations

For Exercises 21–26, perform the elementary row operations on $\left[\begin{array}{cc|c} 1 & 4 & 2 \\ -3 & 6 & 6 \end{array}\right]$. (See Example 2 on p. 950)

21. $R_1 \Leftrightarrow R_2$

22. $-\frac{1}{3}R_2 \rightarrow R_2$

23. $3R_1 \rightarrow R_1$

24. $-3R_1 \rightarrow R_1$

25. $-\frac{1}{3}R_2 + R_1 \rightarrow R_1$

26. $3R_1 + R_2 \rightarrow R_2$

For Exercises 27–32, perform the elementary row operations on $\left[\begin{array}{ccc|c} 1 & 5 & 6 & 2 \\ 2 & 1 & 5 & 1 \\ 4 & -2 & -3 & 10 \end{array}\right]$. (See Example 2 on p. 950)

27. $R_2 \Leftrightarrow R_3$

28. $R_1 \Leftrightarrow R_2$

29. $\frac{1}{4}R_3 \rightarrow R_3$

30. $\frac{1}{2}R_2 \rightarrow R_2$

31. $-2R_1 + R_2 \rightarrow R_2$

32. $-4R_1 + R_3 \rightarrow R_3$

Learn: Use Gaussian Elimination and Gauss-Jordan Elimination

For Exercises 33–36, determine if the matrix is in row-echelon form. If not, explain why.

33. $\left[\begin{array}{cc|c} 1 & 5 & 4 \\ 0 & 2 & 6 \end{array}\right]$

34. $\left[\begin{array}{ccc|c} 1 & 6 & 4 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 3 & 1 & 3 \end{array}\right]$

35. $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 6 \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$

36. $\left[\begin{array}{ccc|c} 1 & 4 & 2 & -6 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

For Exercises 37–40, determine if the matrix is in reduced row-echelon form. If not, explain why.

37. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 4 \\ 1 & 0 & 0 & 5 \end{array}\right]$

38. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$

39. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$

40. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 5 \\ 0 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

For Exercises 41–60, solve the system by using Gaussian elimination or Gauss-Jordan elimination. (See Examples 3–5 on pp. 952–956)

41. $\begin{aligned} 2x + 3y &= -13 \\ x + 4y &= -14 \end{aligned}$

42. $\begin{aligned} -3x + 11y &= 58 \\ x - 3y &= -16 \end{aligned}$

43. $\begin{aligned} 2x - 7y &= -41 \\ 3x - 9y &= -51 \end{aligned}$

44. $\begin{aligned} -2x + 15y &= 6 \\ 3x - 12y &= -9 \end{aligned}$

45. $\begin{aligned} -3(x - 6y) &= -167 - y \\ 14y &= 2x - 122 \end{aligned}$

46. $\begin{aligned} 2(x - y) &= 4x + y - 40 \\ 9y &= 105 - 3x \end{aligned}$

47. $\begin{aligned} 3x + 7y + 22z &= 83 \\ x + 3y + 10z &= 37 \\ -2x - 5y - 18z &= -66 \end{aligned}$

48. $\begin{aligned} 3x + 5y + 9z &= 3 \\ x + 3y + 7z &= 5 \\ -2x - 8y - 15z &= -11 \end{aligned}$

49. $\begin{aligned} -2x + 4y + z &= 7 \\ 4x - 13y + 10z &= 17 \\ 3x - 9y + 6z &= 9 \end{aligned}$

50. $\begin{aligned} 2x - 8y + 54z &= -4 \\ x - 2y + 14z &= -1 \\ x - 3y + 19z &= -3 \end{aligned}$

51. $\begin{aligned} -3x + 4y - 15z &= -44 \\ x - y + 4z &= 13 \\ x - 3y + 14z &= 27 \end{aligned}$

52. $\begin{aligned} -2x + 5y - 4z &= -4 \\ x - 2y + z &= 3 \\ x - 5y + 9z &= -5 \end{aligned}$

53. $\begin{aligned} 2x + 8z &= 7y - 46 \\ x &= 3y - 3z - 18 \\ 6z &= 5y - x - 34 \end{aligned}$

54. $\begin{aligned} 2x &= 7 - y - 3z \\ x + y &= z + 5 \\ 16z &= 2y - x - 4 \end{aligned}$

55. $\begin{aligned} 11y + 65 &= 3x + 13z \\ x + 3z &= 3y + 15 \\ -2x + 4y - 7z &= -25 \end{aligned}$

56. $$\begin{aligned} 2(x-6z) &= 3y+x+17 \\ 2x-19 &= 3y+18z \\ -3x+7y+36z &= -41 \end{aligned}$$

57. $$\begin{aligned} w+3x \qquad -3z &= -5 \\ x \qquad -2z &= -6 \\ -2w-4x+y+2z &= 1 \\ x+y \qquad &= 5 \end{aligned}$$

58. $$\begin{aligned} w-2x+5y \qquad &= 20 \\ -x+2y \qquad &= 9 \\ x-\ y \qquad &= -5 \\ 2w-\ x+7y+z &= 25 \end{aligned}$$

59. $$\begin{aligned} x_1+x_2 \qquad +\ 5x_4 &= -4 \\ x_2 \qquad +\ 2x_4 &= 3 \\ -2x_2+x_3-\ 3x_4 &= -5 \\ 3x_1+3x_2 \qquad +\ 17x_4 &= -10 \end{aligned}$$

60. $$\begin{aligned} x_1 \qquad +\ x_3-\ 5x_4 &= 1 \\ -2x_1+x_2-2x_3+16x_4 &= -3 \\ x_1 \qquad +2x_3-10x_4 &= 5 \\ x_1 \qquad -\ x_3+\ 7x_4 &= -7 \end{aligned}$$

Mixed Exercises

For Exercises 61–64, set up a system of linear equations to represent the scenario. Solve the system by using Gaussian elimination or Gauss-Jordan elimination.

61. Andre borrowed \$20,000 to buy a truck for his business. He borrowed from his parents who charge him 2% simple interest. He borrowed from a credit union that charges 4% simple interest, and he borrowed from a bank that charges 5% simple interest. He borrowed five times as much from his parents as from the bank, and the amount of interest he paid at the end of 1 yr was \$620. How much did he borrow from each source?

62. Sylvia invested a total of \$40,000. She invested part of the money in a certificate of deposit (CD) that earns 2% simple interest per year. She invested in a stock that returns the equivalent of 8% simple interest, and she invested in a bond fund that returns 5%. She invested twice as much in the stock as she did in the CD, and earned a total of \$2300 at the end of 1 yr. How much principal did she put in each investment?

63. Jemila stayed in three different cities (Washington, D.C., Atlanta, Georgia, and Dallas, Texas) for a total of 14 nights. Twice as many nights were spent in Dallas as in Washington. The total cost for 14 nights (excluding tax) was \$2200. Determine the number of nights spent in each city.

City	Cost per Night
Washington	\$200
Atlanta	\$100
Dallas	\$150

64. Three pumps (A, B, and C) work to drain water from a retention pond. Working together, the pumps can pump 1500 gal/hr of water. Pump C works at a rate of 100 gal/hr faster than pump B. In 3 hr, pump C can pump as much water as pumps A and B working together in 2 hr. Find the rate at which each pump works.

For Exercises 65–66, find the partial fraction decomposition for the given rational expression. Use the technique of Gaussian elimination to find *A*, *B*, and *C*.

65. $\dfrac{5x^2-6x-13}{(x+3)(x-2)^2} = \dfrac{A}{x+3} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$

66. $\dfrac{2x^2+17x+3}{(x+5)(x+1)^2} = \dfrac{A}{x+5} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

Write About It

67. Explain why interchanging two rows of an augmented matrix results in an augmented matrix that represents an equivalent system of equations.

68. Explain the difference between a matrix in row-echelon form and reduced row-echelon form.

69. Explain why multiplying a row of an augmented matrix by a nonzero constant results in an augmented matrix that represents an equivalent system of equations.

70. Consider the matrix $\left[\begin{array}{cc|c} 5 & -9 & -57 \\ 1 & -2 & -12 \end{array}\right]$. Identify two row operations that could be used to obtain a leading entry of 1 in the first row. Also indicate which operation would be less cumbersome as a first step toward writing the matrix in reduced row-echelon form.

Technology Connections

For Exercises 71–72, use a calculator to approximate the reduced row-echelon form of the augmented matrix representing the given system. Give the solution set where *x*, *y*, and *z* are rounded to 2 decimal places.

71.
$$\begin{aligned} 0.52x - 3.71y - 4.68z &= 9.18 \\ 0.02x + 0.06y + 0.11z &= 0.56 \\ 0.972x + 0.816y + 0.417z &= 0.184 \end{aligned}$$

72.
$$\begin{aligned} -3.61x + 8.17y - 5.62z &= 30.2 \\ 8.04x - 3.16y + 9.18z &= 28.4 \\ -0.16x + 0.09y + 0.55z &= 4.6 \end{aligned}$$

73. A small grocer finds that the monthly sales y (in \$) can be approximated as a function of the amount spent advertising on the radio x_1 (in \$) and the amount spent advertising in the newspaper x_2 (in \$) according to $y = ax_1 + bx_2 + c$.

The table gives the amounts spent in advertising and the corresponding monthly sales for 3 months.

Radio Advertising, x_1	Newspaper Advertising, x_2	Monthly sales, y
\$2400	\$800	\$36,000
\$2000	\$500	\$30,000
\$3000	\$1000	\$44,000

a. Use the data to write a system of linear equations to solve for a, b, and c.

b. Use a graphing utility to find the reduced row-echelon form of the augmented matrix.

c. Write the model $y = ax_1 + bx_2 + c$.

d. Predict the monthly sales if the grocer spends \$2500 advertising on the radio and \$500 advertising in the newspaper for a given month.

74. The purchase price of a home y (in \$1000) can be approximated based on the annual income of the buyer x_1 (in \$1000) and on the square footage of the home x_2 (in 100 ft²) according to $y = ax_1 + bx_2 + c$.

The table gives the incomes of three buyers, the square footages of the home purchased, and the corresponding purchase prices of the home.

Income (\$1000) x_1	Square Footage (100 ft²) x_2	Price (\$1000) y
80	21	180
150	28	250
75	18	160

a. Use the data to write a system of linear equations to solve for a, b, and c.

b. Use a graphing utility to find the reduced row-echelon form of the augmented matrix.

c. Write the model $y = ax_1 + bx_2 + c$.

d. Predict the purchase price for a buyer who makes \$100,000 per year and wants a 2500 ft² home.

For Exercises 75–76, the given function values satisfy a function defined by $f(x) = ax^2 + bx + c$.

a. Set up a system of equations to solve for a, b, and c.

b. Use a graphing utility to find the reduced row-echelon form of the augmented matrix.

c. Write a function of the form $f(x) = ax^2 + bx + c$ that fits the data.

75.
$$\begin{aligned} f(-3) &= -7.28 \\ f(-1) &= 3.68 \\ f(10) &= 18.2 \end{aligned}$$

76.
$$\begin{aligned} f(3) &= 6.95 \\ f(-2) &= 20.2 \\ f(12) &= 39.8 \end{aligned}$$

Go online for more practice problems.

Lesson 9-2

Inconsistent Systems and Dependent Equations

What Will You Learn?

After completing this lesson, you should be able to:

- Identify Inconsistent Systems
- Solve Systems with Dependent Equations
- Solve Applications of Systems of Equations

Learn: Identify Inconsistent Systems

When we studied systems of linear equations in two and three variables in Chapter 8, we learned that a system may have no solution. Such a system is said to be **inconsistent**, and we recognize an inconsistent system if the system reduces to a contradiction.

Example 1: Identifying an Inconsistent System

Solve the system.

$$\begin{aligned} x - 3y - 17z &= -59 \\ x - 2y - 12z &= -41 \\ -2y - 10z &= 20 \end{aligned}$$

Solution:

$$\begin{aligned} x - 3y - 17z &= -59 \\ x - 2y - 12z &= -41 \\ -2y - 10z &= 20 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -3 & -17 & -59 \\ 1 & -2 & -12 & -41 \\ 0 & -2 & -10 & 20 \end{array}\right]$$

Write the augmented matrix. The leading entry in row 1 is already 1.

$$-R_1 + R_2 \to R_2 \longrightarrow \left[\begin{array}{ccc|c} 1 & -3 & -17 & -59 \\ 0 & 1 & 5 & 18 \\ 0 & -2 & -10 & 20 \end{array}\right]$$

Multiply row 1 by -1 and add the result to row 2.

$$\begin{array}{l} 3R_2 + R_1 \to R_1 \longrightarrow \\ 2R_2 + R_3 \to R_3 \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 5 & 18 \\ 0 & 0 & 0 & 56 \end{array}\right]$$

Multiply row 2 by 3 and add the result to row 1.
Multiply row 2 by 2 and add the result to row 3.

The last row of the matrix cannot be written with a leading 1 along the main diagonal. The last row represents a contradiction, $0 = 56$.

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 5 & 18 \\ 0 & 0 & 0 & 56 \end{array}\right] \xrightarrow[\text{system}]{\text{equivalent}} \begin{cases} x - 2z = -5 \\ y + 5z = 18 \\ \quad 0 = 56 \end{cases}$$

The contradiction indicates that the system is inconsistent and that there is no solution. The solution set is { }.

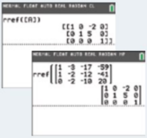
Apply the Skills

1. Solve the system.
$$\begin{aligned} 5x - 9y - 3z &= 3 \\ x - 2y - 7z &= 0 \\ -2x + y + 8z &= -12 \end{aligned}$$

TECHNOLOGY CONNECTIONS

Recognizing an Inconsistent System on a Calculator

We can verify that the system of equations in Example 1 has no solution by using a calculator to find the reduced row-echelon form of the augmented matrix. Notice that the calculator also displays a contradiction in the third row, $0 = 1$.


NORMAL FLOAT AUTO REAL RADIAN CL
NORMAL FLOAT AUTO REAL RADIAN MP

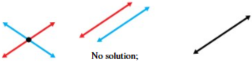
```
rref([A])
                 [[1 0 -2 0]
                  [0 1 5  0]
                  [0 0 0  1]]
```

```
rref([1 -3 -17 -59]
     [1 -2 -12 -41]
     [0 -2 -10  20])
                    [1 0 -2 0]
                    [0 1 5  0]
                    [0 0 0  1]
```

In Example 1, once we reached the contradiction $0 = 56$, we stopped manipulating the augmented matrix. However, we have the option of simplifying the augmented matrix to reduced row-echelon form. To match the result given in the calculator, we would multiply row 3 by $\frac{1}{56}$. Then add -18 times row 3 to row 2, and add 5 times row 3 to row 1.

Learn: Solve Systems with Dependent Equations

Recall that a linear equation in two variables defines a line in a plane. A solution to a system of two linear equations in two variables is a point of intersection of the lines. If the two lines are parallel, the system is inconsistent and has no solution. If the equations in the system represent the same line, we say that the equations are **dependent** and the solution set is the set of all points on the line.



One solution

No solution; Inconsistent system

Infinitely many solutions; Dependent equations

Example 2: Solving a System of Dependent Equations with Two Variables

Solve the system.

$$\begin{aligned} 0.25x - 0.75y &= 1 \\ 3y &= x - 4 \end{aligned}$$

Solution:

$$\begin{aligned} 0.25x - 0.75y = 1 &\xrightarrow{\text{Multiply by 100.}} 25x - 75y = 100 \\ 3y = x - 4 &\xrightarrow[\text{Standard form}]{} -x + 3y = -4 \end{aligned}$$

Write the equations in standard form. One option is to clear decimals in the first equation.

$$\left[\begin{array}{rr|r} 25 & -75 & 100 \\ -1 & 3 & -4 \end{array}\right] \xrightarrow{\frac{1}{25}R_1 \to R_1} \left[\begin{array}{rr|r} 1 & -3 & 4 \\ -1 & 3 & -4 \end{array}\right]$$

To obtain a 1 in the first row, first column, multiply row 1 by $\frac{1}{25}$.

$$\xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{rr|r} 1 & -3 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

To obtain a 0 in the second row, first column, add row 1 to row 2.

The second row of the augmented matrix represents the equation $0 = 0$, which is true regardless of the values of x and y. This means that the original system reduces to the single equation $x - 3y = 4$. That is, the two original equations each represent the same line, and all points on the line are solutions to the system. Solving the equation $x - 3y = 4$ for x or y illustrates the dependency between x and y.

Solving the equation $x - 3y = 4$ for x gives: $x = 3y + 4$ — x depends on the choice of y.

Solving the equation $x - 3y = 4$ for y gives: $y = \frac{x - 4}{3}$ — y depends on the choice of x.

The solution set can be written as

$$\{(3y + 4, y) \mid y \text{ is any real number}\} \text{ or } \left\{\left(x, \frac{x - 4}{3}\right) \middle| \, x \text{ is any real number}\right\}.$$

Good Practices

In Example 2, we wrote the solution set in two ways: one with an arbitrary value of y and one with an arbitrary value of x. However, when using reduced row-echelon form to find the solution set to a system of dependent equations, we generally let the *last* variable in the ordered pair (ordered triple, etc.) be arbitrary. In Example 2, this is $\{(3y + 4, y) \mid y \text{ is any real number}\}$.

Apply the Skills

2. Solve the system.

$$\begin{aligned} 0.3x - 0.1y &= -2 \\ y - 20 &= 3x \end{aligned}$$

A linear equation in three variables represents a plane in space. A solution to a system of equations in three variables is a common point of intersection among all the planes in the system. From Example 1, we have a system with no solution. This means that the planes do not all intersect (Figure 9-1).

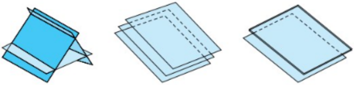


Figure 9-1

A system of linear equations may have infinitely many solutions. In such a case, the equations are dependent. For a system of three equations in three variables, this means that the planes intersect in a common line in space (Figure 9-2 and Figure 9-3), or the three planes all coincide (Figure 9-4).

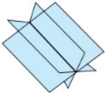


Figure 9-2

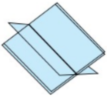


Figure 9-3

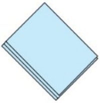


Figure 9-4

Example 3: Solving a System of Dependent Equations with Three Variables

Solve the system.

$$\begin{aligned} x-4y+7z&=14\\ -2x+9y-16z&=-31\\ x-7y+13z&=23 \end{aligned}$$

Solution:

$$\begin{aligned} x-4y+7z&=14\\ -2x+9y-16z&=-31\\ x-7y+13z&=23 \end{aligned} \qquad \left[\begin{array}{rrr|r} 1 & -4 & 7 & 14\\ -2 & 9 & -16 & -31\\ 1 & -7 & 13 & 23 \end{array}\right]$$

Set up the augmented matrix.

$$\xrightarrow[-R_1+R_3\to R_3]{2R_1+R_2\to R_2} \left[\begin{array}{rrr|r} 1 & -4 & 7 & 14\\ 0 & 1 & -2 & -3\\ 0 & -3 & 6 & 9 \end{array}\right] \xrightarrow[3R_2+R_3\to R_3]{4R_2+R_1\to R_1} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last row of the matrix cannot be written with a leading 1 along the main diagonal. The last row represents an identity, $0=0$.

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\text{system}]{\text{equivalent}} \begin{cases} x-z=2\\ y-2z=-3\\ 0=0 \end{cases}$$

The system of three equations and three variables reduces to a system of two equations and three variables. The third equation, $0=0$, is true regardless of the values of x, y, and z. The top two equations each represent a plane in space. Furthermore, two nonparallel planes intersect in a line. All points on the line are solutions to the system, indicating that there are infinitely many solutions.

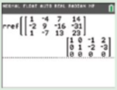
Since the top two equations both contain the variable z, we can express x and y in terms of z.

$$\begin{aligned} x - z &= 2 \xrightarrow{\text{Express } x \text{ in terms of } z.} x = z + 2 \\ y - 2z &= -3 \xrightarrow[\text{Express } y \text{ in terms of } z.]{} y = 2z - 3 \end{aligned}$$

The solution set can be written as $\{(z + 2, 2z - 3, z) | z \text{ is any real number}\}$.

Insights

- A graphing utility can also be used to find the reduced row-echelon form of a system of dependent equations.



```
rref([[1  -4   7   14 ]
      [-2  9  -16  -31]
      [1  -7   13  23 ]]
                      [1 0 -1  2]
                      [0 1 -2 -3]
                      [0 0  0  0]
```

- In Example 3, we can also write the solution set by using a parameter. By choosing any real number t for z, we have:

$$\{(t + 2, 2t - 3, t)\}$$

Apply the Skills

3. Solve the system.

$$\begin{aligned} -4x - 11y + 3z &= -24 \\ x + 3y - z &= 7 \\ 3x + 11y - 5z &= 29 \end{aligned}$$

We can verify the solution to Example 3, by substituting $(z + 2, 2z - 3, z)$ back into each original equation.

$$\begin{aligned} x - 4y + 7z &= 14 \longrightarrow (z + 2) - 4(2z - 3) + 7z \stackrel{?}{=} 14 \longrightarrow 14 = 14 \ \checkmark \\ -2x + 9y - 16z &= -31 \longrightarrow -2(z + 2) + 9(2z - 3) - 16z \stackrel{?}{=} -31 \longrightarrow -31 = -31 \ \checkmark \\ x - 7y + 13z &= 23 \longrightarrow (z + 2) - 7(2z - 3) + 13z \stackrel{?}{=} 23 \longrightarrow 23 = 23 \ \checkmark \end{aligned}$$

In Example 3, the solution set $\{(z + 2, 2z - 3, z) | z \text{ is any real number}\}$ is the general solution representing an infinite number of ordered triples. To find individual solutions, substitute arbitrary real numbers for z, and construct the corresponding ordered triple. For example:

Choose z Arbitrarily	**$(z + 2, 2z - 3, z)$**	**Solution**
If $z = 1$ $\longrightarrow$	$(1 + 2, 2(1) - 3, 1)$	$(3, -1, 1)$
If $z = 2$ $\longrightarrow$	$(2 + 2, 2(2) - 3, 2)$	$(4, 1, 2)$
If $z = 3$ $\longrightarrow$	$(3 + 2, 2(3) - 3, 3)$	$(5, 3, 3)$
If $z = -1$ $\longrightarrow$	$(-1 + 2, 2(-1) - 3, -1)$	$(1, -5, -1)$

A system of linear equations that has the same number of equations as variables is called a **square system**. In Example 3, we were presented with three equations and three variables in the original system. However, after writing the augmented matrix in row-echelon form, we see that the system reduces to a system of two equations and three variables. A system of linear equations cannot have a unique solution unless there are at least the same number of equations as variables.

In Example 4, we investigate a nonsquare system that has fewer equations than variables.

Example 4: Solving a System with Fewer Equations than Variables

Solve the system.
$$\begin{aligned} 3x - 8y + 18z &= 15 \\ x - 3y + 4z &= 6 \end{aligned}$$

Solution:

The two given equations each represent a plane in space. By finding the row-echelon form of the augmented matrix, we can determine the geometrical relationship between the planes.

- If we encounter a contradiction, then the system has no solution and the planes are parallel.
- If we encounter an identity, such as $0 = 0$, then the two equations must represent the same plane.
- Otherwise, the planes must meet in a line.

$$\begin{aligned} 3x - 8y + 18z = 15 &\longrightarrow \\ x - 3y + 4z = 6 &\longrightarrow \end{aligned} \left[\begin{array}{rrr|r} 3 & -8 & 18 & 15 \\ 1 & -3 & 4 & 6 \end{array}\right] \; R_1 \Leftrightarrow R_2 \; \left[\begin{array}{rrr|r} 1 & -3 & 4 & 6 \\ 3 & -8 & 18 & 15 \end{array}\right]$$

$$\xrightarrow{-3R_1 + R_2 \to R_2} \left[\begin{array}{rrr|r} 1 & -3 & 4 & 6 \\ 0 & 1 & 6 & -3 \end{array}\right] \xrightarrow{3R_2 + R_1 \to R_1} \left[\begin{array}{rrr|r} 1 & 0 & 22 & -3 \\ 0 & 1 & 6 & -3 \end{array}\right]$$

The augmented matrix is in reduced row-echelon form. $\left[\begin{array}{rrr|r} 1 & 0 & 22 & -3 \\ 0 & 1 & 6 & -3 \end{array}\right] \longrightarrow \begin{cases} x + 22z = -3 \\ y + 6z = -3 \end{cases}$

The system did not reduce to a contradiction or an identity. Therefore, the two planes must intersect in a line and there must be infinitely many solutions.

To write the general solution, write x and y in terms of z.

$$x + 22z = -3 \xrightarrow{\text{Express } x \text{ in terms of } z.} x = -22z - 3$$

$$y + 6z = -3 \xrightarrow[\text{Express } y \text{ in terms of } z.]{} y = -6z - 3$$

The solution set is $\{(-22z - 3, -6z - 3, z) | z \text{ is any real number}\}$.

Insights

In parametric form, the solution to Example 4 is given by:

$$\{(-22t - 3, -6t - 3, t)\}$$

Apply the Skills

4. Solve the system
$$\begin{aligned} x - 3y - 17z &= -17 \\ -2x + 7y + 38z &= 40 \end{aligned}$$

In Example 5, we investigate a situation in which the solution set to a system of three variables and three equations consists of all points on a common plane in space.

Example 5: Solving a System of Dependent Equations Representing the Same Plane

Solve the system.
$$\begin{aligned} x + 2y + 3z &= 6 \\ -x - 2y - 3z &= -6 \\ 2x + 4y + 6z &= 12 \end{aligned}$$

Solution:

$\boxed{A}$ $x + 2y + 3z = 6$

$\boxed{B}$ $-x - 2y - 3z = -6$

$\boxed{C}$ $2x + 4y + 6z = 12$

Upon inspection, you might notice that each equation is a constant multiple of the others. That is, equation $\boxed{B}$ is -1 times equation $\boxed{A}$. Equation $\boxed{C}$ is 2 times equation $\boxed{A}$. This means that equations $\boxed{A}$, $\boxed{B}$, and $\boxed{C}$ represent the same plane in space.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ -1 & -2 & -3 & -6 \\ 2 & 4 & 6 & 12 \end{array}\right] \begin{array}{c} \xrightarrow{R_1 + R_2 \to R_2} \\ \xrightarrow[-2R_1 + R_3 \to R_3]{} \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The reduced row-echelon form tells us that the system reduces to an equivalent system of one equation with three variables.

The system is equivalent to the single equation $x + 2y + 3z = 6$ and the solution set is the set of all ordered triples that satisfy this equation. Since we have one unique relationship among three variables, one variable is dependent on the other two variables. Letting two variables be arbitrary real numbers, the common equation defines the value of the third variable. For example, solving the equation $x + 2y + 3z = 6$ for x yields: $x = 6 - 2y - 3z$.

The solution set is $\{(6 - 2y - 3z, y, z) | y \text{ and } z \text{ are any real numbers}\}$.

Alternatively, the solution set can be expressed as $\{(x, y, z) | x + 2y + 3z = 6\}$.

Insights

The solution to Example 5 tells us that $x = 6 - 2y - 3z$. That is, x depends on both y and z. If we write the solution set in parametric form, we would need *two* parameters.

$$\{(6 - 2s - 3t, s, t)\}$$

Apply the Skills

5. Solve the system.
$$\begin{aligned} 2x - 3y + 5z &= 3 \\ 4x - 6y + 10z &= 6 \\ 20x - 30y + 50z &= 30 \end{aligned}$$

Learn: Solve Applications of Systems of Equations

A city planner can study the flow of traffic through a network of streets by measuring the flow rates (number of vehicles per unit time) at various points. For traffic to flow freely, we use the principle that the flow rate into an intersection is equal to the flow rate out of the intersection. This must be true for all intersections in the network.

Example 6: Solving an Application Involving Dependent Equations

Consider the network of four one-way streets shown in Figure 9-5. In the figure, x_1, x_2, x_3, and x_4 indicate flow rates (in vehicles per hour) along the stretches of roads AB, BC, CD, and DA, respectively. The other numbers in the figure indicate other flow rates moving into and out of intersections A, B, C, and D.

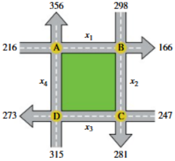


Figure 9-5

a. Set up a system of equations that represents traffic flowing freely.

b. Write the augmented matrix for the system of equations in reduced row-echelon form.

c. If the traffic between intersections D and A is 260 vehicles per hour, determine the flow rates x_1, x_2, and x_3.

d. If the traffic between intersections D and A is between 250 and 300 vehicles per hour, inclusive, determine the flow rates x_1, x_2, and x_3.

Solution:

a.

Intersection	Flow in = Flow out
A	$216 + x_4 = 356 + x_1$
B	$298 + x_1 = 166 + x_2$
C	$x_2 + 247 = 281 + x_3$
D	$x_3 + 315 = 273 + x_4$

The number of cars flowing into each intersection must equal the number of cars leaving the intersection for traffic to flow freely.

b. $$\left.\begin{aligned} -x_1 \qquad\qquad + x_4 &= 140\\ x_1 - x_2 \qquad &= -132\\ x_2 - x_3 \quad &= 34\\ x_3 - x_4 &= -42 \end{aligned}\right\} \to \left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 140\\ 1 & -1 & 0 & 0 & -132\\ 0 & 1 & -1 & 0 & 34\\ 0 & 0 & 1 & -1 & -42 \end{array}\right]$$ Write the equations in standard form and set up the augmented matrix.

$$\xrightarrow{-1R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -140\\ 0 & 1 & 0 & -1 & -8\\ 0 & 1 & -1 & 0 & 34\\ 0 & 0 & 1 & -1 & -42 \end{array}\right] \xrightarrow{-1R_1 + R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -140\\ 0 & -1 & 0 & 1 & 8\\ 0 & 1 & -1 & 0 & 34\\ 0 & 0 & 1 & -1 & -42 \end{array}\right]$$

$$\xrightarrow{-1R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -140\\ 0 & 1 & 0 & -1 & -8\\ 0 & 1 & -1 & 0 & 34\\ 0 & 0 & 1 & -1 & -42 \end{array}\right] \xrightarrow{-1R_2 + R_3 \to R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -140\\ 0 & 1 & 0 & -1 & -8\\ 0 & 0 & -1 & 1 & 42\\ 0 & 0 & 1 & -1 & -42 \end{array}\right]$$

$$\xrightarrow{-1R_3 \to R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -140\\ 0 & 1 & 0 & -1 & -8\\ 0 & 0 & 1 & -1 & -42\\ 0 & 0 & 1 & -1 & -42 \end{array}\right] \xrightarrow{-1R_3 + R_4 \to R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -140\\ 0 & 1 & 0 & -1 & -8\\ 0 & 0 & 1 & -1 & -42\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

c. From the reduced row-echelon form, we see that the flow rates x_1, x_2, and x_3 can all be expressed in terms of the flow rate x_4.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -140\\ 0 & 1 & 0 & -1 & -8\\ 0 & 0 & 1 & -1 & -42\\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \longrightarrow x_1 - x_4 = -140 \longrightarrow x_1 = x_4 - 140\\ \longrightarrow x_2 - x_4 = -8 \longrightarrow x_2 = x_4 - 8\\ \longrightarrow x_3 - x_4 = -42 \longrightarrow x_3 = x_4 - 42 \end{array}$$

If x_4 is 260 vehicles per hour, we have the following values for x_1, x_2, and x_3.

Flow rate between A and B: $x_1 = 260 - 140 \longrightarrow x_1 = 120$ vehicles per hour

Flow rate between B and C: $x_2 = 260 - 8 \longrightarrow x_2 = 252$ vehicles per hour

Flow rate between C and D: $x_3 = 260 - 42 \longrightarrow x_3 = 218$ vehicles per hour

d. If the traffic flow between intersections D and A is given by $250 \le x_4 \le 300$, we can solve the following inequalities to determine the flow rates x_1, x_2, and x_3. From part c, we have

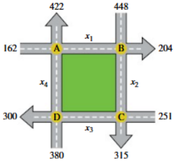
$$\begin{aligned} x_1 = x_4 - 140 &\Rightarrow x_1 + 140 = x_4\\ x_2 = x_4 - 8 &\Rightarrow x_2 + 8 = x_4\\ x_3 = x_4 - 42 &\Rightarrow x_3 + 42 = x_4 \end{aligned}$$

So for $250 \le x_4 \le 300$, we have:

$$\begin{aligned} 250 \le x_1 + 140 \le 300 &\longrightarrow 110 \le x_1 \le 160\\ 250 \le x_2 + 8 \le 300 &\longrightarrow 242 \le x_2 \le 292\\ 250 \le x_3 + 42 \le 300 &\longrightarrow 208 \le x_3 \le 258 \end{aligned}$$

Apply the Skills

6. Refer to the figure. Assume that traffic flows freely with flow rates given in vehicles per hour.



a. If the traffic between intersections D and A is 400 vehicles per hour, determine the flow rates x_1, x_2, and x_3.

b. If the traffic between intersections D and A is between 380 and 420 vehicles per hour, determine the flow rates x_1, x_2, and x_3.

Point of Interest

The principle of equal flow rates into and out of a junction applies to many applications of networks, including communications systems, water systems, and even the circulatory system in the human body.

Practice 9-2

Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, solve for the indicated variable.

R.1. $5x + y = -18$ for y

R.2. $3x - y = -8$ for x

For Exercises R.3–R.4, determine whether the graph of the equation is symmetric with respect to the *x*-axis or *y*-axis.

R.3. $y = -|x| - 6$

R.4. $x = y^2 + 7$

Concept Connections

1. True or false? A system of linear equations in three variables may have no solution.
2. True or false? A system of linear equations in three variables may have exactly one solution.
3. True or false? A system of linear equations in three variables may have exactly two solutions.
4. True or false? A system of linear equations in three variables may have infinitely many solutions.
5. If a system of linear equations has no solution, then the system is said to be ______________.
6. If a system of linear equations has infinitely many solutions, then the equations are said to be ______________.

Learn: Identify Inconsistent Systems and Solve Systems with Dependent Equations

For Exercises 7–14, an augmented matrix is given. Determine the number of solutions to the corresponding system of equations.

7. $\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 5 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & -1 \end{array}\right]$

9. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

10. $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right]$

11. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{array}\right]$

12. $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0 \end{array}\right]$

13. $\left[\begin{array}{ccc|c} 1 & 2 & 5 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

14. $\left[\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

For Exercises 15–18, determine the solution set for the system represented by each augmented matrix.

15. a. $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 0 \end{array}\right]$

b. $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 0 \end{array}\right]$

c. $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 1 \end{array}\right]$

16. a. $\left[\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 1 & 1 \end{array}\right]$

b. $\left[\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 0 & 1 \end{array}\right]$

c. $\left[\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 0 & 0 \end{array}\right]$

17. a. $\left[\begin{array}{ccc|c} 1 & 0 & 6 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 0 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & 0 & 6 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 1 & 0 & 6 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

18. a. $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right]$

For Exercises 19–38, solve the system by using Gaussian elimination or Gauss-Jordan elimination. (See Examples 1–5 on pp. 962–968)

19. $2x + 4y = 5$
$x + 2y = 4$

20. $4x + 16y = 21$
$x + 4y = -1$

21. $2x + 7y = 10$
$\frac{1}{5}x = 1 - \frac{7}{10}y$

22. $4x - 3y = 6$
$y = \frac{4}{3}x - 2$

23. $x - 3y + 14z = -9$
$-2x + 7y - 31z = 21$
$x - 5y + 20z = -14$

24. $x - 3y + 17z = 1$
$x - y + 7z = 2$
$2x - 5y + 29z = 5$

25. $5x + 7y - 11z = 45$
$3x + 5y - 9z = 23$
$x + y - z = 11$

26. $x + 3y + 9z = 12$
$2x + 7y + 22z = 26$
$-5x - 17y - 53z = -64$

27. $2x = 5y - 16z + 40$
$2(x + y) = 4z$
$x - 2y + 7z = 18$

28. $x = 2y + 4z + 5$
$y - 4 = -3z$
$5(x - y) - 9z = 4x - 7$

29. $2x - 5y - 20z = -24$
$x - 3y - 11z = -15$

30. $2x - y - 5z = -3$
$x - 2y - 7z = -12$

31. $2x + 3y + 4z = 12$
$-4x - 6y - 8z = -24$
$x + 1.5y + 2z = 6$

32. $-x + 2y + 7z = 14$
$10x - 20y - 70z = -140$
$-\frac{1}{7}x + \frac{2}{7}y + z = 2$

33. $2x - 3y + 9z = -2$
$x = 5y - 8z - 15$
$3(x - y) + 6z = 2x - 7$

34. $3x + 11y - 3z = -13$
$x + y = z - 15$
$3(x + y) = z + 2x - 7$

35. $-5x + 12y - 20z = -11$
$x + 4z = 3y + 1$

36. $x = 4y + 20z - 1$
$-2x + 5y + 25z = -1$

37. $x_1 - 3x_2 + 9x_3 - 14x_4 = 32$
$x_2 - 3x_3 + 6x_4 = -10$
$x_2 - x_3 + 2x_4 = -4$
$x_1 - 2x_2 + 8x_3 - 12x_4 = 24$

38. $x_1 - 3x_3 - 12x_4 = -15$
$x_2 + x_3 + 6x_4 = 8$
$x_2 - 2x_3 - 6x_4 = -7$
$-2x_1 + 4x_3 + 16x_4 = 22$

For Exercises 39–40, the solution set to a system of dependent equations is given. Write the specific solutions corresponding to the given values of *z*.

39. $\{(2z + 1, z - 4, z) \mid z \text{ is any real number}\}$
a. $z = 1$ **b.** $z = 4$ **c.** $z = -2$

40. $\{(z + 5, 3z - 2, z) \mid z \text{ is any real number}\}$
a. $z = -3$ **b.** $z = 1$ **c.** $z = 0$

For Exercises 41–44, the solution set to a system of dependent equations is given. Write three ordered triples that are solutions to the system. Answers may vary.

41. $\{(4z, 6 - z, z) | z \text{ is any real number}\}$

42. $\{(2z, z - 3, z) | z \text{ is any real number}\}$

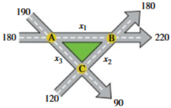
43. $\left\{\left(\frac{6 - 3y - 6z}{2}, y, z\right) \middle| y \text{ and } z \text{ are any real numbers}\right\}$

44. $\{(4y + 2z - 20, y, z) | y \text{ and } z \text{ are any real numbers}\}$

Learn: Solve Applications of Systems of Equations

For Exercises 45–48, assume that traffic flows freely through the intersections A, B, C, and D. The values x_1, x_2, x_3, and x_4 and the other numbers in the figures represent flow rates in vehicles per hour. (See Example 6 on p. 969)

45.



a. Write an equation representing equal flow into and out of intersection A.

b. Write an equation representing equal flow into and out of intersection B.

c. Write an equation representing equal flow into and out of intersection C.

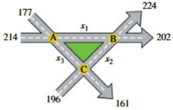
d. Write the system of equations from parts (a)–(c) in standard form.

e. Write the reduced row-echelon form of the augmented matrix representing the system of equations from part (d).

f. If the flow rate between intersections A and C is 120 vehicles per hour, determine the flow rates x_1 and x_2.

g. If the flow rate between intersections A and C is between 100 and 150 vehicles per hour, inclusive, determine the flow rates x_1 and x_2.

46.



a. Write an equation representing equal flow into and out of intersection A.

b. Write an equation representing equal flow into and out of intersection B.

c. Write an equation representing equal flow into and out of intersection C.

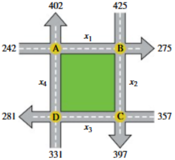
d. Write the system of equations from parts (a)–(c) in standard form.

e. Write the reduced row-echelon form of the augmented matrix representing the system of equations in part (d).

f. If the flow rate between intersections A and C is 156 vehicles per hour, determine the flow rates x_1 and x_2.

g. If the flow rate between intersections A and C is between 100 and 200 vehicles per hour, inclusive, determine the flow rates x_1 and x_2.

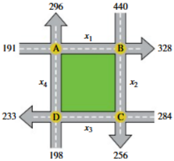
47.



a. Assume that traffic flows at a rate of 220 vehicles per hour on the stretch of road between intersections D and A. Find the flow rates x_1, x_2, and x_3.

b. If traffic flows at a rate of between 200 and 250 vehicles per hour inclusive between intersections D and A, find the flow rates x_1, x_2, and x_3.

48.



a. Assume that traffic flows at a rate of 180 vehicles per hour on the stretch of road between intersections D and A. Find the flow rates x_1, x_2, and x_3.

b. If traffic flows at a rate of between 150 and 200 vehicles per hour inclusive between intersections D and A, find the flow rates x_1, x_2, and x_3.

49. An accountant checks the reported earnings for a theater for three nightly performances against the number of tickets sold.

Night	Children Tickets	Student Tickets	General Admission	Total Revenue
1	80	400	480	\$9280
2	50	350	400	\$7800
3	75	525	600	\$10,500

a. Let *x*, *y*, and *z* represent the cost for children tickets, student tickets, and general admission tickets, respectively. Set up a system of equations to solve for *x*, *y*, and *z*.

b. Set up the augmented matrix for the system and solve the system. (*Hint*: To make the augmented matrix simpler to work with, consider dividing each linear equation by an appropriate constant.)

c. Explain why the auditor knows that there was an error in the record keeping.

50. A concession stand at a city park sells hamburgers, hot dogs, and drinks. Three patrons buy the following food and drink combinations for the following prices.

Patron	Hamburgers	Hot Dogs	Drinks	Total Revenue
1	1	1	5	\$11
2	0	1	2	\$5
3	3	1	11	\$22

a. Let *x*, *y*, and *z* represent the cost for a hamburger, a hot dog, and a drink, respectively. Set up a system of equations to solve for *x*, *y*, and *z*.

b. Set up the augmented matrix for the system and solve the system. c. Explain why the concession stand manager knows that there was an error in the record keeping.

Mixed Exercises

51. The solution set to Exercise 25 is $\{(-2z+16, 3z-5, z) | z \text{ is any real number}\}$. Verify the solution by substituting the ordered triple into each individual equation.

52. The solution set to Exercise 26 is $\{(3z+6, -4z+2, z) | z \text{ is any real number}\}$. Verify the solution by substituting the ordered triple into each individual equation.

The systems in Exercises 53–56 are called homogeneous systems because each has (0, 0, 0) as a solution. However, if a system is made up of dependent equations, it will have infinitely many more solutions. For each system, determine whether (0, 0, 0) is the only solution or if the system has infinitely many solutions. If the system has infinitely many solutions, give the solution set.

53.

$$\begin{aligned} x + 2y - 8z &= 0 \\ -5x - 11y + 43z &= 0 \\ x + 5y - 12z &= 0 \end{aligned}$$

54.
$$\begin{aligned} x - 2y - 7z &= 0 \\ -3x + 8y + 31z &= 0 \\ -2x + 5y + 22z &= 0 \end{aligned}$$

55.
$$\begin{aligned} x - 5y + 13z &= 0 \\ -3x + 17y - 45z &= 0 \\ x - 4y + 10z &= 0 \end{aligned}$$

56.
$$\begin{aligned} -2x + 15y - 83z &= 0 \\ x - 7y + 39z &= 0 \\ x - 5y + 29z &= 0 \end{aligned}$$

Write About It

57. Explain how you can determine from the reduced row-echelon form of a matrix whether the corresponding system of equations is inconsistent.

58. What can you conclude about a system of equations if the corresponding reduced row-echelon form consists of a row entirely of zeros?

59. Consider the following system. By inspection, describe the geometrical relationship among the planes represented by the three equations.

$$\begin{aligned} x + y + z &= 1 \\ 2x + 2y + 2z &= 2 \\ 3x + 3y + 3z &= 3 \end{aligned}$$

60. Explain why a system of two equations with three variables cannot have exactly one ordered triple as its solution.

Technology Connections

For Exercises 61–66, use a graphing utility to find the reduced row-echelon form of the augmented matrix for the system in the given exercise. Use the result to verify the answer to the given exercise.

61. Exercise 23

62. Exercise 24

63. Exercise 25

64. Exercise 26

65. Exercise 31

66. Exercise 32

Go online for more practice problems.

Lesson 9-3

Operations on Matrices

What Will You Learn?

After completing this lesson, you should be able to:

- Determine the Order of a Matrix
- Add and Subtract Matrices
- Multiply a Matrix by a Scalar
- Multiply Matrices
- Apply Operations on Matrices

Learn: Determine the Order of a Matrix

We have seen how the manipulation of an augmented matrix can be used to solve a system of linear equations. Matrices have many other useful mathematical applications, particularly when manipulating tables or databases. In particular, matrices play a useful role in digital photography, film, and computer animation.

The **order of a matrix** is determined by the number of rows and number of columns. A matrix with *m* rows and *n* columns is an $m \times n$ matrix (read as "*m* by *n*" matrix).

Example 1: Determining the Order of a Matrix

Determine the order of each matrix.

a. $\begin{bmatrix} 3 & \pi & 1.7 \\ -1 & 6 & 10 \end{bmatrix}$ **b.** $\begin{bmatrix} -7 \\ 2 \\ 4 \\ \sqrt{2} \end{bmatrix}$ **c.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} x & y & z \end{bmatrix}$

Solution:

a. The matrix has 2 rows and 3 columns. Therefore, it is a 2×3 matrix.

b. The matrix has 4 rows and 1 column. Therefore, it is a 4×1 matrix. A matrix with only 1 column is called a **column matrix**.

c. The matrix has 3 rows and 3 columns. Therefore, it is a 3×3 matrix. A matrix with the same number of rows and columns is called a **square matrix**.

d. The matrix has 1 row and 3 columns. Therefore, it is a 1×3 matrix. A matrix with only 1 row is called a **row matrix**.

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Apply the Skills

1. Determine the order of each matrix.

a. $\begin{bmatrix} -9 & 2 \\ 4.1 & -3 \\ \sqrt{3} & 4 \end{bmatrix}$ **b.** $\begin{bmatrix} -4 \\ 11 \end{bmatrix}$ **c.** $\begin{bmatrix} -0.1 & 0.4 \\ 0.5 & 0.2 \end{bmatrix}$ **d.** $\begin{bmatrix} 105 & 311 \end{bmatrix}$

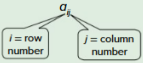
A matrix can be represented generically as follows:

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Using the double subscript notation, a_{43} represents the element in the 4th row, 3rd column. The notation a_{ij} represents the element in the ith row, jth column.

Insights

The notation $[a_{ij}]$ is a generic way to denote a matrix with elements of the form a_{ij}, where i and j represent generic row and column numbers.



Example 2: Using Matrix Notation

Consider $A = \begin{bmatrix} -4 & 5 & \sqrt{2} & -\pi \\ 10 & \frac{1}{2} & 0 & 6 \\ -\frac{7}{6} & 8 & 12 & -9 \end{bmatrix}$, where $A = [a_{ij}]$.

Determine the value of each element.

a. a_{32} **b.** a_{23} **c.** a_{14}

Solution:

a. a_{32} represents the element in the 3rd row, 2nd column: 8

b. a_{23} represents the element in the 2nd row, 3rd column: 0

c. a_{14} represents the element in the 1st row, 4th column: $-\pi$

Apply the Skills

2. Given matrix A from Example 2, determine the value of each element.

a. a_{13} **b.** a_{31} **c.** a_{34}

Learn: Add and Subtract Matrices

We now consider the conditions under which two matrices are equal.

Equality of Matrices

Two matrices are equal if and only if they have the same order and if their corresponding elements are equal.

Example:

- The statement $\begin{bmatrix} 3 & -4 \\ x & z \end{bmatrix} = \begin{bmatrix} 3 & y \\ 9 & 7 \end{bmatrix}$ implies that $x = 9$, $y = -4$, and $z = 7$.
- Conversely, if $x = 9$, $y = -4$, and $z = 7$, then $\begin{bmatrix} 3 & -4 \\ x & z \end{bmatrix} = \begin{bmatrix} 3 & y \\ 9 & 7 \end{bmatrix}$.

Next, we define addition and subtraction of matrices.

Addition and Subtraction of Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be m by n matrices. Then,

$$A + B = [a_{ij} + b_{ij}] \text{ for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

$$A - B = [a_{ij} - b_{ij}] \text{ for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

That is, to add or subtract two matrices, the matrices must have the same order, and the sum or difference is found by adding or subtracting the corresponding elements.

Example 3: Adding and Subtracting Matrices

Given $A = \begin{bmatrix} 3 & -7 & 0 \\ 1 & \frac{4}{3} & -6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -3 & 4 \\ 0 & 1 & -9 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & -5 \\ 4 & 18 \end{bmatrix}$,

find the sum or difference if possible.

a. $A + B$ **b.** $B - A$ **c.** $A + C$

Solution:

a. $A + B = \begin{bmatrix} 3 & -7 & 0 \\ 1 & \frac{4}{3} & -6 \end{bmatrix} + \begin{bmatrix} 8 & -3 & 4 \\ 0 & 1 & -9 \end{bmatrix}$

Matrix A and matrix B both have the same order (2×3), so the sum is defined.

$= \begin{bmatrix} 3+8 & -7+(-3) & 0+4 \\ 1+0 & \frac{4}{3}+1 & -6+(-9) \end{bmatrix}$

Add the elements in the corresponding positions.

$= \begin{bmatrix} 11 & -10 & 4 \\ 1 & \frac{7}{3} & -15 \end{bmatrix}$

Simplify.

b. $B - A = \begin{bmatrix} 8 & -3 & 4 \\ 0 & 1 & -9 \end{bmatrix} - \begin{bmatrix} 3 & -7 & 0 \\ 1 & \frac{4}{3} & -6 \end{bmatrix}$

Matrix *A* and matrix *B* both have the same order (2 × 3), so the difference is defined.

$= \begin{bmatrix} 8-3 & -3-(-7) & 4-0 \\ 0-1 & 1-\frac{4}{3} & -9-(-6) \end{bmatrix}$

Subtract the elements in the corresponding positions.

$= \begin{bmatrix} 5 & 4 & 4 \\ -1 & -\frac{1}{3} & -3 \end{bmatrix}$

Simplify.

c. Matrix *A* is a 2 × 3 matrix, whereas matrix *C* is a 2 × 2 matrix. The matrices are of different orders and therefore cannot be added or subtracted.

Apply the Skills

3. Given $A = \begin{bmatrix} 3 & 4 & -7 \\ 9 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -8 & 0 \\ 1 & 5 \\ -4 & \frac{1}{2} \end{bmatrix}$, and $C = \begin{bmatrix} 4 & -5 \\ 12 & 3 \\ -2 & 2 \end{bmatrix}$, find the sum or difference if possible.

a. $A + B$ **b.** $B + C$ **c.** $C - B$

The opposite of a real number *a* is $-a$. The value $-a$ is also called the additive inverse of *a*. A matrix *A* also has an additive inverse, denoted by $-A$.

Additive Inverse of a Matrix

Given $A = [a_{ij}]$, the **additive inverse of *A***, denoted by $-A$, is $-A = [-a_{ij}]$.

In other words, the additive inverse of a matrix is found by taking the opposite of each element in the matrix.

For example: Given $A = \begin{bmatrix} -4 & 5 & -1 \\ 0 & -3 & 2 \end{bmatrix}$, $-A = \begin{bmatrix} 4 & -5 & 1 \\ 0 & 3 & -2 \end{bmatrix}$.

The number 0 is the identity element under the addition of real numbers because $a + (-a) = 0$. A matrix in which all elements are zero is called a **zero matrix** and is denoted by *O*. The sum of a matrix *A* and its additive inverse $-A$ is the zero matrix of the same order. For example,

$$\underbrace{\begin{bmatrix} -4 & 5 & -1 \\ 0 & -3 & 2 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 4 & -5 & 1 \\ 0 & 3 & -2 \end{bmatrix}}_{(-A)} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{O}$$

2 × 3 zero matrix

Properties of Matrix Addition

Let A, B, and C be matrices of order $m \times n$, and let O be the zero matrix of order $m \times n$. Then,

1. $A + B = B + A$	Commutative property of matrix addition
2. $A + (B + C) = (A + B) + C$	Associative property of matrix addition
3. $A + (-A) = O$	Inverse property of matrix addition
4. $A + O = O + A = A$	Identity property of matrix addition

Learn: Multiply a Matrix by a Scalar

Next we look at the product of a real number k and a matrix. This is called **scalar multiplication**. The real number k is called a **scalar** to distinguish it from a matrix.

Scalar Multiplication

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let k be a real number. Then, $kA = [ka_{ij}]$.

In other words, to multiply a matrix A by a scalar k, multiply each element in the matrix by k.

For example: Given $A = \begin{bmatrix} -4 & 5 & -1 \\ 0 & -3 & 2 \end{bmatrix}$, $5A = \begin{bmatrix} -20 & 25 & -5 \\ 0 & -15 & 10 \end{bmatrix}$.

The following properties of scalar multiplication are similar to those of the multiplication of real numbers.

Properties of Scalar Multiplication

Let A and B be $m \times n$ matrices and let c and d be real numbers. Then,

1. $c(A + B) = cA + cB$	Distributive property of scalar multiplication
2. $(c + d)A = cA + dA$	Distributive property of scalar multiplication
3. $c(dA) = (cd)A$	Associative property of scalar multiplication

Example 4: Multiplying a Matrix by a Scalar

Given $A = \begin{bmatrix} 3 & -5 \\ 0 & 1 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -8 & 9 \\ 4 & -1 \\ 0 & 3 \end{bmatrix}$, find $2A + 4(A + B)$.

Solution:

$$\begin{aligned} 2A + 4(A + B) &= 2A + 4A + 4B \\ &= 6A + 4B \end{aligned}$$

Apply the distributive property of scalar multiplication.

$$\begin{aligned} 6A + 4B = 6\begin{bmatrix} 3 & -5 \\ 0 & 1 \\ -4 & 2 \end{bmatrix} + 4\begin{bmatrix} -8 & 9 \\ 4 & -1 \\ 0 & 3 \end{bmatrix} &= \begin{bmatrix} 18 & -30 \\ 0 & 6 \\ -24 & 12 \end{bmatrix} + \begin{bmatrix} -32 & 36 \\ 16 & -4 \\ 0 & 12 \end{bmatrix} \\ &= \begin{bmatrix} -14 & 6 \\ 16 & 2 \\ -24 & 24 \end{bmatrix} \end{aligned}$$

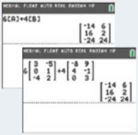
Apply the Skills

4. Given $A = [4 \quad -3 \quad 9]$ and $B = [-2 \quad 0 \quad 3]$, find $-5A - 2(A + B)$.

TECHNOLOGY CONNECTIONS

Adding Matrices and Multiplying a Scalar by a Matrix

Many graphing utilities have the capability to perform operations on matrices. From Example 4, we can enter matrix *A* and matrix *B* into the calculator using the EDIT feature under the MATRIX menu. Then on the home screen enter 6[A] + 4[B] and press ENTER.



NORMAL FLOAT AUTO REAL RADIAN MP
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```
6[A]+4[B]
                 [-14  6]
                 [ 16  2]
                 [-24 24]
```

$$6\begin{bmatrix} 3 & -5 \\ 0 & 1 \\ -4 & 2 \end{bmatrix} + 4\begin{bmatrix} -8 & 9 \\ 4 & -1 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} -14 & 6 \\ 16 & 2 \\ -24 & 24 \end{bmatrix}$$

Alternatively, in “MATHPRINT” mode, enter the matrices on the home screen using ALPHA F3.

As seen, many of the same properties that are true for addition and multiplication of real numbers are true for the addition of matrices and for scalar multiplication. As a result, we can solve a matrix equation involving these operations in the same way as we solve a linear equation.

Example 5: Solving a Matrix Equation

Solve the matrix equation for X, given that $A = \begin{bmatrix} 2 & 5 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -1 \\ -3 & -8 \end{bmatrix}$.

$$4X - A = B$$

Solution:

$4X - A = B$	
$4X = A + B$	Add matrix A to both sides.
$X = \frac{1}{4}(A + B)$	Perform scalar multiplication by $\frac{1}{4}$ on both sides.
$X = \frac{1}{4}\left(\begin{bmatrix} 2 & 5 \\ 1 & -4 \end{bmatrix} + \begin{bmatrix} 6 & -1 \\ -3 & -8 \end{bmatrix}\right)$	Add the matrices within parentheses.
$X = \frac{1}{4}\left(\begin{bmatrix} 8 & 4 \\ -2 & -12 \end{bmatrix}\right)$	Multiply the matrix by the scalar $\frac{1}{4}$.
$X = \begin{bmatrix} 2 & 1 \\ -\frac{1}{2} & -3 \end{bmatrix}$	Simplify.

Check Your Work

We can substitute matrix X back into the equation $4X - A = B$ to verify that it is a solution to the equation.

Apply the Skills

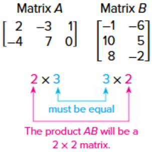
5. Given $A = \begin{bmatrix} 4 & -3 \\ 1 & 11 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 6 & 5 \end{bmatrix}$, solve $2X + A = B$ for X.

Learn: Multiply Matrices

Finding the product of two matrices is more complicated than finding the product of a scalar and a matrix. We will demonstrate the process to multiply two matrices and then offer a formal definition.

Consider $A = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 7 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -6 \\ 10 & 5 \\ 8 & -2 \end{bmatrix}$.

- To multiply AB, we require that the number of columns in A be equal to the number of rows in B.
- The resulting matrix will have dimensions equal to the number of rows of A by the number of columns of B.

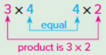


The product will be a 2 × 2 matrix.

$$\begin{bmatrix} 2 & -3 & 1 \\ -4 & 7 & 0 \end{bmatrix}\begin{bmatrix} -1 & -6 \\ 10 & 5 \\ 8 & -2 \end{bmatrix}=\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

Insights

For matrix multiplication, the number of rows of the first matrix does not need to be equal to the number of columns of the second matrix. For example, the product of a 3 × 4 and a 4 × 2 matrix results in a 3 × 2 matrix.



To find the entry in the first row, first column of the product, multiply the corresponding elements in the first row of *A* by the elements in the first column of *B*, and take the sum.

$$\begin{bmatrix} 2 & -3 & 1 \\ -4 & 7 & 0 \end{bmatrix}\begin{bmatrix} -1 & -6 \\ 10 & 5 \\ 8 & -2 \end{bmatrix}=\begin{bmatrix} 2(-1)+(-3)(10)+1(8) & \square \\ \square & \square \end{bmatrix}$$

To find the entry in the first row, second column of the product, take the sum of the products of the corresponding elements in the first row of *A* and second column of *B*.

$$\begin{bmatrix} 2 & -3 & 1 \\ -4 & 7 & 0 \end{bmatrix}\begin{bmatrix} -1 & -6 \\ 10 & 5 \\ 8 & -2 \end{bmatrix}=\begin{bmatrix} 2(-1)+(-3)(10)+1(8) & 2(-6)+(-3)(5)+1(-2) \\ \square & \square \end{bmatrix}$$

To find the entry in the second row, first column of the product, take the sum of the products of the corresponding elements in the second row of *A* and first column of *B*.

$$\begin{bmatrix} 2 & -3 & 1 \\ -4 & 7 & 0 \end{bmatrix}\begin{bmatrix} -1 & -6 \\ 10 & 5 \\ 8 & -2 \end{bmatrix}=\begin{bmatrix} 2(-1)+(-3)(10)+1(8) & 2(-6)+(-3)(5)+1(-2) \\ -4(-1)+7(10)+0(8) & \square \end{bmatrix}$$

To find the entry in the second row, second column of the product, take the sum of the products of the corresponding elements in the second row of *A* and second column of *B*.

$$\begin{bmatrix} 2 & -3 & 1 \\ -4 & 7 & 0 \end{bmatrix}\begin{bmatrix} -1 & -6 \\ 10 & 5 \\ 8 & -2 \end{bmatrix}=\begin{bmatrix} 2(-1)+(-3)(10)+1(8) & 2(-6)+(-3)(5)+1(-2) \\ -4(-1)+7(10)+0(8) & -4(-6)+7(5)+0(-2) \end{bmatrix}$$

Simplifying, we have

$$AB=\begin{bmatrix} 2 & -3 & 1 \\ -4 & 7 & 0 \end{bmatrix}\begin{bmatrix} -1 & -6 \\ 10 & 5 \\ 8 & -2 \end{bmatrix}=\begin{bmatrix} -24 & -29 \\ 74 & 59 \end{bmatrix}$$

From this example, note that each element in the product *AB* is of the form $a_1b_1 + a_2b_2 + \dots + a_nb_n$, where the elements $a_1, a_2, \dots a_n$ are elements in a row of *A* and $b_1, b_2, \dots b_n$ are elements in a column of *B*. The expression $a_1b_1 + a_2b_2 + \dots + a_nb_n$ is called an **inner product**.

Matrix Multiplication

Let A be an $m \times p$ matrix and let B be a $p \times n$ matrix; then the product AB is an $m \times n$ matrix. For the matrix AB, the element in the ith row and jth column is the sum of the products of the corresponding elements in the ith row of A and the jth column of B (the inner product of the ith row and jth column).

Formally, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix}$$

then the elements in the matrix $C = AB$ are given by

$$c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{ip} \cdot b_{pj}$$

Note: If the number of columns in A does not equal the number of rows in B, then it is not possible to compute the product AB.

Example 6: Multiplying Matrices

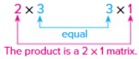
Given $A = \begin{bmatrix} 2 & 5 & 6 \\ -3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 10 & 1 \end{bmatrix}$,

find the following products if possible.

a. AB **b.** BC **c.** AC

Solution:

a. Matrix A Matrix B



Matrix A is a 2×3 matrix and matrix B is a 3×1 matrix. The number of columns in A is equal to the number of rows in B. The resulting matrix will be a 2×1 matrix.

$$= \begin{bmatrix} 2 & 5 & 6 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 2(1) + 5(3) + 6(-4) \\ -3(1) + 0(3) + 1(-4) \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ -7 \end{bmatrix}$$

Multiply the elements in the first row of A by the elements in the first column of B.

Multiply the elements in the second row of A by the elements in the first column of B.

b. Matrix B Matrix C

3×1 1×3

equal

The product is a 3×3 matrix.

Matrix B is a 3×1 matrix and matrix C is a 1×3 matrix. The number of columns in B is equal to the number of rows in C. The resulting matrix will be a 3×3 matrix.

$$BC = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} \begin{bmatrix} -2 & 10 & 1 \end{bmatrix} = \begin{bmatrix} 1(-2) & 1(10) & 1(1) \\ 3(-2) & 3(10) & 3(1) \\ -4(-2) & -4(10) & -4(1) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 10 & 1 \\ -6 & 30 & 3 \\ 8 & -40 & -4 \end{bmatrix}$$

c. Matrix A Matrix C

2×3 1×3

not equal

The number of columns in A does not equal the number of rows in C. Therefore, it is not possible to compute the product AC.

Apply the Skills

6. Given $A = \begin{bmatrix} 2 & -5 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 & 4 \\ 6 & 0 & -8 \end{bmatrix}$, and $C = \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}$, find the products if possible.

a. AB **b.** BC **c.** AC

Example 7: Multiplying Matrices

Given $A = \begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -3 & 6 \end{bmatrix}$, find the following products if possible.

a. AB **b.** BA

Solution:

a. Matrix A Matrix B

2×2 2×2

equal

The product is a 2×2 matrix.

$$AB = \begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1) + (-4)(-3) & 3(0) + (-4)(6) \\ 1(1) + 5(-3) & 1(0) + 5(6) \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -24 \\ -14 & 30 \end{bmatrix}$$

b. Matrix B Matrix A

2×2 2×2

equal

The product is a 2×2 matrix.

$$BA = \begin{bmatrix} 1 & 0 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1(3) + 0(1) & 1(-4) + 0(5) \\ -3(3) + 6(1) & -3(-4) + 6(5) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 \\ -3 & 42 \end{bmatrix}$$

Insights

Example 7 shows that for two matrices A and B, the product AB does not necessarily equal BA. That is, matrix multiplication is *not* commutative.

Apply the Skills

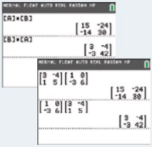
7. Given $A = \begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 6 & 1 \end{bmatrix}$, find the following products if possible.

a. AB **b.** BA

TECHNOLOGY CONNECTIONS

Multiplying Matrices

To multiply two matrices on a graphing utility, first enter the matrices into the calculator using the EDIT feature under the MATRIX menu. Then on the home screen, enter the product. The products AB and BA from Example 7 are shown here. Alternatively, in "MATHPRINT" mode, enter the matrices on the home screen using ALPHA F3. The products AB and BA are shown here.



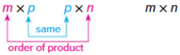
```
NORMAL FLOAT AUTO REAL RADIAN MP
[A]*[B]
                 [ 15 -24]
                 [-14  30]
[B]*[A]
                 [  3  -4]
                 [ -3  42]
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
[3 -4][ 1 0]
[1  5][-3 6]
                 [ 15 -24]
                 [-14  30]
[ 1 0][3 -4]
[-3 6][1  5]
                 [  3  -4]
                 [ -3  42]
```

Learn: Apply Operations on Matrices

In applications, a matrix gives mathematicians a systematic format in which to represent data. Multiplication of matrices is a tool to manipulate these data sets. For example, in business, we might multiply the number of items sold by the unit price per item to determine total revenue.

$$\begin{bmatrix} \text{Quantity} \\ \text{matrix} \end{bmatrix} \begin{bmatrix} \text{Unit price} \\ \text{matrix} \end{bmatrix} = \begin{bmatrix} \text{Total revenue} \\ \text{matrix} \end{bmatrix}$$



Insights

The number of columns p of the quantity matrix must equal the number of rows p of the unit price matrix for the product to be defined.

Example 8: Multiplying Matrices in a Business Application

A company owns two coffee shops. The number of donuts, coffee cakes, hot drinks, and cold drinks sold for each shop is given in matrix Q. The price per item is given in matrix P. Find the product QP and interpret the result.

$$\begin{array}{cc} & \begin{array}{cccc}\textbf{Donuts} & \textbf{Coffee cakes} & \textbf{Hot drinks} & \textbf{Cold drinks}\end{array} \\ Q = & \left[\begin{array}{cccc}162 & 34 & 120 & 44\\ 186 & 50 & 145 & 62\end{array}\right] \begin{array}{l}\textbf{Shop 1}\\ \textbf{Shop 2}\end{array}\end{array} \qquad P = \left[\begin{array}{c}\$0.40\\ \$2.50\\ \$3.50\\ \$1.50\end{array}\right] \begin{array}{l}\textbf{Donuts}\\ \textbf{Coffee cakes}\\ \textbf{Hot drinks}\\ \textbf{Cold drinks}\end{array}$$

Matrix Q is a 2×4 matrix and matrix P is a 4×1 matrix. Matrix Q has 4 columns, which match the number of rows of P, so the product QP is defined. The product matrix will be a 2×1 matrix. The product QP is

162 donuts times $0.40 per donut (Shop 1).

$$\begin{bmatrix}162 & 34 & 120 & 44\\ 186 & 50 & 145 & 62\end{bmatrix}\begin{bmatrix}0.40\\ 2.50\\ 3.50\\ 1.50\end{bmatrix} = \begin{bmatrix}162(0.40) + 34(2.50) + 120(3.50) + 44(1.50)\\ 186(0.40) + 50(2.50) + 145(3.50) + 2(1.50)\end{bmatrix}$$

$$= \begin{bmatrix}\$635.80\\ \$799.90\end{bmatrix}$$

Row 1: Revenue for Shop 1

Row 2: Revenue for Shop 2

The product QP is a matrix representing the total revenue from these four items for each shop.

Apply the Skills

8. A farmer sells organic zucchini, yellow squash, and corn in two different roadside stands. Matrix Q represents the number of pounds of each type of vegetable sold at each stand. Matrix P gives the price per pound of each item. Find the product QP and interpret the result.

$$\begin{array}{cc} & \begin{array}{ccc}\textbf{Zucchini} & \textbf{Yellow squash} & \textbf{Corn}\end{array} \\ Q = & \left[\begin{array}{ccc}42 & 40 & 84\\ 30 & 36 & 90\end{array}\right] \begin{array}{l}\textbf{Stand 1}\\ \textbf{Stand 2}\end{array}\end{array} \qquad P = \left[\begin{array}{c}\$4.00\\ \$3.40\\ \$4.20\end{array}\right] \begin{array}{l}\textbf{Zucchini}\\ \textbf{Yellow squash}\\ \textbf{Corn}\end{array}$$

In Chapter 1 we learned how to apply transformations to the graphs of functions. In Example 9 we see how matrices can also be used to transform images in a rectangular coordinate system. This is particularly helpful to computer programmers who write software for computer gaming.

Example 9: Using Matrix Operations to Transform a Graph

Consider the triangle shown in Figure 9-6.

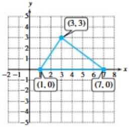


Figure 9-6

The vertices of the triangle can be represented in a matrix where the first row represents the *x*-coordinates and the second row represents the corresponding *y*-coordinates.

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 3 & 0 \end{bmatrix} \begin{array}{l} \longleftarrow x\text{-coordinates} \\ \longleftarrow y\text{-coordinates} \end{array}$$

a. Use addition of matrices to shift the triangle 3 units to the left and 2 units upward.

b. Find the product $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} A$ and determine the effect on the graph.

c. Find the product $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A$ and determine the effect on the graph.

Solution:

a. To shift the graph to the left, subtract 3 from each *x*-coordinate. To shift the graph upward, add 2 to each *y*-coordinate.

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 3 & 0 \end{bmatrix} + \begin{bmatrix} -3 & -3 & -3 \\ 2 & 2 & 2 \end{bmatrix} \begin{array}{l} \longleftarrow \text{This row subtracts 3 from each } x\text{-coordinare} \\ \longleftarrow \text{This row adds 2 to each } y\text{-coordinate} \end{array}$$

$$= \begin{bmatrix} -2 & 0 & 4 \\ 2 & 5 & 2 \end{bmatrix}$$

The vertices of the new triangle are (−2, 2), (0, 5), and (4, 2). See Figure 9-7.

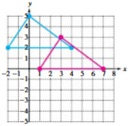


Figure 9-7

b. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 3 & 0 \end{bmatrix}$

$$= \begin{bmatrix} -1(1) + 0(0) & -1(3) + 0(3) & -1(7) + 0(0) \\ 0(1) + 1(0) & 0(3) + 1(3) & 0(7) + 1(0) \end{bmatrix}$$

$= \begin{bmatrix} -1 & -3 & -7 \\ 0 & 3 & 0 \end{bmatrix}$ The corresponding points are $(-1, 0)$, $(-3, 3)$, and $(-7, 0)$.

The triangle has been reflected across the *y*-axis. See Figure 9-8.

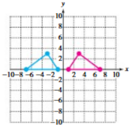


Figure 9-8

c. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 3 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1(1) + 0(0) & 1(3) + 0(3) & 1(7) + 0(0) \\ 0(1) + -1(0) & 0(3) + -1(3) & 0(7) + -1(0) \end{bmatrix}$$

$= \begin{bmatrix} 1 & 3 & 7 \\ 0 & -3 & 0 \end{bmatrix}$ The corresponding points are $(1, 0)$, $(3, -3)$, and $(7, 0)$.

The triangle has been reflected across the *x*-axis. See Figure 9-9.

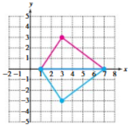


Figure 9-9

Apply the Skills

9. Use matrix *A* from Example 9. Find the product $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} A$ and determine the effect on the graph of the triangle in Figure 9-6.

Practice 9-3

Practice Exercises

Prerequisite Review

R.1. Identify the additive inverse of 9.

R.2. Apply the commutative property of addition. $-9 + x$

R.3. Solve the equation. $5z - 2 = -22$

Concept Connections

1. If the ____________ of a matrix is $p \times q$, then p represents the number of and q represents the number of ____________ and q represents the number of ____________.

2. A matrix with the same number of rows and columns is called a ____________ matrix.

3. What are the requirements for two matrices to be equal?

4. An $m \times n$ matrix whose elements are all zero is called a ____________ matrix.

5. To multiply two matrices A and B, the number of ____________ of A must equal the number of ____________ of B.

6. If A is a 5×3 matrix and B is a 3×7 matrix, then the product AB will be a matrix of order ____________. The product BA (is/is not) defined.

7. True or false: Matrix multiplication is a commutative operation.

8. True or false: If a row matrix A and a column matrix B have the same number of elements, then the product AB is defined.

Learn: Determine the Order of a Matrix

9. What is a row matrix?

10. What is a column matrix?

For Exercises 11–16,

a. Give the order of the matrix.

b. Classify the matrix as a square matrix, row matrix, column matrix, or none of these.

(See Example 1 on p. 977)

11. $\begin{bmatrix} 3 & 5 & -1 \\ \frac{1}{2} & \sqrt{3} & 1.7 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 5 & 6 & 2 \\ -1 & 3 & -\frac{2}{3} & \pi \\ 0 & 1 & -6.1 & 12 \end{bmatrix}$

13. $\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$

14. $\begin{bmatrix} 2.4 & 6.9 \end{bmatrix}$

15. $\begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$

16. $\begin{bmatrix} -4 & 10 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 7 & 0 & 9 & -1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$

For Exercises 17–22, determine the value of the given element of matrix $A = [a_{ij}]$. (See Example 2 on page 978)

$$A = \begin{bmatrix} 3 & -6 & \frac{1}{3} \\ 2 & 4 & 0 \\ \sqrt{5} & 11 & 8.6 \\ \frac{1}{2} & 4 & 2 \end{bmatrix}$$

17. a_{31} **18.** a_{32} **19.** a_{13}

20. a_{23} **21.** a_{43} **22.** a_{42}

Learn: Add and Subtract Matrices

23. Given $A = \begin{bmatrix} 2 & x \\ z & -5 \end{bmatrix}$ and $B = \begin{bmatrix} y & 4 \\ 10 & -5 \end{bmatrix}$, for what values of x, y, and z will $A = B$?

24. Given $A = \begin{bmatrix} 4 & a \\ b & 7 \end{bmatrix}$ and $B = \begin{bmatrix} c & 12 \\ -1 & 6 \end{bmatrix}$, is it possible that $A = B$? Explain.

25. Given $B = \begin{bmatrix} -4 & 6 & 9 \\ \frac{3}{5} & 1 & 7 \end{bmatrix}$, find the additive inverse of B.

26. Given $C = \begin{bmatrix} -1 & 6 \\ \sqrt{3} & 9 \end{bmatrix}$, find the additive inverse of C.

For Exercises 27–32, add or subtract the given matrices if possible. (See Example 3 on p. 979)

$$A = \begin{bmatrix} 6 & -1 \\ 7 & \frac{1}{2} \\ 2 & \sqrt{2} \end{bmatrix} \quad B = \begin{bmatrix} -9 & 2 \\ 6.2 & 2 \\ \frac{1}{3} & \sqrt{8} \end{bmatrix}$$

$$C = \begin{bmatrix} 11 & 4 \\ 1 & -\frac{1}{3} \\ 1 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 3 & 8 \\ -1 & 6 & \frac{1}{6} \end{bmatrix}$$

27. $A + B$

28. $A + C$

29. $C - A + B$

30. $B - A - C$

31. $B + D$

32. $C + D$

Learn: Multiply a Matrix by a Scalar

33. Explain how to multiply a matrix by a scalar.

34. Given matrix A, explain how to find its additive inverse $-A$.

For Exercises 35–42, use $A = \begin{bmatrix} 2 & 4 & -9 \\ 1 & \sqrt{3} & \frac{1}{2} \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 4 \\ 2 & 9 & \frac{2}{3} \end{bmatrix}$. (See Example 4 on p. 982)

35. $3A$

36. $-6B$

37. $-2A - 7B$

38. $4A - 3B$

39. $-4(A + B)$

40. $2(A - B)$

41. $-3A + 5(A - B)$

42. $-8A - 2(A + B)$

For Exercises 43–48, use $A = \begin{bmatrix} 1 & 6 \\ 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -4 \\ 6 & 9 \end{bmatrix}$ and solve for X. (See Example 5 on p. 983)

43. $2X - B = A$

44. $3X + A = B$

45. $A + 5X = B$

46. $B - 4X = A$

47. $2A - B = 10X$

48. $3B - A = -2X$

Learn: Multiply Matrices

49. Given that A is a 4×2 matrix and B is a 2×1 matrix,

- **a.** Is AB defined? If so, what is the order of AB?
- **b.** Is BA defined? If so, what is the order of BA?

50. Given that C is a 3×7 matrix and D is a 7×2 matrix,

- **a.** Is CD defined? If so, what is the order of CD?
- **b.** Is DC defined? If so, what is the order of DC?

51. Given that E is a 5×1 matrix and F is a 1×5 matrix,

- **a.** Is EF defined? If so, what is the order of EF?
- **b.** Is FE defined? If so, what is the order of FE?

52. Given that G is a 1×6 matrix and H is a 6×1 matrix,

- **a.** Is GH defined? If so, what is the order of GH?
- **b.** Is HG defined? If so, what is the order of HG?

For Exercises 53–64, (See Examples 6–7 on pp. 985–986)

a. Find AB if possible.

b. Find BA if possible.

c. Find A^2 if possible.

53. $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}$

54. $A = \begin{bmatrix} 1 & -6 \\ 5 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 \\ 7 & -1 \end{bmatrix}$

55. $A = \begin{bmatrix} 2 & 4 \\ -6 & 3 \\ 1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 & -1 \\ -2 & 0 & 10 \end{bmatrix}$

56. $A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 9 & 2 \\ 0 & 4 \end{bmatrix}$

57. $A = \begin{bmatrix} -9 & 2 & 3 \\ -1 & 5 & 4 \\ 0 & 1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}$

58. $A=\begin{bmatrix}2\\-1\\3\end{bmatrix}$ and $B=\begin{bmatrix}2 & 7 & -1\\0 & 4 & 1\\0 & 3 & -6\end{bmatrix}$

59. $A=\begin{bmatrix}1 & \frac{1}{2}\end{bmatrix}$ and $B=\begin{bmatrix}-\frac{1}{3}\\2\end{bmatrix}$

60. $A=\begin{bmatrix}4\\\frac{3}{4}\\1\end{bmatrix}$ and $B=\begin{bmatrix}-5 & \frac{1}{2} & \frac{1}{3}\end{bmatrix}$

61. $A=\begin{bmatrix}4\\-6\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2 & 5 & 6\end{bmatrix}$

62. $A=[4]$ and $B=\begin{bmatrix}-3 & 4 & 1\end{bmatrix}$

63. $A=[-5]$ and $B=[5]$

64. $A=[6]$ and $B=[-2]$

For Exercises 65–68, find *AB* and *BA*.

65. $A=\begin{bmatrix}3.1 & -2.3\\1.1 & 6.5\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

66. $A=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0.05 & -0.07\\0.16 & 0.09\end{bmatrix}$

67. $A=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}\frac{9}{5} & -3 & \sqrt{6}\\5 & \frac{1}{2} & 2\\3 & 0 & 1\end{bmatrix}$

68. $A=\begin{bmatrix}\frac{2}{3} & 5 & \sqrt{2}\\1 & 0 & 3\\-\frac{1}{4} & 0 & 8\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

Learn: Apply Operations on Matrices

69. Matrix *D* gives the dealer invoice prices for sedan and hatchback models of a car with manual transmission or automatic transmission. Matrix *M* gives the MSRP (manufacturer's suggested retail price) for the cars.

$$\begin{array}{cc} & \begin{array}{cc}\textbf{Sedan} & \textbf{Hatchback}\end{array}\\ D= & \begin{bmatrix}\$29{,}000 & \$27{,}500\\ \$28{,}500 & \$26{,}900\end{bmatrix}\begin{array}{l}\textbf{Manual}\\ \textbf{Automatic}\end{array}\end{array}$$

$$\begin{array}{cc} & \begin{array}{cc}\textbf{Sedan} & \textbf{Hatchback}\end{array}\\ M= & \begin{bmatrix}\$32{,}600 & \$29{,}900\\ \$31{,}900 & \$28{,}900\end{bmatrix}\begin{array}{l}\textbf{Manual}\\ \textbf{Automatic}\end{array}\end{array}$$

a. Compute $M-D$ and interpret the result.

b. A buyer thinks that a fair price is 6% above dealer invoice. Use scalar multiplication to determine a matrix *F* that gives the fair price for these cars for each type of transmission.

70. In matrix *C*, a coffee shop records the cost to produce a cup of standard Columbian coffee and the cost to produce a cup of hot chocolate. Matrix *P* contains the selling prices to the customer.

$$\begin{array}{cc} & \begin{array}{cc}\textbf{Coffee} & \textbf{Chocolate}\end{array}\\ C= & \begin{bmatrix}\$0.90 & \$0.84\\ \$1.26 & \$1.15\\ \$1.64 & \$1.50\end{bmatrix}\begin{array}{l}\textbf{Small}\\ \textbf{Medium}\\ \textbf{Large}\end{array}\end{array}$$

$$\begin{array}{cc} & \begin{array}{cc}\textbf{Coffee} & \textbf{Chocolate}\end{array}\\ P= & \begin{bmatrix}\$3.05 & \$2.25\\ \$3.65 & \$3.05\\ \$4.15 & \$3.65\end{bmatrix}\begin{array}{l}\textbf{Small}\\ \textbf{Medium}\\ \textbf{Large}\end{array}\end{array}$$

a. Compute $P-C$ and interpret its meaning.

b. If the tax rate in a certain city is 7%, use scalar multiplication to find a matrix *F* that gives the final price to the customer (including sales tax) for both beverages for each size. Round each entry to the nearest cent.

71. A street vendor at a parade sells fresh lemonade, soda, bottled water, and iced-tea, and the unit price for each item is given in matrix *P*. The number of units sold of each item is given in matrix *N*. Compute *NP* and interpret the result.

$$\begin{array}{cc} & \begin{array}{cccc}\textbf{Lemonade} & \textbf{Soda} & \textbf{Water} & \textbf{Tea}\end{array}\\ N= & \begin{bmatrix}150 & 270 & 440 & 80\end{bmatrix}\end{array}$$

$$P=\begin{bmatrix}\$2.50\\ \$1.50\\ \$2.00\\ \$2.00\end{bmatrix}\begin{array}{l}\textbf{Lemonade}\\ \textbf{Soda}\\ \textbf{Water}\\ \textbf{Tea}\end{array}$$

72. A math course has 4 exams weighted 20%, 25%, 25%, and 30%, respectively, toward the final grade. Suppose that a student earns grades of 75, 84, 92, and 86, respectively, on the four exams. The weights are given in matrix *W* and the test grades are given in matrix *G*. Compute *WG* and interpret the result.

$$\begin{array}{cc} & \begin{array}{cccc}\textbf{Test 1} & \textbf{Test 2} & \textbf{Test 3} & \textbf{Test 4}\end{array}\\ W= & \begin{bmatrix}0.20 & 0.25 & 0.25 & 0.30\end{bmatrix},\end{array}$$

$$G=\begin{bmatrix}75\\84\\92\\86\end{bmatrix}\begin{array}{l}\textbf{Test 1}\\ \textbf{Test 2}\\ \textbf{Test 3}\\ \textbf{Test 4}\end{array}$$

73. An electronics store sells three models of tablets. The number of each model sold during "Black Friday" weekend is given in matrix *A*. The selling price and profit for each model are given in matrix *B*. **(See Example 8 on p. 988)**

$$\begin{array}{rl} & \begin{matrix} & \textbf{Model} & \\ \textbf{A} & \textbf{B} & \textbf{C} \end{matrix} \\ A = & \begin{bmatrix} 84 & 70 & 32 \\ 62 & 48 & 16 \\ 70 & 40 & 12 \end{bmatrix} \begin{matrix} \textbf{Friday} \\ \textbf{Saturday} \\ \textbf{Sunday} \end{matrix} \end{array}$$

$$\begin{array}{rl} & \begin{matrix} \textbf{Selling Price} & \textbf{Profit} \end{matrix} \\ B = & \begin{bmatrix} \$499 & \$200 \\ \$599 & \$240 \\ \$629 & \$280 \end{bmatrix} \begin{matrix} \textbf{A} \\ \textbf{B Model} \\ \textbf{C} \end{matrix} \end{array}$$

a. Compute *AB* and interpret the result.

b. Determine the total revenue for Sunday.

c. Determine the total profit for the 3-day period for these three models.

74. A gas station manager records the number of gallons of Regular, Plus, and Premium gasoline sold during the week (Monday–Friday) and on the weekends (Saturday–Sunday) in matrix *A*. The selling price and profit for 1 gal of each type of gasoline are given in matrix *B*.

$$\begin{array}{rl} & \begin{matrix} \textbf{Regular} & \textbf{Plus} & \textbf{Premium} \end{matrix} \\ A = & \begin{bmatrix} 4600 & 1850 & 720 \\ 2300 & 620 & 480 \end{bmatrix} \begin{matrix} \textbf{Weekdays} \\ \textbf{Weekend} \end{matrix} \end{array}$$

$$\begin{array}{rl} & \begin{matrix} \textbf{Selling Price} & \textbf{Profit} \end{matrix} \\ B = & \begin{bmatrix} \$3.59 & \$0.21 \\ \$3.79 & \$0.24 \\ \$4.19 & \$0.18 \end{bmatrix} \begin{matrix} \textbf{Regular} \\ \textbf{Plus} \\ \textbf{Premium} \end{matrix} \end{array}$$

a. Compute *AB* and interpret the result.

b. Determine the profit for the weekend.

c. Determine the revenue for the entire week.

75. The labor costs per hour for an electrician, plumber, and air-conditioning/heating expert are given in matrix *L*. The time required from each specialist for three new model homes is given in matrix *T*.

$$\begin{array}{rl} & \textbf{Cost/hr} \\ L = & \begin{bmatrix} \$45 \\ \$38 \\ \$35 \end{bmatrix} \begin{matrix} \textbf{Electrician} \\ \textbf{Plumber} \\ \textbf{AC/heating} \end{matrix} \end{array}$$

$$\begin{array}{rl} & \begin{matrix} & \textbf{Time (hr)} & \\ \textbf{Electrician} & \textbf{Plumber} & \textbf{AC/heating} \end{matrix} \\ T = & \begin{bmatrix} 22 & 16 & 14 \\ 28 & 21 & 18 \\ 18 & 14 & 9 \end{bmatrix} \begin{matrix} \textbf{Model 1} \\ \textbf{Model 2} \\ \textbf{Model 3} \end{matrix} \end{array}$$

a. Which product, *LT* or *TL*, gives the total cost for these three services for each model?

b. Find a matrix that gives the total cost for these three services for each model.

76. The number of calories burned per hour for three activities is given in matrix *N* for a 140-lb person training for a triathlon. The time spent on each activity for two different training days is given in matrix *T*.

$$\begin{array}{rl} & \textbf{Calories/hr} \\ N = & \begin{bmatrix} 540 \\ 400 \\ 360 \end{bmatrix} \begin{matrix} \textbf{Running} \\ \textbf{Bicycling} \\ \textbf{Swimming} \end{matrix} \end{array}$$

$$\begin{array}{rl} & \begin{matrix} & \textbf{Time} & \\ \textbf{Running} & \textbf{Bicycling} & \textbf{Swimming} \end{matrix} \\ T = & \begin{bmatrix} \text{45 min} & \text{1 hr} & \text{30 min} \\ \text{1 hr} & \text{1 hr 30 min} & \text{45 min} \end{bmatrix} \begin{matrix} \textbf{Day 1} \\ \textbf{Day 2} \end{matrix} \end{array}$$

a. Which product, *NT* or *TN*, gives the total number of calories burned from these activities for each day?

b. Find a matrix that gives the total number of calories burned from these activities for each day.

77. A student researches the cost for three cell phone plans. Matrix C contains the cost per text message and the cost per minute over the maximum number of minutes allowed in each plan. Matrix N_1 contains the number of text messages and the number of minutes over the maximum incurred for 1 month. Matrix N_3 represents the number of text messages and number of minutes over the maximum for 3 months.

$$C = \begin{array}{cl} \begin{array}{cc} \textbf{Cost/text} & \textbf{Cost/min} \end{array} & \\ \begin{bmatrix} \$0.25 & \$0.40 \\ \$0 & \$0.40 \\ \$0.10 & \$0 \end{bmatrix} & \begin{array}{l} \textbf{Plan A} \\ \textbf{Plan B} \\ \textbf{Plan C} \end{array} \end{array}$$

$$N_1 = \begin{bmatrix} 24 \\ 100 \end{bmatrix} \begin{array}{l} \textbf{Number of texts} \\ \textbf{Minutes over} \end{array}$$

$$N_3 = \begin{array}{cl} \begin{array}{ccc} \textbf{Month 1} & \textbf{Month 2} & \textbf{Month 3} \end{array} & \\ \begin{bmatrix} 24 & 56 & 30 \\ 100 & 24 & 0 \end{bmatrix} & \begin{array}{l} \textbf{Number of texts} \\ \textbf{Minutes over} \end{array} \end{array}$$

a. Find the product CN_1 and interpret its meaning.

b. Find the product CN_3 and interpret its meaning.

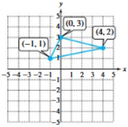
78. Refer to Exercise 77. Suppose that matrix B represents the base cost for each cell phone plan and matrix T represents the tax for each plan.

$$B = \begin{bmatrix} \$39.99 \\ \$49.99 \\ \$59.99 \end{bmatrix} \begin{array}{l} \textbf{Plan A} \\ \textbf{Plan B} \\ \textbf{Plan C} \end{array}$$

$$T = \begin{bmatrix} \$11.96 \\ \$13.04 \\ \$14.91 \end{bmatrix} \begin{array}{l} \textbf{Plan A} \\ \textbf{Plan B} \\ \textbf{Plan C} \end{array}$$

a. Compute $B + CN_1 + T$ and interpret its meaning.

b. Which cell phone plan is the least expensive if the student has 60 text messages and talks 20 min more than the maximum?



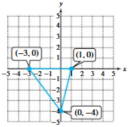
79. a. Write a matrix A that represents the coordinates of the vertices of the triangle. Place the x-coordinate of each point in the first row of A and the corresponding y-coordinate in the second row of A. **(See Example 9 on p. 989)**

b. Use addition of matrices to shift the triangle 2 units to the right and 4 units downward.

c. Find the product $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} A$ and explain the effect on the graph of the triangle.

d. Find the product $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A$ and explain the effect on the graph of the triangle.

e. Find $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A + \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$ and explain the effect on the graph of the triangle.



80. a. Write a matrix A that represents the coordinates of the vertices of the triangle. Place the x-coordinate of each point in the first row of A and the corresponding y-coordinate in the second row of A.

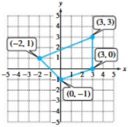
b. Use addition of matrices to shift the triangle 1 unit to the left and 3 units upward.

c. Find the product $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} A$ and explain the effect on the graph of the triangle.

d. Find the product $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A$ and explain the effect on the graph of the triangle.

e. Find $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} A + \begin{bmatrix} 2 & 2 & 2 \\ -5 & -5 & -5 \end{bmatrix}$ and explain the effect on the graph of the triangle.

81. a. Write a matrix A that represents the coordinates of the vertices of the quadrilateral.



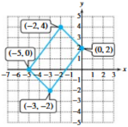
b. What operation on A will shift the graph of the quadrilateral 3 units downward?

c. What operation on A will shift the graph 4 units to the left?

d. Use matrix multiplication to reflect the graph across the x-axis.

e. Use matrix multiplication to reflect the graph across the y-axis.

82. a. Write a matrix A that represents the coordinates of the vertices of the quadrilateral.



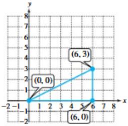
b. What operation on A will shift the graph of the quadrilateral 6 units upward?

c. What operation on A will shift the graph 2 units to the right?

d. Use matrix multiplication to reflect the graph across the x-axis.

e. Use matrix multiplication to reflect the graph across the y-axis.

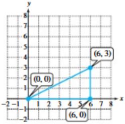
83. a. Write a matrix A that represents the coordinates of the vertices of the triangle.



b. Multiply $\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}A$ and round each entry to 1 decimal place.

c. Graph the figure represented by the matrix from part (b). What effect does this product have on the graph of the triangle?

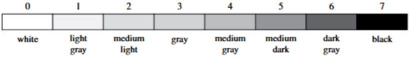
84. a. Write a matrix A that represents the coordinates of the vertices of the triangle.



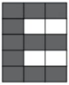
b. Multiply $\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}A$ and round each entry to 1 decimal place.

c. Graph the figure represented by the matrix from part (b). What effect does this product have on the graph of the triangle?

For Exercises 85–86, use the following gray scale.

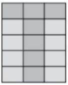


85. a. Write a 5 × 3 matrix that represents the letter E in dark gray on a white background.



b. Use matrix addition to change the pixels so that the letter E is medium dark on a light gray background.

86. a. Write a 5 × 3 matrix that represents the letter T in medium gray on a medium light background.



b. Use matrix addition to change the pixels so that the letter T is dark gray on a light gray background.

Mixed Exercises

For Exercises 87–92, use matrices *A*, *B*, and *C* to prove the given properties. Assume that the elements within *A*, *B*, and *C* are real numbers.

$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ $C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$

87. Commutative property of matrix addition

$$A + B = B + A$$

88. Associative property of matrix addition

$$A + (B + C) = (A + B) + C$$

89. Inverse property of matrix addition

$$A + (-A) = 0$$

90. Identity property of matrix addition

$$A + 0 = A$$

91. Associative property of scalar multiplication

$$s(tA) = (st)A$$

92. Distributive property of scalar multiplication

$$t(A + B) = tA + tB$$

93. Given $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, find A^2, A^3, and A^4. (*Hint*: Recall that $i = \sqrt{-1}$.) Discuss the similarities between A^n and i^n, where n is a positive integer.

94. Given $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, find B^2.

95. a. For real numbers a, b, c, and d, find the product.

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$$

b. Based on the form of the product of part (a), compute the following product mentally.

$$\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

96. Find the product.

$$\begin{bmatrix} a & a \\ b & b \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Write About It

97. Explain why the product of a 3 × 2 matrix by a 3 × 2 matrix is undefined.

98. Explain how to add or subtract matrices.

99. Given a matrix, $A = [a_{ij}]$, explain how to find the additive inverse of A.

100. Given two matrices A and B, how can you determine the order of AB, assuming that the product is defined?

Technology Connections

For Exercises 101–104, refer to matrices *A*, *B*, and *C* and perform the indicated operations on a calculator.

$A = \begin{bmatrix} 1.05 & 3.9 \\ 4.12 & -9.4 \\ -2.4 & 1.5 \end{bmatrix}$ $B = \begin{bmatrix} -10 & 30 \\ 24 & -36 \\ 18 & -8 \end{bmatrix}$

$C = \begin{bmatrix} 6.2 \\ 4.9 \end{bmatrix}$

101. $2.5A - 3.6B$

102. $-6.4(A + B)$

103. $-3AC$

104. $7.5BC$

Go online for more practice problems.

Lesson 9-4

Inverse Matrices and Matrix Equations

What Will You Learn?

After completing this lesson, you should be able to:

- Identify Identity and Inverse Matrices
- Determine the Inverse of a Matrix
- Solve Systems of Linear Equations Using the Inverse of a Matrix

Learn: Identify Identity and Inverse Matrices

The identity element under the multiplication of real numbers is 1 because $a \cdot 1 = a$ and $1 \cdot a = a$. We now investigate a similar property for the product of square matrices. The **identity matrix** I_n for matrix multiplication is the $n \times n$ square matrix with 1's along the main diagonal and 0's for all other elements. For example:

The identity matrix of order 2

The identity matrix of order 3

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For an $n \times n$ square matrix A, we have that

$$AI_n = A \quad \text{and} \quad I_nA = A \quad \text{(Identity property of matrix multiplication)}$$

In Example 1, we illustrate the identity property of matrix multiplication using a 2×2 matrix.

Example 1: Illustrating the Identity Property of Matrix Multiplication

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ show that

a. $AI_2 = A$ **b.** $I_2A = A$

Solution:

a. $AI_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a(1) + b(0) & a(0) + b(1) \\ c(1) + d(0) & c(0) + d(1) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ✓

b. $I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1(a) + 0(c) & 1(b) + 0(d) \\ 0(a) + 1(c) & 0(b) + 1(d) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ✓

Apply the Skills

1. Given $A = \begin{bmatrix} 3 & -4 \\ -2 & 10 \end{bmatrix}$ show that

a. $AI_2 = A$ **b.** $I_2A = A$

For a nonzero real number a, the multiplicative inverse of a is $\frac{1}{a}$ because $a \cdot \frac{1}{a} = 1$ and $\frac{1}{a} \cdot a = 1$. We define the multiplicative inverse of a square matrix in a similar fashion.

Multiplicative Inverse of a Square Matrix

Let A be an $n \times n$ matrix and let I_n be the identity matrix of order n. If there exists an $n \times n$ matrix A^{-1} such that

$$AA^{-1} = I_n \quad \text{and} \quad A^{-1}A = I_n$$

then A^{-1} (read as "A inverse") is the **multiplicative inverse** of A.

Check Your Work

Be sure to interpret the notation A^{-1} as the *inverse* of A, rather than the reciprocal of A. That is, $A^{-1} \neq \frac{1}{A}$.

Example 2: Determining Whether Two Matrices Are Inverses

Determine whether $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$ are inverses.

Solution:

We must show that $AB = I_2$ and $BA = I_2$.

$$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 3(5) + 2(-7) & 3(-2) + 2(3) \\ 7(5) + 5(-7) & 7(-2) + 5(3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$BA = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 5(3) + (-2)(7) & 5(2) + (-2)(5) \\ -7(3) + 3(7) & -7(2) + 3(5) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Apply the Skills

2. Determine whether $A = \begin{bmatrix} 4 & 3 \\ 13 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 10 & -3 \\ -13 & 4 \end{bmatrix}$ are inverses.

Learn: Determine the Inverse of a Matrix

In Example 2, we showed that two given matrices are inverses. Now we look at the task of finding the inverse of a matrix. In Example 3, we will find the inverse of a 2×2 matrix directly from the definition. Then, we develop a general procedure to find the inverse of an $n \times n$ matrix, if the inverse exists.

Example 3: Finding the Inverse of a Matrix

Given $A = \begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix}$, find A^{-1}.

Solution:

We need to find a matrix $A^{-1} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ such that $AA^{-1} = I_2$. That is,

$$\begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2x_1 + 9x_3 & 2x_2 + 9x_4 \\ x_1 + 5x_3 & x_2 + 5x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the matrices to be equal, their corresponding elements must be equal. This results in two systems of equations.

Equating the elements from the 1st column

$$\begin{cases} 2x_1 + 9x_3 = 1 \\ x_1 + 5x_3 = 0 \end{cases} \quad \text{and} \quad \left.\begin{matrix} 2x_2 + 9x_4 = 0 \\ x_2 + 5x_4 = 1 \end{matrix}\right\}$$

Equating the elements from the 2nd column

The corresponding augmented matrices are $\left[\begin{array}{cc|c} 2 & 9 & 1 \\ 1 & 5 & 0 \end{array}\right]$ and $\left[\begin{array}{cc|c} 2 & 9 & 0 \\ 1 & 5 & 1 \end{array}\right]$

We can solve both systems simultaneously by placing the identity matrix to the right of the array of coefficients.

$$\left[\begin{array}{cc|cc} 2 & 9 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right]$$

Apply Gauss-Jordan elimination.

$$\left[\begin{array}{cc|cc} 2 & 9 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 2 & 9 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 5 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{-1R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 5 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right] \xrightarrow{-5R_2 + R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & 5 & -9 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

This result represents the following two matrices and their corresponding systems of equations.

$$\begin{array}{c} x_1 \quad x_3 \\ \downarrow \quad \downarrow \\ \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1 \end{array}\right] \\ x_1 = 5,\ x_3 = -1 \end{array} \quad \text{and} \quad \begin{array}{c} x_2 \quad x_4 \\ \downarrow \quad \downarrow \\ \left[\begin{array}{cc|c} 1 & 0 & -9 \\ 0 & 1 & 2 \end{array}\right] \\ x_2 = -9,\ x_4 = 2 \end{array}$$

Therefore, we have: $A^{-1} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 5 & -9 \\ -1 & 2 \end{bmatrix}$

Check Your Work

The solution to Example 3 can be checked by verifying that $AA^{-1} = I_2$ and $A^{-1}A = I_2$.

Apply the Skills

3. Given $B = \begin{bmatrix} 9 & 7 \\ 5 & 4 \end{bmatrix}$, find B^{-1}.

In Example 3, we transformed the left side of the augmented matrix into the identity matrix. That is, we performed row operations to make the array on the left of the vertical bar have 1's along the main diagonal, and 0's elsewhere. In the process, we make the following important observation:

> The array of elements to the right of the vertical bar is the inverse of *A*.

This observation leads to a general procedure to find the multiplicative inverse of a matrix, provided that the inverse exists.

Finding the Multiplicative Inverse of a Square Matrix

Let A be an $n \times n$ matrix for which A^{-1} exists, and let I_n be the $n \times n$ identity matrix. To find A^{-1},

Step 1 Write a matrix of the form $[A \mid I_n]$.

Step 2 Perform row operations to write the matrix in the form $[I_n \mid B]$.

Step 3 The matrix B is A^{-1}.

It is important to note that not all matrices have a multiplicative inverse. If a matrix A is reducible to a row-equivalent matrix with one or more rows of zeros, then the matrix cannot be written in the form $[I_n \mid B]$. Therefore, the matrix does not have an inverse, and we say that the matrix is **singular**. A matrix that *does* have a multiplicative inverse is said to be **invertible** or **nonsingular**.

In Example 4, we use the general procedure to find the inverse of a 3×3 invertible matrix.

Example 4: Finding the Inverse of a Matrix

Given $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & -2 \\ -3 & -3 & 5 \end{bmatrix}$, find A^{-1} if possible.

Solution:

$$\left[\begin{array}{rrr|rrr} 1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ -3 & -3 & 5 & 0 & 0 & 1 \end{array}\right]$$

Set up the matrix $[A \mid I_3]$. Transform the left side of the matrix to the 3×3 identity matrix.

$$\xrightarrow[3R_1 + R_3 \to R_3]{-1R_1 + R_2 \to R_2} \left[\begin{array}{rrr|rrr} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 6 & 5 & 3 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2 \to R_2} \left[\begin{array}{rrr|rrr} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 6 & 5 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-6R_2 + R_3 \to R_3]{-3R_2 + R_1 \to R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & -3 & -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & 3 & 1 \end{array}\right] \xrightarrow{-1R_3 \to R_3} \left[\begin{array}{rrr|rrr} 1 & 0 & -3 & -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -3 & -1 \end{array}\right]$$

$$\xrightarrow[-1R_3+R_2 \to R_2]{3R_3+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{15}{2} & -3 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{5}{2} & 1 \\ 0 & 0 & 1 & 0 & -3 & -1 \end{array}\right]$$

The left side of the matrix is the 3×3 identity matrix. The right side of the matrix represents the inverse of A.

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{15}{2} & -3 \\ \frac{1}{2} & \frac{5}{2} & 1 \\ 0 & -3 & -1 \end{bmatrix}$$

Check Your Work

As a check, verify that $A \cdot A^{-1} = I_3$ and $A^{-1} \cdot A = I_3$.

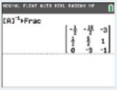
Apply the Skills

4. Given $A = \begin{bmatrix} -2 & 4 & 1 \\ 4 & -13 & 10 \\ 3 & -9 & 6 \end{bmatrix}$ find A^{-1} if possible.

TECHNOLOGY CONNECTIONS

Finding the Inverse of a Nonsingular Square Matrix

To find the inverse of a nonsingular matrix [A] using a calculator, first enter the matrix into the calculator using the MATRIX menu. Then place [A] on the home screen and use the x^{-1} key to take the inverse of [A]. It is very important to realize that in this context, the calculator uses x^{-1} the key for the inverse of a matrix, not the reciprocal of a real number.



[A]$^{-1}$▸Frac

$$\begin{bmatrix} -\frac{1}{2} & -\frac{15}{2} & -3 \\ \frac{1}{2} & \frac{5}{2} & 1 \\ 0 & -3 & -1 \end{bmatrix}$$

In Example 5, we identify a singular matrix—that is, a matrix that has no inverse.

Example 5: Identifying a Singular Matrix

Show that matrix A is singular. $A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Set up the matrix $[A \mid I_3]$.

Transform the left side of the matrix to the 3×3 identity matrix.

$$\xrightarrow[-2R_1+R_3 \to R_3]{-4R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & -4 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 & 1 \end{array}\right]$$

On the left of the vertical bar, the elements in rows 2 and 3 are multiples of each other (in fact they are identical). Therefore, $-1R_2 + R_3 \to R_3$ results in zeros in row 3 to the left of the bar.

$$\xrightarrow{-1R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 \end{array}\right]$$

The left side of the matrix cannot be transformed into the 3 × 3 identity matrix I_3. Therefore, matrix A does not have an inverse.

A is a singular matrix (does not have an inverse).

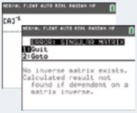
Apply the Skills

5. Show that matrix A is singular. $A = \begin{bmatrix} 1 & -3 & -7 \\ 3 & -7 & -17 \\ 1 & 0 & -1 \end{bmatrix}$

TECHNOLOGY CONNECTIONS

Identifying a Singular Matrix

A calculator will return an error message we attempt to find the inverse of a singular matrix. Consider matrix A from Example 5. The calculator returns the error message shown.


NORMAL FLOAT AUTO REAL RADIAN MP
NORMAL FLOAT AUTO REAL RADIAN MP

```
[A]⁻¹
ERROR: SINGULAR MATRIX
1:Quit
2:Goto
No inverse matrix exists.
Calculated result not
  found if dependent on a
  matrix inverse.
```

To find the inverse of an invertible 2 × 2 matrix, we can take a direct approach and apply the following formula.

Formula for the Inverse of a 2 × 2 Invertible Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an invertible matrix. Then the inverse A^{-1} is given by:

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Insights If A is an invertible matrix then the restriction $ad - bc \neq 0$ is implied.

This formula can be verified by applying the general procedure to find the inverse of A. See Exercise 69.

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow[\text{matrix}]{\text{row-equivalent}} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{\text{inverse of } A} \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Check Your Work

The formula to find the inverse of an invertible 2 × 2 matrix does not apply to matrices of higher order.

Example 6: Finding the Inverse of a 2 × 2 Matrix

Given $A = \begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix}$, find A^{-1} and compare to the result of Example 3.

Solution:

$$\begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix} \quad a = 2, b = 9, c = 1, d = 5$$

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(2)(5) - (9)(1)} \cdot \begin{bmatrix} 5 & -9 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{1} \cdot \begin{bmatrix} 5 & -9 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -9 \\ -1 & 2 \end{bmatrix}$$ This is the same result as was found in Example 3.

Apply the Skills

6. Given $B = \begin{bmatrix} 9 & 7 \\ 5 & 4 \end{bmatrix}$, find B^{-1}.

Learn: Solve Systems of Linear Equations Using the Inverse of a Matrix

A system of linear equations written in standard form can be represented by using matrix multiplication. For example:

$$\begin{aligned} 3x - 2y + z &= 4 \\ x + 4y - z &= 2 \\ 2x + 5y - 2z &= 1 \end{aligned} \xrightarrow[\text{matrix equation}]{\text{corresponding}} \underbrace{\begin{bmatrix} 3 & -2 & 1 \\ 1 & 4 & -1 \\ 2 & 5 & -2 \end{bmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{column matrix of variables}} = \underbrace{\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}}_{\text{column matrix of constants}}$$

The matrix on the left is called the **coefficient matrix**. It consists of the coefficients of the variable terms.

If we denote the coefficient matrix by A, the column matrix of variables as X, and the column matrix of constant terms as B, we can write the equation as $AX = B$. If A^{-1} exists, then we can solve the equation for X by multiplying both sides by A^{-1}.

$$\begin{aligned} AX &= B && \text{To solve this equation, the goal is to isolate } X. \\ A^{-1}AX &= A^{-1}B && \text{Multiply both sides by } A^{-1} \text{ (provided that } A^{-1} \text{ exists).} \\ (A^{-1}A)X &= A^{-1}B && \\ I_n X &= A^{-1}B && \text{The product of a matrix and its inverse is the identity matrix.} \\ X &= A^{-1}B && \end{aligned}$$

This process is summarized as follows:

Solving Systems of Linear Equations Using Inverse Matrices

Suppose that $AX = B$ represents a system of n linear equations in n variables with a unique solution. Then,

$$X = A^{-1}B$$

where A is the coefficient matrix, B is the matrix of constants, and X is the matrix of variables.

Check Your Work

The technique of using inverse matrices to solve a system of linear equations does not apply if matrix A does not have an inverse. This occurs when the system of equations is inconsistent or the equations are dependent.

Example 7: Solving a System of Equations by Using an Inverse Matrix

Use the inverse of the coefficient matrix to solve the system.

$$\begin{aligned} x + 3y \phantom{{}-2z} &= -10 \\ x + y - 2z &= -4 \\ -3x - 3y + 5z &= 11 \end{aligned}$$

Solution:

$$\begin{aligned} x + 3y \phantom{{}-2z} &= -10 \\ x + y - 2z &= -4 \\ -3x - 3y + 5z &= 11 \end{aligned} \longrightarrow \overbrace{\begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & -2 \\ -3 & -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \\ 11 \end{bmatrix}}^{AX = B}$$

From Example 4, we have that $A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{15}{2} & -3 \\ \frac{1}{2} & \frac{5}{2} & 1 \\ 0 & -3 & -1 \end{bmatrix}$.

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{15}{2} & -3 \\ \frac{1}{2} & \frac{5}{2} & 1 \\ 0 & -3 & -1 \end{bmatrix} \begin{bmatrix} -10 \\ -4 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}\right)(-10) + \left(-\frac{15}{2}\right)(-4) + (-3)(11) \\ \left(\frac{1}{2}\right)(-10) + \left(\frac{5}{2}\right)(-4) + (1)(11) \\ (0)(-10) + (-3)(-4) + (-1)(11) \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$ Therefore, $x = 2$, $y = -4$, and $z = 1$. The solution set is $\{(2, -4, 1)\}$.

Apply the Skills

7. Use the inverse of the coefficient matrix found in Skill Practice 4 to solve the system.

$$\begin{aligned} -2x + 4y + z &= 0 \\ 4x - 13y + 10z &= 24 \\ 3x - 9y + 6z &= 15 \end{aligned}$$

Practice 9-4

Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, match the statement with the property that describes it.

- **a.** Identity property of addition
- **b.** Inverse property of multiplication
- **c.** Identity property of multiplication
- **d.** Inverse property of addition

R.1. $\frac{1}{7} \cdot 7 = 1$

R.2. $15 \cdot 1 = 15$

R.3. $-6 + 6 = 0$

R.4. $7 + 0 = 7$

Concept Connections

1. The symbol I_n represents the ____________ matrix of order n.

2. I_n is an $n \times n$ matrix with ____________'s along the main diagonal and ____________ elsewhere.

3. Given an $n \times n$ matrix A, if there exists a matrix A^{-1} such that $A \cdot A^{-1} = I_n$ and $A^{-1} \cdot A = I_n$, then A^{-1} is called the ____________ of A.

4. A matrix that does not have an inverse is called a ____________ matrix. A matrix that does have an inverse is said to be invertible or ____________.

5. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an invertible matrix. Then a formula for the inverse A^{-1} is given by ____________.

6. Suppose that the matrix equation $AX = B$ represents a system of n linear equations in n variables with a unique solution. Then $X =$ ____________.

Learn: Identify Identity and Inverse Matrices

7. Write the matrix I_2.

8. Write the matrix I_3.

For Exercises 9–12, verify that

a. $AI_n = A$

b. $I_nA = A$ **(See Example 1 on p. 998)**

9. $\begin{bmatrix} -\frac{7}{8} & \sqrt{5} \\ 5.1 & 8 \end{bmatrix}$

10. $A = \begin{bmatrix} \sqrt{3} & 1 \\ \pi & 4 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & -3 & 4 \\ 9 & 5 & 3 \\ 11 & -6 & -4 \end{bmatrix}$

12. $A = \begin{bmatrix} -3 & 9 & 1 \\ 0 & 4 & -1 \\ 5 & 0 & 3 \end{bmatrix}$

For Exercises 13–18, determine whether *A* and *B* are inverses. (See Example 2 on p. 999)

13. $A = \begin{bmatrix} 10 & -3 \\ 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{4} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{5}{4} \end{bmatrix}$

14. $A = \begin{bmatrix} 4 & -1 \\ -8 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{3}{8} & \frac{1}{16} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

15. $A = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

16. $A = \begin{bmatrix} -1 & 2 & 1 \\ -5 & 8 & 2 \\ 7 & -11 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 4 & 3 \\ 1 & -3 & -2 \end{bmatrix}$

17. $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ -7 & 6 \end{bmatrix}$

18. $A = \begin{bmatrix} 3 & 2 \\ 10 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & 10 \\ 11 & -15 \end{bmatrix}$

Learn: Determine the Inverse of a Matrix

For Exercises 19–34, determine the inverse of the given matrix if possible. Otherwise, state that the matrix is singular. (See Examples 3–6 on pp. 1000–1004)

19. $A = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix}$

20. $A = \begin{bmatrix} 5 & -3 \\ 10 & -7 \end{bmatrix}$

21. $A = \begin{bmatrix} -8 & -2 \\ 10 & 5 \end{bmatrix}$

22. $A = \begin{bmatrix} 0 & 1 \\ 6 & -2 \end{bmatrix}$

23. $A = \begin{bmatrix} 3 & 7 \\ 6 & 14 \end{bmatrix}$

24. $A = \begin{bmatrix} 2 & -5 \\ -6 & 15 \end{bmatrix}$

25. $A = \begin{bmatrix} 2 & -7 & 8 \\ 1 & -3 & 3 \\ 1 & -5 & 6 \end{bmatrix}$

26. $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$

27. $A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

28. $A = \begin{bmatrix} -2 & 5 & -4 \\ 1 & -2 & 1 \\ 1 & -5 & 9 \end{bmatrix}$

29. $A = \begin{bmatrix} 5 & 7 & -11 \\ 3 & 5 & -9 \\ 1 & 1 & -1 \end{bmatrix}$

30. $A = \begin{bmatrix} 1 & 3 & 9 \\ 2 & 7 & 22 \\ -5 & -17 & -53 \end{bmatrix}$

31. $A = \begin{bmatrix} 0 & 1 & 1 \\ \frac{5}{3} & -1 & -3 \\ -1 & 1 & 2 \end{bmatrix}$

32. $A = \begin{bmatrix} 3 & -1 & -\frac{1}{2} \\ 2 & -1 & \frac{1}{2} \\ -5 & 2 & \frac{1}{2} \end{bmatrix}$

33. $A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & -1 & 7 & 1 \end{bmatrix}$

34. $A = \begin{bmatrix} 1 & 3 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ -2 & -4 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

Learn: Solve Systems of Linear Equations Using the Inverse of a Matrix

For Exercises 35–38, write the system of equations as a matrix equation of the form *AX* = *B*, where *A* is the coefficient matrix, *X* is the column matrix of variables, and *B* is the column matrix of constants.

35. $\begin{aligned} 3x - 4y &= -1 \\ 2x + y &= 14 \end{aligned}$

36. $\begin{aligned} -6x - y &= 1 \\ 2x + 3y &= 13 \end{aligned}$

37. $\begin{aligned} 9x - 6y + 4z &= 27 \\ 4x \qquad - z &= 1 \\ 3y + z &= 0 \end{aligned}$

38. $\begin{aligned} -3x \qquad + 8z &= 4 \\ 6y - z &= 7 \\ 2x - y + 6z &= -15 \end{aligned}$

For Exercises 39–50, solve the system by using the inverse of the coefficient matrix. (See Example 7 on p. 1005)

39. $\begin{aligned} -4x - 3y &= -4 \\ 6x + 5y &= 8 \end{aligned}$

(See Exercise 19 for A^{-1}.)

40. $\begin{aligned} 5x - 3y &= -20 \\ 10x - 7y &= -50 \end{aligned}$

(See Exercise 20 for A^{-1}.)

41. $\begin{aligned} 3x + 7y &= -5 \\ 4x + 9y &= -7 \end{aligned}$

42. $\begin{aligned} 3x - 8y &= -1 \\ -4x + 11y &= 2 \end{aligned}$

43. $\begin{aligned} 2x - 7y + 8z &= 1 \\ x - 3y + 3z &= 0 \\ x - 5y + 6z &= 2 \end{aligned}$

(See Exercise 25 for A^{-1}.)

44. $$\begin{aligned} x - y + z &= -4 \\ x + y - 2z &= 12 \\ -x + z &= -5 \end{aligned}$$

(See Exercise 26 for A^{-1}.)

45. $$\begin{aligned} x - 2y + 2z &= -12 \\ 2x - 3y + z &= -10 \\ y - z &= 6 \end{aligned}$$

(See Exercise 27 for A^{-1}.)

46. $$\begin{aligned} -2x + 5y - 4z &= -20 \\ x - 2y + z &= 8 \\ x - 5y + 9z &= 24 \end{aligned}$$

(See Exercise 28 for A^{-1}.)

47. $$\begin{aligned} r - 2s + t &= 2 \\ -r + 4s + t &= 3 \\ 2r - 2s - t &= -1 \end{aligned}$$

48. $$\begin{aligned} a - b + 3c &= 2 \\ 2a + b + 2c &= 2 \\ -2a - 2b + c &= 3 \end{aligned}$$

49. $$\begin{aligned} w - 2x + 5y &= 3 \\ -x + 2y &= 1 \\ x - y &= -1 \\ 2w - x + 7y + z &= 5 \end{aligned}$$

(See Exercise 33 for A^{-1}.)

50. $$\begin{aligned} w + 3x - 3z &= 8 \\ x - 2z &= 4 \\ -2w - 4x + y + 2z &= -6 \\ x + y &= 0 \end{aligned}$$

(See Exercise 34 for A^{-1}.)

Mixed Exercises

For Exercises 51–58, determine whether the statement is true or false. If a statement is false, explain why.

51. A 3 × 2 matrix has a multiplicative inverse.

52. A 2 × 3 matrix has a multiplicative inverse.

53. Every square matrix has an inverse.

54. Every singular matrix has an inverse.

55. Every nonsingular matrix has an inverse.

56. The matrix $\begin{bmatrix} -a & -b \\ a & b \end{bmatrix}$ is invertible.

57. The matrix $\begin{bmatrix} x & y \\ 2x & 2y \end{bmatrix}$ is invertible.

58. The inverse of the matrix I_4 is itself.

59. Find a 2 × 2 matrix that is its own inverse. Answers will vary.

60. Given an invertible 2 × 2 matrix A and the nonzero real number k, find the inverse of kA in terms of A^{-1}.

61. Given $A = \begin{bmatrix} 3 & 2 \\ 5 & 6 \end{bmatrix}$,

a. Find A^{-1}. **b.** Find $(A^{-1})^{-1}$.

62. Given $B = \begin{bmatrix} -3 & 2 \\ -5 & 4 \end{bmatrix}$,

a. Find B^{-1}. **b.** Find $(B^{-1})^{-1}$.

63. Given $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where a and b are nonzero real numbers, find A^{-1}.

64. Given $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b, and c are nonzero real numbers, find A^{-1}.

65. Use $A = \begin{bmatrix} -4 & 11 \\ -1 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} -3 & 11 \\ -1 & 4 \end{bmatrix}$ to solve the system.

a. $$\begin{aligned} -4x + 11y &= -66 \\ 3y - x &= -19 \end{aligned}$$

b. $$\begin{aligned} -4x + 11y &= \frac{16}{3} \\ -x + 3y &= \frac{3}{2} \end{aligned}$$

66. Use $B = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ -2 & 0 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -3 & -6 & -2 \\ -2 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$ to solve the system.

a.
$$\begin{aligned} x - 2y &= 8 \\ y - z &= -8 \\ -2x + 3z &= 10 \end{aligned}$$

b.
$$\begin{aligned} x - 2y &= -3 \\ y - z &= 0 \\ -2x + 3z &= 3 \end{aligned}$$

Write About It

67. Explain how you can determine if two $n \times n$ matrices, A and B, are inverses.

68. Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, for what conditions on a, b, c, and d will the matrix not have an inverse?

Expanding Your Skills

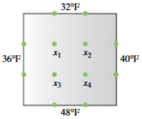
69. Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, perform row operations on the matrix $[A \mid I_n]$ to find A^{-1}. This confirms the formula for the inverse of an invertible 2×2 matrix. (See page 1003.)

70. Show by counterexample that $(AB)^{-1} \neq A^{-1}B^{-1}$. That is, find two matrices A and B for which $(AB)^{-1} \neq A^{-1}B^{-1}$.

Physicists know that if each edge of a thin conducting plate is kept at a constant temperature, then the temperature at the interior points is the mean (average) of the four surrounding points equidistant from the interior point. Use this principle in Exercise 71 to find the temperature at points x_1, x_2, x_3, and x_4.

71. *Hint*: Set up four linear equations to represent the temperature at points x_1, x_2, x_3, and x_4. Then solve the system. For example, one equation would be:

$$x_1 = \frac{1}{4}(36 + 32 + x_2 + x_3)$$



Technology Connections

For Exercises 72–73, use a graphing utility to find the inverse of the given matrix. Round the elements in the inverse to 2 decimal places.

72. $A = \begin{bmatrix} 0.04 & 0.13 & -0.08 & -0.43 \\ 0.19 & 0.33 & 0.06 & -0.84 \\ 0.01 & 0.08 & -0.11 & 0.46 \\ 0.37 & -1.42 & 0.03 & 0.52 \end{bmatrix}$

73. $A = \begin{bmatrix} 3.5 & 2.1 & 1.6 & 2.4 \\ -4.5 & 3.9 & -9.1 & 3.8 \\ 5.7 & 6.8 & -1.4 & -7.7 \\ 3.3 & 2.8 & 4.6 & -1.0 \end{bmatrix}$

For Exercises 74–75, use a graphing calculator and the inverse of the coefficient matrix to find the solution to the given system. Round to 2 decimal places.

74.
$$\begin{aligned} (\log 2)x + (\sqrt{7})y + z &= 4.1 \\ (e^2)x - (\sqrt{3})y - \pi z &= -3.7 \\ (\ln 10)x + y - 2.2z &= 7.2 \end{aligned}$$

75.
$$\begin{aligned} (\sqrt{11})x + y - (\ln 5)z &= 52.3 \\ (\sqrt{7})x - \pi y + (e^3)z &= -27.5 \\ -x + (\log 81)y - z &= 69.8 \end{aligned}$$

Point of Interest

Navajo code talkers

Cryptography (from the Greek *crypt* for "hidden" or "secret," and *graphy* for "write" or "study") is the study of coding and decoding messages. During World War II, coded messages were used by all sides to relay information. Some methods were alphanumeric ciphers and others, such as the Native American "code talkers," used a language unknown to adversaries to speak in open communication over the radio.

Alan Turing

Breaking the codes used by Germany and Japan was crucial to winning the war, and it required the work of thousands of men and women, including the author's uncle. One of the more famous codebreakers of World War II was the British mathematician Alan Turing. Turing successfully worked on breaking the cipher codes produced by Nazi Germany's Enigma machine.

One style of encoding a message uses matrices. The process begins by assigning a number (1–26) to each letter of the alphabet and a 0 to designate a space. Other numbers may be assigned to punctuation or special characters (e.g., % = 27, ! = 28).

TABLE 9-2 Code Numbers

A	1	I	9	Q	17	Y	25
B	2	J	10	R	18	Z	26
C	3	K	11	S	19	Space	0
D	4	L	12	T	20		
E	5	M	13	U	21		
F	6	N	14	V	22		
G	7	O	15	W	23		
H	8	P	16	X	24		

Using Table 9-2, the message MEET AT TWO would become:

M	E	E	T		A	T		T	W	O
13	5	5	20	0	1	20	0	20	23	15

This gives a list of numbers to place in a matrix:

13, 5, 5, 20, 0, 1, 20, 0, 20, 23, 15

Exactly *how* the list of numbers is arranged depends on the encoding and decoding matrices. An invertible matrix must be used to encode the message, and the inverse of the encoding matrix is used to decode the message.

For example, if the coding matrix is $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, then the decoding matrix is $A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$.

If the coding matrix is $B = \begin{bmatrix} 8 & -5 & -2 \\ -11 & 7 & 3 \\ -34 & 21 & 9 \end{bmatrix}$, then the decoding matrix is $B^{-1} = \begin{bmatrix} 0 & 3 & -1 \\ -3 & 4 & -2 \\ 7 & 2 & 1 \end{bmatrix}$.

To use A as the encoding matrix, arrange the numbers in the message matrix M in columns of 2. Notice that a zero in the last row is used to complete the matrix.

$$\begin{bmatrix} \text{M} & \text{E} & \text{E} & \text{T} & \_ & \text{A} \\ \text{T} & \_ & \text{T} & \text{W} & \text{O} & \_ \end{bmatrix} \text{ becomes } M = \begin{bmatrix} 13 & 5 & 5 & 20 & 0 & 1 \\ 20 & 0 & 20 & 23 & 15 & 0 \end{bmatrix}$$

Saul Loeb/AFP/Getty Images, Archivio GBB/Alamy Stock Photo

Next, multiply:

$$AM = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 13 & 5 & 5 & 20 & 0 & 1 \\ 20 & 0 & 20 & 23 & 15 & 0 \end{bmatrix} = \begin{bmatrix} 46 & 10 & 30 & 63 & 15 & 2 \\ 79 & 15 & 55 & 106 & 30 & 3 \end{bmatrix}$$

The encoded matrix, AM, may be decoded by using the inverse matrix A^{-1}. Multiply:

$$A^{-1}(AM) = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 46 & 10 & 30 & 63 & 15 & 2 \\ 79 & 15 & 55 & 106 & 30 & 3 \end{bmatrix} = M$$

To use B as the encoding matrix, arrange the message in columns of 3.

$$\begin{bmatrix} \text{M} & \text{E} & \text{E} & \text{T} \\ \_ & \text{A} & \text{T} & \_ \\ \text{T} & \text{W} & \text{O} & \_ \end{bmatrix} \text{ becomes } M = \begin{bmatrix} 13 & 5 & 5 & 20 \\ 0 & 1 & 20 & 0 \\ 20 & 23 & 15 & 0 \end{bmatrix}$$

Multiply BM to encode and $B^{-1}(BM)$ to decode.

$$BM = \begin{bmatrix} 8 & -5 & -2 \\ -11 & 7 & 3 \\ -34 & 21 & 9 \end{bmatrix} \begin{bmatrix} 13 & 5 & 5 & 20 \\ 0 & 1 & 20 & 0 \\ 20 & 23 & 15 & 0 \end{bmatrix} = \begin{bmatrix} 64 & -11 & -90 & 160 \\ -83 & 21 & 130 & -220 \\ -262 & 58 & 385 & -680 \end{bmatrix}$$

For Exercises 76–77, a message is represented in a matrix by using the numerical code given in Table 9-2. The message is then encoded by forming a new matrix, *P*, as the product of an encoding matrix, *A*, and the original matrix.

a. Determine the decoding matrix, A^{-1}.

b. Decode the message.

76. $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$, $P = \begin{bmatrix} 29 & 17 & 78 & 10 & 104 & 15 \\ 70 & 42 & 185 & 25 & 249 & 35 \end{bmatrix}$

77. $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & -1 & 1 \\ 0 & 1 & 4 \end{bmatrix}$, $P = \begin{bmatrix} 73 & 44 & 59 \\ -43 & -14 & -15 \\ 29 & 24 & 39 \end{bmatrix}$

Further Reading

Hodges, Andrew. *Alan Tuning: The Enigma*. Princeton University Press; Revised Edition, 2014.

Mundy, Liza. *Code Girls: The Untold Story of the American Women Code Breakers of World War II*. Hachette Books, 2018.

Nez, Chester and Avila, Judith. *Code Talker: The First and Only Memoir By One of the Original Navajo Code Talkers of WWII*. Dutton Caliber, 2012.

Singh, Simon. The Code Book: *The Science of Secrecy from Ancient Egypt to Quantum Cryptography.* Anchor, 2000.

Go online for more practice problems.

Lesson 9-5

Determinants and Cramer's Rule

What Will You Learn?

After completing this lesson, you should be able to:

- Evaluate the Determinant of a 2 × 2 Matrix
- Evaluate the Determinant of an $n \times n$ Matrix
- Apply Cramer's Rule

Learn: Evaluate the Determinant of a 2 × 2 Matrix

In this lesson, we present yet another method to solve a linear system with one unique solution. This method, called Cramer's rule, enables us to write a formula to solve a system based on the coefficients of the equations in the system.

Before studying Cramer's rule, we need to become familiar with the determinant of a square matrix. Associated with every square matrix is a real number called the *determinant* of the matrix. A determinant of a square matrix A, denoted $|A|$, is written by enclosing the elements of the matrix within two vertical bars. For example:

$$\text{Given } A = \begin{bmatrix} 3 & -1 \\ 4 & 0 \end{bmatrix}, \quad \text{then} \quad |A| = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix}$$

Check Your Work

Notice that a determinant is denoted using vertical bars, whereas a matrix is denoted using brackets.

Determinants have many applications in mathematics, including solving systems of linear equations, finding the area of a triangle, determining whether three points are collinear, and finding an equation of a line through two points (see Exercises 77–84). The determinant of a 2 × 2 matrix is defined as follows:

Determinant of a 2 × 2 Matrix

The **determinant** of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. That is,

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Example 1: Evaluating a 2 × 2 Determinant

Evaluate the determinant of each matrix.

a. $A = \begin{bmatrix} 6 & -2 \\ 5 & \frac{1}{3} \end{bmatrix}$ **b.** $B = \begin{bmatrix} 2 & -11 \\ 0 & 0 \end{bmatrix}$

Solution:

a. $|A| = \begin{vmatrix} 6 & -2 \\ 5 & \frac{1}{3} \end{vmatrix} = \overset{a \cdot d}{(6)\left(\tfrac{1}{3}\right)} \overset{-}{-} \overset{b \cdot c}{(-2)(5)} = 12$ $a = 6, b = -2, c = 5, \text{ and } d = \frac{1}{3}$

b. $|B| = \begin{vmatrix} 2 & -11 \\ 0 & 0 \end{vmatrix} = \overset{a \cdot d}{(2)(0)} \overset{-}{-} \overset{b \cdot c}{(-11)(0)} = 0$ $a = 2, b = -11, c = 0, \text{ and } d = 0$

Insights

Example 1(b) illustrates that the determinant of a matrix having a row of all zeros is 0. The same is true for a matrix having a column of all zeros.

Apply the Skills

1. Evaluate the determinant.

a. $\begin{vmatrix} 3 & -\frac{1}{2} \\ 10 & 2 \end{vmatrix}$ **b.** $\begin{vmatrix} 9 & 0 \\ -4 & 0 \end{vmatrix}$

Learn: Evaluate the Determinant of an $n \times n$ Matrix

We now develop the tools to compute the determinant of a higher-order square matrix. To find the determinant of an $n \times n$ matrix $A = [a_{ij}]$ we first need to define the minor of an element of the matrix. For any element a_{ij}, the **minor** M_{ij} of that element is the determinant of the resulting matrix obtained by deleting the ith row and jth column. For example, consider the matrix:

$$A = [a_{ij}] = \begin{bmatrix} 5 & -1 & 6 \\ 0 & -7 & 1 \\ 4 & 2 & 6 \end{bmatrix}$$

The element a_{11} is 5. The minor of this element is found by deleting the first row and first column and then evaluating the determinant of the remaining 2 × 2 matrix:

$$\begin{bmatrix} 5 & -1 & 6 \\ 0 & -7 & 1 \\ 4 & 2 & 6 \end{bmatrix} \qquad M_{11} = \begin{vmatrix} -7 & 1 \\ 2 & 6 \end{vmatrix} = (-7)(6) - (1)(2) = -44$$

The element a_{32} is 2. The minor of this element is found by deleting the third row and second column and then evaluating the determinant of the remaining 2 × 2 matrix:

$$\begin{bmatrix} 5 & -1 & 6 \\ 0 & -7 & 1 \\ 4 & 2 & 6 \end{bmatrix} \qquad M_{32} = \begin{vmatrix} 5 & 6 \\ 0 & 1 \end{vmatrix} = (5)(1) - (6)(0) = 5$$

Example 2: Determining the Minor for Elements in a Matrix

Find the minor for each element in the first column of the matrix.

$$\begin{bmatrix} 3 & 4 & -1 \\ 2 & -4 & 5 \\ 0 & 1 & -6 \end{bmatrix}$$

Solution:

For the element a_{11}:

$$\begin{bmatrix} 3 & 4 & -1 \\ 2 & -4 & 5 \\ 0 & 1 & -6 \end{bmatrix} \qquad M_{11} = \begin{vmatrix} -4 & 5 \\ 1 & -6 \end{vmatrix} = (-4)(-6) - (5)(1) = 19$$

For the element a_{21}:

$$\begin{bmatrix} 3 & 4 & -1 \\ 2 & -4 & 5 \\ 0 & 1 & -6 \end{bmatrix} \qquad M_{21} = \begin{vmatrix} 4 & -1 \\ 1 & -6 \end{vmatrix} = (4)(-6) - (-1)(1) = -23$$

For the element a_{31}:

$$\begin{bmatrix} 3 & 4 & -1 \\ 2 & -4 & 5 \\ 0 & 1 & -6 \end{bmatrix} \qquad M_{31} = \begin{vmatrix} 4 & -1 \\ -4 & 5 \end{vmatrix} = (4)(5) - (-1)(-4) = 16$$

Apply the Skills

2. Find the minor for each element in the second row of the matrix from Example 2.

Next, we define the determinant for a 3 × 3 matrix.

Determinant of a 3 × 3 Matrix

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \text{ or equivalently,}$$

$$= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2) \text{ or}$$

$$= a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

From this definition, we see that the determinant of the given 3 × 3 matrix can be written as

$$a_1 \cdot (\text{minor of } a_1) - a_2 \cdot (\text{minor of } a_2) + a_3 \cdot (\text{minor of } a_3)$$

Example 3: Evaluating the Determinant of a 3 × 3 Matrix

Evaluate the determinant of the matrix. $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ -5 & 5 & -1 \end{bmatrix}$

Solution:

$$|A| = \begin{vmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ -5 & 5 & -1 \end{vmatrix} = 2 \cdot \overbrace{\begin{vmatrix} -3 & 0 \\ 5 & -1 \end{vmatrix}}^{\text{minor of 2}} - (1)\overbrace{\begin{vmatrix} 4 & 2 \\ 5 & -1 \end{vmatrix}}^{\text{minor of 1}} + (-5) \cdot \overbrace{\begin{vmatrix} 4 & 2 \\ -3 & 0 \end{vmatrix}}^{\text{minor of }-5}$$

$$= 2[(-3)(-1) - (0)(5)] - 1[(4)(-1) - (2)(5)] + (-5)[(4)(0) - (2)(-3)]$$

$$= 2(3) - 1(-14) - 5(6)$$

$$= -10$$

Apply the Skills

3. Evaluate the determinant. $\begin{vmatrix} -2 & 4 & 9 \\ 5 & -1 & 2 \\ 1 & 1 & 6 \end{vmatrix}$

Although we defined the determinant of a 3 × 3 matrix by expanding the minors of the elements in the first column, *any row or column can be used*. However, we must choose the correct sign to apply to the product of factors of each term. The following array of signs is helpful.

row 1, column 1
$1 + 1 = 2$ (even)

row 1, column 2
$1 + 2 = 3$ (odd)

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Notice that the sign is positive if the sum of the row and column numbers is even. The sign is negative if the sum of the row and column numbers is odd.

To evaluate the determinant of an $n \times n$ matrix, first choose any row or column. For each element a_{ij} in the selected row or column, multiply the minor by 1 or –1, depending on whether the sum of the row and column numbers is even or odd. That is, for the element a_{ij}, we have $(-1)^{i+j}M_{ij}$. This product is called the *cofactor* of the element a_{ij}.

Cofactor of an Element of a Matrix

Given a square matrix $A = [a_{ij}]$, the **cofactor** of a_{ij} is $(-1)^{i+j}M_{ij}$, where M_{ij} is the minor of a_{ij}.

Using the definition of the cofactor of an element of an $n \times n$ matrix, we can now present a generic method to find the determinant of the matrix.

Evaluating the Determinant of an $n \times n$ Matrix by Expanding Cofactors

Step 1 Choose any row or column.

Step 2 Multiply each element in the selected row or column by its cofactor.

Step 3 The value of the determinant is the sum of the products from step 2.

It is important to note that this three-step process to evaluate a determinant works for any $n \times n$ matrix, including a 2×2 matrix. For example, given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we can evaluate the determinant of A by expanding cofactors about the first row. The cofactor of a is d, and the cofactor of b is $-c$. Therefore,

$$\begin{aligned} |A| &= a(d) + b(-c) \\ &= ad - bc \text{ as expected.} \end{aligned}$$

Example 4: Evaluating the Determinant of a 3×3 Matrix

Evaluate the determinant of the matrix from Example 3 by expanding cofactors about the elements in the third row.

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ -5 & 5 & -1 \end{bmatrix}$$

Solution:

$$\begin{vmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ -5 & 5 & -1 \end{vmatrix} = (-5)\overbrace{(-1)^{3+1}\begin{vmatrix} 4 & 2 \\ -3 & 0 \end{vmatrix}}^{\text{cofactor of } -5} + (5)\overbrace{(-1)^{3+2}\begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix}}^{\text{cofactor of } 5} + (-1)\overbrace{(-1)^{3+3}\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix}}^{\text{cofactor of } -1}$$

$$\begin{aligned} &= (-5)(1)[(4)(0) - (2)(-3)] + (5)(-1)[(2)(0) - (2)(1)] + (-1)(1)[(2)(-3) - (4)(1)] \\ &= -5(6) - 5(-2) - 1(-10) \\ &= -30 + 10 + 10 \\ &= -10 \end{aligned}$$

This is the same result as in Example 3.

Apply the Skills

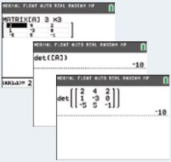
4. Evaluate the determinant by expanding cofactors about the elements in the third column.

$$\begin{vmatrix} -2 & 4 & 9 \\ 5 & -1 & 2 \\ 1 & 1 & 6 \end{vmatrix}$$

TECHNOLOGY CONNECTIONS

Evaluating a Determinant

To find the determinant of a matrix on a graphing calculator, first enter the matrix in the calculator by using the MATRIX menu and EDIT menu. Select a name for the matrix such as [A]. Then access the determinant function det(under the MATRIX and MATH menus. Enter det([A]) in the home screen and press ENTER.



```
MATRIX[A] 3 ×3
[ 2   4   2 ]
[ 1  -3   0 ]
[-5   5  -1 ]
```

```
det([A])
                    -10
```

```
    ⎡ 2  4  2⎤
det ⎢ 1 -3  0⎥
    ⎣-5  5 -1⎦
                    -10
```

Alternatively, if the calculator is in “MATHPRINT” mode, enter the matrix by selecting ALPHA F3.

Insights

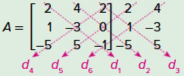
The determinant of a 3 × 3 matrix can also be evaluated by using the “method of diagonals.”

Step 1: Recopy columns 1 and 2 to the right of the matrix.

Step 2: Multiply the elements on the diagonals labeled d_1 through d_6 (each diagonal has three elements).

Step 3: The value of the determinant is $(d_1 + d_2 + d_3) - (d_4 + d_5 + d_6)$.

The determinant from Example 4 is evaluated as follows:



$$
\begin{aligned}
&= [(2)(-3)(-1) + (4)(0)(-5) + (2)(1)(5)] \\
&\quad - [(2)(-3)(-5) + (2)(0)(5) + (4)(1)(-1)] \\
&= -10
\end{aligned}
$$

Some students find the method of diagonals to be a faster technique to find the determinant of a 3 × 3 matrix. However, it is critical to note that the method of diagonals only works for the determinant of a 3 × 3 matrix.

Perhaps one of the most important applications of the determinant of a matrix is to determine whether the matrix has an inverse. In Lesson 9.4, we found the inverse of a 2 × 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by applying the formula $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. This is equivalent to $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Furthermore, the inverse exists if $ad - bc \neq 0$ or equivalently, if $|A| \neq 0$. This is true in general for any $n \times n$ matrix.

Using Determinants to Determine if a Matrix Is Invertible

Let A be an $n \times n$ matrix. Then A is invertible if and only if $|A| \neq 0$.

In Example 5, we will use the determinant of A to determine if A has an inverse.

Example 5: Using a Determinant to Determine if a Matrix Is Invertible

Use $|A|$ to determine if A is invertible. $A = \begin{bmatrix} -2 & 0 & 1 & 2 \\ 6 & 2 & -2 & 5 \\ 5 & 3 & -1 & 1 \\ 0 & 4 & 2 & 1 \end{bmatrix}$

Solution:

To evaluate a determinant, we can simplify the arithmetic by expanding around the row or column that contains the greatest number of 0 elements. In this case, we have chosen row 1.

$$\begin{vmatrix} -2 & 0 & 1 & 2 \\ 6 & 2 & -2 & 5 \\ 5 & 3 & -1 & 1 \\ 0 & 4 & 2 & 1 \end{vmatrix} = -2(-1)^{1+1}\begin{vmatrix} 2 & -2 & 5 \\ 3 & -1 & 1 \\ 4 & 2 & 1 \end{vmatrix} + 0(-1)^{1+2}\begin{vmatrix} 6 & -2 & 5 \\ 5 & -1 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$+ 1(-1)^{1+3}\begin{vmatrix} 6 & 2 & 5 \\ 5 & 3 & 1 \\ 0 & 4 & 1 \end{vmatrix} + 2(-1)^{1+4}\begin{vmatrix} 6 & 2 & 2 \\ 5 & 3 & -1 \\ 0 & 4 & 2 \end{vmatrix}$$

The second term is zero. Evaluating the determinants in the first, third, and fourth terms (shown in red), we have:

$$\begin{vmatrix} -2 & 0 & 1 & 2 \\ 6 & 2 & -2 & 5 \\ 5 & 3 & -1 & 1 \\ 0 & 4 & 2 & 1 \end{vmatrix} = -2(1)(42) + 0 + 1(1)(84) + 2(-1)(0) = 0$$

The determinant of A is zero. Therefore, A is not invertible.

Apply the Skills

5. Use $|A|$ to determine if A is invertible. $A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 4 & 0 & 3 & 1 \\ 2 & 0 & 0 & 5 \\ 3 & -1 & 0 & 9 \end{bmatrix}$

Learn: Apply Cramer's Rule

We have learned several methods to solve a system of linear equations: the substitution method, the addition method, Gaussian elimination, Gauss-Jordan elimination, and the application of matrix inverses. We now present another method called Cramer's rule.

Cramer's rule involves finding the ratio of several determinants derived from the coefficients of the equations within the system. For example, consider the following system of equations.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

Using the addition method to solve for x, we have

$$\begin{array}{ll} a_1x + b_1y = c_1 & \xrightarrow{\text{Multiply by } b_2.} \\ a_2x + b_2y = c_2 & \xrightarrow{\text{Multiply by } -b_1.} \end{array} \qquad \begin{array}{rl} a_1b_2x + b_1b_2y &= c_1b_2 \\ -a_2b_1x - b_1b_2y &= -c_2b_1 \\ \hline (a_1b_2 - a_2b_1)x \qquad &= c_1b_2 - c_2b_1 \end{array}$$

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Using similar logic, we can show that $y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

These results are summarized as Cramer's rule for a system of linear equations in two variables.

Cramer's Rule for a System of Two Linear Equations in Two Variables

Given the system $\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2, \end{array}$

let $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, and $D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

Then if $D \neq 0$, the system has the unique solution: $x = \dfrac{D_x}{D}$ and $y = \dfrac{D_y}{D}$

Good Practices

Here are some memory tips to remember the patterns presented in Cramer's rule.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

1. The determinant D is the determinant of the coefficients of x and y.

Coefficients of
x terms y terms

$$D = \begin{vmatrix} \overset{\downarrow}{a_1} & \overset{\downarrow}{b_1} \\ a_2 & b_2 \end{vmatrix}$$

2. The determinant D_x has the column of x term coefficients replaced by c_1 and c_2.

x-coefficients replaced by c_1 and c_2

$$D_x = \begin{vmatrix} \overset{\downarrow}{c_1} & b_1 \\ c_2 & b_2 \end{vmatrix}$$

3. The determinant D_y has the column of y term coefficients replaced by c_1 and c_2.

y-coefficients replaced by c_1 and c_2

$$D_y = \begin{vmatrix} a_1 & \overset{\downarrow}{c_1} \\ a_2 & c_2 \end{vmatrix}$$

Example 6: Solving a 2 × 2 System by Using Cramer's Rule

Solve the system by using Cramer's rule.
$$\begin{aligned} -11x + 6y &= 4 \\ 2x - 5y &= -3 \end{aligned}$$

Solution:

For this system, $a_1 = -11$, $b_1 = 6$, $c_1 = 4$, $a_2 = 2$, $b_2 = -5$, and $c_2 = -3$.

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \longrightarrow D = \begin{vmatrix} -11 & 6 \\ 2 & -5 \end{vmatrix} = (-11)(-5) - (6)(2) = 43$$

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \longrightarrow D_x = \begin{vmatrix} 4 & 6 \\ -3 & -5 \end{vmatrix} = (4)(-5) - (6)(-3) = -2$$

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \longrightarrow D_y = \begin{vmatrix} -11 & 4 \\ 2 & -3 \end{vmatrix} = (-11)(-3) - (4)(2) = 25$$

Therefore, $x = \frac{D_x}{D} = \frac{-2}{43}$ and $y = \frac{D_y}{D} = \frac{25}{43}$.

The solution set is $\left\{\left(-\frac{2}{43}, \frac{25}{43}\right)\right\}$.

Apply the Skills

6. Solve the system by using Cramer's rule.
$$\begin{aligned} 3x - 4y &= 9 \\ -5x + 6y &= 2 \end{aligned}$$

The patterns associated with Cramer's rule can be generalized to solve a system of *n* linear equations in *n* variables.

Cramer's Rule for a System of *n* Linear Equations in *n* Variables

Consider the following system of *n* linear equations in *n* variables.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

If the system has a unique solution, then the solution is $(x_1, x_2, \cdots, x_i, \cdots, x_n)$, where

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \cdots, x_i = \frac{D_i}{D}, \cdots, x_n = \frac{D_n}{D}.$$

$D \neq 0$ is the determinant of the coefficient matrix, and D_i is the determinant formed by replacing the *i*th column of the coefficient matrix by the column of constants $b_1, b_2, \cdots, b_n$.

In Example 7, we apply Cramer's rule to a system of equations with three variables.

Example 7: Solving a 3 × 3 System by Using Cramer's Rule

Solve the system by using Cramer's rule.

$$\begin{aligned} 2x - 3y + 5z &= 11 \\ -5x + 7y - 2z &= -6 \\ 9x - 2y + 3z &= 4 \end{aligned}$$

Solution:

Evaluate the determinants D, D_x, D_y, and D_z.

$$D = \begin{vmatrix} 2 & -3 & 5 \\ -5 & 7 & -2 \\ 9 & -2 & 3 \end{vmatrix} = -222 \qquad D_x = \begin{vmatrix} 11 & -3 & 5 \\ -6 & 7 & -2 \\ 4 & -2 & 3 \end{vmatrix} = 77$$

$$D_y = \begin{vmatrix} 2 & 11 & 5 \\ -5 & -6 & -2 \\ 9 & 4 & 3 \end{vmatrix} = 117 \qquad D_z = \begin{vmatrix} 2 & -3 & 11 \\ -5 & 7 & -6 \\ 9 & -2 & 4 \end{vmatrix} = -449$$

Therefore, $x = \frac{D_x}{D} = -\frac{77}{222}$, $y = \frac{D_y}{D} = -\frac{117}{222}$ or $-\frac{39}{74}$, and $z = \frac{D_z}{D} = \frac{449}{222}$.

The solution set is $\left\{\left(-\frac{77}{222}, -\frac{39}{74}, \frac{449}{222}\right)\right\}$.

Check Your Work

It is important to note that each equation in the system of equations should be written in standard form before applying Cramer's rule.

Apply the Skills

7. Solve the system by using Cramer's rule.

$$\begin{aligned} 5x + 3y - 3z &= -14 \\ 3x - 4y + z &= 2 \\ x + 7y + z &= 6 \end{aligned}$$

Although Cramer's rule may seem cumbersome for solving a 3 × 3 system of linear equations, it provides convenient formulas that can be programmed into a computer or calculator to solve the system. However, it is important to remember that Cramer's rule does not apply if $D = 0$. In such a case, the system of equations is either inconsistent (has no solution) or the equations are dependent (the system has infinitely many solutions).

Example 8: Identifying Whether Cramer's Rule Applies

Solve the system by using Cramer's rule if possible. Otherwise, use a different method.

$$\begin{aligned} x + 3y &= 6 \\ -2x - 6y &= -12 \end{aligned}$$

Solution:

Evaluate D. $D = \begin{vmatrix} 1 & 3 \\ -2 & -6 \end{vmatrix} = 1(-6) - 3(-2) = -6 + 6 = 0$

Since $D = 0$, Cramer's rule does not apply. Using Gauss-Jordan elimination, we have

$$\left[\begin{array}{cc|c} 1 & 3 & 6 \\ -2 & -6 & -12 \end{array}\right] \xrightarrow{2R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

The last row of the augmented matrix represents the equation $0 = 0$, which is true for all values of x and y. The system reduces to the single equation $x + 3y = 6$, and there are infinitely many solutions.

The solution set is $\{(6 - 3y, y) | y \text{ is any real number}\}$.

Apply the Skills

8. Solve the system by using Cramer's rule if possible. Otherwise, use a different method.

$$\begin{aligned} x + 4y &= 2 \\ 3x + 12y &= 4 \end{aligned}$$

Practice 9-5

Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, simplify the exponential expression.

R.1. $(-1)^4$ **R.2.** $(-1)^7$

R.3. Write an equation of the line that passes through the points $(-3, 5)$ and $(-2, 4)$. Write the equation in slope-intercept form.

Concept Connections

1. Associated with every square matrix A is a real number denoted by $|A|$ called the __________ of A.

2. For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} =$ __________.

3. Given $A = [a_{ij}]$, the __________ of the element a_{ij} is the determinant obtained by deleting the ith row and jth column.

4. Given $A = [a_{ij}]$, then the value $(-1)^{i+j} M_{ij}$ is called the __________ of the element a_{ij}.

5. The determinant of a 3×3 matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is given by

$$|A| = a_1\begin{vmatrix} \square & \square \\ \square & \square \end{vmatrix} - a_2\begin{vmatrix} \square & \square \\ \square & \square \end{vmatrix} + a_3\begin{vmatrix} \square & \square \\ \square & \square \end{vmatrix}$$

6. Suppose that the given system has one solution.

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Cramer's rule gives the solution as $x = \frac{\square}{\square}$ and $y = \frac{\square}{\square}$, where

$D = \begin{vmatrix} \square & \square \\ \square & \square \end{vmatrix}$, $D_x = \begin{vmatrix} \square & \square \\ \square & \square \end{vmatrix}$, and

$D_y = \begin{vmatrix} \square & \square \\ \square & \square \end{vmatrix}$.

Learn: Evaluate the Determinant of a 2 × 2 Matrix

For Exercises 7–16, evaluate the determinant of the matrix. (See Example 1 on p. 1014)

7. $A = \begin{bmatrix} 3 & -2 \\ 6 & 5 \end{bmatrix}$

8. $B = \begin{bmatrix} 7 & 12 \\ -1 & 4 \end{bmatrix}$

9. $C = \begin{bmatrix} \frac{2}{3} & \frac{1}{5} \\ 10 & 12 \end{bmatrix}$

10. $D = \begin{bmatrix} \frac{8}{9} & 4 \\ \frac{5}{2} & 18 \end{bmatrix}$

11. $E = \begin{bmatrix} -3 & 0 \\ 4 & 0 \end{bmatrix}$

12. $F = \begin{bmatrix} 0 & 9 \\ 0 & 4 \end{bmatrix}$

13. $G = \begin{bmatrix} x & 4 \\ 9 & x \end{bmatrix}$

14. $H = \begin{bmatrix} y & 16 \\ 4 & y \end{bmatrix}$

15. $T = \begin{bmatrix} e^x & e^{2x} \\ 4 & -e^x \end{bmatrix}$

16. $V = \begin{bmatrix} \log x & \log x \\ 2 & 5 \end{bmatrix}$

Learn: Evaluate the Determinant of an *n* × *n* Matrix

For Exercises 17–22, refer to the matrix

$$\mathbf{A} = [\mathbf{a}_{ij}] = \begin{bmatrix} -6 & 11 & 8 \\ 4 & -2 & -5 \\ -3 & 7 & 10 \end{bmatrix}.$$

a. Find the minor of the given element. **(See Example 2 on p. 1015)**

b. Find the cofactor of the given element.

17. a_{12}

18. a_{23}

19. a_{31}

20. a_{13}

21. a_{22}

22. a_{33}

For Exercises 23–32, evaluate the determinant of the matrix and state whether the matrix is invertible. (See Examples 3–5 on pp. 1016–1019)

23. $A = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -1 & 2 \\ 5 & 8 & 0 \end{bmatrix}$

24. $B = \begin{bmatrix} 9 & 5 & -1 \\ 2 & 0 & 4 \\ 7 & -2 & 0 \end{bmatrix}$

25. $C = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 3 & 4 \\ 8 & -1 & 7 \end{bmatrix}$

26. $D = \begin{bmatrix} -3 & 1 & -2 \\ 10 & 5 & 8 \\ 6 & 7 & -4 \end{bmatrix}$

27. $E = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$

28. $F = \begin{bmatrix} 1 & -3 & 17 \\ 1 & -1 & 7 \\ 2 & -5 & 29 \end{bmatrix}$

29. $G = \begin{bmatrix} 5 & 6 & 4 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix}$

30. $H = \begin{bmatrix} 8 & 0 & 5 & 1 \\ 0 & 3 & 4 & -2 \\ 2 & 6 & 3 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$

31. $T = \begin{bmatrix} 3 & 8 & 1 & 4 \\ -2 & 4 & 0 & 5 \\ -1 & 1 & 0 & -1 \\ 0 & 5 & 2 & 3 \end{bmatrix}$

32. $W = \begin{bmatrix} 2 & 5 & 2 & 4 \\ 0 & 0 & -3 & 1 \\ 4 & 8 & 0 & 1 \\ -1 & 2 & 0 & 5 \end{bmatrix}$

Learn: Apply Cramer's Rule

For Exercises 33–48, solve the system if possible by using Cramer's rule. If Cramer's rule does not apply, solve the system by using another method. (See Examples 6–8 on pp. 1021–1022)

33. $2x + 10y = 11$
$3x - 5y = 6$

34. $-5x - 8y = 3$
$4x + 7y = 13$

35. $-10x + 4y = 7$
$6x = 7y + 2$

36. $11x + 6y = 8$
$2x = 9y + 5$

37. $3(x - y) = y + 8$
$y = \frac{3}{4}x - 2$

38. $5(x + y) = 7x + 4$
$4x = 10y - 8$

39. $y = -3x + 7$
$\frac{1}{2}x + \frac{1}{6}y = 1$

40. $x = 4y + 5$
$3(x - 4) = 12y$

41. $11x - 3z = 1$
$2y + 9z = 6$
$4x + 5y = -9$

42. $-2x + 6y = 9$
$5y + 7z = 1$
$4x - 3z = -8$

43. $2x - 5y + z = 11$
$3x + 7y - 4z = 8$
$x - 9y + 2z = 4$

44. $-5x - 6y + 8z = 1$
$2x + y - 4z = 5$
$3x - 4y - z = -2$

45. $2x - 3y + z = 6$
$-4x + 6y - 2z = -12$
$6x - 9y + 3z = 18$

46. $-x + y - 3z = 4$
$3x - 3y + 9z = -12$
$-2x + 2y - 6z = 8$

47. $x - 2y + 3z = -1$
$5x - 7y + 3z = 1$
$x - 5z = 2$

48. $x + 3y - 5z = 10$
$-2x - 4y + 8z = -14$
$x + y - 3z = 5$

For Exercises 49–50, use Cramer's rule to solve for the indicated variable.

49.
$$\begin{aligned} x_1 + 2x_2 + 3x_3 - 4x_4 &= 3 \\ 5x_2 \qquad\quad + x_4 &= 9 \\ x_1 \qquad + 4x_3 \qquad &= -1 \\ 5x_3 - 2x_4 &= 8 \end{aligned}$$
Solve for x_2.

50.
$$\begin{aligned} -2x_1 - x_2 + x_3 + 3x_4 &= 10 \\ x_1 \qquad + 5x_3 \qquad &= 4 \\ 2x_2 \qquad\quad + x_4 &= -1 \\ 4x_3 + 2x_4 &= 7 \end{aligned}$$
Solve for x_3.

Mixed Exercises

For Exercises 51–52, suppose that the graph of a parabola $f(x) = ax^2 + bx + c$ passes through the given points, (x, y).

a. Write a system of linear equations of the form $ax + by + cz = d$ using the given

b. Solve the system using determinants and give the equation of the parabola.

51. (3, 5), (5, –1), (7, –3)

52. (–2, 4), (–1, 3), (0, 4)

For Exercises 53–54, three noncollinear points, (x, y), are given.

a. Find the general form of the circle, $x^2 + y^2 + Dx + Ey + F = 0$, that passes through the points by setting up a system of equations and solving for D, E, and F using Cramer's rule.

b. Write the equation in the form $(x - h)^2 + (y - k)^2 = r^2$ and determine the center and radius.

53. (3, 5), (5, –1), (7, –3)

54. (–2, 4), (–1, 3), (0, 4)

For Exercises 55–56, suppose that the graph of the cubic polynomial $p(x) = x^3 + bx^2 + cx + d$ passes through the given points, (x, y). Find the equation of the polynomial. (Consider checking your answer using a graphing utility.)

55. (3, 5), (5, –1), (7, –3)

56. (–2, 4), (–1, 3), (0, 4)

Given a square matrix A, elementary row operations (or column operations) performed on A affect the value of $|A|$ in the following ways:

- Interchanging any two rows (or columns) of A will change the sign of $|A|$.
- Multiplying a row (or column) of A by a constant real number k multiplies $|A|$ by k.
- Adding a multiple of a row (or column) of A to another row (or column) of A does not change the value of $|A|$.

For Exercises 57–62, demonstrate these three properties.

57. Given $A = \begin{bmatrix} 5 & 2 \\ -3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 6 \\ 5 & 2 \end{bmatrix}$,

a. Evaluate $|A|$.

b. Evaluate $|B|$.

c. How are A and B related and how are $|A|$ and $|B|$ related?

58. Given $A = \begin{bmatrix} 2 & -7 \\ 4 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 10 \\ 2 & -1 \end{bmatrix}$,

a. Evaluate $|A|$.

b. Evaluate $|B|$.

c. How are A and B related and how are $|A|$ and $|B|$ related?

59. Given $A = \begin{bmatrix} 1 & -3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -6 \\ 4 & 1 \end{bmatrix}$,

a. Evaluate $|A|$.

b. Evaluate $|B|$.

c. How are A and B related and how are $|A|$ and $|B|$ related?

60. Given $A = \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & -3 \\ 5 & 7 \end{bmatrix}$,

a. Evaluate $|A|$.

b. Evaluate $|B|$.

c. How are A and B related and how are $|A|$ and $|B|$ related?

61. Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 6 & 10 \end{bmatrix}$,

a. Evaluate $|A|$.

b. Evaluate $|B|$.

c. How are A and B related and how are $|A|$ and $|B|$ related?

62. Given $A = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$,

a. Evaluate $|A|$.

b. Evaluate $|B|$.

c. How are A and B related and how are $|A|$ and $|B|$ related?

Given a square matrix A, if either of the following conditions is true, then $|A| = 0$.

- **A row (or column) of A consists entirely of zeros.**
- **One row (or column) is a constant multiple of another row (or column).**

For Exercises 63–66, demonstrate these two properties.

63. Given $A = \begin{bmatrix} 3 & 5 \\ 6 & 10 \end{bmatrix}$, find $|A|$.

64. Given $A = \begin{bmatrix} -2 & 7 \\ -6 & 21 \end{bmatrix}$, find $|A|$.

65. Given $A = \begin{bmatrix} 4 & -5 & 0 \\ 3 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, find $|A|$.

66. Given $A = \begin{bmatrix} 5 & 3 & 1 \\ -1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, find $|A|$.

67. Evaluate $|I_2|$.

68. Evaluate $|I_3|$.

69. Evaluate $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.

70. Evaluate $\begin{vmatrix} x & 0 \\ 0 & x \end{vmatrix}$.

71. If A and B are square matrices, then the product property of determinants indicates that $|AB| = |A| \cdot |B|$. Use matrix A and matrix B to demonstrate this property.

$$A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -5 & 1 \\ 3 & 2 \end{bmatrix}$$

72. The **transpose** of a square matrix A, denoted as A^T, is a square matrix that results by writing the rows of A as the columns of A^T.

a. Given $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 8 & 4 \\ 3 & 7 & 6 \end{bmatrix}$, find A^T.

b. Show that $|A| = |A^T|$.

Write About It

73. What is the difference between the minor of an element a_{ij} and the cofactor of the element?

74. Explain the difference between the notation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

75. The determinant of a square matrix can be computed by expanding the cofactors of the elements in any row or column. How would you choose which row or column?

76. Consider the system shown here. Describe the pattern associated with constructing the determinants used with Cramer's rule.

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Expanding Your Skills

Determinants can be used to determine whether three points are collinear (lie on the same line). Given the ordered pairs (x_1, y_1), (x_2, y_2), and (x_3, y_3), the points are collinear if the determinant below equals zero. For Exercises 77–80, determine if the points are collinear.

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

77. (3, 6), (6, 10), (−3, −2)

78. (−2, 1), (−4, −4), (4, 16)

79. (4, −3), (5, −7), (8, −14)

80. (0, 6), (1, 4), (4, −6)

The equation below represents an equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2). For Exercises 81–82,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

a. Use the determinant equation to write an equation of the line passing through the given points.

b. Write the equation of the line in slope-intercept form.

81. (−3, 2) and (−4, 6)

82. (−4, 1) and (−5, 4)

For Exercises 83–84, use the formula below to find the area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3). Choose the + or − sign so that the value of the area is positive.

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

83. (1, 0), (7, −2), (4, −5)

84. (−2, 1), (−1, −6), (−8, −5)

The given determinant equation represents the equation of a circle passing through the distinct, noncollinear points (x_1, y_1), (x_2, y_2), and (x_3, y_3). For Exercises 85–86, use the determinant equation to write the equation of a circle in the form $(x - h)^2 + (y - k)^2 = r^2$ passing through the given points.

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

85. (4, −6), (5, 1), (8, 2)

86. (6, 2), (−1, 3), (2, −6)

Technology Connections

For Exercises 87–90, use a graphing utility to evaluate the determinant of the matrix. Round to the nearest whole unit.

87. $\begin{bmatrix} \sqrt{3} & e & 1.6 \\ \log 5 & -2\pi & \ln 3 \\ -4 & 8.4 & -\sqrt{6} \end{bmatrix}$

88. $\begin{bmatrix} 8.9 & -2.3 & 3.8 \\ -1.7 & 0.9 & 4.6 \\ 2.7 & 10.1 & 14.9 \end{bmatrix}$

89. $A = \begin{bmatrix} -0.4 & 1.5 & 9 & 11.3 \\ -3.5 & 0.2 & -1.1 & 3 \\ 8 & 9.4 & -5.4 & 2 \\ -1 & 4.6 & 10.8 & -9.7 \end{bmatrix}$

90. $B = \begin{bmatrix} -2\pi & e^2 & 9.1 & \log 2 \\ \log 50 & -\sqrt{11} & 4.3 & \pi \\ -4.9 & 0 & e^2 & 8.1 \\ \sqrt{7} & \ln 7 & -9.7 & 0 \end{bmatrix}$

Problem Recognition Exercises

Using Multiple Methods to Solve Systems of Linear Equations

For Exercises 1–4, solve the system of equations using

- **a.** The substitution method or the addition method (see Lessons 8.1 and 8.2).
- **b.** Gaussian elimination (see Lesson 9.1).
- **c.** Gauss-Jordan elimination (see Lesson 9.1).
- **d.** The inverse of the coefficient matrix (see Lesson 9.4).
- **e.** Cramer's rule (see Lesson 9.5).

1. $\begin{aligned} x &= -3y - 10 \\ -3x - 7y &= 22 \end{aligned}$

2. $\begin{aligned} 2x &= 2 - 8y \\ 3x + 10y &= 5 \end{aligned}$

3. $\begin{aligned} x + 2y - z &= 0 \\ 2x \phantom{{}+ 2y} + z &= 4 \\ 2x - y + 2z &= 5 \end{aligned}$

4. $\begin{aligned} x + 4y + 2z &= 10 \\ 2y + z &= 4 \\ x + y \phantom{{}+ 2z} &= 2 \end{aligned}$

For Exercises 5–8,

- **a.** Evaluate the determinant of the coefficient matrix.
- **b.** Based on the value of the determinant from part (a), can an inverse matrix or Cramer's rule be used to solve the system?
- **c.** Solve the system using an appropriate method.

5. $\begin{aligned} 1.5x - 2y &= 3 \\ -3x + 4y &= 12 \end{aligned}$

6. $\begin{aligned} 5x - 2y &= 1 \\ x - 0.4y &= 4 \end{aligned}$

7. $\begin{aligned} x - 3y + 7z &= 1 \\ -2x + 5y - 11z &= -3 \\ x - 5y + 13z &= -1 \end{aligned}$

8. $\begin{aligned} x - 2y + 3z &= -7 \\ -2x + y \phantom{{}+ 3z} &= -1 \\ x \phantom{{}- 2y} - z &= 3 \end{aligned}$

Go online for more practice problems.

Chapter 9 Review

Chapter 9 Key Concepts

Lesson 9-1 Solving Systems of Linear Equations Using Matrices

Key Concepts	Reference
A system of linear equations can be represented by an **augmented matrix.**	p. 948
Elementary row operations can be used to write a matrix in row-echelon form or reduced row-echelon form. **1.** Interchange two rows. **2.** Multiply a row by a nonzero constant. **3.** Add a nonzero multiple of one row to another row.	p. 950
The method of **Gaussian elimination** uses elementary row operations to write an augmented matrix in row-echelon form so that the system can be solved by back substitution.	pp. 951–952
With the method of **Gauss-Jordan elimination,** the augmented matrix is written in *reduced* row-echelon form so that the solution can be found by inspection.	p. 955

Lesson 9-2 Inconsistent Systems and Dependent Equations

Key Concepts	Reference
A system of equations that has no solution is called an **inconsistent system.** An inconsistent system is detected algebraically if a contradiction is reached when solving the system.	p. 962
A system of linear equations may have infinitely many solutions. In such a case, the equations are said to be **dependent.** Dependent equations are detected algebraically if an identity is reached when solving the system.	p. 963

Lesson 9-3 Operations on Matrices

Key Concepts	Reference
An $m \times n$ matrix has m rows and n columns.	p. 977
Adding and subtracting matrices: If A and B represent two matrices of the same order, then $A + B$ and $A - B$ are found by adding or subtracting the corresponding elements.	p. 979
Scalar multiplication: Let $A = [a_{ij}]$ be an $m \times n$ matrix and let k be a real number. Then, $kA = [ka_{ij}]$.	p. 981
Matrix multiplication: Let A be an $m \times p$ matrix and let B be a $p \times n$ matrix, then the product AB is an $m \times n$ matrix. For the matrix AB, the element in the ith row and jth column is the sum of the products of the corresponding elements in the ith row of A and the jth column of B (the inner product of the ith row and jth column). *Note*: If the number of columns in A does not equal the number of rows in B, then it is not possible to compute the product AB.	p. 985

Lesson 9-4 Inverse Matrices and Matrix Equations

Key Concepts	Reference
The **identity matrix** I_n for the multiplication of matrices is the $n \times n$ square matrix with 1's along the main diagonal and 0's for all other elements.	p. 998
Inverse of a square matrix: Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix A^{-1} such that $AA^{-1} = I_n \quad \text{and} \quad A^{-1}A = I_n$ then A^{-1} is the **multiplicative inverse** of A.	p. 999
A matrix that does not have an inverse is called a **singular matrix.** A matrix that does have an inverse is said to be **invertible** or **nonsingular.**	p. 1001
Finding the inverse of a nonsingular matrix: Let A be an $n \times n$ matrix for which A^{-1} exists. To find A^{-1}: **Step 1** Write a matrix of the form $[A \mid I_n]$. **Step 2** Perform row operations to write the matrix in the form $[I_n \mid B]$. **Step 3** The matrix B is A^{-1}.	p. 1001
The **inverse of a nonsingular 2 × 2 matrix *A*** can also be found as follows: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, or equivalently $A^{-1} = \frac{1}{\lvert A \rvert}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.	p. 1003
Solving a system using inverse matrices: Suppose that $AX = B$ represents a system of n linear equations in n variables with a unique solution. Then, $X = A^{-1}B$ where A is the coefficient matrix, B is the matrix of constants, and X is the matrix of variables.	p. 1005

Lesson 9-5 Determinants and Cramer's Rule

Key Concepts	Reference
Associated with every square matrix is a real number called the determinant of the matrix.	p. 1013
Determinant of a 2 × 2 matrix: Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of A is denoted by $\lvert A \rvert$ or $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the value is $ad - bc$.	p. 1013
Given an $n \times n$ matrix $A = [a_{ij}]$, the **minor** M_{ij} of an element a_{ij} is the determinant of the resulting matrix obtained by deleting the *i*th row and *j*th column.	p. 1014
The **cofactor** of a_{ij} is $(-1)^{i+j}M_{ij}$ where M_{ij} is the minor of a_{ij}.	p. 1016
Determinant of a 3 × 3 matrix: $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$	p. 1015
Find the determinant of an $n \times n$ matrix: **Step 1** Choose any row or column. **Step 2** Multiply each element in the selected row or column by its cofactor. **Step 3** The value of the determinant is the sum of the products from step 2.	p. 1017
Cramer's rule for a system of two equations and two variables: Given $\begin{matrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{matrix}$ if $D \neq 0$, then $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$, where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, and $D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.	p. 1020
The patterns associated with Cramer's rule can be generalized to solve a system of n linear equations in n variables.	p. 1021
When applying Cramer's rule, if $D = 0$, then the system has either no solution or infinitely many solutions.	p. 1022

Go online for a detailed chapter summary.

Chapter 9 Test

For Exercises 1–3, perform the elementary row operations on the matrix $A = \left[\begin{array}{rrr|r} \mathbf{3} & \mathbf{1} & \mathbf{4} & \mathbf{-2} \\ \mathbf{1} & \mathbf{5} & \mathbf{-3} & \mathbf{1} \\ \mathbf{0} & \mathbf{4} & \mathbf{2} & \mathbf{6} \end{array}\right]$.

1. $R_1 \Leftrightarrow R_2$
2. $-3R_2 + R_1 \rightarrow R_1$
3. $\frac{1}{4}R_3 \rightarrow R_3$
4. Explain why the matrix is not in reduced row-echelon form.

$$\left[\begin{array}{rrr|r} 1 & 0 & 6 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

For Exercises 5–7, write the solution set for the system represented by each augmented matrix.

5. $\left[\begin{array}{rr|r} 1 & 4 & 2 \\ 0 & 1 & 3 \end{array}\right]$
6. $\left[\begin{array}{rrr|r} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$
7. $\left[\begin{array}{rrr|r} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

For Exercises 8–9, refer to the matrix

$$A = \begin{bmatrix} \mathbf{-1} & \mathbf{3} & \mathbf{0} \\ \mathbf{2} & \mathbf{5} & \mathbf{-4} \\ \mathbf{6} & \mathbf{9} & \mathbf{-8} \end{bmatrix}.$$

a. Find the minor of the given element.

b. Find the cofactor of the given element.

8. a_{12}
9. a_{31}

For Exercises 10–11, evaluate the determinant of the matrix.

10. $\begin{bmatrix} 3 & 7 \\ 4 & -1 \end{bmatrix}$
11. $\begin{bmatrix} -3 & 4 & 7 \\ 1 & -2 & 3 \\ 6 & 5 & 0 \end{bmatrix}$
12. Use the determinant of A to determine whether A has an inverse.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 11 \\ 2 & 5 & 10 \end{bmatrix}$ $|A| = 0$. Therefore, A is singular (does not have an inverse).

For Exercises 13–16, solve the system by using Gaussian elimination or Gauss-Jordan elimination.

13. $-3(x + y) = 3y - 12$
    $-3x = 4y - 6$
14. $-3x = 11 - 18y$
    $x - 6y = 2$
15. $x - 2y = 5z + 4$
    $6y + 18z = 2x - 8$
    $-3x + 8y + 20z = -18$
16. $2x + 6y + 30z = 2$
    $x + 2y + 11z = 0$
    $-3x - 6y - 33z = 0$
17. Solve the system by using Cramer's rule.
    $3x - 5y = 7$
    $11x + 2y = 8$
18. Use Cramer's rule to solve for x.
    $2x = 3y - 4z + 11$
    $9x + y = z - 1$
    $3x - 4y = 7$
19. Solve the equation $A - 4X = B$ for X, given that $A = \begin{bmatrix} 2 & 5 \\ -2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -3 \\ 14 & 5 \end{bmatrix}$.

For Exercises 20–23, perform the indicated operations if possible.

$$A = \begin{bmatrix} 4 & 1 & -3 \\ 2 & 4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 9 \\ 0 & -1 \\ 3 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & -4 \\ 2 & -1 & 8 \end{bmatrix}$$

20. $2A - 3C$

21. $A + B$

22. AB

23. BA

24. Determine whether A and B are inverses.

$$A = \begin{bmatrix} -1 & \frac{2}{3} & -\frac{2}{3} \\ 1 & -\frac{1}{3} & \frac{1}{3} \\ 1 & -\frac{2}{3} & \frac{5}{3} \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

For Exercises 25–27, determine the inverse of the matrix if possible. Otherwise, state that the matrix is singular.

25. $A = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$

26. $A = \begin{bmatrix} 2 & 5 & 10 \\ 1 & 3 & 7 \\ 1 & 4 & 11 \end{bmatrix}$

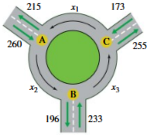
27. $A = \begin{bmatrix} 3 & -1 & -1 \\ 2 & -1 & 1 \\ -5 & 2 & 1 \end{bmatrix}$

For Exercises 28–29, solve the system by using the inverse of the coefficient matrix.

28. $3x + 2y = 13$
$5x + 4y = 25$ (See Exercise 25 for A^{-1}.)

29. $3x - y - z = 8$
$2x + z = y$ (See Exercise 27 for A^{-1}.)
$-5x + 2y + z = -11$

30. a. Assume that traffic flows freely around the traffic circle. The flow rates given are measured in vehicles per hour. If the flow rate x_3 is 210 vehicles per hour, determine the flow rates x_1 and x_2.



b. If traffic between intersections B and C flows at a rate of between 200 and 250 vehicles per hour, inclusive, find the range of values for x_1 and x_2.

31. Matrix C represents the number of calories burned per hour of exercise riding a bicycle, running, and walking for individuals of two different weights. Matrix N represents the number of hours spent working out with each type of activity for a given week. Find the product CN and interpret its meaning.

$$\begin{array}{cccc} & \textbf{Bike} & \textbf{Run} & \textbf{Walk} & \\ C = & \left[\begin{matrix} 400 \\ 550 \end{matrix}\right. & \begin{matrix} 500 \\ 780 \end{matrix} & \left.\begin{matrix} 320 \\ 480 \end{matrix}\right] & \begin{matrix} \textbf{120-lb person} \\ \textbf{180-lb person} \end{matrix} \end{array}$$

$$N = \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix} \begin{matrix} \textbf{Bike} \\ \textbf{Run} \\ \textbf{Walk} \end{matrix}$$

32. Given three points (x_1, y_1), (x_2, y_2), and (x_3, y_3), the points are collinear if the following determinant is equal to zero.

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Determine if the points $(4, -11)$, $(-1, -1)$, and $(-5, 7)$ are collinear.

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Chapter 9 Cumulative Review Exercises

1. Given $f(x) = x^3$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

2. Given $f(x) = -2(x-4)^2 - 5$,

- **a.** Determine the vertex of the graph of the parabola.
- **b.** Determine the intervals over which the graph is increasing.
- **c.** Determine the intervals over which the graph is decreasing.
- **d.** Write the domain in interval notation.
- **e.** Write the range in interval notation.

3. Given $f(x) = \sqrt{2x-3}$, find the average rate of change on the interval $\left[2, \frac{7}{2}\right]$

4. Simplify each expression.

- **a.** i^{57}
- **b.** i^{22}

5. Divide and write the answer in the form $a + bi$. $\frac{3+4i}{2-3i}$

6. Find the distance between the points $(1, -7)$ and $(4, 2)$.

7. Determine the center and radius of the circle defined by $\left(x + \frac{5}{3}\right)^2 + y^2 = 11$.

8. Technetium-99m (abbreviated ^{99m}Tc) is a short-lived gamma-ray emitter that is used in nuclear medicine. Suppose that a patient is given a small amount of ^{99m}Tc for a diagnostic test. Further suppose that the amount of ^{99m}Tc decays exponentially according to the model $Q(t) = Q_0 e^{-kt}$. The value Q_0 is the initial amount administered, and $Q(t)$ represents the amount of ^{99m}Tc remaining after t hours. The value of k is the decay constant.

- **a.** If 94% of the technetium has decayed after 24 hr (that is, 6% remains), determine the value of k. Round to 3 decimal places.
- **b.** If 30 mCi is initially given to a patient for blood pool imaging of the heart, determine the amount remaining after 10 hr. Round to 1 decimal place.
- **c.** Determine the amount of time required for the amount of ^{99m}Tc to fall below 1% of the original amount given. Round to 1 decimal place.

For Exercises 9–12,

a. Graph the function.

b. Write the domain in interval notation.

c. Write the range in interval notation.

9. $f(x) = e^{x-2} + 1$

10. $g(x) = \ln(x + 4)$

11. $h(x) = \sqrt{-x}$

12. $k(x) = \begin{cases} (x-1)^2 - 2 & \text{for } x \leq 1 \\ x + 1 & \text{for } x > 1 \end{cases}$

For Exercises 13–14, solve the system of equations.

13. $3(x - y) = y - 3$

$-2x + \frac{1}{2}y = -11$

14. $2x - y + 5z = -2$

$x \quad + 4z = 0$

$x + y = 2 - 7z$

15. Determine if the graph of the equation $|x| + y^2 = 9$ is symmetric with respect to the x-axis, y-axis, origin, or none of these.

16. Determine if the function defined by $f(x) = 6x^3 - 4x$ is even, odd, or neither.

For Exercises 17–20, refer to the given matrices and perform the indicated operations.

$A = \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 \\ 0 & 7 \end{bmatrix}$ $C = \begin{bmatrix} -1 & 4 & 0 \\ 4 & 2 & 6 \\ 3 & -2 & 0 \end{bmatrix}$

17. AB

18. $|C|$

19. $4A - B$

20. A^{-1}

For Exercises 21–23, find the union or intersection of the given sets. Write the answers in set-builder notation.

$A = \{x | 3 < x\}$, $B = \{x | -2 < x \leq 4\}$, $C = \{x | x \geq 5\}$

21. $A \cup B$

22. $A \cap B$

23. $A \cap C$

For Exercises 24–27, solve the equation or inequality. Write the solution set to the inequalities in interval notation if possible.

24. $2x - 3 - 5\sqrt{2x - 3} + 4 = 0$

25. $|-x + 4| = |x - 4|$

26. $\frac{x + 4}{2x - 1} \geq 1$

27. $\log_2(3x - 4) = \log_2(x + 1) + 1$

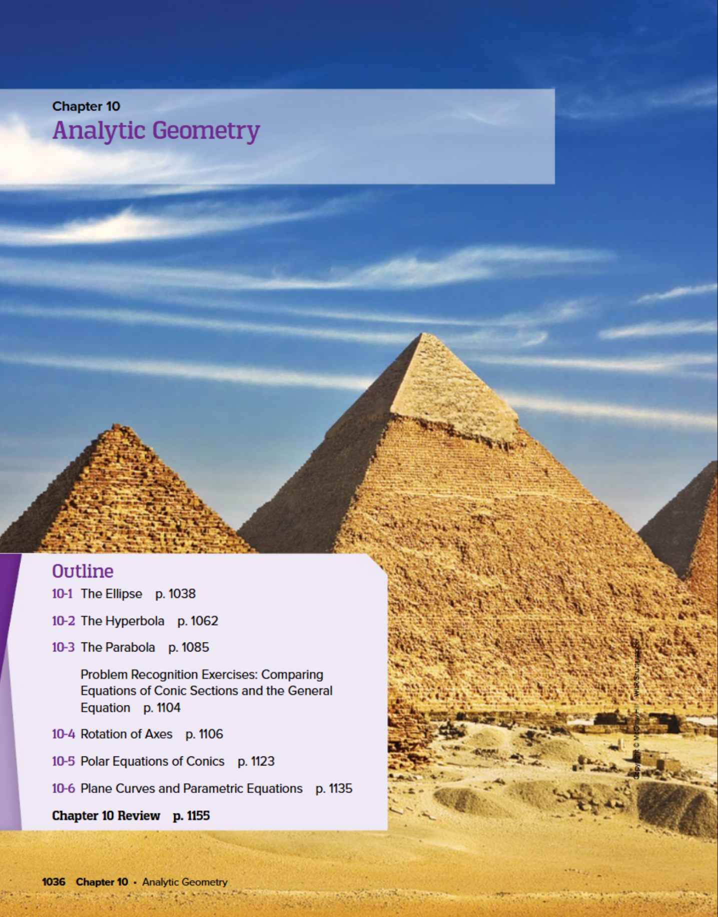
28. Find all zeros of $f(x) = 2x^3 - x^2 + 18x - 9$.

29. Find the asymptotes. $r(x) = \frac{2x - 1}{x^2 - 9}$

30. Write the expression as a single logarithm.

$-3\log x - \frac{1}{2}\log y + 5\log z$

Go online to answer these questions.



Chapter 10

Analytic Geometry

Outline

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Launch Activity: Mathematical Modeling

The Pyramidal Cake

In History class, Troy and Abed learned about the Great Pyramid of Giza in Egypt. As one of the Seven Wonders of the Ancient World, they were amazed that the pyramid is still largely intact. To pay homage to the Great Pyramid, Troy and Abed baked pyramid-shaped cakes for all their classes. When they took the cake to their math class, the teacher used it as a learning opportunity before everyone could eat their slice. As Troy was cutting slices of the cake, the teacher asked Abed to make observations on the knife cuts used to divide the cake for the class.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer this question: How many 2-dimensional shapes can be created by slicing the pyramidal cake and at what angle does the knife need to cut to create each shape?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

Go online to answer these questions.

Lesson 10-1

The Ellipse

What Will You Learn?

After completing this lesson, you should be able to:

- Graph an Ellipse Centered at the Origin
- Graph an Ellipse Centered at (***h***, ***k***)
- Use Ellipses in Applications
- Determine and Apply the Eccentricity of an Ellipse

Learn: Graph an Ellipse Centered at the Origin

In this chapter, we investigate four types of curves called conic sections. Specifically, these are the circle (covered in detail in Lesson 1.2), the ellipse, the hyperbola, and the parabola. Conic sections derive their names because each is the intersection of a plane and a double-napped cone (Figure 10-1).

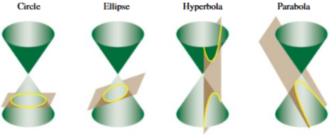


Figure 10-1

Recall from Lesson 1.2 that a circle is the set of all points in a plane, *r* units from a fixed point (the center of the circle), where *r* is the radius of the circle. We derived the standard form of an equation of a circle from the distance formula by noting that the distance between the center of the circle (h, k) and a point on the circle (x, y) is *r* units.

Standard Form of an Equation of a Circle

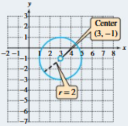
Given a circle centered at (h, k) with radius *r*, the standard form of an equation of the circle is given by

$$(x-h)^2+(y-k)^2=r^2 \text{ for } r>0.$$

Example: $(x-3)^2+(y+1)^2=4$

Standard form: $(x-3)^2+[y-(-1)]^2=2^2$

Center: $(3, -1)$, Radius: 2



We now study a second type of conic section called an ellipse. An ellipse is an oval-shaped curve that appears in applications in many fields, including architecture, acoustics, medicine, and astronomy. To determine an equation of an ellipse, we first need to formalize the definition.

Definition of an Ellipse

An **ellipse** is the set of all points (x, y) in a plane such that the sum of the distances between (x, y) and two fixed points is a constant. The fixed points are called the **foci** (plural of **focus**) of the ellipse.

To visualize the definition of an ellipse, consider the following application. Suppose Sonya wants to cut an elliptical rug from a rectangular rug to avoid a stain made by the family dog. Two tacks are placed along the center horizontal line. Then the ends of a slack piece of rope are tied to each tack. With the rope pulled tight, Sonya traces out a curve. This curve is an ellipse, and the tacks are located at the foci of the ellipse (Figure 10-2).

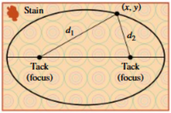


Figure 10-2

An ellipse may be elongated in any direction; however, we will study only those that are elongated horizontally and vertically. Figure 10-3 and Figure 10-4 show ellipses centered at the origin with foci on the x-axis and y-axis, respectively. The line through the foci intersects the ellipse at two points called **vertices.** The line segment with endpoints at the vertices is called the **major axis.** The **center** of the ellipse is the midpoint of the major axis. The line segment perpendicular to the major axis and passing through the center of the ellipse with endpoints on the ellipse is called the **minor axis.**

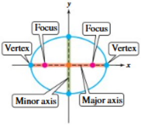


Figure 10-3

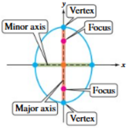


Figure 10-4

We will derive the standard form of an equation of an ellipse centered at the origin with foci on the x-axis. In Figure 10-5, the foci are labeled as $F_1(-c, 0)$ and $F_2(c, 0)$, and the vertices are labeled as $V_1(-a, 0)$ and $V_2(a, 0)$. Let (x, y) be an arbitrary point on the ellipse, and let the distances between (x, y) and the two foci be d_1 and d_2.

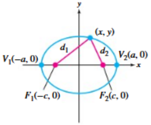


Figure 10-5

From Figure 10-5, the distance from F_1 to V_2 is $(a + c)$. The distance from F_2 to V_2 is $(a - c)$. By the definition of an ellipse, the sum of the distances between a point on the ellipse (such as V_2) and the two foci must be constant. Adding $(a + c)$ and $(a - c)$, we see that this constant is $2a$.

$$(a + c) + (a - c) = 2a$$

The sum of the distances between a point on an ellipse and the two foci is equal to the length of the major axis, $2a$.

The sum of the distances between (x, y) and the two foci is equal to $2a$.

$$\begin{aligned}
d_1 \qquad\qquad &+ \qquad\qquad d_2 &= 2a& \\
\sqrt{[(x-(-c)]^2 + (y-0)^2} &+ \sqrt{(x-c)^2 + (y-0)^2} &= 2a& \quad \text{Apply the distance formula.} \\
\sqrt{(x+c)^2 + y^2} &+ \sqrt{(x-c)^2 + y^2} &= 2a& \quad \text{Simplify within parentheses.}
\end{aligned}$$

In Exercise 109, we guide you through a series of algebraic steps to eliminate the radicals. Then collecting the variable terms on one side and the constant terms on the other side, the equation becomes

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2) \qquad (1)$$

From Figure 10-5, we see that the distance between F_1 and F_2 is $2c$. Furthermore, since $d_1 + d_2 = 2a$, we have that $2c < 2a$, which implies that $c < a$ and that $a^2 - c^2 > 0$. Letting $b^2 = a^2 - c^2$, Equation (1) becomes

$$\begin{aligned}
b^2x^2 + a^2y^2 &= a^2b^2 & \\
\frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} &= \frac{a^2b^2}{a^2b^2} & \quad \text{Divide both sides by } a^2b^2. \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 & \quad \text{Standard form of an equation of an ellipse}
\end{aligned}$$

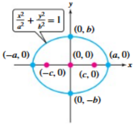
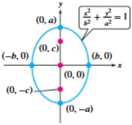
This equation represents an ellipse centered at the origin with foci on the x-axis. The length of the major axis is $2a$ and the length of the minor axis is $2b$. The value a is also referred to as the length of the **semimajor axis** (the distance from the center of an ellipse to a vertex). The value b is referred to as the length of the **semiminor axis** (half the length of the minor axis).

In a similar manner, we can develop the standard form of an equation of an ellipse centered at the origin with foci on the y-axis. These two equations are summarized in the following box.

Standard Forms of an Equation of an Ellipse Centered at the Origin

The standard forms of an equation of an ellipse centered at the origin are as follows, assuming:

1. $a > b > 0$.
2. The length of the major axis is $2a$.
3. The length of the minor axis is $2b$.

	Major Axis: *x*-axis	**Major Axis: *y*-axis**
Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
Center	(0, 0)	(0, 0)
Foci **(*Note*: $c^2 = a^2 - b^2$)**	$(c, 0)$ and $(-c, 0)$	$(0, c)$ and $(0, -c)$
Vertices **Endpoints: major axis**	$(a, 0)$ and $(-a, 0)$	$(0, a)$ and $(0, -a)$
Endpoints: minor axis	$(0, b)$ and $(0, -b)$	$(b, 0)$ and $(-b, 0)$
Graph	 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ y x (0, b) (−a, 0) (0, 0) (a, 0) (−c, 0) (c, 0) (0, −b)	 y x (0, a) $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ (0, c) (−b, 0) (b, 0) (0, 0) (0, −c) (0, −a)

Check Your Work

It is important to note that the foci are not actually on the graph of an elliptical curve. However, the foci are used to define the curve.

For an ellipse centered at the origin, the *x*- and *y*-intercepts are the endpoints of the major and minor axes. For example, for an ellipse centered at (0, 0) with foci on the *x*-axis, we have:

Standard form:

x-intercept: substitute 0 for *y*.

y-intercept: substitute 0 for *x*.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow \frac{x^2}{a^2}+\frac{(0)^2}{b^2}=1 \qquad \frac{(0)^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{x^2}{a^2}=1 \qquad \frac{y^2}{b^2}=1$$

$$x^2=a^2 \qquad y^2=b^2$$

$$x=\pm a \qquad y=\pm b$$

In Example 1, we graph an ellipse and identify the key features.

Example 1: Graphing an Ellipse Centered at the Origin

Given $\frac{x^2}{16} + \frac{y^2}{9} = 1$, graph the ellipse, and identify the center, vertices, and foci.

Solution:

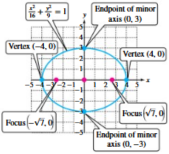
$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is in the standard form of an ellipse. Since 16 > 9, $a^2 = 16$ and $b^2 = 9$. Since the greater number in the denominator is found in the x^2 term, the ellipse is elongated horizontally and the foci are on the x-axis.

$$\frac{x^2}{\underbrace{(4)^2}_{a=4}} + \frac{y^2}{\underbrace{(3)^2}_{b=3}} = 1$$

$a^2 = 16$, and because $a > 0$, we have that $a = 4$.
$b^2 = 9$, and because $b > 0$, we have that $b = 3$.

To graph the ellipse, plot the vertices (endpoints of the major axis) and the endpoints of the minor axis, and sketch the ellipse through the points.



Center: (0, 0)

Vertices: $(a, 0)$ and $(-a, 0)$
(4, 0) and (−4, 0)

Minor axis endpoints: $(0, b)$ and $(0, -b)$
(0, 3) and (0, −3)

To find the foci, $(c, 0)$ and $(-c, 0)$, use the relationship $c^2 = a^2 - b^2$:

$c^2 = (4)^2 - (3)^2 = 16 - 9 = 7$

$c^2 = 7$ (Recall that $c > 0$.)

$c = \sqrt{7} \approx 2.65$

Foci: $(c, 0)$ and $(-c, 0)$

$(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$

It is important to note that the center and foci of the ellipse are not actually part of the curve. For this reason, we graphed the curve in blue and plotted the foci and center in red.

Apply the Skills

1. Graph the ellipse, and identify the center, vertices, minor axis endpoints, and foci. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

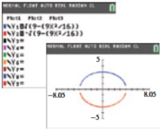
Point of Interest

The National Center for the Performing Arts in Beijing, People's Republic of China, is a semielliptical dome made of titanium and glass and surrounded by an artificial lake. On a calm day, the reflection from the lake gives the illusion of a full ellipse.

The graph of an ellipse does not define y as a function of x (the graph fails the vertical line test). However, an ellipse can be defined by the union of two functions: an upper semiellipse and a lower semiellipse. This is helpful for graphing an ellipse on a graphing utility.

From Example 1, we have

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \longrightarrow y = \pm\sqrt{9 - \frac{9x^2}{16}}$$



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NORMAL FLOAT AUTO REAL RADIAN CL
 Plot1  Plot2  Plot3
\Y1=√(9-(9X²/16))
\Y2=-√(9-(9X²/16))
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
```

```
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```

The functions represent the top and bottom semiellipses, respectively.

$y = \sqrt{9 - \frac{9x^2}{16}}$	top semiellipse
$y = -\sqrt{9 - \frac{9x^2}{16}}$	bottom semiellipse

Example 2: Graphing an Ellipse Centered at the Origin

Given $25x^2 + 9y^2 = 225$, graph the ellipse and identify the center, vertices, minor axis endpoints, and foci.

Solution:

$25x^2 + 9y^2 = 225$	For the equation to be in the standard form for an ellipse, we require the constant term to be 1.
$\frac{25x^2}{225} + \frac{9y^2}{225} = \frac{225}{225}$	Divide both sides by 225.
$\frac{x^2}{9} + \frac{y^2}{25} = 1$	Since $25 > 9$, $a^2 = 25$ and $b^2 = 9$. Since the greater number in the denominator is found in the y^2 term, the ellipse is elongated vertically and the foci are on the y-axis.
$\frac{x^2}{\underbrace{(3)^2}_{b=3}} + \frac{y^2}{\underbrace{(5)^2}_{a=5}} = 1$	$a^2 = 25$, and because $a > 0$, we have that $a = 5$. $b^2 = 9$, and because $b > 0$, we have that $b = 3$.

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Center: (0, 0)

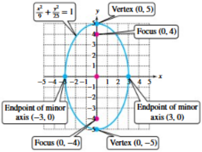
Vertices: $(0, a)$ and $(0, -a)$
$(0, 5)$ and $(0, -5)$

Minor axis endpoints: $(b, 0)$ and $(-b, 0)$
$(3, 0)$ and $(-3, 0)$

To find the foci, $(0, c)$ and $(0, -c)$, use the relationship $c^2 = a^2 - b^2$:

$$\begin{aligned} c^2 &= (5)^2 - (3)^2 = 25 - 9 = 16 \\ c^2 &= 16 \\ c &= 4 \quad \text{(Recall that } c > 0.\text{)} \end{aligned}$$

Foci: $(0, c)$ and $(0, -c)$
$(0, 4)$ and $(0, -4)$



Insights

For an equation of an ellipse written in standard form, the term containing the larger denominator indicates the direction of elongation of the graph. In Example 2, the larger denominator of 25 is located in the y term. Notice that the ellipse is elongated vertically.

Apply the Skills

2. Graph the ellipse and identify the center, vertices, minor axis endpoints, and foci.

$$25x^2 + 9y^2 = 900$$

Learn: Graph an Ellipse Centered at (*h*, *k*)

In Lesson 1.2, we learned that the standard form of an equation of a circle centered at the origin with radius r is given by $x^2 + y^2 = r^2$. For a circle centered at (h, k), we have the equation

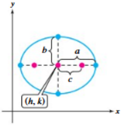
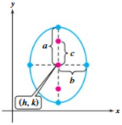
$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Circle centered at } (h, k) \text{ with radius } r$$

Replacing x by $x - h$ and y by $y - k$ shifts the graph of $x^2 + y^2 = r^2$ a total of h units horizontally and k units vertically. The same is true for an equation of an ellipse.

Standard Forms of an Equation of an Ellipse Centered at (*h*, *k*)

The standard forms of an equation of an ellipse centered at (*h*, *k*) are as follows. Assume that

1. $a > b > 0$.
2. The length of the major axis is $2a$.
3. The length of the minor axis is $2b$.

	Major Axis Horizontal	**Major Axis Vertical**
Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
Center	(h, k)	(h, k)
Foci **(*Note:* $c^2 = a^2 - b^2$)**	$(h + c, k)$ and $(h - c, k)$	$(h, k + c)$ and $(h, k - c)$
Vertices **Endpoints: major axis**	$(h + a, k)$ and $(h - a, k)$	$(h, k + a)$ and $(h, k - a)$
Endpoints: minor axis	$(h, k + b)$ and $(h, k - b)$	$(h + b, k)$ and $(h - b, k)$
Graph		

To graph an ellipse centered at (*h*, *k*), it is helpful to locate the center first as well as to identify the orientation of the ellipse (elongated horizontally or elongated vertically). Then use the values of *a* and *b* to locate the endpoints of the major and minor axes. This is demonstrated in Example 3.

Example 3: Graphing an Ellipse Centered at (*h*, *k*)

Graph the ellipse and identify the center, vertices, endpoints of the minor axis, and foci.

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25} = 1$$

Solution:

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25} = 1$$

The standard form of an equation of an ellipse is given. The quantity $(y + 2)^2$ is equivalent to $[y - (-2)]^2$. Therefore, the center of the ellipse is $(3, -2)$.

$\underbrace{\frac{(x-3)^2}{(4)^2}}_{b=4} + \underbrace{\frac{[y-(-2)]^2}{(5)^2}}_{a=5} = 1$ Since $25 > 16$, $a^2 = 25$ and $b^2 = 16$.
$a = 5$ and $b = 4$.
Since the greater number in the denominator is found in the y^2 term, the ellipse is elongated vertically.

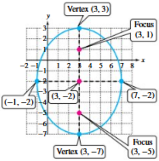
Center: $(3, -2)$

The vertices are $a = 5$ units above and below the center.

Vertices: $(3, -2 + 5)$ and $(3, -2 - 5)$ $(3, 3)$ and $(3, -7)$

The minor axis endpoints are $b = 4$ units to the right and left of the center.

Minor axis endpoints: $(3 + 4, -2)$ and $(3 - 4, -2)$
$(7, -2)$ and $(-1, -2)$



To find the foci, use the relationship $c^2 = a^2 - b^2$:

$c^2 = (5)^2 - (4)^2 = 25 - 16 = 9$
$c = 3$ (Recall that $c > 0$.)

The foci are $c = 3$ units above and below the center.

Foci: $(3, -2 + 3)$ and $(3, -2 - 3)$
$(3, 1)$ and $(3, -5)$.

Apply the Skills

3. Graph the ellipse and identify the center, vertices, foci, and endpoints of the minor axis. $\frac{(x+1)^2}{25} + \frac{(y-2)^2}{9} = 1$

The equation of the ellipse from Example 3 can be expanded by expanding the binomials and combining like terms.

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25} = 1 \qquad \text{Standard form}$$

$$400\left[\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25}\right] = 400(1) \qquad \text{Clear fractions.}$$

$$25(x-3)^2 + 16(y+2)^2 = 400$$

$$25(x^2 - 6x + 9) + 16(y^2 + 4y + 4) = 400 \qquad \text{Expand the binomials.}$$

$$25x^2 + 16y^2 - 150x + 64y - 111 = 0 \qquad \text{Expanded form}$$

Given an equation of an ellipse in expanded form, we can reverse this process by completing the square to write the equation in standard form. The benefit of writing an equation in standard form is that we can easily identify the center of the ellipse and the values of a and b.

Example 4: Writing an Equation of an Ellipse in Standard Form

Given $3x^2 + 7y^2 + 6x - 28y + 10 = 0$,

a. Write the equation of the ellipse in standard form.

b. Identify the center, vertices, foci, and endpoints of the minor axis.

Solution:

a. $3x^2 + 7y^2 + 6x - 28y + 10 = 0$

Group the x terms and group the y terms.

$$3x^2 + 6x + 7y^2 - 28y = -10$$

Move the constant term to the right side of the equation. In this case, we substract 10 from both sides.

$$3(x^2 + 2x + \underline{\quad}) + 7(y^2 - 4y + \underline{\quad}) = -10$$

Factor out the leading coefficient from the x and y terms. Leave room to complete the square within parentheses.

Complete the square.

Note that $\left[\frac{1}{2}(2)\right]^2 = 1$ and $\left[\frac{1}{2}(-4)\right]^2 = 4$.

$$3(\overbrace{x^2 + 2x + 1}^{3\cdot 1\text{ is added to the left side.}}) + 7(\overbrace{y^2 - 4y + 4}^{7\cdot 4\text{ is added to the left side.}}) = -10 + \overset{3\cdot 1}{\downarrow}\!\!\!\!\!3 + \overset{7\cdot 4}{\downarrow}\!\!\!\!\!28$$

To balance the equation add $3\cdot 1$ and $7\cdot 4$ to the right side.

The values 1 and 4 are added inside parentheses. However, as a result of the factors of 3 and 7 in front of the parentheses, $3\cdot 1$ and $7\cdot 4$ are actually added to the left side of the equation.

$$3(x + 1)^2 + 7(y - 2)^2 = 21$$

To write the equation in standard form, we require that the constant on the right side of the equation be 1.

$$\frac{3(x + 1)^2}{21} + \frac{7(y - 2)^2}{21} = \frac{21}{21}$$

Divide both sides by 21.

$$\frac{(x + 1)^2}{7} + \frac{(y - 2)^2}{3} = 1$$

Standard form

b. $$\frac{[x - (-1)]^2}{7} + \frac{(y - 2)^2}{3} = 1$$

The center is $(-1, 2)$.

Since $7 > 3$, $a^2 = 7$ and $b^2 = 3$. The greater number in the denominator is in the x^2 term.

$$\frac{[x - (-1)]^2}{\underset{\boxed{a = \sqrt{7}}}{(\sqrt{7})^2}} + \frac{(y - 2)^2}{\underset{\boxed{b = \sqrt{3}}}{(\sqrt{3})^2}} = 1$$

Therefore, the ellipse is elongated horizontally.

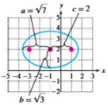
$a^2 = 7$. Because $a > 0$, $a = \sqrt{7}$.

$b^2 = 3$. Because $b > 0$, $b = \sqrt{3}$.

$c^2 = a^2 - b^2 = 7 - 3 = 4$

$c^2 = 4$, and because $c > 0$, we have $c = 2$.

The center is $(-1, 2)$.
The vertices are $(-1 + \sqrt{7}, 2)$ and $(-1 - \sqrt{7}, 2)$.
The foci are $(1, 2)$ and $(-3, 2)$.
The endpoints of the minor axis are
$(-1, 2 + \sqrt{3})$ and $(-1, 2 - \sqrt{3})$.



Apply the Skills

4. Repeat Example 4 with $2x^2 + 11y^2 - 12x + 44y + 40 = 0$.

Now that we know the standard form of an equation of an ellipse, we can build an equation based on known information about the ellipse. This is demonstrated in Example 5.

Example 5: Finding an Equation of an Ellipse from Given Information

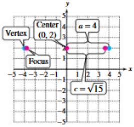
Determine the standard form of an equation of an ellipse with vertices $(4, 2)$ and $(-4, 2)$ and foci $(\sqrt{15}, 2)$ and $(-\sqrt{15}, 2)$.

Solution:

Plotting the given points tells us that the ellipse is elongated horizontally. Therefore, the standard form is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The center is midway between the vertices.



Therefore, the center is $(0, 2)$, indicating that $h = 0$ and $k = 2$.

The vertices are located 4 units away from the center. This means that $a = 4$.

The foci are located $\sqrt{15}$ units from the center, indicating that $c = \sqrt{15}$.

The only missing piece of information is the value of b.

We can find b from the relationship $c^2 = a^2 - b^2$ or equivalently $b^2 = a^2 - c^2$.

$$b^2 = (4)^2 - (\sqrt{15})^2$$
$$b^2 = 1$$
$$b = 1 \quad \text{(Recall that } b > 0.)$$

$$\frac{(x-0)^2}{(4)^2} + \frac{(y-2)^2}{(1)^2} = 1 \quad \text{Substitute } h = 0, k = 2, a = 4, \text{ and } b = 1 \text{ into the standard form.}$$

$$\frac{x^2}{16} + \frac{(y-2)^2}{1} = 1 \quad \text{or} \quad \frac{x^2}{16} + (y-2)^2 = 1 \qquad \text{Standard form}$$

Apply the Skills

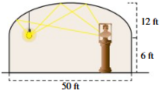
5. Determine the standard form of an equation of an ellipse with vertices (3, 3) and (−3, 3) and foci $(2\sqrt{2}, 3)$ and $(-2\sqrt{2}, 3)$.

Learn: Use Ellipses in Applications

The ellipse and its three-dimensional counterpart, the ellipsoid (an object in the shape of a blimp), have many useful applications that arise from the reflective property of the ellipse. For example, consider an elliptical-shaped dome. Sound or light waves emitted from one focus are reflected off the surface to the other focus. This principle is applied in Example 6.

Example 6: Applying the Reflective Property of an Ellipse

A room in a museum has a semielliptical dome with a major axis of 50 ft and a semiminor axis of 12 ft. A light is placed at one focus of the ellipse and the light that reflects off the ceiling will be directed to the other focus. The museum curator wants to place a sculpture so that the sculpture is at the other focus where it will be illuminated by the light.



a. Choose a coordinate system so that the ellipse is oriented with horizontal major axis with the center placed at (0, 0). Write an equation of the semiellipse.

b. Approximately where should the sculpture be placed?

Solution:

a. A graph of the semiellipse is shown in Figure 10-6. The length of the semimajor axis is 25 ft ($a = 25$), and the length of the semiminor axis is 12 ft ($b = 12$). An equation of the full ellipse is $\frac{x^2}{25^2} + \frac{y^2}{12^2} = 1$ or $\frac{x^2}{625} + \frac{y^2}{144} = 1$.

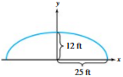


Figure 10-6

We want the top half only, which is given by $\frac{x^2}{625} + \frac{y^2}{144} = 1$ for $y \geq 0$, or equivalently $y = 12\sqrt{1 - \frac{x^2}{625}}$.

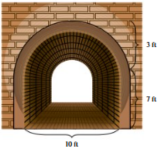
b. To find the location of the sculpture, we must find the location of the other focus.

$$c^2 = a^2 - b^2$$
$$c^2 = (25)^2 - (12)^2 = 625 - 144 = 481$$
$$c = \pm\sqrt{481} \approx \pm 21.9$$

The sculpture should be located approximately 21.9 ft from the center of the room along the major axis, opposite the light source, and 6 ft up.

Apply the Skills

6. A tunnel has vertical sides of 7 ft with a semielliptical top. The width of the tunnel is 10 ft, and the height at the top is 10 ft.
   a. Write an equation of the semiellipse. For convenience, place the coordinate system with (0, 0) at the center of the ellipse.
   b. To construct the tunnel, an engineer needs to find the location of the foci. How far from the center are the foci?



Learn: Determine and Apply the Eccentricity of an Ellipse

The degree of elongation of an ellipse is measured by the eccentricity *e* of the ellipse.

Eccentricity of an Ellipse

For an ellipse defined by $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$,

the **eccentricity *e*** is given by

$$e = \frac{c}{a},$$

where $a > b > 0$, $c > 0$, and $c^2 = a^2 - b^2$.

Note: The eccentricity of an ellipse is a number between 0 and 1; that is, $0 < e < 1$.

With this definition, we see that the eccentricity of an ellipse is the ratio of the distance between a focus and the center to the distance between a vertex and the center; that is, $e = \frac{c}{a}$.

- If the foci are near the vertices (that is, *a* is only slightly greater than *c*), then the ellipse is more elongated (Figure 10-7).
- If the foci are near the center, then the ellipse is more circular (Figure 10-8).

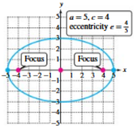


Figure 10-7

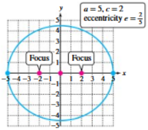


Figure 10-8

The orbits of the planets in our solar system are elliptical with the Sun at one focus. The eccentricities given in Table 10-1 indicate that the orbit of Venus is the most circular and the orbit of Pluto is the most elongated. (*Source*: NASA, www.nasa.gov)

TABLE 10-1 Eccentricity Values for the Orbits of the Planets

Planet	Eccentricity
Mercury	0.2056
Venus	0.0067 ←
Earth	0.0167
Mars	0.0935
Jupiter	0.0489
Saturn	0.0565
Uranus	0.0457
Neptune	0.0113
Pluto (dwarf planet)	0.2488 ←

The orbit of Venus is the most circular.

The orbit of Pluto is the most elongated.

Example 7: Using Eccentricity in an Application

The point in a planet's orbit where it is closest to the Sun is called **perihelion** and the point where it is farthest away is called **aphelion** (Figure 10-9). Suppose that Mars is 2.0662×10^8 km from the Sun at perihelion. Use the eccentricity from Table 10-1 to find the distance between Mars and the Sun at aphelion.

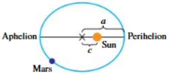


Figure 10-9

Solution:

For convenience, we have labeled the center of the ellipse as point *C* and the focus at which the Sun resides as point *S*. Aphelion and perihelion are labeled *A* and *P*, respectively (Figure 10-10).

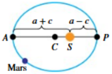


Figure 10-10

We need to find the distance between Mars and the Sun at aphelion. This is given by $a + c$. Since a and c are two unknown values, we need two independent relationships between a and c.

The distance at perihelion and the eccentricity provide this information.

(1) $a - c = 2.0662 \times 10^8$ (distance at perihelion)

(2) $\frac{c}{a} = 0.0935 \rightarrow c = 0.0935a$ (eccentricity)

This a nonlinear system of equations.

$a - 0.0935a = 2.0662 \times 10^8$ Use the substitution method.

$0.9065a = 2.0662 \times 10^8$ Combine like terms.

$a \approx 2.2793 \times 10^8$ Divide by 0.9065.

$c = 0.0935(2.2793 \times 10^8)$ Back substitute.

$c = 2.1311 \times 10^7$

So, $a + c = 2.2793 \times 10^8 + 2.1311 \times 10^7$

$= 2.4924 \times 10^8$

The distance between Mars and the Sun at aphelion (the farthest point from the Sun) is approximately 2.4924×10^8 km.

Apply the Skills

7. Pluto is 7.376×10^9 km at aphelion (farthest point from the Sun). Use the eccentricity of Pluto's orbit of 0.2488 to find the closest distance between Pluto and the Sun.

Practice 10-1

Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, determine the center and radius of the circle.

R.1. $x^2 + (y + 5)^2 = 8$

R.2. $x^2 + y^2 = 25$

R.3. $(x - 4)^2 + (y + 3)^2 = 81$

For Exercises R.4–R.5, find the value of *n* so that the expression is a perfect square trinomial. Then factor the trinomial.

R.4. $a^2 - 8a + n$

R.5. $y^2 + 7y + n$

R.6. Complete the square and write the equation of the circle in standard form.
$x^2 + y^2 + 10x - 10y + 41 = 0$

Concept Connections

1. The circle, the ellipse, the hyperbola, and the parabola are categories of ____________ sections.

2. An ____________ is a set of points (x, y) in a plane such that the sum of the distances between (x, y) and two fixed points called ____________ is a constant.

3. The line through the foci intersects an ellipse at two points called ____________.

4. The line segment with endpoints at the vertices of an ellipse is called the ____________ axis.

5. The line segment perpendicular to the major axis, with endpoints on the ellipse, and passing through the center of the ellipse, is called the ____________ axis.

6. Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$, the ordered pairs representing the vertices are ____________ and ____________ The ordered pairs representing the endpoints of the minor axis are ____________ and ____________.

7. Given $\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$, where $a > b > 0$, the ordered pairs representing the endpoints of the vertices are ____________ and ____________. The ordered pairs representing the endpoints of the minor axis are ____________ and ____________.

8. When referring to the standard form of an equation of an ellipse, the ____________ *e* is defined as $e = \frac{\square}{\square}$.

Lesson Graph an Ellipse Centered at the Origin

For Exercises 9–10,

a. Use the distance formula to find the distances d_1, d_2, d_3, and d_4.

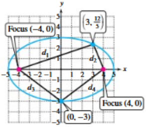
b. Find the sum $d_1 + d_2$.

c. Find the sum $d_3 + d_4$.

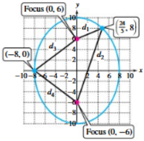
d. How do the sums from parts (b) and (c) compare?

e. How do the sums of the distances from parts (b) and (c) relate to the length of the major axis?

9.



10.



For Exercises 11–12, from the equation of the ellipse, determine if the major axis is horizontal or vertical.

11. a. $\frac{x^2}{2}+\frac{y^2}{5}=1$ **b.** $\frac{x^2}{5}+\frac{y^2}{2}=1$

12. a. $\frac{x^2}{11}+\frac{y^2}{10}=1$ **b.** $\frac{x^2}{10}+\frac{y^2}{11}=1$

For Exercises 13–22,

a. Identify the center of the ellipse.

b. Determine the value of *a*.

c. Determine the value of *b*.

d. Identify the vertices.

e. Identify the endpoints of the minor axis.

f. Identify the foci.

g. Determine the length of the major axis.

h. Determine the length of the minor axis.

i. Graph the ellipse. (**See Examples 1–2, pp. 1042–1043**)

13. $\frac{x^2}{100}+\frac{y^2}{25}=1$

14. $\frac{x^2}{64}+\frac{y^2}{49}=1$

15. $\frac{x^2}{25}+\frac{y^2}{100}=1$

16. $\frac{x^2}{49}+\frac{y^2}{64}=1$

17. $4x^2+25y^2=100$

18. $9x^2+64y^2=576$

19. $-36x^2-4y^2=-36$

20. $-64x^2-16y^2=-64$

21. $\frac{x^2}{12}+\frac{y^2}{5}=1$

22. $\frac{x^2}{18}+\frac{y^2}{7}=1$

Learn: Graph an Ellipse Centered at (h, k)

For Exercises 23–32,

a. Identify the center of the ellipse.

b. Identify the vertices.

c. Identify the endpoints of the minor axis.

d. Identify the foci.

e. Graph the ellipse. (**See Example 3, p. 1045**)

23. $\frac{(x-1)^2}{25}+\frac{(y+6)^2}{16}=1$

24. $\frac{(x-3)^2}{25}+\frac{(y+4)^2}{9}=1$

25. $\frac{(x+4)^2}{49}+\frac{(y-2)^2}{64}=1$

26. $\frac{(x+1)^2}{36}+\frac{(y-5)^2}{81}=1$

27. $(x-6)^2+\frac{y^2}{9}=1$

28. $(x-2)^2+\frac{y^2}{4}=1$

29. $x^2+9(y+1)^2=81$

30. $x^2+4(y+4)^2=100$

31. $\frac{4(x-3)^2}{25}+\frac{16(y-2)^2}{49}=1$

32. $\frac{4(x-4)^2}{81}+\frac{16(y-3)^2}{225}=1$

For Exercises 33–42,

a. Write the equation of the ellipse in standard form. (**See Example 4, p. 1047**)

b. Identify the center, vertices, endpoints of the minor axis, and foci.

33. $3x^2+5y^2+12x-60y+177=0$

34. $7x^2+11y^2+70x-66y+197=0$

35. $3x^2+2y^2-30x-4y+59=0$

36. $5x^2+2y^2-40x-12y+78=0$

37. $4x^2+y^2+14y+45=0$

38. $x^2+49y^2-6x-40=0$

39. $36x^2+100y^2-180x+800y+925=0$

40. $4x^2+9y^2-12x+18y-18=0$

41. $4x^2+9y^2-8x+3=0$

42. $25x^2+16y^2+64y+63=0$

For Exercises 43–56, write the standard form of an equation of an ellipse subject to the given conditions. (See Example 5, p. 1048)

43. Vertices: $(4, 0)$ and $(-4, 0)$; Foci: $(3, 0)$ and $(-3, 0)$

44. Vertices: $(6, 0)$ and $(-6, 0)$; Foci: $(5, 0)$ and $(-5, 0)$

45. Endpoints of minor axis: $(\sqrt{17}, 0)$ and $(-\sqrt{17}, 0)$; Foci: $(0, 9)$ and $(0, -9)$

46. Endpoints of minor axis: $(\sqrt{21}, 0)$ and $(-\sqrt{21}, 0)$; Foci: $(0, 5)$ and $(0, -5)$

47. Major axis parallel to the *x*-axis; Center: $(2, 3)$; Length of major axis: 14 units; Length of minor axis: 10 units

48. Major axis parallel to the y-axis; Center: $(1, 5)$;
Length of major axis: 22 units;
Length of minor axis: 14 units

49. Foci: $(0, 1)$ and $(8, 1)$;
Length of minor axis: 6 units

50. Foci: $(-3, 3)$ and $(7, 3)$;
Length of minor axis: 8 units

51. Vertices: $(0, 5)$ and $(0, -5)$;
Passes through $\left(\frac{16}{5}, 3\right)$

52. Vertices: $(0, 13)$ and $(0, -13)$;
Passes through $\left(\frac{25}{13}, 12\right)$

53. Vertices: $(3, 4)$ and $(3, -4)$;
Foci: $(3, \sqrt{11})$ and $(3, -\sqrt{11})$

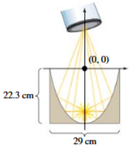
54. Vertices: $(4, 5)$ and $(-4, 5)$;
Foci: $(\sqrt{6}, 5)$ and $(-\sqrt{6}, 5)$

55. Vertices: $(2, 1)$ and $(-12, 1)$;
Foci: $(-5 - \sqrt{33}, 1)$ and $(-5 + \sqrt{33}, 1)$

56. Vertices: $(-8, 4)$ and $(2, 4)$;
Foci: $(-3 - \sqrt{21}, 4)$ and $(-3 + \sqrt{21}, 4)$

Learn: Use Ellipses in Applications

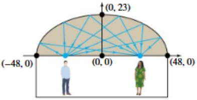
57. The reflective property of an ellipse is used in lithotripsy. Lithotripsy is a technique for treating kidney stones without surgery. Instead, high-energy shock waves are emitted from one focus of an elliptical shell and reflected painlessly to a patient's kidney stone located at the other focus. The vibration from the shock waves shatters the stone into pieces small enough to pass through the patient's urine.



A vertical cross section of a lithotripter is in the shape of a semiellipse with the dimensions shown. Approximate the distance from the center along the major axis where the patient's kidney stone should be located so that the shock waves will target the stone. Round to 2 decimal places.
(**See Example 6, p. 1049**)

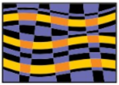
58. The reflective property of an ellipse is the principle behind "whispering galleries." These are rooms with elliptically shaped ceilings such that a person standing at one focus can hear even the slightest whisper spoken by another person standing at the other focus.

Suppose that a dome has a semielliptical ceiling, 96 ft long and 23 ft high. Choose a coordinate system so that the center of the semiellipse is $(0, 0)$ with vertices $(-48, 0)$ and $(48, 0)$ and with the top of the ceiling at $(0, 23)$.



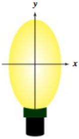
Approximately how far from the center along the major axis should each person be standing to hear the "whispering" effect? Round to 1 decimal place.

59. A homeowner wants to make an elliptical rug from a 12-ft by 10-ft rectangular piece of carpeting.



- **a.** What lengths of the major and minor axes would maximize the area of the new rug?
- **b.** Write an equation of the ellipse with maximum area. Use a coordinate system with the origin at the center of the rug and horizontal major axis.
- **c.** To cut the rectangular piece of carpeting, the homeowner needs to know the location of the foci. Tacks will then be attached at the foci, and a piece of string the length of the major axis will be fastened to the tacks. Drawing the string tight, the homeowner will use a piece of chalk to trace the ellipse. At what coordinates should the tacks be located? Describe the location.

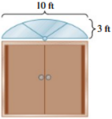
60. Coordinate axes are superimposed on a photo of an elliptical lightbulb.



- **a.** If the vertices are 1.5 in. from the center and the width of the bulb is 1 in. at its widest point, write an equation of the elliptical curve along the outside edge of the bulb (do not consider the metal fixture at the bottom as part of the elliptical shape).
- **b.** If the company logo is to be placed at one of the foci, at what coordinates could the logo be placed? Describe the location.

61. Charles and Bernice ("Ray") Eames were American designers who made major contributions to modern architecture and furniture design. Suppose that a manufacturer wants to make an Eames elliptical coffee table 90 in. long and 30 in. wide out of an 8 ft by 4 ft piece of birch plywood. If the center of a piece of plywood is positioned at (0, 0), determine the distance from the center at which the foci should be located to draw the ellipse.

62. A window above a door is to be made in the shape of a semiellipse. If the window is 10 ft at the base and 3 ft high at the center, determine the distance from the center at which the foci are located.



Learn: Determine and Apply the Eccentricity of an Ellipse

63. Choose one: The eccentricity of an ellipse is a number

- **a.** less than 0.
- **b.** between 0 and 1.
- **c.** greater than 1.

64. Choose one: An ellipse with eccentricity close to 0 will appear

- **a.** more elongated.
- **b.** more circular.

65. Choose one: An ellipse with eccentricity close to 1 will appear

- **a.** more elongated.
- **b.** more circular.

66. Choose one: If the foci of an ellipse are close to the center of the ellipse, then the graph will appear

- **a.** more elongated.
- **b.** more circular.

For Exercises 67–72, determine the eccentricity of the ellipse.

67. $\frac{x^2}{169} + \frac{y^2}{25} = 1$

68. $\frac{x^2}{100} + \frac{y^2}{64} = 1$

69. $\frac{\left(x + \frac{4}{5}\right)^2}{144} + \frac{\left(y - \frac{5}{3}\right)^2}{225} = 1$

70. $\frac{\left(x - \frac{1}{3}\right)^2}{9} + \frac{\left(y + \frac{7}{9}\right)^2}{25} = 1$

71. $\frac{x^2}{12} + \frac{(y-3)^2}{6} = 1$

72. $\frac{(x+7)^2}{18} + \frac{y^2}{12} = 1$

73. Halley's Comet and the Earth both orbit the Sun in elliptical paths with the Sun at one focus. The eccentricity of the comet's orbit is 0.967 and the eccentricity of the Earth's orbit is 0.0167. The eccentricity for the Earth is close to zero, whereas the eccentricity for Halley's Comet is close to 1. Based on this information, how do the orbits compare?

74. Halley's Comet and the comet Hale-Bopp both orbit the Sun in elliptical paths with the Sun at one focus. The eccentricity of Halley's Comet's orbit is 0.967 and the eccentricity of comet Hale-Bopp's orbit is 0.995. Which comet has a more elongated orbit?

75. The Moon's orbit around the Earth is elliptical with the Earth at one focus and with eccentricity 0.0549. If the distance between the Moon and Earth at the closest point is 363,300 km, determine the distance at the farthest point. Round to the nearest 100 km. (**See Example 7, p. 1051**)

76. The planet Saturn orbits the Sun in an elliptical path with the Sun at one focus. The eccentricity of the orbit is 0.0565, and the distance between the Sun and Saturn at perihelion (the closest point) is 1.353×10^9 km. Determine the distance at aphelion (the farthest point). Round to the nearest million kilometers.

77. The Roman Coliseum is an elliptical stone and concrete amphitheater in the center of Rome, built between 70 A.D. and 80 A.D. The Coliseum seated approximately 50,000 spectators and was used among other things for gladiatorial contests.

a. Using a vertical major axis, write an equation of the ellipse representing the center arena if the maximum length is 287 ft and the maximum width is 180 ft. Place the origin at the center of the arena.

b. Approximate the eccentricity of the center arena. Round to 2 decimal places.

c. Find an equation of the outer ellipse if the maximum length is 615 ft and the maximum width is 510 ft.

d. Approximate the eccentricity of the outer ellipse. Round to 2 decimal places.

e. Explain how you know that the outer ellipse is more circular than the inner ellipse.

78. *The Ellipse*, also called *President's Park South,* is a park in Washington, D.C. The lawn area is elliptical with a major axis of 1058 ft and minor axis of 903 ft.

a. Find an equation of the elliptical boundary. Take the horizontal axes to be the major axis and locate the origin of the coordinate system at the center of the ellipse.

b. Approximate the eccentricity of the ellipse. Round to 2 decimal places.

Micheal Evans/Life File/Photodisc/ Getty Images

For Exercises 79–82, write the standard form of an equation of the ellipse subject to the following conditions.

79. Center: (0, 0); Eccentricity: $\frac{15}{17}$;
Major axis vertical of length 34 units

80. Center: (0, 0); Eccentricity: $\frac{40}{41}$;
Major axis vertical of length 82 units

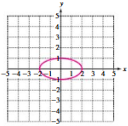
81. Foci: (0, –1) and (8, –1);
Eccentricity: $\frac{4}{5}$

82. Foci: (0, –1) and (–6, –1);
Eccentricity: $\frac{3}{5}$

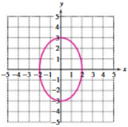
Mixed Exercises

For Exercises 83–86, write the equation of the ellipse in standard form. Determine the eccentricity and coordinates of the foci.

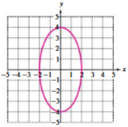
83. a.



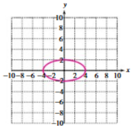
b.



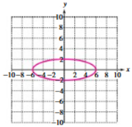
c.



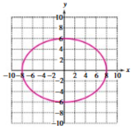
84. a.



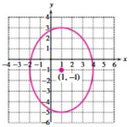
b.



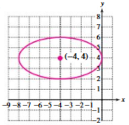
c.



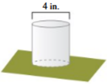
85.

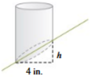


86.



87. A circular vent pipe is placed on a flat roof.



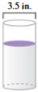


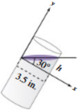
a. Write an equation of the circular cross section that the pipe makes with the roof. Assume that the origin is placed at the center of the circle.

b. Now suppose that the pipe is placed on a roof with a slope of $\frac{3}{5}$. What shape will the cross section of the pipe form with the plane of the roof?

c. Determine the length of the major and minor axes. Find the exact value and approximate to 1 decimal place if necessary.

88. A cylindrical glass of water with diameter 3.5 in. sits on a horizontal countertop.





a. Write an equation of the circular surface of the water. Assume that the origin is placed at the center of the circle.

b. If the glass is tipped 30°, what shape will the surface of the water have?

c. With the glass tipped 30°, the waterline makes a slope of $\frac{1}{2}$ with the coordinate system shown. Determine the length of the major and minor axes. Round to 1 decimal place.

89. The graph of $\frac{x^2}{9} + \frac{y^2}{4} = 1$ represents an ellipse. Determine the part of the ellipse represented by the given equation.

a. $y = 2\sqrt{1 - \frac{x^2}{9}}$ **b.** $y = -2\sqrt{1 - \frac{x^2}{9}}$

c. $x = 3\sqrt{1 - \frac{y^2}{4}}$ **d.** $x = -3\sqrt{1 - \frac{y^2}{4}}$

90. The graph of $\frac{x^2}{16} + \frac{y^2}{81} = 1$ represents an ellipse. Determine the part of the ellipse represented by the given equation.

a. $x = -4\sqrt{1 - \frac{y^2}{81}}$ **b.** $x = 4\sqrt{1 - \frac{y^2}{81}}$

c. $y = 9\sqrt{1 - \frac{x^2}{16}}$ **d.** $y = -9\sqrt{1 - \frac{x^2}{16}}$

For Exercises 91–94, solve the system of equations.

91. $\frac{x^2}{25} + \frac{y^2}{9} = 1$
$3x + 5y = 15$

92. $13y = 12x - 156$
$\frac{x^2}{169} + \frac{y^2}{144} = 1$

93. $\frac{x^2}{4} + \frac{y^2}{16} = 1$
$y = -x^2 + 4$

94. $\frac{x^2}{64} + \frac{y^2}{289} = 1$
$y = \frac{13}{64}x^2 - 13$

For Exercises 95–96, graph the equations on the same coordinate system and determine the points of intersection.

95. $\frac{x^2}{4} + \frac{y^2}{9} = 1$
$\frac{x^2}{9} + \frac{y^2}{4} = 1$

96. $\frac{x^2}{4} + \frac{y^2}{9} = 1$
$\frac{x^2}{4} + \frac{y^2}{4} = 1$

Given an ellipse with major axis of length $2a$ and minor axis of length $2b$, the area is given by $A = \pi ab$. The perimeter is *approximated* by $P \approx \pi\sqrt{2(a^2 + b^2)}$. For Exercises 97–98,

a. Determine the area of the ellipse.

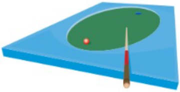
b. Approximate the perimeter.

97. $\frac{x^2}{8} + \frac{(y+3)^2}{4} = 1$

98. $\frac{(x-1)^2}{9} + \frac{y^2}{11} = 1$

Write About It

99. A pool table is in the shape of an ellipse with one pocket located at one focus of the ellipse. If a ball is located at the other focus, explain why a player can strike the ball in any direction to have the ball land in the pocket.



100. Given an equation of an ellipse in standard form, how do you determine whether the major axis is horizontal or vertical?

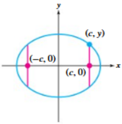
101. Explain the difference between the graphs of the two equations.

$4x^2 + 9y^2 = 36 \qquad 4x + 9y = 36$

102. Discuss the solution set of the equation. $\frac{(x-4)^2}{9} + \frac{(x+2)^2}{16} = -1$

Expanding Your Skills

103. A line segment with endpoints on an ellipse, perpendicular to the major axis, and passing through a focus, is called a *latus rectum* of the ellipse. Show that the length of a latus rectum is $\frac{2b^2}{a}$ for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [*Hint*: Substitute (c, y) into the equation and solve for y. Recall that $c^2 = a^2 - b^2$.]



104. A line segment with endpoints on an ellipse and passing through a focus of the ellipse is called a focal chord. Given the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

- **a.** Show that one focus of the ellipse lies on the line $y = \frac{4}{3}x + 4$.
- **b.** Determine the points of intersection between the ellipse and the line.
- **c.** Approximate the length of the focal chord that lies on the line $y = \frac{4}{3}x + 4$. Round to 2 decimal places.

105. This exercise guides you through the steps to find an equation of the tangent line to a point (x_1, y_1) on an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

- **a.** From calculus, we know that the slope of the tangent line to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is given by $m = -\frac{b^2x_1}{a^2y_1}$. Write the point-slope form of an equation of the tangent line to the ellipse through (x_1, y_1).
- **b.** Simplify the equation from part (a) to eliminate the parentheses and clear the denominator.
- **c.** Divide each term in the equation from part (b) by a^2b^2.
- **d.** The point (x_1, y_1) is on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Therefore, substituting (x_1, y_1) into the equation of the ellipse will yield $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$. Use this equation and the result from part (c) to obtain the equation of the tangent line $\left(\frac{x_1}{a^2}\right)x + \left(\frac{y_1}{b^2}\right)y = 1$.

From Exercise 105, we learned that the tangent line through a point (x_1, y_1) on an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by $\left(\frac{x_1}{a^2}\right)x + \left(\frac{y_1}{b^2}\right)y = 1$. Likewise, we can show that the tangent line through a point (x_1, y_1) on an ellipse of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ is given by $\left(\frac{x_1}{b^2}\right)x + \left(\frac{y_1}{a^2}\right)y = 1$.

For Exercises 106–108, use these results to find an equation of the tangent line at the given point on the curve. Write the answer in slope-intercept form, if possible.

106. $\frac{x^2}{100} + \frac{y^2}{25} = 1$

- **a.** $(8, -3)$ **b.** $(10, 0)$

107. $\frac{x^2}{10} + \frac{y^2}{15} = 1$

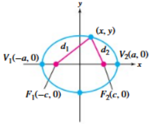
- **a.** $(2, 3)$ **b.** $(0, -\sqrt{15})$

108. $x^2 + y^2 = 10$ (*Hint*: A circle is a special case of an ellipse where $a = b$.)

- **a.** $(2, \sqrt{6})$ **b.** $(3, 1)$

109. This exercise guides you through the steps to find the standard form of an equation of an ellipse centered at the origin with foci on the x-axis.

a. Refer to the figure to verify that the distance from F_1 to V_2 is $(a + c)$ and the distance from F_2 to V_2 is $(a - c)$. Verify that the sum of these distances is $2a$.



b. Write an expression that represents the sum of the distances from F_1 to (x, y) and from F_2 to (x, y). Then set this expression equal to $2a$.

c. Given the equation $\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$, isolate the leftmost radical and square both sides of the equation. Show that the equation can be written as $a\sqrt{(x - c)^2 + y^2} = a^2 - xc$.

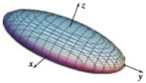
d. Square both sides of the equation $a\sqrt{(x - c)^2 + y^2} = a^2 - xc$ and show that the equation can be written as $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$. (*Hint*: Collect variable terms on the left side of the equation and constant terms on the right side.)

e. Replace $a^2 - c^2$ by b^2. Then divide both sides of the equation by a^2b^2. Verify that the resulting equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

110. Find the points on the ellipse that are twice the distance from one focus to the other.

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

111. An ellipsoid is a three-dimensional surface that resembles the shape of the blimps we see at sporting events. Mathematically, an equation of an ellipsoid centered at the origin of a three-dimensional coordinate system is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.



a. Explain how the formula for an ellipsoid is similar to a two-dimensional formula for an ellipse centered at the origin.

b. The graph of $\frac{x^2}{9} + \frac{y^2}{36} + \frac{z^2}{4} = 1$ can be generated using computer software (see figure).
Write an equation that results if $z = 0$. What does this equation represent?

c. Write the equation that results if $x = 0$. What does this equation represent?

d. Write the equation that results if $y = 0$. What does this equation represent?

Technology Connections

For Exercises 112–115, graph the ellipse from the given exercise on a square viewing window.

112. Exercise 16

113. Exercise 15

114. Exercise 36

115. Exercise 35

Go online for more practice problems.

Lesson 10-2

The Hyperbola

What Will You Learn?

After completing this lesson, you should be able to:

- Graph a Hyperbola Centered at the Origin
- Graph a Hyperbola Centered at (h, k)
- Determine the Eccentricity of a Hyperbola
- Use Hyperbolas in Applications

Learn: Graph a Hyperbola Centered at the Origin

In this lesson, we will study another type of conic section called the hyperbola (Figure 10-11).

Definition of a Hyperbola

A **hyperbola** is the set of all points (x, y) in a plane such that the difference in distances between (x, y) and two fixed points (foci) is a positive constant.

Insights

The definition of a hyperbola differs from the definition of an ellipse in that the curve is the set of points for which the difference (rather than the sum) of d_1 and d_2 is a constant.

In Figure 10-12, we present a hyperbola with foci on the x-axis. By definition, the curve is the set of points such that $d_1 - d_2$ is a positive constant.

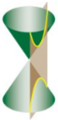


Figure 10-11

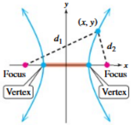


Figure 10-12

Notice that the hyperbola consists of two parts called **branches**. The points where the hyperbola intersects the line through the foci are called the **vertices**. The line segment between the vertices (highlighted in red) is called the **transverse axis**. The midpoint of the transverse axis is the **center** of the hyperbola.

To derive the standard form of an equation of a hyperbola, consider a hyperbola centered at the origin with foci on the x-axis (Figure 10-13). The points $F_1(-c, 0)$ and $F_2(c, 0)$ are the foci and the vertices are $V_1(-a, 0)$ and $V_2(a, 0)$.

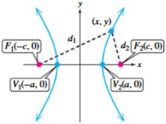


Figure 10-13

The distance between F_1 and V_2 is $c + a$. The distance between F_2 and V_2 is $c - a$. By the definition of a hyperbola, the positive difference between a point on the hyperbola (such as V_2) and the two foci is a positive constant. Subtracting $(c + a)$ and $(c - a)$, we have

$$(c + a) - (c - a) = 2a$$

Therefore, the difference of the distances between (x, y) and the two foci is equal to $2a$.

$$\begin{aligned} d_1 \quad - \quad d_2 &= 2a \\ \sqrt{[(x-(-c)]^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} &= 2a \quad \text{Apply the distance formula.} \\ \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= 2a \quad \text{Simplify within parentheses.} \end{aligned}$$

In Exercise 90, we guide you through a series of algebraic steps to eliminate the radicals. Then, collecting the variable terms on the left side and the constant terms on the right side, the equation becomes

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \qquad (1)$$

From Figure 10-13, we see that $d_1 - d_2 = 2a < 2c$. Therefore, $a < c$ and $c^2 - a^2 > 0$. Substituting $b^2 = c^2 - a^2$ in Equation (1) we have

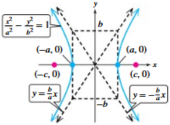
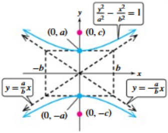
$$\begin{aligned} &b^2x^2 - a^2y^2 = a^2b^2 \\ &\frac{b^2x^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2} \qquad \text{Divide both sides by } a^2b^2. \\ &\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{Standard form} \end{aligned}$$

This equation represents a hyperbola centered at the origin with foci on the x-axis. In a similar manner, we can develop the standard equation of a hyperbola centered at the origin with foci on the y-axis. These two equations are summarized on the following page.

Standard Forms of an Equation of a Hyperbola Centered at the Origin

The standard forms of an equation of a hyperbola centered at the origin are as follows.

Assume that $a > 0$ and $b > 0$.

	Transverse Axis: *x*-axis	**Transverse Axis: *y*-axis**
Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Center	(0, 0)	(0, 0)
Foci ($c^2 = a^2 + b^2$)	$(c, 0)$ and $(-c, 0)$	$(0, c)$ and $(0, -c)$
Vertices	$(a, 0)$ and $(-a, 0)$	$(0, a)$ and $(0, -a)$
Asymptotes	$y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$	$y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$
Graph	 y x b −b $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (−a, 0) (a, 0) (−c, 0) (c, 0) $y = \frac{b}{a}x$ $y = -\frac{b}{a}x$ **Figure 10-14**	 y x $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ (0, a) (0, c) −b b $y = \frac{a}{b}x$ $y = -\frac{a}{b}x$ (0, −a) (0, −c) **Figure 10-15**

Insights

- Notice that we do not have the restriction that $a > b$ as with the ellipse.
- Notice that an equation of a hyperbola shows a *difference* between the square terms, whereas an equation of an ellipse shows a sum of the square terms.

From the standard forms of an equation of a hyperbola, notice that the variable term with the positive coefficient always contains a^2 in the denominator. This term also indicates the orientation of the hyperbola.

- If the variable term with the positive coefficient is the x^2 term, then the transverse axis is horizontal and the branches of the hyperbola open horizontally.
- If the variable term with the positive coefficient is the y^2 term, then the transverse axis is vertical and the branches of the hyperbola open vertically.

As x or y increase, the branches of a hyperbola approach a pair of asymptotes. For example, given the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (horizontal transverse axis),

$$\begin{aligned} y &= \pm b\sqrt{\frac{x^2}{a^2} - 1} \\ &= \pm b\sqrt{\frac{x^2 - a^2}{a^2}} \\ &= \pm \frac{b}{a}\sqrt{x^2 - a^2} \end{aligned}$$

As $x \to \infty$, the term a^2 is insignificant (very small relative to x^2), so the radical approaches $\sqrt{x^2} = x$. So, for large values of x, the graph of the hyperbola approaches $y = \pm\frac{b}{a}x$. Likewise, as $x \to -\infty$, the graph of the hyperbola approaches $y = \pm\frac{b}{a}x$.

As with the graph of a rational function, the asymptotes are lines of reference that help graph the curve. To identify the asymptotes, first sketch a **reference rectangle** with dimensions $2a$ by $2b$, centered at the center of the hyperbola (Figures 10-14 and 10-15). The asymptotes are the lines through the opposite corners of the rectangle.

The length of the transverse axis is $2a$. The line segment passing through the center of the hyperbola, perpendicular to the transverse axis, and with endpoints on the reference rectangle is called the **conjugate axis**. The length of the conjugate axis is $2b$.

To graph a hyperbola, we suggest the following guidelines.

Graphing a Hyperbola

Step 1 Identify the center and vertices.

Step 2 Draw the reference rectangle centered at the center of the hyperbola, with dimensions $2a$ and $2b$ as shown in Figures 10-14 and 10-15.

Step 3 Draw the asymptotes through the opposite corners of the rectangle.

Step 4 Sketch each branch of the hyperbola starting at the vertices and approaching the asymptotes.

Example 1: Graphing a Hyperbola Centered at the Origin

Given $\frac{x^2}{4} - \frac{y^2}{9} = 1$,

a. Graph the hyperbola.

b. Identify the center, vertices, foci, and equations of the asymptotes.

Solution:

a. Graph the hyperbola.

Step 1: Identify the center and vertices: $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

The equation is in the standard form of a hyperbola centered at (0, 0).

The x^2 term has the positive coefficient, indicating that the transverse axis is horizontal. The denominator of the x^2 term is a^2.

$$\frac{x^2}{\underbrace{(2)^2}_{a=2}} - \frac{y^2}{\underbrace{(3)^2}_{b=3}} = 1$$

$a^2 = 4$, and because $a > 0$, we have that $a = 2$.

$b^2 = 9$, and because $b > 0$, we have that $b = 3$.

Vertices: $(a, 0)$ and $(-a, 0)$
(2, 0) and (−2, 0)

Steps 2–3: Draw the reference rectangle and asymptotes.

The reference rectangle is centered at the center of the hyperbola, (0, 0).

Draw the rectangle with width $2a = 2(2) = 4$ in the direction of the transverse axis.

The height of the rectangle is the length of the conjugate axis $2b = 2(3) = 6$.

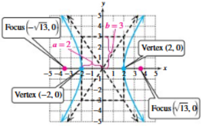
Draw the asymptotes through the opposite corners of the reference rectangle.

Step 4: Sketch each branch of the hyperbola starting at the vertices and approaching the asymptotes.

Note that $c^2 = a^2 + b^2 = 4 + 9 = 13$.

Since $c > 0$, $c = \sqrt{13}$.

Foci: $(c, 0)$ and $(-c, 0)$
$(\sqrt{13}, 0)$ and
$(-\sqrt{13}, 0)$



b. The center is (0, 0).

The vertices are (−2, 0) and (2, 0).

The foci are $(-\sqrt{13}, 0)$ and $(\sqrt{13}, 0)$.

The equations of the asymptotes are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

Therefore, the asymptotes are $y = \frac{3}{2}x$ and $y = -\frac{3}{2}x$.

Insights

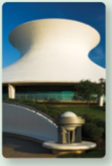
If you find it difficult to memorize the equations of the asymptotes, the equations can be found by using the point-slope formula and the points defining the opposite corners of the reference rectangle.

Apply the Skills

1. Repeat Example 1 with the equation: $\frac{x^2}{9} - \frac{y^2}{4} = 1$

Point of Interest

The James S. McDonnell Planetarium in St. Louis, Missouri, has vertical cross sections in the shape of a hyperbola.



In Example 2, we graph a hyperbola in which the transverse axis is vertical and the conjugate axis is horizontal.

Example 2: Graphing a Hyperbola Centered at the Origin

Given $16y^2 - 9x^2 = 144$,

a. Graph the hyperbola.

b. Identify the center, vertices, foci, and equations of the asymptotes.

Solution:

a. Graph the hyperbola.

Step 1: Identify the center and vertices: $16y^2 - 9x^2 = 144$.

For the equation to be in the standard form for a hyperbola, we require the constant term to be 1.

$$\frac{16y^2}{144} - \frac{9x^2}{144} = \frac{144}{144}$$ Divide both sides by 144.

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

The hyperbola is centered at (0, 0).

The y^2 term has the positive coefficient, indicating that the transverse axis is vertical. The denominator of the y^2 term is a^2.

$$\frac{y^2}{\underbrace{(3)^2}_{a=3}} - \frac{x^2}{\underbrace{(4)^2}_{b=4}} = 1$$

$a^2 = 9$, and because $a > 0$, we have that $a = 3$.

$b^2 = 16$, and because $b > 0$, we have that $b = 4$.

Vertices: $(0, a)$ and $(0, -a)$
$(0, 3)$ and $(0, -3)$

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Steps 2–3: Draw the reference rectangle and asymptotes.

The reference rectangle is centered at the center of the hyperbola, (0, 0).

Draw the rectangle with width $2a = 2(3) = 6$ in the direction of the transverse axis.

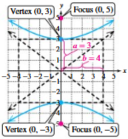
The height of the rectangle is the length of the conjugate axis $2b = 2(4) = 8$.

Draw the asymptotes through the opposite corners of the reference rectangle.

Step 4: Sketch each branch of the hyperbola starting at the vertices and approaching the asymptotes.

Note that $c^2 = a^2 + b^2 = 9 + 16 = 25$ and $c = 5$.

Foci: $(0, c)$ and $(0, -c)$
$(0, 5)$ and $(0, -5)$



b. The center is (0, 0).

The vertices are (0, 3) and (0, −3).

The foci are (0, −5) and (0, 5).

The equations of the asymptotes are $y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$.

Therefore, the asymptotes are $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$.

Apply the Skills

2. Repeat Example 2 with the equation: $4y^2 - x^2 = 16$

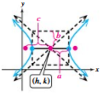
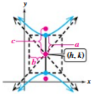
Learn: Graph a Hyperbola Centered at (h, k)

In the standard form of a hyperbola, if x and y are replaced by $x - h$ and $y - k$, then the graph of the hyperbola is shifted h units horizontally and k units vertically. The hyperbola is then centered at (h, k).

Standard Forms of an Equation of a Hyperbola Centered at (h, k)

The standard forms of an equation of a hyperbola centered at (h, k) are as follows.

Assume that $a > 0$ and $b > 0$.

	Transverse Axis Horizontal	Transverse Axis Vertical
Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Center	(h, k)	(h, k)
Foci (*Note:* $c^2 = a^2 + b^2$)	$(h + c, k)$ and $(h - c, k)$	$(h, k + c)$ and $(h, k - c)$
Vertices	$(h + a, k)$ and $(h - a, k)$	$(h, k + a)$ and $(h, k - a)$
Asymptotes	$y - k = \pm\frac{b}{a}(x - h)$	$y - k = \pm\frac{a}{b}(x - h)$
Graph		

To graph a hyperbola centered at (h, k), first locate the center and identify the orientation of the hyperbola (transverse axis horizontal or vertical). Then use the values of a and b to draw the reference rectangle. Draw the asymptotes through the corners of the reference rectangle. Finally, sketch the hyperbola starting at the vertices and approaching the asymptotes.

Example 3: Graphing a Hyperbola Centered at (h, k)

Given $-\frac{(x-2)^2}{9} + (y+3)^2 = 1$,

a. Graph the hyperbola.

b. Identify the center, vertices, foci, and equations of the asymptotes.

Solution:

a. Graph the hyperbola.

Step 1: Identify the center and vertices: $-\frac{(x-2)^2}{9} + \frac{(y+3)^2}{1} = 1$.

$$\frac{(y-[-3])^2}{1} - \frac{(x-2)^2}{9} = 1$$

The equation is in the standard form of a hyperbola centered at $(h, k) = (2, -3)$.

The y^2 term has a positive coefficient. Therefore, the transverse axis is vertical. The denominator of the y^2 term is a^2.

$$\frac{[y-(-3)]^2}{\underbrace{(1)^2}_{a=1}} - \frac{(x-2)^2}{\underbrace{(3)^2}_{b=3}} = 1$$

We have $a^2 = 1$ and $b^2 = 9$.

Therefore, $a = 1$ and $b = 3$.

The vertices are a units above and below the center.

Vertices: $(h, k + a)$ and $(h, k - a)$

$(2, -3 + 1)$ and $(2, -3 - 1)$

$(2, -2)$ and $(2, -4)$

Steps 2–3: Draw the reference rectangle and asymptotes.

Draw the rectangle centered at $(2, -3)$ with vertical sides of length $2a = 2(1) = 2$ and horizontal sides of length $2b = 2(3) = 6$.

Draw the asymptotes through the opposite corners of the reference rectangle.

Step 4: Sketch the hyperbola.

To find the foci, use the relationship:

$c^2 = a^2 + b^2$

$c^2 = 1 + 9 = 10$

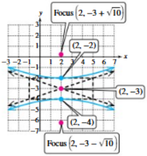
$c = \sqrt{10}$ (Recall that $c > 0$.)

The foci are c units above and below the center.

Foci: $(h, k + c)$ and $(h, k - c)$

$\left(2, -3 + \sqrt{10}\right)$ and

$\left(2, -3 - \sqrt{10}\right)$.



b. The center is $(2, -3)$.

The vertices are $(2, -2)$ and $(2, -4)$.

The foci are $\left(2, -3 + \sqrt{10}\right)$ and $\left(2, -3 - \sqrt{10}\right)$.

The equations of the asymptotes can be found by using the point-slope formula and the points through the opposite vertices of the reference rectangle.

Alternatively, the asymptotes can be found by using $y - k = \pm\frac{a}{b}(x - h)$, where h and k are the coordinates of the center.

$$y - k = \frac{a}{b}(x - h) \qquad y - k = -\frac{a}{b}(x - h)$$

$$y - (-3) = \frac{1}{3}(x - 2) \qquad y - (-3) = -\frac{1}{3}(x - 2)$$

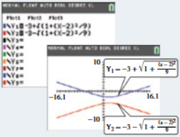
$$y = \frac{1}{3}x - \frac{11}{3} \qquad y = -\frac{1}{3}x - \frac{7}{3}$$

Apply the Skills

3. Repeat Example 3 with the equation: $-\frac{(x+1)^2}{4} + \frac{(y-4)^2}{25} = 1$

TECHNOLOGY CONNECTIONS

Graphing a Hyperbola


NORMAL FLOAT AUTO REAL DEGREE CL
Plot1 Plot2 Plot3
\Y1 = -3+√(1+(X-2)²/9)
\Y2 = -3-√(1+(X-2)²/9)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
NORMAL FLOAT AUTO REAL DEGREE CL
10
-16.1
16.1
-10
$Y_1 = -3 + \sqrt{1 + \frac{(x-2)^2}{9}}$
$Y_2 = -3 - \sqrt{1 + \frac{(x-2)^2}{9}}$

To graph the hyperbola from Example 3 on a graphing utility, first solve the equation for *y*. We have

$$-\frac{(x-2)^2}{9} + (y+3)^2 = 1$$

$$y = -3 \pm \sqrt{1 + \frac{(x-2)^2}{9}}$$

The functions represent the top and bottom branches of the hyperbola, respectively.

The equation of the hyperbola from Example 3 can be expanded by expanding the binomials and combining like terms.

$$-\frac{(x-2)^2}{9} + (y+3)^2 = 1 \qquad \text{Standard form}$$

$$9 \cdot \left[-\frac{(x-2)^2}{9} + (y+3)^2\right] = 9 \cdot (1) \qquad \text{Clear fractions.}$$

$$-(x-2)^2 + 9(y+3)^2 = 9$$

$$-x^2 + 4x - 4 + 9y^2 + 54y + 81 = 9 \qquad \text{Expand the binomials.}$$

$$-x^2 + 9y^2 + 4x + 54y + 68 = 0 \qquad \text{Expanded form}$$

Given an equation of a hyperbola in expanded form, we can reverse this process by completing the square to write the equation in standard form. This is shown in Example 4.

Example 4: Writing an Equation of a Hyperbola in Standard Form

Write the equation of the hyperbola in standard form.

$$11x^2 - 2y^2 + 66x - 4y + 75 = 0$$

Solution:

$$11x^2 - 2y^2 + 66x - 4y + 75 = 0 \qquad \text{Group the } x \text{ variable terms and group the } y \text{ variable terms.}$$

$$11x^2 + 66x - 2y^2 - 4y = -75 \qquad \text{Move the constant term to the right side of the equation.}$$

$$11(x^2 + 6x \quad) - 2(y^2 + 2y \quad) = -75 \qquad \text{Factor out the leading coefficient from the } x \text{ and } y \text{ terms. Leave room to complete the square within parentheses.}$$

Complete the square.

Note that $\left[\frac{1}{2}(6)\right]^2 = 9$ and $\left[\frac{1}{2}(2)\right]^2 = 1$.

$$\overbrace{11(x^2 + 6x + 9)}^{11 \cdot 9 \text{ is added to the left side.}} - \overbrace{2(y^2 + 2y + 1)}^{-2 \cdot 1 \text{ is added to the left side.}} = -75 + \overset{11 \cdot 9}{99} - \overset{-2 \cdot 1}{2}$$

The values 9 and 1 were added inside parentheses. However, 11 · 9 and −2 · 1 were actually added to the left side as a result of the factors in front of the parentheses.

$$11(x + 3)^2 - 2(y + 1)^2 = 22$$

To write the equation in standard form, we require a 1 on the right side of the equation.

$$\frac{11(x + 3)^2}{22} - \frac{2(y + 1)^2}{22} = \frac{22}{22}$$

Divide both sides by 22.

$$\frac{(x + 3)^2}{2} - \frac{(y + 1)^2}{11} = 1$$

Standard form

Apply the Skills

4. Write the equation of the hyperbola in standard form.

$$3x^2 - 7y^2 + 30x + 56y - 58 = 0$$

In Example 5, we determine the standard form of an equation of a hyperbola based on known information about the hyperbola.

Example 5: Finding an Equation of a Hyperbola

Determine the standard form of an equation of a hyperbola with vertices (−4, 0) and (8, 0) and foci (−8, 0) and (12, 0).

Solution:

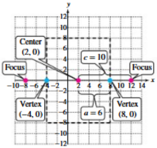
Plotting the given points tells us that the foci are aligned horizontally. Therefore, the transverse axis of the hyperbola is horizontal. The standard form is

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

The center is midway between the vertices. Therefore, the center is (2, 0), indicating that $h = 2$ and $k = 0$.

The vertices are located 6 units away from the center. This means that $a = 6$.

The foci are located 10 units from the center, indicating that $c = 10$.



The only missing piece of information is the value of b. We can find b from the relationship $c^2 = a^2 + b^2$ or equivalently $b^2 = c^2 - a^2$.

$$b^2 = (10)^2 - (6)^2$$

$$b^2 = 64$$

$$b = 8 \qquad \text{(Recall that } b > 0.\text{)}$$

$$\frac{(x-2)^2}{(6)^2} - \frac{(y-0)^2}{(8)^2} = 1 \qquad \text{Substitute } h = 2, k = 0, a = 6, \text{ and } b = 8.$$

$$\frac{(x-2)^2}{36} - \frac{y^2}{64} = 1 \qquad \text{Standard form}$$

Apply the Skills

5. Determine the standard form of an equation of a hyperbola with vertices $(-9, 0)$ and $(1, 0)$ and foci $(-17, 0)$ and $(9, 0)$.

Learn: Determine the Eccentricity of a Hyperbola

The branches of a hyperbola may be wider or narrower depending on the eccentricity of the hyperbola. Like the ellipse, the eccentricity of a hyperbola is given by $e = \frac{c}{a}$.

Eccentricity of a Hyperbola

For a hyperbola defined by $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$,

the **eccentricity *e*** is given by

$$e = \frac{c}{a},$$

where $a > 0$, $c > 0$, and $c^2 = a^2 + b^2$.

Note: The eccentricity of a hyperbola is a number greater than 1; that is, $e > 1$.

In Example 6, we compare the graphs of two hyperbolas with different eccentricities.

Example 6: Finding the Eccentricity of a Hyperbola

Determine the eccentricity of the hyperbola.

a. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ **b.** $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Solution:

a. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ The x^2 term has a positive coefficient. Therefore, $a^2 = 16$.

$$\frac{x^2}{\underset{\boxed{a = 4}}{(4)^2}} - \frac{y^2}{\underset{\boxed{b = 3}}{(3)^2}} = 1$$

$a^2 = 16$, and because $a > 0$, we have that $a = 4$.

$b^2 = 9$, and because $b > 0$, we have that $b = 3$.

$c^2 = a^2 + b^2 = (4)^2 + (3)^2 = 25$

$c = 5$ (Recall that $c > 0$.)

Therefore, $e = \frac{c}{a} = \frac{5}{4}$.

b. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

$\frac{x^2}{(3)^2} - \frac{y^2}{(4)^2} = 1$

$\boxed{a = 3}$ $\boxed{b = 4}$

The x^2 term has a positive coefficient. Therefore, $a^2 = 9$.

$a^2 = 9$, and because $a > 0$, we have that $a = 3$.

$b^2 = 16$, and because $b > 0$, we have that $b = 4$.

$c^2 = a^2 + b^2 = (3)^2 + (4)^2 = 25$

$c = 5$ (Recall that $c > 0$.)

Therefore, $e = \frac{c}{a} = \frac{5}{3}$.

The graphs of the hyperbolas are shown in Figures 10-16 and 10-17. Compare the graphs, and notice that the hyperbola in Figure 10-17 has a greater eccentricity and that the branches open wider.

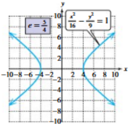


Figure 10-16

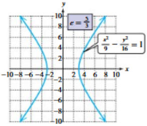


Figure 10-17

Apply the Skills

6. Determine the eccentricity of the hyperbola.

a. $\frac{y^2}{25} - \frac{x^2}{144} = 1$ **b.** $\frac{y^2}{144} - \frac{x^2}{25} = 1$

Learn: Use Hyperbolas in Applications

Applications of hyperbolas arise in many fields. For example, astronomers know that some comets, such as Halley's comet, follow an elliptical path around the Sun. Other comets have a hyperbolic orbit because they are not captured by the Sun's gravity (Figure 10-18).

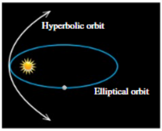


Figure 10-18

Hyperbolas also provide the mathematical basis for LORAN (LOng RAnge Navigation), a technique used for navigation before global positioning systems were introduced. A ship using LORAN measures the difference in time between synchronized radio signals from two different transmitters.

Using the relationship $d = rt$, the difference in distances between the ship and the two transmitters ($d_1 - d_2$) can be computed. Therefore, the position of the ship is located on the hyperbolic path with the two transmitters at the foci (Figure 10-19).

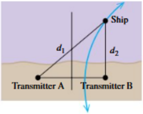


Figure 10-19

The same principle used in LORAN can also be used with sound waves to locate the source of sound heard from two different receivers. This is demonstrated in Example 7.

Example 7: Using a Hyperbola in an Application

Suppose that two people located 3 mi (15,840 ft) apart at points A and B hear a clap of thunder (Figure 10-20). The time difference between the sound heard at point A and the sound heard at point B is 10 sec. If sound travels 1100 ft/sec, find an equation of the hyperbola (with foci at A and B) on which the point of origination of the sound must lie.

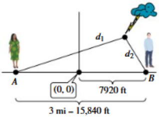


Figure 10-20

Solution:

First, we choose a coordinate system with the origin midway between the two observers. The two observers are at the points (7920, 0) and (–7920, 0). Let d_1 and d_2 represent the distances between each observer and the source of the sound (Figure 10-20).

The difference in distance $d_1 - d_2$ can be computed from the relationship $d = rt$.

$$d_1 - d_2 = (1100 \text{ ft/s})(10 \text{ s}) = 11{,}000 \text{ ft}$$

By the definition of a hyperbola, the source of the sound must be at a point on the hyperbola for which $d_1 - d_2 = 11{,}000$. The center of the hyperbola is (0, 0), and the transverse axis is horizontal with foci (7920, 0) and (–7920, 0).

The standard form of the equation of the hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ The value $d_1 - d_2 = 2a$

$11{,}000 = 2a$

$a = 5500$ ft

From the location of the foci, $c = 7920$ ft.

$c^2 = a^2 + b^2$. Taking $b > 0$, we have $b = \sqrt{c^2 - a^2}$.

$b = \sqrt{(7920)^2 - (5500)^2} \approx 5699$ ft

The source of the sound must be at a point on the curve $\frac{x^2}{(5500)^2} - \frac{y^2}{(5699)^2} = 1$.

Apply the Skills

7. Repeat Example 7 with two observers 4 mi apart, who hear a clap of thunder 16 sec apart. (*Hint*: 5280 ft = 1 mi.)

The hyperbola, like the ellipse, has an interesting property of reflection. Light directed toward one focus of a hyperbolic mirror is reflected toward the other focus of the hyperbola (Figure 10-21). This property is used in the design of Cassegrain reflecting telescopes. The telescope is a combination of a primary parabolic mirror and a secondary hyperbolic mirror (Figure 10-22). Light from distant stars is reflected off the parabolic mirror toward the focus of the hyperbolic mirror. The light is then reflected again toward an eyepiece located at the other focus of the hyperbola.

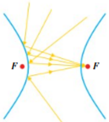


Figure 10-21

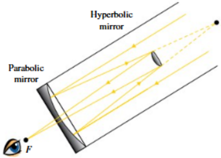


Figure 10-22

Practice 10-2

Practice Exercises

Prerequisite Review

R.1. Use the distance formula to find the distance between the points (6, 12) and (–1, 12).

For Exercises R.2–R.3, write an equation of the line satisfying the given conditions. Write the answer in slope-intercept form.

R.2. The line passes through the point (–8, 8) and has a slope of –8.

R.3. The line passes through (–8, 8) and (1, –2).

Concept Connections

1. A ____________ is the set of points (x, y) in a plane such that the difference in distances between (x, y) and two fixed points (called ____________) is a positive constant.

2. The points where a hyperbola intersects the line through the foci are called the ____________.

3. The line segment between the vertices of a hyperbola is called the ____________ axis.

4. The line segment perpendicular to the transverse axis passing through the center of a hyperbola, and with endpoints on the reference rectangle, is called the ____________ axis.

5. The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represents a ____________ hyperbola with a (horizontal/vertical) transverse axis. The vertices are given by the ordered pairs ____________ and ____________. The asymptotes are given by the equations ____________ and ____________.

6. The equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ represents a ____________ hyperbola with a (horizontal/vertical) transverse axis. The vertices are given by the ordered pairs ____________ and ____________. The asymptotes are given by the equations ____________ and ____________.

Learn: Graph a Hyperbola Centered at the Origin

For Exercises 7–8,

a. Use the distance formula to find the distances d_1, d_2, d_3, and d_4.

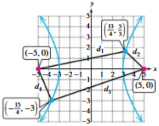
b. Find the difference of the distances: $d_1 - d_2$.

c. Find the difference of the distances: $d_3 - d_4$.

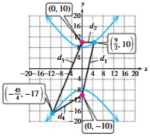
d. How do the results from parts (b) and (c) compare?

e. How do the results of part (b) and part (c) compare to the length of the transverse axis?

7.



8.



For Exercises 9–12, determine whether the transverse axis and foci of the hyperbola are on the *x*-axis or the *y*-axis.

9. $\frac{x^2}{15} - \frac{y^2}{20} = 1$

10. $\frac{y^2}{12} - \frac{x^2}{18} = 1$

11. $-x^2 + \frac{y^2}{3} = 1$

12. $-\frac{y^2}{16} + x^2 = 1$

For Exercises 13–22,

a. Identify the center.

b. Identify the vertices.

c. Identify the foci.

d. Write equations for the asymptotes.

e. Graph the hyperbola. (**See Examples 1–2, pp. 1065–1067**)

13. $\frac{x^2}{16} - \frac{y^2}{25} = 1$

14. $\frac{x^2}{25} - \frac{y^2}{36} = 1$

15. $\frac{y^2}{4} - \frac{x^2}{36} = 1$

16. $\frac{y^2}{9} - \frac{x^2}{49} = 1$

17. $25y^2 - 81x^2 = 2025$

18. $49y^2 - 16x^2 = 784$

19. $-5x^2 + 7y^2 = -35$

20. $-7x^2 + 11y^2 = -77$

21. $\frac{4x^2}{25} - \frac{16y^2}{49} = 1$

22. $\frac{4x^2}{81} - \frac{16y^2}{225} = 1$

Learn: Graph a Hyperbola Centered at (h, k)

For Exercises 23–32,

a. Identify the center.

b. Identify the vertices.

c. Identify the foci.

d. Write equations for the asymptotes.

e. Graph the hyperbola. (**See Example 3, p. 1069**)

23. $\frac{(x-4)^2}{9} - \frac{(y+2)^2}{16} = 1$

24. $\frac{(x-3)^2}{36} - \frac{(y+1)^2}{64} = 1$

25. $\frac{(y-5)^2}{49} - \frac{(x+3)^2}{25} = 1$

26. $\frac{(y-4)^2}{36} - \frac{(x+5)^2}{16} = 1$

27. $100(y-7)^2 - 81(x+4)^2 = -8100$

28. $49(y-3)^2 - 100(x+6)^2 = -4900$

29. $y^2 - \frac{(x-3)^2}{12} = 1$

30. $y^2 - \frac{(x-2)^2}{18} = 1$

31. $x^2 - \frac{(y-4)^2}{8} = 1$

32. $x^2 - \frac{(y-6)^2}{24} = 1$

For Exercises 33–40,

a. Write the equation of the hyperbola in standard form. (**See Example 4, p. 1071**)

b. Identify the center, vertices, and foci.

33. $7x^2 - 5y^2 + 42x + 10y + 23 = 0$

34. $5x^2 - 3y^2 + 10x + 24y - 73 = 0$

35. $-5x^2 + 9y^2 + 20x - 72y + 79 = 0$

36. $-7x^2 + 16y^2 - 70x + 96y - 143 = 0$

37. $4x^2 - y^2 - 10y - 29 = 0$

38. $9x^2 - y^2 - 14y - 58 = 0$

39. $-36x^2 + 64y^2 + 108x + 256y - 401 = 0$

40. $-144x^2 + 25y^2 + 720x - 50y - 4475 = 0$

For Exercises 41–50, write the standard form of the equation of the hyperbola subject to the given conditions. (See Example 5, p. 1072)

41. Vertices: $(12, 0)$, $(-12, 0)$; Foci: $(13, 0)$, $(-13, 0)$

42. Vertices: $(40, 0)$, $(-40, 0)$; Foci: $(41, 0)$, $(-41, 0)$

43. Vertices: $(0, 12)$, $(0, -12)$; Asymptotes: $y = \pm\frac{4}{3}x$

44. Vertices: $(0, 15)$, $(0, -15)$; Asymptotes: $y = \pm\frac{15}{8}x$

45. Vertices: $(-3, -3)$, $(7, -3)$; Slope of the asymptotes: $\pm\frac{7}{5}$

46. Vertices: $(2, 1)$, $(-10, 1)$; Slope of the asymptotes: $\pm\frac{5}{6}$

47. Vertices: $(-3, 0)$, $(-3, 8)$; Foci: $(-3, 4 + \sqrt{21})$, $(-3, 4 - \sqrt{21})$

48. Vertices: $(2, -1)$, $(2, -11)$; Foci: $(2, -6 + \sqrt{31})$, $(2, -6 - \sqrt{31})$

49. Corners of the reference rectangle: $(8, 7)$, $(-6, 7)$, $(8, -3)$, $(-6, -3)$; Horizontal transverse axis

50. Corners of the reference rectangle: $(7, 6)$, $(-1, 6)$, $(7, 0)$, $(-1, 0)$; Horizontal transverse axis

Learn: Determine the Eccentricity of a Hyperbola

For Exercises 51–52,

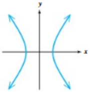
a. Determine the eccentricity of each hyperbola. (**See Example 6, p. 1073**)

b. Based on the eccentricity, match the equation with its graph. The scaling is the same for both graphs

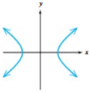
51. Equation 1: $\frac{x^2}{144} - \frac{y^2}{81} = 1$

Equation 2: $\frac{x^2}{81} - \frac{y^2}{144} = 1$

A.



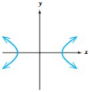
B.



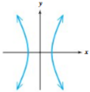
52. Equation 1: $\frac{x^2}{225} - \frac{y^2}{64} = 1$

Equation 2: $\frac{x^2}{64} - \frac{y^2}{225} = 1$

A.



B.



For Exercises 53–54, determine the eccentricity.

53. $\frac{\left(y - \frac{2}{3}\right)^2}{1600} - \frac{\left(x + \frac{7}{4}\right)^2}{81} = 1$

54. $\frac{(y - 3.8)^2}{49} - \frac{(x - 2.7)^2}{576} = 1$

55. Determine the eccentricity of a hyperbola with a horizontal transverse axis of length 24 units and asymptotes $y = \pm\frac{3}{4}x$.

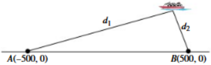
56. Determine the eccentricity of a hyperbola with a vertical transverse axis of length 48 units and asymptotes $y = \pm\frac{12}{5}x$.

57. Determine the standard form of an equation of a hyperbola with eccentricity $\frac{5}{4}$ and vertices $(-1, -1)$ and $(7, -1)$.

58. Determine the standard form of an equation of a hyperbola with eccentricity $\frac{13}{12}$ and vertices $(-2, 8)$ and $(-2, -16)$.

Learn: Use Hyperbolas in Applications

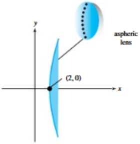
59. Two radio transmitters are 1000 mi apart at points *A* and *B* along a coastline. Using LORAN on the ship, the time difference between the radio signals is 4 milliseconds (0.004 sec). If radio signals travel 186 mi/millisecond, find an equation of the hyperbola with foci at *A* and *B*, on which the ship is located. (**See Example 7, p. 1075**)



60. Suppose that two microphones 1500 m apart at points *A* (–750, 0) and *B* (750, 0) detect the sound of a bird call. The time difference between the sound detected at point *A* and the sound detected at point *B* is 4 sec. If sound travels at approximately 330 m/sec, find an equation of the hyperbola with foci at *A* and *B* defining the points where the bird may be located.

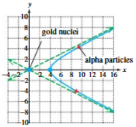
61. In some designs of eyeglasses, the surface is "aspheric," meaning that the contour varies slightly from spherical. An aspheric lens is often used to correct for spherical aberration—a distortion due to increased refraction of light rays when they strike the lens near its edge. Aspheric lenses are often designed with hyperbolic cross sections.

Write an equation of the cross section of the hyperbolic lens shown if the center is (0, 0), one vertex is (2, 0), and the focal length (distance between center and foci) is $\sqrt{85}$. Assume that all units are in millimeters.

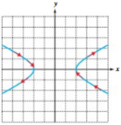


62. In 1911, Ernest Rutherford discovered the nucleus of the atom. Experiments leading to this discovery involved the scattering of alpha particles by the heavy nuclei in gold foil. When alpha particles are thrust toward the gold nuclei, the particles are deflected and follow a hyperbolic path.

Suppose that the minimum distance that the alpha particles get to the gold nuclei is 4 microns (1 micron is one-millionth of a meter) and that the hyperbolic path has asymptotes of $y = \pm\frac{1}{2}x$. Determine an equation of the path of the particles shown.

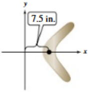


63. Atomic particles with like charges tend to repel one another. Suppose that two beams of like-charged particles are hurled toward each other from two parallel atomic accelerators. The path defined by the particles is $x^2 - 4y^2 = 36$, where x and y are measured in microns. What is the minimum distance between the particles?



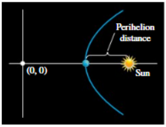
64. A returning boomerang is a V-shaped throwing device made from two wings that are set at a slight tilt and that have an airfoil design. One side is rounded and the other side is flat, similar to an airplane propeller. When thrown properly, a boomerang follows a circular flight path and should theoretically return close to the point of release.

The boomerang pictured is approximately in the shape of one branch of a hyperbola (although the two wings are in slightly different planes). To construct the hyperbola, an engineer needs to know the location of the foci. Determine the location of the focus to the right of the center if the vertex is 7.5 in. from the center and the equations of the asymptotes are $y = \pm\frac{4}{5}x$. Round the coordinates to the nearest tenth of an inch.



65. In September 2009, Australian astronomer Robert H. McNaught discovered comet C/2009 R_1 (McNaught). The orbit of this comet is hyperbolic with the Sun at one focus. Because the orbit is not elliptical, the comet will not be captured by the Sun's gravitational pull and instead will pass by the Sun only once. The comet reached perihelion on July 2, 2010. (*Source*: Minor Planet Center, http://minorplanetcenter.net/)

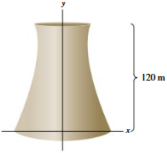
The path of the comet can be modeled by the equation $\frac{x^2}{(1191.2)^2} - \frac{y^2}{(30.9)^2} = 1$ where x and y are measured in AU (astronomical units).



a. Determine the distance (in AU) at perihelion. Round to 1 decimal place.

b. Using the rounded value from part (a), if 1 AU ≈ 93,000,000 mi, find the distance in miles.

66. The cross section of a cooling tower of a nuclear power plant is in the shape of a hyperbola and can be modeled by the equation $\frac{x^2}{625} - \frac{(y-80)^2}{2500} = 1$ where x and y are measured in meters. The top of the tower is 120 m above the base.



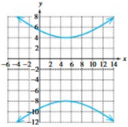
a. Determine the diameter of the tower at the base. Round to the nearest meter.

b. Determine the diameter of the tower at the top. Round to the nearest meter.

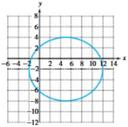
Mixed Exercises

For Exercises 67–70, identify the equation as representing an ellipse or a hyperbola, and match the equation with the graph.

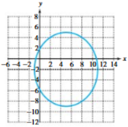
A.



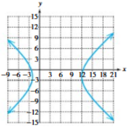
B.



C.



D.



67. $\frac{(x-5)^2}{49} + \frac{(y+2)^2}{36} = 1$

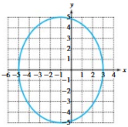
68. $\frac{(x-5)^2}{36} + \frac{(y+2)^2}{49} = 1$

69. $\frac{(x-5)^2}{49} - \frac{(y+2)^2}{36} = 1$

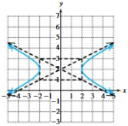
70. $-\frac{(x-5)^2}{49} + \frac{(y+2)^2}{36} = 1$

For Exercises 71–74, find the standard form of the equation of the ellipse or hyperbola shown.

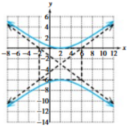
71.



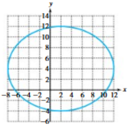
72.



73.



74.



For Exercises 75–76, graph the equations on the same coordinate system and determine the points of intersection.

75. $\frac{x^2}{4} - \frac{y^2}{9} = 1$

$y^2 - \frac{x^2}{4} = 1$

76. $xy = 2$

$\frac{y^2}{2} - x^2 = 1$

For Exercises 77–78, solve the system of equations.

77. $9x^2 - 4y^2 = 36$

$-13x^2 + 8y^2 = 8$

78. $x^2 + 4y^2 = 36$

$4x^2 - y^2 = 8$

Write About It

79. Given an equation of a hyperbola in standard form, how do you determine whether the transverse axis is horizontal or vertical?

80. Discuss the solution set of the equation. $\frac{x^2}{4} - \frac{y^2}{9} = 0$

Expanding Your Skills

81. What is the eccentricity of a hyperbola if the asymptotes are perpendicular?

82. a. Describe the graph of $y = 4\sqrt{1 + \frac{x^2}{9}}$.

b. Does the equation $y = 4\sqrt{1 + \frac{x^2}{9}}$ define y as a function of x?

c. Sketch $y = 4\sqrt{1 + \frac{x^2}{9}}$.

d. Determine the interval(s) over which the function is increasing.

e. Determine the interval(s) over which the function is decreasing.

An equation of the tangent line through a point (x_1, y_1) on a hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by $\left(\frac{x_1}{a^2}\right)x - \left(\frac{y_1}{b^2}\right)y = 1$. The proof is similar to the proof offered in Exercise 105 of Lesson 10-1. For Exercises 83–85, use this result to

a. Find an equation of the tangent line at the given point on the curve. Write the answer in slope-intercept form, if possible.

b. Find an equation of the *normal* line at the given point on the curve. The **normal line** is the line perpendicular to the tangent line at the given point. Write the answer in slope-intercept form, if possible.

83. $\frac{x^2}{9} - \frac{y^2}{16} = 1$; $\left(-\frac{15}{4}, -3\right)$

84. $\frac{x^2}{9} - \frac{y^2}{4} = 1$; $(3\sqrt{5}, 4)$

85. $\frac{x^2}{9} - y^2 = 1$; $\left(5, -\frac{4}{3}\right)$

Two conic sections with the same foci are called *confocal*. For Exercises 86–88, match each conic section with its confocal curve, A, B, or C. Then identify the common foci.

A. $\frac{x^2}{12} + \frac{y^2}{6} = 1$

B. $\frac{x^2}{25} + \frac{y^2}{16} = 1$

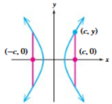
C. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

86. $\frac{x^2}{4} - \frac{y^2}{21} = 1$

87. $\frac{x^2}{5} - y^2 = 1$

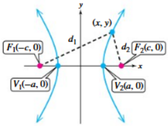
88. $\frac{x^2}{16} + \frac{y^2}{7} = 1$

89. A line segment with endpoints on a hyperbola, perpendicular to the transverse axis, and passing through a focus is called a *latus rectum* of the hyperbola (shown in red). Show that the length of a latus rectum is $\frac{2b^2}{a}$ for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. (*Hint*: Substitute (c, y) into the equation and solve for y. Recall that $c^2 = a^2 + b^2$.)



90. This exercise guides you through the steps to find the standard form of an equation of a hyperbola centered at the origin with foci on the x-axis.

a. Refer to the figure to verify that the distance between F_1 and V_2 is $(c + a)$ and the distance between F_2 and V_2 is $(c - a)$. Verify that the *difference* between these distances is $2a$.



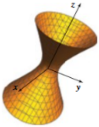
b. Write an expression that represents the difference of the distances from F_1 to (x, y) and from F_2 to (x, y). Then set this expression equal to $2a$.

c. Given the equation $\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = 2a$, isolate the leftmost radical and square both sides of the equation. Show that the equation can be written as $cx - a^2 = a\sqrt{(x - c)^2 + y^2}$.

d. Square both sides of the equation $cx - a^2 = a\sqrt{(x - c)^2 + y^2}$ and show that the equation can be written as $(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$. (*Hint*: Collect variable terms on the left side of the equation and constant terms on the right side.)

e. Replace $c^2 - a^2$ by b^2. Then divide both sides of the equation by a^2b^2. Verify that the resulting equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

91. A hyperboloid of one sheet is a three-dimensional surface generated by an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. The surface has hyperbolic cross sections and either circular cross sections or elliptical cross sections.

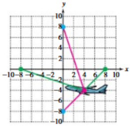


a. Write the equation with $z = 0$. What type of curve is represented by this equation?

b. Write the equation with $x = 0$. What type of curve is represented by this equation?

c. Write the equation with $y = 0$. What type of curve is represented by this equation?

92. a. Radio signals emitted from points (8, 0) and (–8, 0) indicate that a plane is 8 mi closer to (8, 0) than to (–8, 0). Find an equation of the hyperbola that passes through the plane's location and with foci (8, 0) and (–8, 0). All units are in miles.



b. At the same time, radio signals emitted from points (0, 8) and (0, –8) indicate that the plane is 4 mi farther from (0, 8) than from (0, –8). Find an equation of the hyperbola that passes through the plane's location and with foci (0, 8) and (0, –8).

c. From the figure, the plane is located in the fourth quadrant of the coordinate system. Solve the system of equations defining the two hyperbolas for the point of intersection in the fourth quadrant. This is the location of the plane. Then round the coordinates to the nearest tenth of a mile.

Technology Connections

For Exercises 93–96, graph the hyperbola from the given exercise.

93. Exercise 15

94. Exercise 16

95. Exercise 37

96. Exercise 38

GO ONLINE **Go online for more practice problems.**

Lesson 10-3

The Parabola

What Will You Learn?

After completing this lesson, you should be able to:

- Identify the Focus and Directrix of a Parabola
- Graph a Parabola with Vertex at the Origin
- Graph a Parabola with Vertex (h, k)
- Use Parabolas in Applications

Learn: Identify the Focus and Directrix of a Parabola

A parabola is another type of conic section generated by the cross section of a cone intersected by a plane (Figure 10-23). An equation of the form $y = ax^2 + bx + c$ $(a \neq 0)$ is a parabola opening upward if $a > 0$ and opening downward if $a < 0$. We now extend our study of parabolas to include those that open to the left and right. To do so, we will define a parabola geometrically.

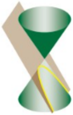


Figure 10-23

Definition of a Parabola

A **parabola** is the set of all points in a plane that are equidistant from a fixed line (called the **directrix**) and a fixed point (called the **focus**).

Figure 10-24 shows a parabola opening upward. The line perpendicular to the directrix and passing through the focus is called the **axis of symmetry**. The **vertex** of the parabola is the point of intersection of the parabola and the axis of symmetry.

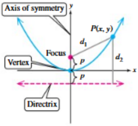


Figure 10-24

A point $P(x, y)$ is on the parabola if $d_1 = d_2$ where d_1 is the distance between the focus and P, and d_2 is the perpendicular distance between P and the directrix.

To develop an algebraic equation that represents a parabola, we will consider a parabola with vertex at the origin and with a vertical line of symmetry (that is, the parabola opens upward or downward). See Figure 10-25. In the discussion that follows, the same logic can be applied to a parabola with a horizontal line of symmetry (the parabola opens to the left or to the right).

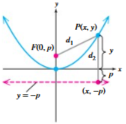


Figure 10-25

The distance between the vertex and the focus of a parabola is called the **focal length** and is often represented by $|p|$. Suppose that we choose a coordinate system with the vertex at the origin and focus $F(0, p)$. Because the distance between any point on the parabola and the focus must equal the distance between the point and the directrix, the directrix is a horizontal line $|p|$ units from the vertex. The equation of the directrix is $y = -p$.

Let $P(x, y)$ represent any point on the parabola. Then the distance between $F(0, p)$ and $P(x, y)$ can be found by using the distance formula.

$$d_1 = \sqrt{(x - 0)^2 + (y - p)^2}$$

The distance from $P(x, y)$ to the directrix is the vertical distance between $P(x, y)$ and the point $(x, -p)$ on the directrix.

$$d_2 = y + p$$

Insights

The shortest distance between P and the directrix is the *perpendicular distance*. This is the length of the line segment perpendicular to the directrix with one endpoint at P and the other endpoint on the directrix.

Equating d_1 and d_2, we have

$$\sqrt{x^2 + (y - p)^2} = y + p$$

$$x^2 + (y - p)^2 = (y + p)^2 \qquad \text{Square both sides.}$$

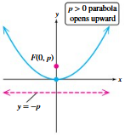
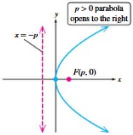
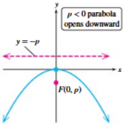
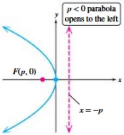
$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 \qquad \text{Expand the binomials.}$$

$$x^2 = 4py$$

Standard form
If $p > 0$, the parabola opens upward.
If $p < 0$, the parabola opens downward.

The equation $x^2 = 4py$ represents a parabola with vertex at the origin and focus on the y-axis. Using similar logic, we can derive the standard form of an equation of a parabola with focus on the x-axis.

Standard Forms of an Equation of a Parabola with Vertex at the Origin

	Axis of Symmetry: *y*-axis	**Axis of Symmetry: *x*-axis**
Equation	$x^2 = 4py$	$y^2 = 4px$
Vertex	(0, 0)	(0, 0)
Focus	$(0, p)$	$(p, 0)$
Directrix	$y = -p$	$x = -p$
Axis of symmetry	$x = 0$	$y = 0$
Graph ($p > 0$)	 $p > 0$ parabola opens upward y x F(0, p) y = −p	 $p > 0$ parabola opens to the right y x x = −p F(p, 0)
Graph ($p < 0$)	 $p < 0$ parabola opens downward y x y = −p F(0, p)	 $p < 0$ parabola opens to the left y x F(p, 0) x = −p

Insights

Notice that an equation of a parabola has only one variable squared, whereas an equation of an ellipse or hyperbola has both x and y appearing to the second power.

Insights

The orientation of a parabola (whether it opens vertically or horizontally) can be determined by noting the linear term in the equation.

For the equation $x^2 = 4py$, the y term is linear (to the first power). Therefore, the parabola will open in the y direction (upward or downward).

For the equation $y^2 = 4px$, the x term is linear, so the parabola will open to the left or right.

From Lesson 2.1, we became accustomed to writing an equation of a quadratic function with vertex at the origin in the form $y = ax^2$. The benefit of writing the equation as $x^2 = 4py$ is that we can identify the value of p, which gives us the distance $|p|$ between the vertex and the focus.

Finding the location of the focus is particularly important in applications. For example, a flashlight has a mirror with cross section in the shape of a parabola (Figure 10-26). The bulb is located at the focus. The light emitted from the bulb will reflect off the mirror in lines parallel to the axis of symmetry to form a beam of light. This is a result of the reflective property of a parabola.

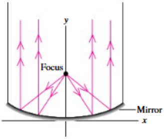


Figure 10-26

Example 1: Determining the Focus and Directrix of a Parabola

An engineer designs a flashlight with a parabolic mirror. A coordinate system is chosen so that the vertex of a cross section through the center of the mirror is located at (0, 0). The equation of the parabola is modeled by $x^2 = 8y$, where x and y are measured in centimeters.

a. Where should the bulb be placed to make use of the reflective property of the parabola? That is, find the location of the focus.

b. Write an equation of the directrix.

Solution:

a. $x^2 = 8y$ is in the form $x^2 = 4py$ with $4p = 8$.

$4p = 8$

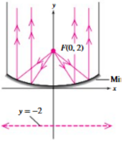
$p = 2$ Since the equation is in the form $x^2 = 4py$, the focus is on the y-axis. Furthermore, since $p > 0$, the parabola opens upward, and the focus is $|p|$ units above the vertex.

The focus is (0, 2).

The bulb should be placed 2 cm above the vertex.

b. The directrix is the horizontal line $|p|$ units below the vertex.

The equation of the directrix is $y = -2$.



Apply the Skills

1. Given a parabola defined by $x^2 = 4y$, find the focus and an equation of the directrix.

Learn: Graph a Parabola with Vertex at the Origin

In Examples 2 and 3, we practice graphing parabolas with vertex at the origin. To sketch the graph, we will identify the location of the vertex and the orientation of the parabola (up or down versus left or right). To determine how "wide" to graph the parabola, we will determine the length of the line segment passing through the focus, perpendicular to the axis of symmetry with endpoints on the parabola. This line segment is called the **latus rectum.**

For the parabola defined by $x^2 = 4py$, the endpoints of the latus rectum will have the same y-coordinate as the focus. Therefore, to find the endpoints of the latus rectum, we can substitute p for y in the equation (Figure 10-27).

$$x^2 = 4py$$

$$x^2 = 4p(p) \qquad \text{Substitute } p \text{ for } y.$$

$$x^2 = 4p^2$$

$$x = \pm 2|p|$$

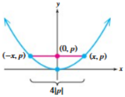


Figure 10-27

Therefore, the endpoints of the latus rectum are located $2|p|$ units to the left and right of the focus at $(-2|p|, p)$ and $(2|p|, p)$. The length of the latus rectum is called the **focal diameter** and is equal to $4|p|$.

In Example 2, we graph a parabola by using the vertex, orientation of the parabola, and focal diameter.

Example 2: Graphing a Parabola with Vertex at the Origin

Given $x^2 = -12y$,

- **a.** Identify the vertex, focus, and focal diameter.
- **b.** Identify the endpoints of the latus rectum and graph the parabola.
- **c.** Write equations for the directrix and axis of symmetry.

Solution:

a. $x^2 = -12y$ The equation is in the standard form $x^2 = 4py$. From the standard form, we see that the focus is on the y-axis and that the parabola will open upward or downward. The vertex is (0, 0), with $4p = -12$.

$4p = -12$

$p = -3$ Because $p < 0$, the parabola will open downward.

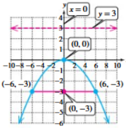
The vertex is (0, 0).

The focus is $(0, p)$, which is $(0, -3)$.

The focal diameter is $4|p|$, which is $4|(-3)| = 12$.

b. Plot the vertex and locate the focus. Note that the focus is shown in red because it is not actually part of the graph. The focal diameter is 12 units. Therefore, plot points 6 units to the left and right of the focus. These are the endpoints of the latus rectum: $(6, -3)$ and $(-6, -3)$.

c. Equation of the directrix: $y = 3$
Equation of the axis of symmetry: $x = 0$



Apply the Skills

2. Repeat Example 2 with the equation $x^2 = -8y$.

Insights

The focal diameter gives the length of the latus rectum. This is helpful for graphing a parabola because it tells us the "width" of the parabola at the focus.

In Example 3, we graph a parabola opening horizontally.

Example 3: Graphing a Parabola with Vertex at the Origin

Given $2y^2 = 32x$,

- **a.** Identify the vertex, focus, and focal diameter.
- **b.** Identify the endpoints of the latus rectum and graph the parabola.
- **c.** Write equations for the directrix and axis of symmetry.

Solution:

a. $2y^2 = 32x$ Divide both sides by 2 to obtain a coefficient of 1 on the square term.

$y^2 = 16x$ The equation is in the standard form $y^2 = 4px$. From the standard form, we see that the focus is on the x-axis, and that the parabola will open to the left or right. The vertex is $(0, 0)$ and $4p = 16$.

$4p = 16$

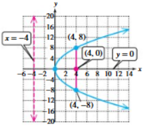
$p = 4$ Because $p > 0$, the parabola will open to the right.

The vertex is $(0, 0)$.

The focus is $(p, 0)$, which is $(4, 0)$.

The focal diameter is $4|p|$, which is $4|(4)| = 16$.

b. Plot the vertex and locate the focus. The focal diameter is 16 units. Therefore, plot points 8 units above and 8 units below the focus. These are the endpoints of the latus rectum, $(4, 8)$ and $(4, -8)$.



c. Equation of the directrix: $x = -4$
Equation of the axis of symmetry: $y = 0$

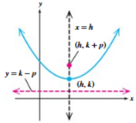
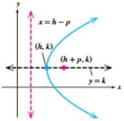
Apply the Skills

3. Repeat Example 3 with the equation: $y^2 = -12x$

Learn: Graph a Parabola with Vertex (h, k)

In the standard form of a parabola, if x and y are replaced by $x - h$ and $y - k$, then the graph of the parabola is shifted h units horizontally and k units vertically. The vertex is (h, k).

Standard Forms of an Equation of a Parabola with Vertex (h, k)

	Vertical Axis of Symmetry	Horizontal Axis of Symmetry
Equation	$(x - h)^2 = 4p(y - k)$	$(y - k)^2 = 4p(x - \text{h})$
Vertex	(h, k)	(h, k)
Focus	$(h, k + p)$	$(h + p, k)$
Directrix	$y = k - p$	$x = h - p$
Axis of symmetry	$x = h$	$y = k$
Graph ($p > 0$)		
Note:	If $p < 0$, the parabola opens downward.	If $p < 0$, the parabola opens to the left.

To graph a parabola with vertex (h, k), first locate the vertex and identify the orientation of the parabola (vertical axis of symmetry or horizontal axis of symmetry). Then identify the focus, directrix, and endpoints of the latus rectum. Finally, sketch the parabola through the vertex and endpoints of the latus rectum.

Example 4: Graphing a Parabola with Vertex (h, k)

Given $(y + 3)^2 = -4(x - 2)$,

a. Identify the vertex, focus, and focal diameter.

b. Identify the endpoints of the latus rectum and graph the parabola.

c. Write equations for the directrix and axis of symmetry.

Solution:

a. $(y + 3)^2 = -4(x - 2)$ The equation is in the form:

$(y - (-3))^2 = -4(x - 2)$ $(y - k)^2 = 4p(x - h)$, where $h = 2$ and $k = -3$.

The vertex is $(2, -3)$, and $4p = -4$.

The x term is linear. Therefore, the parabola will open in the x direction (left or right).

$4p = -4$ This implies that $p = -1$.

$p = -1$ Since $p < 0$, the parabola will open to the left.

Since $p = -1$, the focus is 1 unit to the left of the vertex:

$(h + p, k) = (2 + (-1), -3) = (1, -3)$.

The focal diameter is $4|p| = 4|(-1)| = 4$.

The vertex is $(2, -3)$.

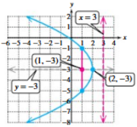
The focus is $(1, -3)$.

The focal diameter is 4.

b. Plot the vertex and locate the focus. The focal diameter is 4 units. Therefore, plot points 2 units above and below the focus. These are the endpoints of the latus rectum: $(1, -1)$ and $(1, -5)$.

Sketch the curve through the vertex and endpoints of the latus rectum.

c. Equation of the directrix: $x = 3$
Equation of the axis of symmetry: $y = -3$



Apply the Skills

4. Repeat Example 4 with the equation: $(y - 2)^2 = -8(x - 4)$

TECHNOLOGY CONNECTIONS

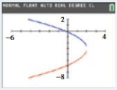
Graphing a Parabola

To graph the parabola from Example 4 on a graphing utility, first solve the equation for y.

$$(y + 3)^2 = -4(x - 2)$$

$$y = -3 \pm \sqrt{-4(x - 2)} \quad \text{Solve for } y.$$

The functions represent the top and bottom branches of the parabola, respectively.



Sometimes an equation of a parabola is given in expanded form. We can complete the square to write the equation in standard form. The process involves completing the square on either the x or y term, depending on which variable is squared in the equation. This is demonstrated in Example 5.

Example 5: Writing an Equation of a Parabola in Standard Form

Given the equation of the parabola $4x^2 - 20x - 8y + 57 = 0$,

- **a.** Write the equation in standard form.
- **b.** Identify the vertex, focus, and focal diameter.
- **c.** Graph the parabola.

Solution:

a. $4x^2 - 20x - 8y + 57 = 0$

$4(x^2 - 5x \quad) = 8y - 57$

The x term is squared. Therefore, we complete the square on x on the left side. Move the y term and constant term to the right.

$4 \cdot \frac{25}{4}$ is added to the left side.

$4 \cdot \frac{25}{4}$

Complete the square. Note that $\left[\frac{1}{2} \cdot (-5)\right]^2 = \left[-\frac{5}{2}\right]^2 = \frac{25}{4}$.

$$4\left(x^2 - 5x + \frac{25}{4}\right) = 8y - 57 + 25$$

The value $\frac{25}{4}$ was added inside the parentheses. However, $4 \cdot \frac{25}{4}$ was actually added to the left side as a result of the factor 4 in front of the parentheses.

$$4\left(x - \frac{5}{2}\right)^2 = 8y - 32$$

Factor the left side. To write the equation in standard form, we require the coefficient of the square term to be 1.

$$\frac{4\left(x - \frac{5}{2}\right)^2}{4} = \frac{8y - 32}{4}$$

Divide both sides by 4.

$$\left(x - \frac{5}{2}\right)^2 = 2y - 8$$

$$\left(x - \frac{5}{2}\right)^2 = 2(y - 4)$$

Factor out the leading coefficient on the right side of the equation. The equation is in the form $(x - h)^2 = 4p(y - k)$, where $h = \frac{5}{2}$, $k = 4$, and $4p = 2$. The vertex is $\left(\frac{5}{2}, 4\right)$.

b. $4p = 2$

$p = \frac{1}{2}$

The y term is linear. Therefore, the parabola will open in the y direction (upward or downward). Furthermore, since $p > 0$, the parabola will open upward.

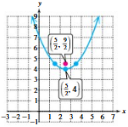
The vertex is $\left(\frac{5}{2}, 4\right)$. Since $p = \frac{1}{2}$, the focus is $\frac{1}{2}$ unit above the vertex:

The focus is $\left(\frac{5}{2}, \frac{9}{2}\right)$. $(h, k + p) = \left(\frac{5}{2}, 4 + \frac{1}{2}\right) = \left(\frac{5}{2}, \frac{8}{2} + \frac{1}{2}\right) = \left(\frac{5}{2}, \frac{9}{2}\right)$

The focal diameter is 2. The focal diameter is $4|p| = 4\left|\frac{1}{2}\right| = 2$.

c. Plot the vertex and locate the focus. The focal diameter is 2 units. Therefore, plot points 1 unit to the left and right of the focus. These are the points $\left(\frac{3}{2}, \frac{9}{2}\right)$ and $\left(\frac{7}{2}, \frac{9}{2}\right)$.

Sketch the curve.



Apply the Skills

5. Repeat Example 5 with the equation: $4x^2 - 12x - 12y + 21 = 0$

In Example 6, we determine the standard form of an equation of a parabola based on known information about the parabola.

Example 6: Writing an Equation of a Parabola

Determine the standard form of an equation of the parabola with focus (1, 2) and directrix $x = 7$.

Solution:

The directrix is a vertical line $x = 7$ and the focus is 6 units to the left of the directrix (Figure 10-28). The vertex is halfway between the directrix and focus. Therefore, the vertex is 3 units to the right of the focus at (4, 2). This indicates that $h = 4$ and $k = 2$.

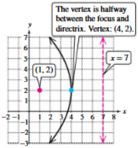


Figure 10-28

The parabola must open away from the directrix and toward the focus, which in this case is to the left. The standard equation is

$$(y - k)^2 = 4p(x - h)$$

The distance between the focus and vertex is 3. Therefore, $|p| = 3$. Because the parabola opens to the left, $p < 0$, indicating that $p = -3$.

The equation is: $(y - k)^2 = 4p(x - h)$

$$(y - 2)^2 = 4(-3)(x - 4)$$

$$(y - 2)^2 = -12(x - 4)$$

Apply the Skills

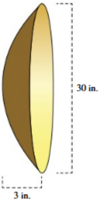
6. Determine the standard form of an equation of a parabola with focus (−2, 3) and directrix $y = 7$.

Learn: Use Parabolas in Applications

We complete this lesson by determining an equation of a parabola used to construct a satellite dish. A parabolic satellite dish uses the reflective property of a parabola to gather radio waves and direct the signal to a common focus to make the signal stronger.

Example 7: Applying an Equation of a Parabola

A satellite dish is in the shape of a paraboloid. Cross sections taken parallel to the direction the dish opens are parabolic. Cross sections taken perpendicular to the direction the dish opens are circular. The diameter of the dish is 30 in. and the depth is 3 in.



a. Find an equation of a parabolic cross section through the vertex of the dish.

b. Where should the receiver be placed?

Solution:

a. Orient the dish with the vertex at the origin of a rectangular coordinate system with the dish opening along the positive x-axis (Figure 10-29).

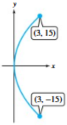


Figure 10-29

An equation of a parabolic cross section through the origin is of the form $y^2 = 4px$. To solve for the value of p, we can use a known point on the parabola such as (3, 15) or (3, −15). Using (3, 15), we have:

$$\begin{aligned} y^2 &= 4px \\ (15)^2 &= 4p(3) \\ 225 &= 12p \\ p &= 18.75 \end{aligned}$$

Therefore, an equation of a parabolic cross section through the origin is

$$\begin{aligned} y^2 &= 4px \\ y^2 &= 4(18.75)x \\ y^2 &= 75x \end{aligned}$$

b. The receiver should be placed at the focus of the parabola. Since the focal length is $|p| = |18.75| = 18.75$, the receiver should be placed at (18.75, 0).

Apply the Skills

7. Repeat Example 7 with a radio telescope of diameter 70 m and depth 14 m.

Practice 10-3

Practice Exercises

Prerequisite Review

R.1. Use the distance formula to find the distance between the points $(-1, -2)$ and $(-2, -9)$.

For Exercises R.2–R.3, write an equation of the line satisfying the given conditions.

R.2. The line is parallel to the line $y = 4$ and passes through $(-4, 3)$.

R.3. The line is perpendicular to the x-axis and passes through $(2, 1)$.

Concept Connections

1. A __________ is the set of all points in a plane that are equidistant from a fixed line (called the __________) and a fixed point (called the __________).

2. The line perpendicular to the directrix and passing through the focus of a parabola is called the axis of __________.

3. The __________ of a parabola is the point of intersection of the parabola and the axis of symmetry.

4. The distance between the vertex and the focus of a parabola is called the __________ length and is often represented by $|p|$.

5. Given $y^2 = 4px$ with $p < 0$, the parabola opens (upward/downward/left/right).

6. Given $x^2 = 4py$ with $p > 0$, the parabola opens (upward/downward/left/right).

7. The line segment perpendicular to the axis of symmetry, passing through the focus and with endpoints on the parabola, is called the __________ __________.

8. The length of the latus rectum is called the __________ diameter.

9. Given $(x - h)^2 = 4p(y - k)$, the ordered pairs representing the vertex and focus are __________ and __________, respectively. The directrix is the line defined by the equation __________.

10. Given $(y - k)^2 = 4p(x - h)$, the ordered pairs representing the vertex and focus are __________ and __________, respectively. The directrix is the line defined by the equation __________.

11. If the directrix is horizontal, then the parabola opens (horizontally/vertically).

12. If the line of symmetry is horizontal, then the parabola opens (horizontally/vertically).

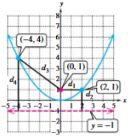
For Exercises 13–14, the graph of a parabola is given.

a. Determine the distances d_1, d_2, d_3, and d_4.

b. Compare d_1 and d_2.

c. Compare d_3 and d_4.

13.



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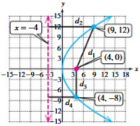


Lesson Identify the Focus and Directrix of a Parabola

For Exercises 15–22, a model of the form $x^2 = 4py$ or $y^2 = 4px$ is given.

a. Determine the value of p.

b. Identify the focus of the parabola.

c. Write an equation for the directrix. (**See Example 1, p. 1088**)

15. $x^2 = 24y$

16. $x^2 = 12y$

17. $y^2 = 36x$

18. $y^2 = 48x$

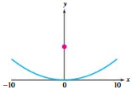
19. $x^2 = -5y$

20. $x^2 = -11y$

21. $-x = y^2$

22. $-3x = y^2$

23. A 20-in. satellite dish for a television has parabolic cross sections. A coordinate system is chosen so that the vertex of a cross section through the center of the dish is located at (0, 0). The equation of the parabola is modeled by $x^2 = 25.2y$, where x and y are measured in inches.



a. Where should the receiver be placed to maximize signal strength? That is, where is the focus? (**See Example 1, p. 1088**)

b. Determine the equation of the directrix.

24. Solar cookers provide an alternative form of cooking in regions of the world where consistent sources of fuel are not readily available. Suppose that a 36-in. solar cooker has parabolic cross sections. A coordinate system is chosen with the origin placed at the vertex of a cross section through the center of the mirror. The equation of the parabola is modeled by $x^2 = 82y$, where x and y are measured in inches.

a. Where should a pot be placed to maximize heat? That is, where is the focus?

b. Determine the equation of the directrix.

25. If a cross section of the parabolic mirror in a flashlight has an equation $y^2 = 2x$, where should the bulb be placed?

26. A cross section of the parabolic mirror in a car headlight is modeled by $y^2 = 12x$. Where should the bulb be placed?

Learn: Graph a Parabola with Vertex at the Origin

For Exercises 27–34, an equation of a parabola $x^2 = 4py$ or $y^2 = 4px$ is given.

a. Identify the vertex, value of p, focus, and focal diameter of the parabola.

b. Identify the endpoints of the latus rectum.

c. Graph the parabola.

d. Write equations for the directrix and axis of symmetry. (**See Examples 2–3, pp. 1090–1091**)

27. $x^2 = -4y$

28. $x^2 = -20y$

29. $10y^2 = 80x$

30. $3y^2 = 12x$

31. $4x^2 = 40y$

32. $2x^2 = 14y$

33. $y^2 = -x$

34. $y^2 = -2x$

Learn: Graph a Parabola with Vertex (h, k)

For Exercises 35–44, an equation of a parabola $(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$ is given.

a. Identify the vertex, value of p, focus, and focal diameter of the parabola.

b. Identify the endpoints of the latus rectum.

c. Graph the parabola.

d. Write equations for the directrix and axis of symmetry. (**See Example 4, p. 1092**)

35. $(y+1)^2 = -12(x-4)$

36. $(y+4)^2 = -16(x-2)$

37. $(x-1)^2 = -4(y+5)$

38. $(x-5)^2 = -8(y+2)$

39. $(x+3)^2 = 2\left(y-\frac{3}{2}\right)$

40. $(x+2)^2 = \left(y-\frac{7}{4}\right)$

41. $-(y-3)^2 = 7\left(x-\frac{1}{4}\right)$

42. $-(y-6)^2 = 9\left(x-\frac{3}{4}\right)$

43. $2(y-3) = \frac{1}{10}(x+6)^2$

44. $8(y-2) = \frac{1}{2}(x+3)^2$

For Exercises 45–52, an equation of a parabola is given.

a. Write the equation of the parabola in standard form.

b. Identify the vertex, focus, and focal diameter. (**See Example 5, p. 1094**)

45. $x^2 - 6x - 4y + 5 = 0$

46. $x^2 - 4x - 8y - 20 = 0$

47. $y^2 + 4y + 8x + 52 = 0$

48. $y^2 + 8y + 4x + 36 = 0$

49. $4x^2 - 28x + 24y + 73 = 0$

50. $4x^2 + 36x + 40y + 1 = 0$

51. $16y^2 + 24y - 16x + 57 = 0$

52. $16y^2 - 56y - 16x + 81 = 0$

For Exercises 53–58, fill in the blanks. Let $|p|$ represent the focal length (distance between the vertex and focus).

53. If the directrix of a parabola is given by $y = 4$ and the focus is $(1, 10)$, then the vertex is given by the ordered pair _______ and the value of p is _______.

54. If the directrix of a parabola is given by $x = 6$ and the focus is $(2, 1)$, then the vertex is given by the ordered pair _______ and the value of p is _______.

55. If the vertex of a parabola is $(-3, 0)$ and the focus is $(-7, 0)$, then the directrix is given by the equation _______, and the value of p is _______.

56. If the vertex of a parabola is $(4, -2)$ and the focus is $(4, -7)$, then the directrix is given by the equation _______, and the value of p is _______.

57. If the focal length of a parabola is 3 units, then the length of the latus rectum is _______ units.

58. If the focal diameter of a parabola is 8 units, then the focal length is _______ units.

59. If the focal length of a parabola is 6 units and the vertex is $(3, 2)$, is it possible to determine whether the parabola opens upward, downward, to the right, or to the left? Explain.

60. If the axis of symmetry of a parabola is given by $x = 4$, is it possible for the directrix to have the equation $x = 3$? Explain.

For Exercises 61–68, determine the standard form of an equation of the parabola subject to the given conditions. (See Example 6, p. 1095)

61. Vertex: $(0, 0)$; Directrix: $x = 4$

62. Vertex: $(0, 0)$; Directrix: $y = -2$

63. Focus: $(2, 4)$; Vertex: $(2, 1)$

64. Focus: $(5, 3)$: Vertex: $(3, 3)$

65. Focus: $(-6, -2)$; Directrix: $y = 0$

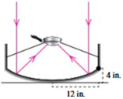
66. Focus: (–4, 5); Directrix: $x = 0$

67. Vertex: (2, 3); Parabola passes through (6, 5) (*Hint*: There are two possible answers.)

68. Vertex: (–4, 2); Parabola passes through (8, 14) (*Hint*: There are two possible answers.)

Learn: Use Parabolas in Applications

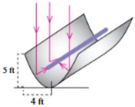
69. Suppose that a solar cooker has a parabolic mirror (see figure).



a. Use a coordinate system with origin at the vertex of the mirror and write an equation of a parabolic cross section of the mirror.

b. Where should a pot be placed so that it receives maximum heat? (**See Example 7, p. 1096**)

70. A solar water heater is made from a long sheet of metal bent so that the cross sections are parabolic. A long tube of water is placed inside the curved surface so that the height of the tube is equal to the focal length of the parabolic cross section. In this way, water in the tube is exposed to maximum heat.

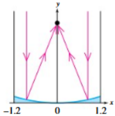


a. Determine the focal length of the parabolic cross sections so that the engineer knows where to place the tube.

b. Use a coordinate system with origin at the vertex of a parabolic cross section and write an equation of the parabola.

71. The Hubble Space Telescope was launched into space in 1990 and now orbits the Earth at 5 mi/sec at a distance of 353 mi above the Earth. From its location in space, the Hubble is free from the distortion of the Earth's atmosphere, enabling it to return magnificent images from distant stars and galaxies. The quality of the Hubble's images results from a large parabolic mirror, 2.4 m (7.9 ft) in diameter, that collects light from space.

Suppose that a coordinate system is chosen so that the vertex of a cross section through the center of the mirror is located at (0, 0). Furthermore, the focal length is 57.6 m.



a. Assume that x and y are measured in meters. Write an equation of the parabolic cross section of the mirror for $-1.2 \le x \le 1.2$. See figure.

b. How thick is the mirror at the edge? That is, what is the y value for $x = 1.2$?

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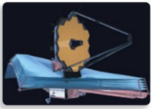
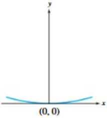
72. The James Webb Space Telescope (JWST) is a new space telescope that will enable scientists to see the earliest stars formed in the universe. The JWST will orbit the Sun in an orbit roughly 1.5 million km from the Earth, in a position with the Earth between the telescope and the Sun. The primary mirror of the JWST will consist of 18 hexagonally shaped segments that when fitted together will be 6.5 m in diameter (over 2.5 times the diameter of the Hubble). With a larger mirror, the JWST will be able to collect more light to "see" deeper into space.

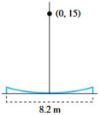


The primary mirror when pieced together will function as a parabolic mirror with a focal length of 131.4 m. Suppose that a coordinate system is chosen with (0, 0) at the center of a cross section of the primary mirror. Write an equation of the parabolic cross section of the mirror for $-3.25 \leq x \leq 3.25$. Assume that x and y are measured in meters.



73. The Subaru telescope is a large optical-infrared telescope at the summit of Mauna Kea, Hawaii. The telescope has a parabolic mirror 8.2 m in diameter with a focal length of 15 m.

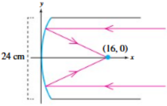
a. Suppose that a cross section of the mirror is taken through the vertex, and that a coordinate system is set up with (0, 0) placed at the vertex. If the focus is (0, 15), find an equation representing the curve.



b. Determine the vertical displacement of the mirror relative to horizontal at the edge of the mirror. That is, find the y value at a point 4.1 m to the left or right of the vertex.

c. What is the average slope between the vertex of the parabola and the point on the curve at the right edge?

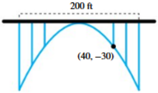
74. A parabolic mirror on a telescope has a focal length of 16 cm.



a. For the coordinate system shown, write an equation of the parabolic cross section of the mirror.

b. Determine the displacement of the mirror relative to the y-axis at the edge of the mirror. That is, find the x value at a point 12 cm above or below the vertex.

75. A parabolic arch forms a structure that allows equal vertical loading along its length.

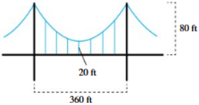


a. Take the origin at the vertex of the parabolic arch, and write an equation of the parabola.

b. Determine the focal length.

c. Determine the length of the vertical support 60 ft from the center.

76. A cable hanging freely between two vertical support beams forms a curve called a catenary. The shape of a catenary resembles a parabola, but mathematically the two functions are quite different.



a. On a graphing utility, graph a catenary defined by $y = \frac{1}{2}(e^x + e^{-x})$ and graph the parabola defined by $y = x^2 + 1$.

b. A catenary and a parabola are so similar in shape that we can often use a parabolic curve to approximate the shape of a catenary. For example, a bridge has cables suspended from a larger approximately parabolic cable. Take the origin at a point on the road directly below the vertex and write an equation of the parabolic cable.

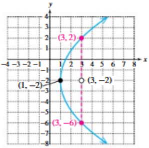
c. Determine the focal length of the parabolic cable.

d. Determine the length of the vertical support cable 100 ft from the vertex. Round to the nearest tenth of a foot.

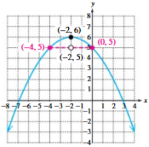
Mixed Exercises

For Exercises 77–80, write the equation of the parabola in standard form.

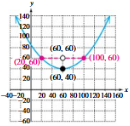
77.



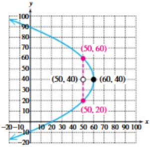
78.



79.



80.



For Exercises 81–84, each equation represents the graph of a parabola. Write the equation in standard form and identify the vertex and focus.

81. $y = \frac{x^2}{24} + \frac{x}{6} + \frac{37}{6}$

82. $y = \frac{x^2}{20} - \frac{x}{2} + \frac{9}{4}$

83. $x = \frac{y^2}{36} - \frac{y}{9} - \frac{26}{9}$

84. $x = -\frac{y^2}{24} - \frac{y}{12} + \frac{47}{24}$

For Exercises 85–88, graph the equations on the same coordinate system and determine the points of intersection.

85. $y = \frac{1}{4}x^2 - x + 1$
$(x - 2)^2 = -4(y - 8)$

86. $(y - 3)^2 = 4(x + 1)$
$(x + 2)^2 + (y - 3)^2 = 8$

87. $(x + 1)^2 = -8(y - 2)$
$y = |x + 1| - 4$

88. $(y - 2)^2 = -(x + 1)$
$(y - 2)^2 - (x + 1)^2 = 0$

For Exercises 89–90, solve the system of nonlinear equations.

89. $(y + 3)^2 = 4(x - 4)$
$2x + y = 9$

90. $(y - 2)^2 = -8(x - 1)$
$2x - y = -8$

Write About It

91. Given an equation of a parabola $(x - h)^2 = 4p(y - k)$ or $(y - k)^2 = 4p(x - h)$, how can you determine whether the parabola opens vertically or horizontally?

92. Give an example of the reflective property of a parabola.

Expanding Your Skills

For Exercises 93–96, each equation represents the graph of a parabola. Write the equation in standard form, and identify the vertex and focus.

93. $(x - 3)^2 + (y - 10)^2 = (x + 1)^2$

94. $(x - 2)^2 + (y - 4)^2 = (x + 16)^2$

95. $(y - 3)^2 + (x - 9)^2 = (y - 1)^2$

96. $(y + 4)^2 + (x + 1)^2 = (y - 18)^2$

A line is tangent to a parabola if it intersects the parabola at one point *and* is not parallel to the axis of symmetry.

- **For a parabola given by $x^2 = 4py$, an equation of the tangent line through the point (x_0, y_0) is $y = \frac{x_0}{2p}x - y_0$.**
- **For a parabola given by $y^2 = 4px$, an equation of the tangent line through the point (x_0, y_0) is $y = \frac{2p}{y_0}x + \frac{y_0}{2}$.**

For Exercises 97–100, find an equation of the tangent line (or tangent lines) to the parabola at the indicated point or value. Write the answer in slope-intercept form.

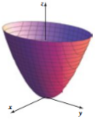
97. $x^2 = 4y$
a. at the point $\left(5, \frac{25}{4}\right)$ **b.** at $x = 3$

98. $x^2 = -8y$
a. at the point $\left(-2, -\frac{1}{2}\right)$ **b.** at $x = 3$

99. $y^2 = 20x$
a. at the point $(5, -10)$ **b.** at $x = \frac{4}{5}$

100. $y^2 = -3x$
a. at the point $(-12, -6)$ **b.** at $x = -\frac{16}{3}$

101. The surface defined by the equation $z = 4x^2 + y^2$ is called an elliptical paraboloid.



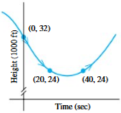
a. Write the equation with $x = 0$. What type of curve is represented by this equation?

b. Write the equation with $y = 0$. What type of curve is represented by this equation?

c. Write the equation with $z = 0$. What type of curve is represented by this equation?

Technology Connections

102. A jet flies in a parabolic arc to simulate partial weightlessness. The curve shown in the figure represents the plane's height y (in 1000 ft) versus the time t (in sec).



a. For each ordered pair, substitute the t and y values into the model $y = at^2 + bt + c$ to form a linear equation with three unknowns a, b, and c. Together, these form a system of three linear equations with three unknowns.

b. Use a graphing utility to solve for a, b, and c.

c. Substitute the known values of a, b, and c into the model $y = at^2 + bt + c$.

d. Determine the vertex of the parabola.

e. Determine the focal length of the parabola.

For Exercises 103–106, graph the parabola from the given exercise.

103. Exercise 29

104. Exercise 30

105. Exercise 41

106. Exercise 42

Problem Recognition Exercises

Comparing Equations of Conic Sections and the General Equation

For Exercises 1–8, identify each equation as representing a circle, an ellipse, a hyperbola, or a parabola.

- If the equation represents a circle, identify the center and radius.
- If the equation represents an ellipse, identify the center, vertices, endpoints of the minor axis, foci, and eccentricity.
- If the equation represents a hyperbola, identify the center, vertices, foci, equations of the asymptotes, and eccentricity.
- If the equation represents a parabola, identify the vertex, focus, equation of the directrix, and equation of the axis of symmetry.

1. $\frac{(x-2)^2}{16} - \frac{(y+2)^2}{9} = 1$

2. $9x^2 + 25(y-2)^2 = 225$

3. $(y-5)^2 = -(x+2)$

4. $(x-3)^2 + (y+7)^2 = 25$

5. $16(x+1)^2 + y^2 = 16$

6. $(x-1)^2 = 10(y+3)$

7. $-(x+1)^2 - (y+6)^2 = -16$

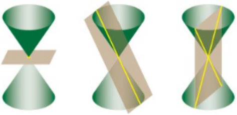
8. $-\frac{(x+4)^2}{25} + \frac{(y-1)^2}{144} = 1$

Suppose that the binomials within an equation of a conic section are expanded and all terms are collected on the left side of the equation. The result will be an equation of the conic section written in **general form**:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where A and C are not both zero.

Furthermore, given an equation of a conic section in general form, we can complete the square to identify the type of conic that the equation represents. In some cases, however, the graph of an equation in general form is a **degenerate conic section**. Geometrically, a degenerate case occurs when a plane intersects a double-napped cone in a point, a line, or a pair of intersecting lines.



General Equation of a Conic Section with No Rotation

The graph of the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where A and C are not both zero, is a conic section or a degenerate conic. If the graph represents a conic section, then the graph is

1. A circle if $A = C$.
2. An ellipse if $AC > 0$ (A and C have the same sign).
3. A hyperbola if $AC < 0$ (A and C have opposite signs).
4. A parabola if $AC = 0$ (A or C equals zero).

Insights

The general form of an equation of a conic section also has a term Bxy, which rotates the curve. Some courses in trigonometry address rotated conics.

For Exercises 9–16, determine whether the equation represents a circle, an ellipse, a hyperbola, or a parabola. Write the equation in standard form.

9. $9x^2 - 16y^2 - 36x - 64y - 172 = 0$

10. $9x^2 + 25y^2 - 100y - 125 = 0$

11. $y^2 - 10y + x + 27 = 0$

12. $x^2 + y^2 - 6x + 14y + 33 = 0$

13. $16x^2 + y^2 + 32x = 0$

14. $x^2 - 2x - 10y - 29 = 0$

15. $x^2 + y^2 + 2x + 12y + 21 = 0$

16. $-144x^2 + 25y^2 - 1152x - 50y - 5879 = 0$

For Exercises 17–22, an equation of a degenerate conic section is given. Complete the square and describe the graph of each equation.

17. $x^2 + y^2 - 4x + 2y + 5 = 0$

18. $9x^2 + 4y^2 - 24y + 36 = 0$

19. $4x^2 - y^2 - 32x - 4y + 60 = 0$

20. $9x^2 - 4y^2 - 90x - 8y + 221 = 0$

21. $9x^2 + 4y^2 - 16y + 52 = 0$

22. $x^2 + y^2 + 12x + 37 = 0$

23. Given $Ax^2 + Cy^2 + Dx + Ey + F = 0$, show that the equation represents a circle if $A = C \neq 0$ and $D^2 + E^2 - 4AF > 0$.

24. Consider $Ax^2 + Cy^2 + Dx + Ey + F = 0$, for $A \neq 0$ and $C \neq 0$.

- **a.** For A and C of the same sign, show that the equation represents an ellipse if $\frac{D^2}{4A} + \frac{E^2}{4C} - F$ has the same sign as A.
- **b.** For A and C of opposite signs, show that the equation represents a hyperbola if $\frac{D^2}{4A} + \frac{E^2}{4C} - F \neq 0$.

25. Given $Ax^2 + Cy^2 + Dx + Ey + F = 0$, show that the equation represents a parabola if $A = 0$ but $C \neq 0$ and $D \neq 0$.

Go online for more practice problems.

Lesson 10-4

Rotation of Axes

What Will You Learn?

After completing this lesson, you should be able to:

- Identify Conic Sections with No Rotation
- Use the Rotation of Axes Formulas
- Transform Equations by Rotation of Axes
- Use the Discriminant to Identify Conic Sections Containing a *Bxy* Term

Learn: Identify Conic Sections with No Rotation

In Lessons 10.1–10.3, we analyzed equations of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \qquad (1)$$

Equation (1) represents a conic section or a degenerate case. For nondegenerate cases, we can complete the square on the quadratic expressions in x and y to determine the type of conic section represented by the equation. Alternatively we can identify the type of conic section directly from the equation based on the values of A and C.

General Equation of a Conic with No Rotation

The graph of the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where A and C are not both zero, is a conic section or a degenerate case. If the graph represents a conic section, then the graph is

1. A circle if $A = C$.
2. An ellipse if $AC > 0$ (A and C have the same sign).
3. A hyperbola if $AC < 0$ (A and C have opposite signs).
4. A parabola if $AC = 0$ (A or C equals zero).

Insights

Note that the equation above is the general equation of a conic *with no rotation*. The general form of an equation of a conic section also has a term *Bxy*, which rotates the curve.

Example 1: Identifying Conic Sections with No Rotation

The equations represent nondegenerate conic sections. Identify the type of conic section.

a. $x^2 - y^2 - 4x = 0$

b. $y^2 - 12x - 6y + 23 = 0$

c. $36x^2 + 36y^2 - 18x + 24y + 5 = 0$

d. $4x^2 + y^2 + 24x - 2y + 3 = 0$

Solution:

a. $x^2 - y^2 - 4x = 0$

The coefficients of the square terms are $A = 1$ and $C = -1$. Since A and C have opposite signs ($AC < 0$), the graph is a hyperbola.

b. $y^2 - 12x - 6y + 23 = 0$

The equation does not have an x^2 term. Since $A = 0$, the graph is a parabola.

c. $36x^2 + 36y^2 - 18x + 24y + 5 = 0$

Since A and C are equal ($A = C = 36$), the graph is a circle.

d. $4x^2 + y^2 + 24x - 2y + 3 = 0$

The coefficients of the square terms are $A = 4$ and $C = 1$. Since A and C have the same sign ($AC > 0$), the graph is an ellipse.

Apply the Skills

1. The equations represent nondegenerate conic sections. Identify the type of conic section.

a. $x^2 + y^2 - 4x - 14 = 0$

b. $4x^2 - 9y^2 + 54y - 117 = 0$

c. $4x^2 - 16x - y + 15 = 0$

d. $25x^2 + 4y^2 + 250x - 16y + 541 = 0$

Learn: Use the Rotation of Axes Formulas

We now consider the most general second-degree polynomial equation in two variables.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (2)$$

Equation (2) differs from equation (1) $Ax^2 + Cy^2 + Dx + Ey + F = 0$ because of the existence of the Bxy term. We will see that the graph of (2) is also a conic section (or degenerate case). Excluding degenerate cases, the term Bxy introduces a rotation of the graph through an angle θ (Figure 10-30). To analyze the graph, we can rotate the x- and y-axes about the origin through an angle θ to form a new coordinate system with axes x' and y' (Figure 10-31). Relative to the x'- and y'-axes, the graph has no rotation.

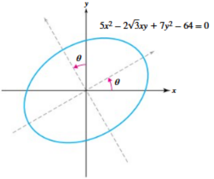


Figure 10-30

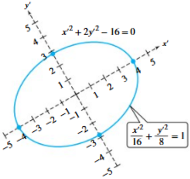


Figure 10-31

The goal in this section is to transform the original equation (2) in the variables x and y to a new equation in x' and y' with $B = 0$.

Suppose that a point $P(x, y)$ relative to the xy-plane has coordinate $P(x', y')$ relative to the $x'y'$-plane (Figure 10-32). Let r represent the distance between P and the origin and let α be the angle made by line segment $\overline{OP}$ and the positive x'-axis.

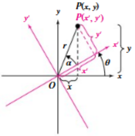


Figure 10-32

From Figure 10-32, we have:

(3) $x' = r\cos\alpha$

(4) $y' = r\sin\alpha$

(5) $x = r\cos(\theta + \alpha)$

(6) $y = r\sin(\theta + \alpha)$

Applying the sum formula for cosine in equation (5), we have

$$x = r\cos(\theta + \alpha)$$

$$= r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$

$$= (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta$$

$$= x'\cos\theta - y'\sin\theta \qquad \text{By equations (3) and (4)}$$

Applying the sum formula for sine in equation (6), we have

$$y = r\sin(\theta + \alpha)$$

$$= r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$

$$= (r\cos\alpha)\sin\theta + (r\sin\alpha)\cos\theta$$

$$= x'\sin\theta + y'\cos\theta \qquad \text{By equations (3) and (4)}$$

Treating the equations $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ as a system of equations, we can also solve for x' and y' in terms of x and y (see Exercise 51).

Rotation of Axes

Suppose that the x- and y-axes in a rectangular coordinate system are rotated about the origin through an angle θ to form the x'- and y'-axes. Then the coordinates of $P(x, y)$ in the xy-plane are related to $P(x', y')$ in the $x'y'$-plane as follows.

x and y in terms of x' and y'	x' and y' in terms of x and y
(7) $x = x'\cos\theta - y'\sin\theta$	(9) $x' = x\cos\theta + y\sin\theta$
(8) $y = x'\sin\theta + y'\cos\theta$	(10) $y' = -x\sin\theta + y\cos\theta$

Example 2: Identifying Coordinates of a Point after Rotating the Coordinate Axes

Suppose that the x- and y-axes are rotated 60° about the origin to form the x'- and y'-axes. Given $P(6, -2)$ in the xy-plane, find the coordinates of P in the $x'y'$-plane.

Solution:

$P(6, -2)$ corresponds to $x = 6$ and $y = -2$.

Using rotation equations (9) and (10), we have:

$$x' = x\cos\theta + y\sin\theta$$

$$= 6\cos 60^\circ + (-2)\sin 60^\circ$$

$$= 6\left(\frac{1}{2}\right) + (-2)\left(\frac{\sqrt{3}}{2}\right) = 3 - \sqrt{3}$$

$$y' = -x\sin\theta + y\cos\theta$$

$$= -6\sin 60^\circ - 2\cos 60^\circ$$

$$= -6\left(\frac{\sqrt{3}}{2}\right) + (-2)\left(\frac{1}{2}\right) = 3\sqrt{3} - 1$$

$P(x', y')$ is $(3 - \sqrt{3}, -3\sqrt{3} - 1) \approx (1.3, -6.2)$.

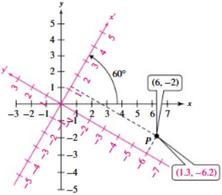


Figure 10-33

Insights

In Figure 10-33, the coordinates of point *P* are given relative to the *xy*-plane and the *x′y′*-plane.

Apply the Skills

2. Suppose that the *x*- and *y*-axes are rotated 45° about the origin to form the *x′*- and *y′*-axes. Given $P(-4, 2)$ in the *xy*-plane, find the coordinate of *P* in the *x′y′*-plane.

The equation $y = \frac{1}{x}$ was first graphed in Lesson 1.6 (Figure 10-34). The graph looks like a hyperbola rotated through some angle about the origin. This is indeed the case. The equation can be written as $xy = 1$ and the left side of the equation is the *Bxy* term in a general second-degree polynomial equation. In Example 3, we show that the graph of $xy = 1$ is a hyperbola rotated 45° from the *x*-axis.

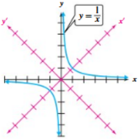


Figure 10-34

Example 3: Rotating Axes and Transforming an Equation

Assume that the x- and y-axes are rotated through an angle of 45° about the origin to form the x'- and y'-axes. Write the equation $xy = 1$ in $x'y'$-coordinates. Discuss the result.

Solution:

Use the rotation formulas (7) and (8) and replace x by $x'\cos\theta - y'\sin\theta$ and y by $x'\sin\theta + y'\cos\theta$.

$$xy = 1$$

$$(x'\cos 45^\circ - y'\sin 45^\circ)(x'\sin 45^\circ + y'\cos 45^\circ) = 1$$

$$\left[x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right)\right]\left[x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right)\right] = 1$$

$$\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)(x' - y')(x' + y') = 1$$ Factor $\frac{\sqrt{2}}{2}$ from each binomial factor.

$$\frac{1}{2}(x'^2 - y'^2) = 1$$ Multiply conjugates.

$$x'^2 - y'^2 - 2 = 0$$ Multiply both sides by 2 and collect terms on the left.

The equation is of the form $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$. Notice that relative to the $x'y'$-coordinate system, there is no rotation term $B'x'y'$. Furthermore, $A' = 1$ and $C' = -1$, so the product $A'C' = -1 < 0$, indicating that the graph is a hyperbola.

We can write the equation in the standard form of a hyperbola, $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1$.

$$\frac{x'^2}{2} - \frac{y'^2}{2} = 1 \quad \text{where } a^2 = 2 \text{ and } b^2 = 2.$$

The center of the hyperbola is at the origin and the vertices are on the x'-axis at $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$. The equations of the asymptotes are given by $y' = \pm\frac{b}{a}x'$, which is $y' = \pm\frac{\sqrt{2}}{\sqrt{2}}x'$ or simply $y' = \pm x'$. These are the x- and y-axes in the original coordinate system (Figure 10-35).

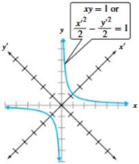


Figure 10-35

Apply the Skills

3. Write the equation $xy = 8$ in terms of x' and y' given that the angle of rotation of the x'- and y'-axes is 45° from the x- and y-axes.

Learn: Transform Equations by Rotation of Axes

The technique used in Example 3 can be used to transform any equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (2)$$

to an equation in x' and y' with no $x'y'$ term by choosing an appropriate angle of rotation, θ. To find θ, substitute $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ into (2).

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (2)$$

$$\begin{aligned} &A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) \\ &\qquad + C(x'\sin\theta + y'\cos\theta)^2 + D(x'\cos\theta - y'\sin\theta) \\ &\qquad + E(x'\sin\theta + y'\cos\theta) + F = 0 \end{aligned}$$

Expanding binomial factors and collecting like terms results in an equation of the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

where

$$\begin{aligned} A' &= A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta \\ B' &= 2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) \\ C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\ D' &= D\cos\theta + E\sin\theta \\ E' &= -D\sin\theta + E\cos\theta \\ F' &= F \end{aligned}$$

To eliminate the $x'y'$ term, we require $B' = 0$. Therefore,

$$2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) = 0$$

Recall that $2\sin\theta\cos\theta = \sin 2\theta$ and $\cos^2\theta - \sin^2\theta = \cos 2\theta$.

$$(C - A)\sin 2\theta + B\cos 2\theta = 0$$

$$B\cos 2\theta = (A - C)\sin 2\theta$$

$$\cot 2\theta = \frac{A - C}{B}$$

Divide both sides by $B\sin 2\theta$.

There are an infinite number of solutions to the equation $\cot 2\theta = \frac{A - C}{B}$, however, we choose θ to be an acute angle. The cotangent function is used rather than the tangent function because $\cot^{-1}\left(\frac{A - C}{B}\right)$ returns $0 < 2\theta < \pi$ and therefore, $0 < \theta < \frac{\pi}{2}$.

Formula for Determining an Angle of Rotation

To write the general second-degree polynomial equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (2)$$

with no xy term, rotate the axes through an acute angle θ given by

$$\cot 2\theta = \frac{A - C}{B}$$

The resulting equation is $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$, where the coefficients are found by substituting $x'\cos\theta - y'\sin\theta$ for x and $x'\sin\theta + y'\cos\theta$ for y into (2).

In Example 4, we will eliminate the *xy* term in the equation $5x^2 - 2\sqrt{3}\,xy + 7y^2 - 64 = 0$ and verify that it represents the graph of an ellipse as shown in Figures 10-30 and 10-31 (*on page 1108*).

Example 4: Eliminating the *xy* Term in an Equation

Given $5x^2 - 2\sqrt{3}\,xy + 7y^2 - 64 = 0$,

a. Determine an acute angle of rotation to eliminate the *xy* term.

b. Use a rotation of axes to eliminate the *xy* term in the equation.

c. Identify the type of curve represented by the equation.

d. Sketch the graph.

Solution:

a. To eliminate the *xy* term in the equation $5x^2 - 2\sqrt{3}\,xy + 7y^2 - 64 = 0$, we need to rotate the *x*- and *y*-axes through an angle θ that satisfies

$$\cot 2\theta = \frac{A-C}{B} \Rightarrow \cot 2\theta = \frac{5-7}{-2\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow 2\theta = 60° \Rightarrow \theta = 30°$$

b. Using rotation formulas (7) and (8), we have

$$x = x'\cos\theta - y'\sin\theta = x'\cos 30° - y'\sin 30° = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$$

$$y = x'\sin\theta + y'\cos\theta = x'\sin 30° + y'\cos 30° = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$$

Substitute $x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$ and $y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$ into the original equation.

$$\boxed{A}\quad 5\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)^2 - 2\sqrt{3}\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right) + 7\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)^2 - 64 = 0$$

To facilitate the "bookkeeping," consider simplifying the first three terms on the left side of equation $\boxed{A}$ separately.

$$\begin{aligned}5\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)^2 &= 5\left(\frac{3}{4}x'^2 - \frac{\sqrt{3}}{2}x'y' + \frac{1}{4}y'^2\right)\\ &= \frac{15}{4}x'^2 - \frac{5\sqrt{3}}{2}x'y' + \frac{5}{4}y'^2\end{aligned}$$

$$\begin{aligned}-2\sqrt{3}\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right) &= -2\sqrt{3}\left(\frac{\sqrt{3}}{4}x'^2 + \frac{3}{4}x'y' - \frac{1}{4}x'y' - \frac{\sqrt{3}}{4}y'^2\right)\\ &= -2\sqrt{3}\left(\frac{\sqrt{3}}{4}x'^2 + \frac{1}{2}x'y' - \frac{\sqrt{3}}{4}y'^2\right)\\ &= -\frac{3}{2}x'^2 - \sqrt{3}x'y' + \frac{3}{2}y'^2\end{aligned}$$

$$\begin{aligned}7\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)^2 &= 7\left(\frac{1}{4}x'^2 + \frac{\sqrt{3}}{2}x'y' + \frac{3}{4}y'^2\right)\\ &= \frac{7}{4}x'^2 + \frac{7\sqrt{3}}{2}x'y' + \frac{21}{4}y'^2\end{aligned}$$

Equation $\boxed{A}$ becomes

$$\begin{aligned}\frac{15}{4}x'^2 - \frac{5\sqrt{3}}{2}x'y' + \frac{5}{4}y'^2 - \frac{3}{2}x'^2 - \sqrt{3}x'y' + \frac{3}{2}y'^2 + \frac{7}{4}x'^2 + \frac{7\sqrt{3}}{2}x'y' + \frac{21}{4}y'^2 - 64 &= 0\\ 4x'^2 + 8y'^2 - 64 &= 0\\ x'^2 + 2y'^2 - 16 &= 0\end{aligned}$$

c. The equation $x'^2 + 2y'^2 - 16 = 0$ is of the form $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$ with $A' = 1$ and $C' = 2$. Since A' and C' have the same sign ($A'C' > 0$), the equation is an ellipse.

d. To graph the equation, divide both sides of the equation by 16 and write the equation in standard form.

$$\frac{x'^2}{16} + \frac{2y'^2}{16} - \frac{16}{16} = \frac{0}{16}$$

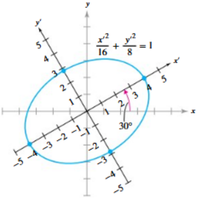
$$\frac{x'^2}{16} + \frac{y'^2}{8} = 1$$

Relative to the $x'y'$-plane, the equation is an ellipse with center (0, 0).

$a = \sqrt{16} = 4$ and $b = \sqrt{8} = 2\sqrt{2}$

The vertices are (4, 0) and (–4, 0) in the $x'y'$-plane.

The endpoints of the minor axis are $(0, 2\sqrt{2})$ and $(0, -2\sqrt{2})$ in the $x'y'$-plane.



Apply the Skills

4. Given $5x^2 + 2\sqrt{3}xy + 7y^2 - 32 = 0$,

a. Determine an acute angle of rotation to eliminate the xy term.

b. Use a rotation of axes to eliminate the xy term in the equation.

c. Identify the type of curve represented by the equation.

d. Sketch the graph.

Learn: Use the Discriminant to Identify Conic Sections Containing a *Bxy* Term

In Example 4, we have an equation of a conic section of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (2)$$

To eliminate the rotation term Bxy, we rotated the x- and y-axes through a suitable angle θ to form the x'- and y'-axes and an equation of the form $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$. Since the new equation has no rotation term, we then used the values of A' and C' to identify the type of conic section represented by the equation.

This method to identify a conic section involves tedious algebra. Fortunately, we now present a means to identify the type of conic section represented by equation (2) by using the coefficients A, B, and C.

Suppose that the x- and y-axes are rotated about the origin through an angle θ to form the x'- and y'-axes. As a result, the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is changed into the equation $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$. From the equations on page 1114 relating A, B, and C to A', B', and C', we can show that

$$B^2 - 4AC = B'^2 - 4A'C'.$$

When θ is chosen such that $B' = 0$, we have

$$B^2 - 4AC = -4A'C'.$$

Excluding degenerate cases, the type of conic section represented by the equation $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$ can be determined by the product $A'C'$.

The equation $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$ represents

An ellipse (or a circle) if $A'C' > 0$.

A hyperbola if $A'C' < 0$.

A parabola if $A'C' = 0$.

Since $B^2 - 4AC = -4A'C'$, nonzero values of $B^2 - 4AC$ have the *opposite* sign of $A'C'$. Therefore, we can make the following statements about the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

Identify Conic by the Discriminant

Excluding degenerate cases, the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ represents

An ellipse (or a circle) if $B^2 - 4AC < 0$.

A hyperbola if $B^2 - 4AC > 0$.

A parabola if $B^2 - 4AC = 0$.

The quantity $B^2 - 4AC$ is called the **discriminant** of the equation.

Insights

The discriminant $B^2 - 4AC = B'^2 - 4A'C'$ is unchanged under rotation. That is, the value of $B^2 - 4AC$ and $B'^2 - 4A'C'$ are the same regardless of the angle of rotation. For this reason, we say that the discriminant is **invariant** under rotation.

In Examples 5 and 6, we use the discriminant to identify the type of conic section represented by a general second-degree polynomial equation. Then we graph the equation.

Example 5: Graphing a Rotated Conic Section

The equation $x^2 + 6xy + 9y^2 - 7\sqrt{10}\,x - 11\sqrt{10}\,y + 30 = 0$ is a conic section (nondegenerate case).

- **a.** Use the discriminant to determine the type of conic section represented.
- **b.** Use a suitable rotation to eliminate the xy term.
- **c.** Sketch the graph.

Solution:

a. $x^2 + 6xy + 9y^2 - 7\sqrt{10}\,x - 11\sqrt{10}\,y + 30 = 0$ $\qquad A = 1, B = 6$, and $C = 9$

$B^2 - 4AC = (6)^2 - 4(1)(9) = 0$, so the equation represents a parabola.

b. We need to rotate the x- and y-axes through an angle θ that satisfies $\cot 2\theta = \frac{A-C}{B} \Rightarrow \cot 2\theta = \frac{1-9}{6} = -\frac{4}{3}$.

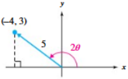


Figure 10-36

In order to apply rotation formulas (7) and (8), we need to know the value of $\sin\theta$ and $\cos\theta$. Since θ is an acute angle and $\cot 2\theta < 0$, angle 2θ is a second quadrant angle (Figure 10-36). Therefore, $\cos 2\theta = -\frac{4}{5}$, and using the half-angle formulas, we have

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1+\left(-\frac{4}{5}\right)}{2}} = \sqrt{\frac{1}{10}} = \frac{\sqrt{10}}{10}$$

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1-\left(-\frac{4}{5}\right)}{2}} = \sqrt{\frac{9}{10}} = \frac{3\sqrt{10}}{10}$$

From the equation $\cos\theta = \frac{\sqrt{10}}{10}$, we have $\theta \approx 71.6°$.

We now apply rotation formulas (7) and (8).

$$x = x'\cos\theta - y'\sin\theta = \frac{\sqrt{10}}{10}x' - \frac{3\sqrt{10}}{10}y' = \frac{\sqrt{10}}{10}(x' - 3y')$$

$$y = x'\sin\theta + y'\cos\theta = \frac{3\sqrt{10}}{10}x' + \frac{\sqrt{10}}{10}y' = \frac{\sqrt{10}}{10}(3x' + y')$$

Before substituting formulas (7) and (8) into the original equation, we will write expressions for x^2, y^2, and xy in terms of x' and y'.

$$x^2 = \left[\frac{\sqrt{10}}{10}(x' - 3y')\right]^2 = \frac{1}{10}(x'^2 - 6x'y' + 9y'^2)$$

$$y^2 = \left[\frac{\sqrt{10}}{10}(3x' + y')\right]^2 = \frac{1}{10}(9x'^2 + 6x'y' + y'^2)$$

$$xy = \left[\frac{\sqrt{10}}{10}(x' - 3y')\right]\left[\frac{\sqrt{10}}{10}(3x' + y')\right] = \frac{1}{10}(3x'^2 - 8x'y' - 3y'^2)$$

The equation $x^2 + 6xy + 9y^2 - 7\sqrt{10}\,x - 11\sqrt{10}\,y + 30 = 0$ becomes

$$\frac{1}{10}(x'^2 - 6x'y' + 9y'^2) + 6 \cdot \frac{1}{10}(3x'^2 - 8x'y' - 3y'^2) + 9 \cdot \frac{1}{10}(9x'^2 + 6x'y' + y'^2)$$
$$-7\sqrt{10} \cdot \frac{\sqrt{10}}{10}(x' - 3y') - 11\sqrt{10} \cdot \frac{\sqrt{10}}{10}(3x' + y') + 30 = 0$$

Multiplying both sides by 10 gives

$$(x'^2 - 6x'y' + 9y'^2) + 6(3x'^2 - 8x'y' - 3y'^2) + 9(9x'^2 + 6x'y' + y'^2) - 70(x' - 3y')$$
$$- 110(3x' + y') + 300 = 0$$

Applying the distributive property and combining like terms gives

$$100x'^2 - 400x' + 100y' + 300 = 0$$

Dividing both sides by 100 results in $x'^2 - 4x' + y' + 3 = 0$.

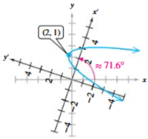
c. We can find the vertex of the parabola by completing the square.

$x'^2 - 4x' + 4 = -y' - 3 + 4$

$(x' - 2)^2 = -(y' - 1)$

Draw the x'- and y'-axes 71.6° counterclockwise from the x- and y-axes.

In the $x'y'$-plane, the vertex is (2, 1) and the axis of symmetry is parallel to the y'-axis.



Apply the Skills

5. The equation $x^2 + 4xy + 4y^2 + 6\sqrt{5}\,x + 7\sqrt{5}\,y + 20 = 0$ is a conic section (nondegenerate case).

- **a.** Determine the type of conic section represented by the equation.
- **b.** Use a suitable rotation to eliminate the xy term.
- **c.** Sketch the graph.

Example 6: Graphing a Rotated Conic Section with a Graphing Utility

The equation $27x^2 - 120xy - 92y^2 - 96x + 40y + 520 = 0$ represents a conic section (nondegenerate case).

- **a.** Identify the type of conic section.
- **b.** Graph the equation on a graphing utility.

Solution:

a. $27x^2 - 120xy - 92y^2 - 96x + 40y + 520 = 0$

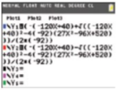
Since $B^2 - 4AC = (-120)^2 - 4(27)(-92) = 24{,}336 > 0$, the equation represents a hyperbola.

b. The equation is quadratic in the y variable, and we can write it in the form $ay^2 + by + c = 0$.

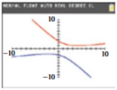
$$-92y^2 + (-120x + 40)y + (27x^2 - 96x + 520) = 0$$

Apply the quadratic formula with $a = -92$, $b = -120x + 40$, and $c = 27x^2 - 96x + 520$.

$$y = \frac{-(-120x + 40) \pm \sqrt{(-120x + 40)^2 - 4(-92)(27x^2 - 96x + 520)}}{2(-92)}$$



```
 Plot1  Plot2  Plot3
■\Y1■(-(-120X+40)+√(((-120X
+40)²-4(-92)(27X²-96X+520)
))/(2*(-92))
■\Y2■(-(-120X+40)-√(((-120X
+40)²-4(-92)(27X²-96X+520)
))/(2*(-92))
■\Y3=
■\Y4=
■\Y5=
```



Insights

For the purpose of entering the equation from Example 6 into a calculator, we do not need to simplify the radical.

Apply the Skills

6. The equation $44x^2 + 600xy - 551y^2 - 624x - 260y - 2028 = 0$ represents a conic section (nondegenerate case).

a. Identify the type of conic section.

b. Graph the equation on a graphing utility.

Practice 10-4

Practice Exercises

Prerequisite Review

R.1. Given $\cos \alpha = \frac{12}{13}$ for $\frac{3\pi}{2} < \alpha < 2\pi$, find the exact values of the following.

a. $\sin \frac{\alpha}{2}$ **b.** $\cos \frac{\alpha}{2}$

R.2. Use a calculator to approximate the degree measure (to 1 decimal place) of θ subject to the given conditions.

$\sin \theta = \frac{12}{19}$ and $90° \le \theta \le 180°$

R.3. Find the exact value or state that the expression is undefined.

$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

R.4. Solve by using the quadratic formula.

$w^2 - 3w - 5 = 0$

R.5. Solve for the indicated variable.

$v = 2\pi rq + \pi r^2 q$ for r.

R.6. Determine the value of n that makes the polynomial a perfect square trinomial. Then factor as the square of a binomial.

$v^2 + 30v + n$

Concept Connections

1. Suppose that the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ represents a conic section (nondegenerative case). In terms of A and C, the equation represents a circle if __________, an ellipse if __________, a hyperbola if __________, and a parabola if __________.

2. Suppose that the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ represents a conic section (nondegenerative case). In terms of the discriminant, $B^2 - 4AC$, the equation represents an ellipse if __________, a hyperbola if __________, and a parabola if __________.

3. Suppose that the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ represents a conic section (nondegenerative case). Which term determines whether the conic section is rotated?

4. The conic section defined by $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ will have no rotation relative to the new coordinate axes x' and y' if the x- and y-axes are rotated through an angle θ satisfying the relationship $\cot 2\theta =$ __________ (answer in terms of A, B, and C).

Objective 1: Identify Conic Sections with No Rotation

For Exercises 5–8, the given equation represents a conic section (nondegenerative case). Identify the type of conic section. (See Example 1, p. 1107)

5. **a.** $8x^2 - 2y^2 - 3x + 2y - 6 = 0$

b. $-6y^2 + 4x - 12y - 24 = 0$

c. $-9x^2 - 4y^2 - 18x + 12y = 0$

6. **a.** $-5x^2 + 8y + 4 = 0$

b. $3x^2 + 3y^2 - 4x + 2y - 8 = 0$

c. $-2x^2 + y^2 - 4y + 1 = 0$

7. **a.** $0.1x^2 + 0.6x - 1.6 = 0.2y - 0.1y^2$

b. $2x^2 - 7xy = -y^2 + 4x - 2y - 1$

c. $8x + 2y = y^2 + 4$

8. **a.** $6x^2 + 3x + 10y = 10y^2 + 8$

b. $3x^2 + 18xy = 5x + 2y + 9$

c. $4x^2 + 8x - 5 = -y^2 + 6y + 3$

Learn: Use the Rotation of Axes Formulas

For Exercises 9–14, assume that the x- and y-axes are rotated through angle θ about the origin to form the x'- and y'-axes. (See Example 2, p. 1109)

9. Given $\theta = 60°$ and $P(2, -10)$ in the xy-plane, find the coordinates of P in the $x'y'$-plane.

10. Given $\theta = 30°$ and $P(-4, 8)$ in the xy-plane, find the coordinates of P in the $x'y'$-plane.

11. Given $\theta = 45°$ and $P(-3, -6)$ in the $x'y'$-plane, find the coordinates of P in the xy-plane.

12. Given $\theta = 45^\circ$ and $P(-\sqrt{2}, 5\sqrt{2})$ in the $x'y'$-plane, find the coordinates of P in the xy-plane.

13. Given $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$ and $P(-5, -15)$ in the xy-plane, find the coordinates of P in the $x'y'$-plane.

14. Given $\cos\theta = \frac{5}{13}$ and $\sin\theta = \frac{12}{13}$ and $P(-39, 26)$ in the xy-plane, find the coordinates of P in the $x'y'$-plane.

Learn: Transform Equations by Rotation of Axes

For Exercises 15–18, assume that the x- and y-axes are rotated through angle θ about the origin to form the x'- and y'-axes. Write the equation in $x'y'$-coordinates. (See Example 3, p. 1111)

15. $xy + 2 = 0, \theta = 45^\circ$

16. $xy - 4 = 0, \theta = 45^\circ$

17. $7x^2 - 2\sqrt{3}xy + 5y^2 - 48 = 0, \theta = 60^\circ$

18. $13x^2 + 6\sqrt{3}xy + 7y^2 - 32 = 0, \theta = 30^\circ$

For Exercises 19–22, assume that $0 < 2\theta < \pi$. Find the exact values of $\cos\theta$ and $\sin\theta$. Then approximate the value of θ to the nearest tenth of a degree if necessary.

19. $\cot 2\theta = -\frac{\sqrt{3}}{3}$

20. $\cot 2\theta = \frac{\sqrt{3}}{3}$

21. $\cot 2\theta = \frac{24}{7}$

22. $\cot 2\theta = -\frac{8}{15}$

For Exercises 23–28,

a. Determine an acute angle of rotation to eliminate the xy term. (**See Example 4, p. 1113**)

b. Use a rotation of axes to eliminate the xy term to write the equation in the form $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$.

c. Identify the type of curve represented by the equation.

d. Write the equation in standard form and sketch the graph.

23. $3x^2 - 2xy + 3y^2 - 16 = 0$

24. $2x^2 + 2xy + 2y^2 - 9 = 0$

25. $11x^2 - 10\sqrt{3}xy + y^2 - 16 = 0$

26. $3x^2 - 10\sqrt{3}xy + 13y^2 + 18 = 0$

27. $x^2 - 2\sqrt{3}xy + 3y^2 - 16\sqrt{3}x - 16y = 0$

28. $x^2 + 2\sqrt{3}xy + 3y^2 - 8\sqrt{3}x + 8y = 0$

Learn: Use the Discriminant to Identify Conic Sections Containing a *Bxy* Term

29. The equation $3x^2 + 6xy + 3y^2 - 9x + 12y - 4 = 0$ has coefficients $A = 3$ and $C = 3$. Although $A = C$, the graph of the equation is not a circle, but rather a parabola. Why?

30. The equation $9x^2 + 10xy + y^2 - 3x + 2y - 4 = 0$ has coefficients $A = 9$ and $C = 1$. Although $AC > 0$ (A and C have the same sign), the graph of the equation is not an ellipse, but rather a hyperbola. Why?

For Exercises 31–34, the given equation represents a conic section (nondegenerative case). Identify the type of conic section. (See Examples 5–6, pp. 1116–1117)

31. a. $4x^2 - 12xy + 6y^2 + 2x - 3y - 8 = 0$

b. $4x^2 - 12xy + 9y^2 + 2x - 3y - 8 = 0$

32. a. $-3x^2 + xy - 4y^2 + 2x - 4y - 4 = 0$

b. $-3x^2 + 10xy - 4y^2 + 2x - 4y - 4 = 0$

33. a. $x^2 + 6xy + 4x = 3y^2 - 2y + 10$

b. $6y^2 - 3y = 8 - 2x - 4x^2$

34. a. $x^2 - 3xy + 12x = -6y^2 - 12y + 1$

b. $-3x^2 + 12x = 4y + 4$

For Exercises 35–42, the equation represents a conic section (nondegenerative case).

a. Use the discriminant to determine the type of conic section represented. (**See Example 5**)

b. Use a rotation of axes to eliminate the xy term to write the equation in the form $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$.

c. Write the equation in standard form and sketch the graph.

35. $3x^2 + 26\sqrt{3}xy - 23y^2 - 144 = 0$

36. $13x^2 + 6\sqrt{3}xy + 7y^2 - 64 = 0$

37. $16x^2 - 24xy + 9y^2 + 140x - 230y + 625 = 0$

38. $4x^2 - 6xy - 4y^2 + \sqrt{10}x + 3\sqrt{10}y - 25 = 0$

39. $40x^2 + 20xy + 25y^2 + 24\sqrt{5}x - 48\sqrt{5}y = 0$

40. $x^2 + 6xy + 9y^2 - 9\sqrt{10}x - 17\sqrt{10}y + 90 = 0$

41. $5x^2 + 6xy + 5y^2 - 12\sqrt{2}x - 4\sqrt{2}y - 16 = 0$

42. $7x^2 + 50xy + 7y^2 + 78\sqrt{2}x + 114\sqrt{2}y - 18 = 0$

For Exercises 43–48, the equation represents a conic section (nondegenerative case).

a. Identify the type of conic section. (**See Example 6, p. 1117**)

b. Graph the equation on a graphing utility.

43. $4x^2 - 4xy + 5y^2 - 20 = 0$

44. $6x^2 + 4\sqrt{3}xy + 2y^2 - 18x + 18\sqrt{3}y - 72 = 0$

45. $2x^2 - 6xy + 3y^2 - 4x + 12y - 9 = 0$

46. $5x^2 - 3xy + 2y^2 - 6 = 0$

47. $4x^2 + 8xy + 4y^2 - 2x - 5y - 2 = 0$

48. $4x^2 + 8\sqrt{3}xy + 3y^2 + 2x - 12y - 6 = 0$

Mixed Exercises

49. Given $y = \frac{\sqrt{3}}{3}x$

a. What angle does the line $y = \frac{\sqrt{3}}{3}x$ make with the positive x-axis?

b. Suppose that the x- and y-axes are rotated through an angle of 30° to form the x'- and y'-axes. Write the equation $y = \frac{\sqrt{3}}{3}x$ in terms of x' and y'.

c. Suppose that the x- and y-axes are rotated through an angle of 60° to form the x'- and y'-axes. Write the equation $y = \frac{\sqrt{3}}{3}x$ in terms of x' and y'.

50. Given $y = \frac{\sqrt{2}}{2}x$

a. What angle does the line $y = \frac{\sqrt{2}}{2}x$ make with the positive x-axis?

b. Suppose that the x- and y-axes are rotated through an angle of 45° to form the x'- and y'-axes. Write the equation $y = \frac{\sqrt{2}}{2}x$ in terms of x' and y'.

c. Suppose that the x- and y-axes are rotated through an angle of 90° to form the x'- and y'-axes. Write the equation $y = \frac{\sqrt{2}}{2}x$ in terms of x' and y'.

51. Solve the system for x' and y'. (*Hint*: Multiply the first equation by $\cos\theta$ and the second equation by $\sin\theta$ to eliminate the y' term.)

$x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

52. Consider the parabola with axis of symmetry $y = x$ and passing through the points (1, 0), (0, 1), and (1, 1) in the xy-plane. Find an equation of the parabola

a. In the $x'y'$-plane, where the x'-axis is aligned with the line $y = x$.

b. In the xy-plane.

53. Discuss the graph of $x^2 - 2xy + y^2 - 1 = 0$.

54. Discuss the graph of $x^2 - 8xy + y^2 = 0$.

Write About It

55. Given $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, what effect does the term Bxy have on the graph of the equation?

56. Given $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, explain how to find the sine and cosine of the angle θ by which the conic section is rotated.

57. Suppose that the x'- and y'-axes are formed by a 30°-counterclockwise rotation from the x- and y-axes. If a point $P(2, 1)$ is located in the xy-plane, in which quadrant in the $x'y'$-plane will P be located? Why?

58. Given an equation in x and y, many graphing utilities require that the y variable be isolated. Explain how to isolate the y variable in the equation $2x^2 + 3xy + 5y^2 - 4x + 2y - 6 = 0$.

Expanding Your Skills

59. Use the relationships between A, B, and C and A', B', and C' on page 1115 to show that $A + C$ is invariant under rotation. That is, show that $A + C = A' + C'$.

60. Show that the distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is invariant under rotation. That is, show that the distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ equals the distance between $P_1(x'_1, y'_1)$ and $P_2(x'_2, y'_2)$.

61. For a positive real number r, explain why the equation $x^2 + y^2 = r^2$ is or is not invariant under rotation.

62. Show that $x^2 + y^2 = r^2$ is invariant under rotation.

63. Recall from Lesson 9.3 that a point $P(x, y)$ can be represented by the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. By applying matrix addition, subtraction, or multiplication, we can translate the point or rotate the point to a new location. Given $P(x, y)$, the product $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ gives the coordinates of the point rotated by an angle θ about the origin.

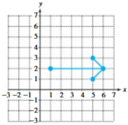
a. Write a product of matrices that rotates the point (6, 2) counterclockwise 60° about the origin.

b. Compute the product in part (a).

c. Graph the point (6, 2) and the point found in part (b) to illustrate the rotation. Round the coordinates of the rotated point to 1 decimal place.

64. The set of points $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ can be represented as the matrix $\begin{bmatrix} x_1 & x_2 & \cdots & y_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$.

a. Write a matrix to represent the points defining the given figure.



b. Suppose a computer programmer wants to rotate the matrix 90° counterclockwise. Write a product of matrices that will perform this rotation and compute the result.

c. Plot the original points and arrow along with the rotated points and arrow.

Go online for more practice problems.

Lesson 10-5

Polar Equations of Conics

What Will You Learn?

After completing this lesson, you should be able to:

- Identify and Graph Conics in Polar Form
- Rotate Conics in Polar Form

Learn: Identify and Graph Conics in Polar Form

In Lesson 10.3, we defined a parabola as the set of points in a plane equidistant from a fixed point (focus) and a fixed line (directrix). However, the ellipse and hyperbola were defined in terms of two points (their foci). We now give a unified approach in which all three categories of conics are defined in terms of a fixed point and fixed line.

Alternative Definitions of Conics

Let *l* be a fixed line (**directrix**) in a plane and let *F* be a fixed point (**focus**) in the plane, not on the line. A **conic** is the set of points *P* in the plane such that the ratio of the distance between *P* and *F* to the distance from *P* to *l* is a constant positive real number *e* (the **eccentricity**). That is,

$$\frac{\overbrace{d(P,F)}^{\text{Distance between } P \text{ and } F}}{\underbrace{d(P,l)}_{\text{Distance between } P \text{ and } l}} = e \tag{1}$$

Furthermore,

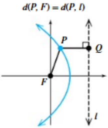
***1.** If $e = 1$, the conic is a parabola.

2. If $e < 1$, the conic is an ellipse.

3. If $e > 1$, the conic is a hyperbola.

*We show that these statements are true in online appendix Lesson A.2.

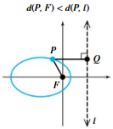
A polar coordinate system with the pole placed at the focus F is well suited to illustrate these alternative definitions of conics. For example, in Figures 10-37 through 10-39 we show a parabola, ellipse, and hyperbola with a vertical directrix. The point Q is the point on line l closest to P (that is, line segment $\overline{QP}$ is the per-pendicular distance from P to l).



Parabola $e = 1$

$e = \frac{d(P, F)}{d(P, l)} = 1$

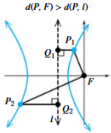
Figure 10-37



Ellipse $0 < e < 1$

$e = \frac{d(P, F)}{d(P, l)} < 1$

Figure 10-38



Hyperbola $e > 1$

$e = \frac{d(P, F)}{d(P, l)} > 1$

Figure 10-39

To develop polar equations of conics for the case of a vertical directrix, begin by placing the origin (pole) at the focus F. Then place the directrix l parallel to the y-axis $\left(\text{line } \theta = \frac{\pi}{2}\right)$ at a distance of $d > 0$ units to the right of the y-axis (Figure 10-40). Let Q represent the point on l closest to P.

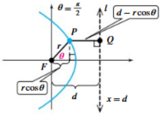


Figure 10-40

Insights

The graph shown in Figure 10-40 is a parabola. However, the same orientation of the directrix relative to points F, P, and Q holds true for the graphs of an ellipse and hyperbola.

The directrix is perpendicular to the polar axis and is represented in rectangular coordinates as $x = d$. Suppose that point P has rectangular coordinates (x, y) and polar coordinates (r, θ). Then from Figure 10-40, $d(P, F) = r$ and $d(P, Q) = d - r\cos\theta$.

The condition $\frac{d(P, F)}{d(P, Q)} = e$ can be written as

$d(P, F) = e \cdot d(P, Q)$ or as $r = e(d - r\cos\theta)$.

Solving for r gives $r = \frac{ed}{1 + e\cos\theta}$.

- If the directrix is a vertical line $d > 0$ units to the *left* of the pole, then an equation of the directrix is $x = -d$, and the equation of the conic is $r = \frac{ed}{1 - e\cos\theta}$.

- If the directrix is horizontal (parallel to the polar axis) $d > 0$ units above or below the pole, then $y = d$ and $y = -d$ are equations of the directrix, and the equation of the conic is

$$r = \frac{ed}{1 + e\sin\theta} \quad \text{or} \quad r = \frac{ed}{1 - e\sin\theta}.$$

Polar Equations of Conics

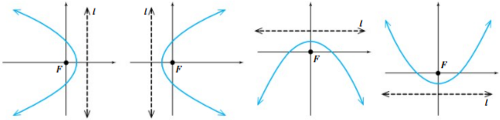
A polar equation of the form

$$(2)\ r = \frac{ed}{1 \pm e\cos\theta} \qquad \text{or} \qquad (3)\ r = \frac{ed}{1 \pm e\sin\theta}$$

is a conic with focus at the pole and eccentricity $e > 0$. The value $|d|$ is the distance between the focus at the pole and the directrix. The conic is

1. A parabola if $e = 1$ (axis of symmetry perpendicular to the directrix).
2. An ellipse if $0 < e < 1$ (major axis perpendicular to the directrix).
3. A hyperbola if $e > 1$ (transverse axis perpendicular to the directrix).

Given a polar equation of a conic, the location of the directrix and the orientation of the curve for $d > 0$ are summarized in Figure 10-41. If $d < 0$, the curve is reflected across the pole.



$r = \frac{ed}{1 + e\cos\theta}$ $r = \frac{ed}{1 - e\cos\theta}$ $r = \frac{ed}{1 + e\sin\theta}$ $r = \frac{ed}{1 - e\sin\theta}$

Figure 10-41 **Location of directrix and orientation of conic ($d > 0$)**

Insights

The graphs shown in Figure 10-41 are parabolas. However, the same orientation of the conic holds in each case for an ellipse and a hyperbola.

From the graphs of the equations in Figure 10-41, we make the following observations.

- If the equation contains $\cos\theta$, the directrix is vertical. If the equation contains $\sin\theta$, the directrix is horizontal.
- The axis of symmetry (parabola), major axis (ellipse), or transverse axis (hyperbola) is always perpendicular to the directrix.
- If the sign between terms in the denominator is positive, then the directrix is to the right of or above the pole.

- If the sign between terms in the denominator is negative, then the directrix is to the left of or below the pole.

To graph a conic expressed in polar form, consider using the following guidelines.

Guidelines to Graph a Conic in Polar Form

1. Write the polar equation in the form $r = \frac{ed}{1 \pm e\cos\theta}$ or $r = \frac{ed}{1 \pm e\sin\theta}$.
2. Determine the value of the eccentricity e and identify the type of conic.
3. Determine the orientation of the conic.
4. Plot points at $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$.
5. Identify key characteristics of the conic as desired (center, vertices, length of major/minor/transverse/conjugate axes, directrix).

Example 1: Graphing a Parabola from Its Polar Form

Graph the equation. $r = \frac{4}{2 - 2\sin\theta}$

Solution:

$$r = \frac{4}{2 - 2\sin\theta} = \frac{\frac{1}{2} \cdot (4)}{\frac{1}{2} \cdot (2 - 2\sin\theta)} = \frac{2}{1 - \sin\theta}$$

We want 1 here.

To fit the model of equation (3), $r = \frac{ed}{1 - e\sin\theta}$, we need a 1 as the constant term in the denominator. Multiply numerator and denominator by $\frac{1}{2}$.

$$r = \frac{2}{1 - 1\sin\theta}$$

$ed = 2$

$e = 1$

The equation is in the form $r = \frac{ed}{1 - e\sin\theta}$ with $e = 1$, and $ed = 2$.

Substitute $e = 1$ into the equation $ed = 2$ to get

$(1)d = 2$, or $d = 2$.

- Since $e = 1$, the graph is a parabola.
- The equation contains $\sin\theta$, which means that the directrix is horizontal.
- The subtraction sign in the denominator further indicates that the directrix is below the pole and has an equation in rectangular coordinates of $y = -d$ or $y = -2$.
- The axis of symmetry is perpendicular to the directrix, so the axis of symmetry is vertical and the parabola opens upward toward the focus.
- Evaluating $r = \frac{4}{2 - 2\sin\theta}$ for $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$ gives the ordered pairs $(2, 0)$, $(2, \pi)$, and $\left(1, \frac{3\pi}{2}\right)$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
r	2	Undefined	2	1

A sketch of the equation is given in Figure 10-42. The focus is at the pole (0, 0). The vertex is equidistant from the focus and the directrix, which implies that the vertex V is $\left(1, \frac{3\pi}{2}\right)$ or equivalently in rectangular coordinates $(0, -1)$.

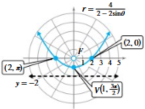


Figure 10-42

Apply the Skills

1. Graph the equation. $r = \frac{8}{2 - 2\cos\theta}$

We can confirm the results of Example 1 by converting the equation $r = \frac{4}{2 - 2\sin\theta}$ to rectangular coordinates.

$r = \frac{4}{2 - 2\sin\theta}$	
$2r - 2r\sin\theta = 4$	Multiply both sides by $2 - 2\sin\theta$.
$2r = 2r\sin\theta + 4$	Add $2r\sin\theta$ to both sides.
$r = r\sin\theta + 2$	Divide by 2.
$r^2 = (r\sin\theta + 2)^2$	Square both sides. The purpose of squaring both sides is to use the relationship $r^2 = x^2 + y^2$.
$r^2 = (r\sin\theta)^2 + 4r\sin\theta + 4$	Substitute $x^2 + y^2$ for r^2 and y for $r\sin\theta$.
$x^2 + y^2 = y^2 + 4y + 4$	
$x^2 = 4y + 4$	
$x^2 = 4(y + 1)$	Write the equation in the standard form of a parabola, $(x - h)^2 = 4p(y - k)$.

The equation $x^2 = 4(y + 1)$ is a parabola opening upward with vertex $(0, -1)$. To find the focal length $|p|$ we have $4|p| = 4$, which implies that $|p| = 1$. So, the focus is (0, 0), and the directrix is the horizontal line 1 unit below the vertex, $y = -2$. The focal diameter (length of the latus rectum) is $4|p| = 4(1) = 4$, indicating that the points $(-2, 0)$ and $(2, 0)$ are on the parabola. The graph of the equation in rectangular coordinates is shown in Figure 10-43.

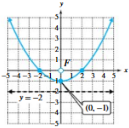


Figure 10-43

Example 2: Graphing a Hyperbola from Its Polar Form

Graph the equation. $r = \frac{12}{3 - 6\cos\theta}$

Solution:

$$r = \frac{12}{\underset{\text{We want 1 here.}}{3} - 6\cos\theta} = \frac{\frac{1}{3}\cdot(12)}{\frac{1}{3}\cdot(3 - 6\cos\theta)} = \frac{4}{1 - 2\cos\theta}$$

To fit the model of equation (2), $r = \frac{ed}{1 - e\cos\theta}$, we need a 1 as the constant term in the denominator. Multiply numerator and denominator by $\frac{1}{3}$.

$$r = \frac{\overset{ed = 4}{4}}{1 - \underset{e = 2}{2}\cos\theta}$$

The equation is in the form $r = \frac{ed}{1 - e\cos\theta}$ with $e = 2$ and $ed = 4$.

Since $ed = (2)d = 4$, we have $d = 2$.

- Since $e = 2 > 1$, the graph is a hyperbola.
- The equation contains $\cos\theta$, which means that the directrix is vertical.
- The subtraction sign in the denominator further indicates that the directrix is to the left of the pole.
- The transverse axis is perpendicular to the directrix, indicating that the transverse axis is horizontal and the branches of the hyperbola open to the left and right.
- Evaluating $r = \frac{12}{3 - 6\cos\theta}$ for $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$ gives the ordered pairs $(-4, 0)$, $\left(4, \frac{\pi}{2}\right)$, $\left(\frac{4}{3}, \pi\right)$, and $\left(4, \frac{3\pi}{2}\right)$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
r	-4	4	$\frac{4}{3}$	4

- Since the transverse axis must pass through the focus at the pole, the transverse axis is the polar axis itself (the line $\theta = 0$).
- The vertices V_1 and V_2 are on the transverse axis, and therefore, are the points on the hyperbola for which $\theta = 0$ or $\theta = \pi$. The vertices are $(-4, 0)$ and $\left(\frac{4}{3}, \pi\right)$.
- The center of the hyperbola is halfway between the vertices. Since the vertex $(-4, 0)$ is also represented by $(4, \pi)$, the center C is halfway between $(4, \pi)$ and $\left(\frac{4}{3}, \pi\right)$, which is $\left(\frac{8}{3}, \pi\right)$.

A sketch of the hyperbola is given in Figure 10-44.

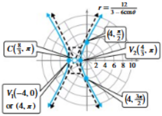


Figure 10-44

The distance between the vertices $(4, \pi)$ and $\left(\frac{4}{3}, \pi\right)$ is the length of the transverse axis. So, $2a = 4 - \frac{4}{3} = \frac{8}{3}$, which means that $a = \frac{4}{3}$.

Since $e = \frac{c}{a}$, then $c = ae = \frac{4}{3}\cdot 2 = \frac{8}{3}$.

For a hyperbola, $c^2 = a^2 + b^2$, so $b = \sqrt{c^2 - a^2} = \sqrt{\left(\frac{8}{3}\right)^2 - \left(\frac{4}{3}\right)^2} = \frac{4\sqrt{3}}{3} \approx 2.3$.

With the values of a and b known, we can sketch the representative rectangle through which the asymptotes of the graph are drawn (Figure 10-44).

Insights

In Example 2, $r = \frac{4}{1 - 2\cos\theta}$ is a hyperbola with vertical directrix to the left of the pole. We know that $e = 2$ and $ed = 4$. So, $ed = (2)d = 4$ or $d = 2$.

An equation of the directrix in rectangular coordinates is $x = -d$, which is $x = -2$.

Apply the Skills

2. Graph the equation. $r = \frac{18}{3 + 6\sin\theta}$

The polar equation of the hyperbola from Example 2 can be converted to rectangular coordinates to verify the results of Example 2 (see Exercise 34).

Example 3: Graphing an Ellipse from Its Polar Form

Graph the equation. $r = \frac{9}{5 + 4\sin\theta}$

Solution:

$$r = \frac{9}{5 + 4\sin\theta} = \frac{\frac{1}{5}\cdot(9)}{\frac{1}{5}\cdot(5 + 4\sin\theta)} = \frac{\frac{9}{5}}{1 + \frac{4}{5}\sin\theta}$$

We want 1 here.

To fit the model of equation (3), $r = \frac{ed}{1 + e\sin\theta}$, we need a 1 as the constant term in the denominator. Multiply numerator and denominator by $\frac{1}{5}$.

$$r = \frac{\frac{9}{5}}{1 + \frac{4}{5}\sin\theta}$$

$ed = \frac{9}{5}$

$e = \frac{4}{5}$

The equation is in the form $r = \frac{ed}{1 + e\sin\theta}$ with $e = \frac{4}{5}$ and $ed = \frac{9}{5}$.

- Since $e = \frac{4}{5} < 1$, the graph is an ellipse.
- The equation contains $\sin\theta$, which means that the directrix is horizontal.
- The addition sign in the denominator further indicates that the directrix is above the pole.
- The major axis is perpendicular to the directrix, and therefore must be vertical.
- Evaluating $r = \frac{9}{5 + 4\sin\theta}$ for $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$ gives the ordered pairs $\left(\frac{9}{5}, 0\right)$, $\left(1, \frac{\pi}{2}\right)$, and $\left(\frac{9}{5}, \pi\right)$, $\left(9, \frac{3\pi}{2}\right)$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
r	$\frac{9}{5}$	1	$\frac{9}{5}$	9

- Since the major axis is vertical and must pass through the focus at the pole, the major axis is the line $\theta = \frac{\pi}{2}$ (y-axis).

- The vertices V_1 and V_2 are on the major axis, and so are the points on the ellipse for which $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. The vertices are $\left(1, \frac{\pi}{2}\right)$ and $\left(9, \frac{3\pi}{2}\right)$.
- The center of the ellipse is halfway between the vertices. Since the vertex $\left(9, \frac{3\pi}{2}\right)$ is also represented by $\left(-9, \frac{\pi}{2}\right)$, the center C is halfway between $\left(1, \frac{\pi}{2}\right)$ and $\left(-9, \frac{\pi}{2}\right)$, which is the point $\left(-4, \frac{\pi}{2}\right)$.

A sketch of the ellipse is given in Figure 10-45.

The distance between the vertices $\left(1, \frac{\pi}{2}\right)$ and $\left(-9, \frac{\pi}{2}\right)$ is the length of the major axis. So, $2a = 1 - (-9) = 10$, which means that $a = 5$.

Since $e = \frac{c}{a}$, then $c = ae = 5 \cdot \frac{4}{5} = 4$.

For an ellipse, $c^2 = a^2 - b^2$, so $b = \sqrt{a^2 - c^2} = \sqrt{5^2 - 4^2} = 3$.

Therefore, the length of the minor axis is $2b = 2(3) = 6$.

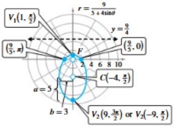

$r = \frac{9}{5 + 4\sin\theta}$
$V_1\left(1, \frac{\pi}{2}\right)$
$y = \frac{9}{4}$
$\left(\frac{9}{5}, \pi\right)$
$\left(\frac{9}{5}, 0\right)$
F
2
4
6
8
10
$C\left(-4, \frac{\pi}{2}\right)$
$a = 5$
$b = 3$
$V_2\left(9, \frac{3\pi}{2}\right)$ or $V_2\left(-9, \frac{\pi}{2}\right)$

Figure 10-45

Insights

From Example 3, $r = \dfrac{\frac{9}{5}}{1 + \frac{4}{5}\sin\theta}$ is an ellipse with horizontal directrix above the pole.

We know that $e = \frac{4}{5}$ and $ed = \frac{9}{5}$. So, $\frac{4}{5}d = \frac{9}{5}$ or $d = \frac{9}{4}$.

An equation of the directrix in rectangular coordinates is $y = d$, which is $y = \frac{9}{4}$.

Apply the Skills

3. Graph the equation. $r = \frac{16}{5 - 3\cos\theta}$

The polar equation of the ellipse from Example 3 can be converted to rectangular coordinates to verify the results of Example 3 (see Exercise 33).

Learn: Rotate Conics in Polar Form

The rotation of a conic about the origin is much more convenient when the conic is expressed in polar form versus rectangular form. Consider the graph of $r = f(\theta)$. Replacing θ by $(\theta - \alpha)$ results in

- A counterclockwise rotation (rotation through a positive angle) if $\alpha > 0$.
- A clockwise rotation (rotation through a negative angle) if $\alpha < 0$.

In Example 4, we rotate the ellipse from Example 3.

Example 4: Rotating a Conic in Polar Form

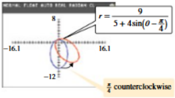
Write an equation of the ellipse $r = \frac{9}{5 + 4\sin\theta}$ rotated about the origin through the given angle. Then graph the result on a graphing utility.

a. Counterclockwise $\frac{\pi}{4}$

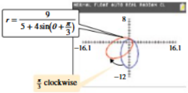
b. Clockwise $\frac{\pi}{3}$

Solution:

a. Substitute $\theta - \frac{\pi}{4}$ for θ.


NORMAL FLOAT AUTO REAL RADIAN CL
$r = \frac{9}{5 + 4\sin\left(\theta - \frac{\pi}{4}\right)}$
8
−16.1
16.1
−12
$\frac{\pi}{4}$ counterclockwise

b. Substitute $\theta + \frac{\pi}{3}$ for θ.


NORMAL FLOAT AUTO REAL RADIAN CL
$r = \frac{9}{5 + 4\sin\left(\theta + \frac{\pi}{3}\right)}$
8
−16.1
16.1
−12
$\frac{\pi}{3}$ clockwise

Apply the Skills

4. Rotate the ellipse $r = \frac{16}{5 - 3\cos\theta}$ and graph the result on a graphing utility.

a. Counterclockwise $\frac{2\pi}{3}$

b. Clockwise $\frac{\pi}{4}$

Insights

Notice the similarity in the process to rotate a polar graph about the origin to the process to shift a graph horizontally or vertically in a rectangular coordinate system.

For example, consider an equation of an ellipse in rectangular form,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

If $h > 0$, the ellipse is shifted horizontally in the positive x direction.

If $h < 0$, the ellipse is shifted horizontally in the negative x direction.

Practice 10-5

Practice Exercises

Prerequisite Review

R.1. Given $5x^2 + 11y^2 + 40x - 132y + 421 = 0$,

- **a.** Write the equation of the ellipse in standard form.
- **b.** Identify the center, vertices, endpoints of the minor axis, and foci.

R.2. Given $11x^2 - 5y^2 + 22x + 20y - 64 = 0$,

- **a.** Write the equation of the hyperbola in standard form.
- **b.** Identify the center, vertices, and foci.

R.3. Given $y^2 - 2y + 4x - 15 = 0$,

- **a.** Write the equation of the parabola in standard form.
- **b.** Identify the vertex, focus, and directrix.

For Exercises R.4–R.5, convert the polar equation to rectangular form and identify the type of curve represented.

R.4. $r = -6\cos\theta$

R.5. $r = \frac{4}{\sin\theta}$

Concept Connections

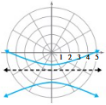
1. Consider a conic defined by the set of points P satisfying the ratio $\frac{d(P, F)}{d(P, l)} = e$, where F is the focus at the pole and l is the directrix. The conic is a parabola if ____________, an ellipse if ____________, and a hyperbola if ____________.
2. For a conic defined by $r = \frac{ed}{1 - e\cos\theta}$, the directrix is (horizontal/vertical) with equation ____________.
3. For a conic defined by $r = \frac{ed}{1 + e\sin\theta}$, the directrix is (horizontal/vertical) with equation ____________.
4. Given an equation of a conic in polar form $r = f(\theta)$, substituting $(\theta - \alpha)$ results in a (clockwise/counterclockwise) rotation about the origin if $\alpha > 0$. The rotation is (clockwise/counterclockwise) if $\alpha < 0$.

Lesson Identify and Graph Conics in Polar Form

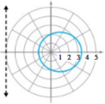
For Exercises 5–10,

- **a.** Determine the value of e and d.
- **b.** Identify the type of conic represented by the equation.
- **c.** Write an equation for the directrix in rectangular coordinates.
- **d.** Match the equation with its graph. Choose from A–F.

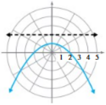
A.



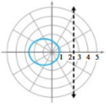
B.



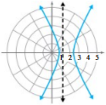
C.



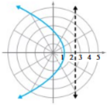
D.



E.



F.



5. $r=\frac{10}{5+5\sin\theta}$

6. $r=\frac{10}{2-5\sin\theta}$

7. $r=\frac{10}{5-2\cos\theta}$

8. $r=\frac{5}{2+2\cos\theta}$

9. $r=\frac{5}{2+4\cos\theta}$

10. $r=\frac{5}{4+2\cos\theta}$

For Exercises 11–22, graph the conic. (See Examples 1–3, pp. 1126–1129)

11. $r=\frac{5}{1+\cos\theta}$

12. $r=\frac{3}{1+\sin\theta}$

13. $r=\frac{5}{2+3\sin\theta}$

14. $r=\frac{8}{3-5\cos\theta}$

15. $r=\frac{6}{2+\sin\theta}$

16. $r=\frac{9}{3+\cos\theta}$

17. $r=\frac{-6}{-2+2\sin\theta}$

18. $r=\frac{-15}{5\cos\theta-5}$

19. $r=\frac{10\sec\theta}{5\sec\theta-3}$

20. $r=\frac{10\csc\theta}{4\csc\theta-3}$

21. $5r\sin\theta+3r=16$

22. $4r-5r\cos\theta=9$

For Exercises 23–24, refer to the polar equation of the hyperbola.

a. Determine the center in rectangular coordinates.

b. Determine equations of the asymptotes in rectangular coordinates. (**See Example 2, p. 1128**)

23. $r=\frac{5}{2+3\sin\theta}$ (Exercise 13)

24. $r=\frac{8}{3-5\cos\theta}$ (Exercise 14)

For Exercises 25–26, refer to the polar equation of the ellipse.

a. Determine the center in rectangular coordinates.

b. Determine the length of the major axis.

c. Determine the length of the minor axis.

25. $r=\frac{10\sec\theta}{5\sec\theta-3}$ (Exercise 19)

26. $r=\frac{10\csc\theta}{4\csc\theta-3}$ (Exercise 20)

Learn: Rotate Conics in Polar Form

For Exercises 27–30, write an equation of the conic rotated about the origin through the given angle. Then graph the result on a graphing utility. (See Example 4, p. 1131)

27. $r=\frac{1}{1-\sin\theta}$; Clockwise $\frac{\pi}{3}$

28. $r=\frac{8}{5+6\cos\theta}$; Clockwise $\frac{\pi}{4}$

29. $r=\frac{12}{4-3\cos\theta}$; Counterclockwise $\frac{7\pi}{6}$

30. $r=\frac{5}{2+2\sin\theta}$; Counterclockwise $\frac{4\pi}{3}$

31. a. Write a polar equation to represent the vertical line defined by $x=2$ in rectangular form.

b. Use the polar equation from part (a) to write a polar equation of the line rotated about the origin 20° clockwise.

c. Graph the equations from parts (a) and (b).

32. a. Write a polar equation to represent the horizontal line defined by $y=4$ in rectangular form.

b. Use the polar equation from part (a) to write a polar equation of the line rotated about the origin 15° counterclockwise.

c. Graph the equations from parts (a) and (b).

Mixed Exercises

For Exercises 33–36, convert the equation to rectangular coordinates. Compare the result to the indicated example or exercise.

33. $r=\frac{9}{5+4\sin\theta}$; Example 3

34. $r=\frac{12}{3-6\cos\theta}$; Example 2

35. $r=\frac{5}{1+\cos\theta}$; Exercise 11

36. $r=\frac{6}{2+\sin\theta}$; Exercise 15

For Exercises 37–40, write a polar equation of a conic subject to the given conditions.

37. $e=1$, Directrix $y=2$

38. $e=\frac{5}{4}$, Directrix $x=-2$

39. $e = \frac{3}{5}$, Directrix is perpendicular to the polar axis and 10 units to the right of the pole.

40. $e = 1$, Directrix is parallel to the polar axis and 3 units below the pole.

Write About It

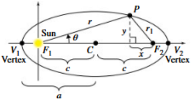
41. The equations $r_1 = \frac{6}{2 - 3\cos\theta}$ and $r_2 = \frac{6}{3 - 2\cos\theta}$ look similar. Discuss the difference in their graphs.

42. Discuss the difference between the graphs of $r_1 = \frac{8}{4 - \cos\theta}$ and $r_2 = \frac{8}{4 - \sin\theta}$.

Expanding Your Skills

43. Recall from Lesson 10.1 that the planets travel in elliptical orbits with the Sun at one focus. Assume that the Sun is located at the pole and the major axis of the ellipse is aligned with the polar axis as shown in the figure. We can represent the ellipse in polar form in terms of the eccentricity e and length of the semimajor axis a.

Let c represent the distance between the center C and either focus, F_1 and F_2. Let r and r_1 represent the distances between the foci and point P on the ellipse.



a. Show that $c = ea$.

b. Show that $r_1 = \sqrt{4e^2a^2 - 4ear\cos\theta + r^2}$.

c. From the definition of an ellipse, $r + r_1 = 2a$. Use this relationship to show that the ellipse is represented in polar form by $r = \frac{a(1 - e^2)}{1 - e\cos\theta}$.

44. Expressing a planet's elliptical orbit in polar form with the Sun at one focus is particularly convenient for finding the planet's distance at perihelion and aphelion (closest and farthest points between the planet and the Sun). Refer to the figure in Exercise 43. Write expressions in terms of a and e representing the distances R_p and R_a between the planet and Sun at perihelion and aphelion, respectively.

For Exercises 45–46, use the results of Exercises 43–44 to

a. Find a polar equation of the planet's orbit.

b. Find the distances R_p and R_a between the planet and Sun at perihelion and aphelion, respectively. Round to the nearest million kilometers.

45. Mercury: $a = 5.790 \times 10^7$ km, $e = 0.2056$

46. Mars: $a = 2.279 \times 10^8$ km, $e = 0.0935$

Technology Connections

47. Graph the equations $r_1 = \frac{2}{1 - 0.6\cos\theta}$ and $r_2 = \frac{-2}{1 - 0.6\cos\theta}$ on the same viewing window. How does the negative value of d affect the graph of the second equation?

48. Graph the equations $r = \frac{1}{1 - e\sin\theta}$ for $e = 0.2, 0.4, 0.6$, and 0.8 on the same viewing window. How does the value of e affect the ellipse?

49. Graph the equations $r = \frac{1}{1 - e\sin\theta}$ for $e = 0.8, 1$, and 1.2 on a viewing window with $-16.1 \le x \le 16.1$ and $-10 \le y \le 10$. Comment on the effect of e on the graph.

GO ONLINE **Go online for more practice problems.**

Lesson 10-6

Plane Curves and Parametric Equations

What Will You Learn?

After completing this lesson, you should be able to:

- Graph a Plane Curve by Plotting Points
- Eliminate the Parameter
- Write Parametric Equations to Represent a Curve
- Apply Parametric Equations

Learn: Graph a Plane Curve by Plotting Points

Suppose that a target in a video game starts at the lower left corner of the screen and moves across the screen along the path shown (Figure 10-46).

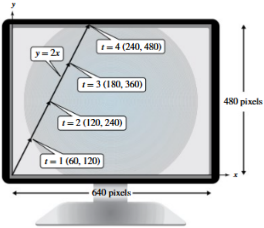


Figure 10-46

Placing the origin at the lower left corner of the screen, the path of the target is given by $y = 2x$. However, a player who wants to shoot the target must also know *when* the object is at a given point (x, y). That is, to adequately model the motion of the target, we must know its x- and y-coordinates at a given time t. The variable t is called a *parameter*. From the information given in Figure 10-46, we can express x and y (in pixels) in terms of $t \geq 0$ (in seconds) as follows:

$$x = 60t \text{ and } y = 120t$$

These equations are called parametric equations because x and y are defined in terms of a parameter. Furthermore, the collection of points $(x(t), y(t))$ defined by these equations is called a plane curve.

Plane Curves and Parametric Equations

Let f and g be functions of t over an interval I. Then the set of points defined by $(x, y) = (f(t), g(t))$ is called a **plane curve**. The equations

$$x = f(t) \text{ and } y = g(t)$$

are called **parametric equations** of the curve, and t is called the **parameter**.

Insights

In many applications of parametric equations, the parameter often represents the time t during which an object is in motion or an angle θ through which an object moves.

One method to graph a plane curve defined by parametric equations is to create a table of points. Choose arbitrary values of the parameter t and compute the corresponding values of x and y. Plot the points (x, y) in the order of increasing t and connect the points with a curve. Then use arrows to indicate the **orientation** of the curve. This is the direction of the path for increasing values of t.

Example 1: Graphing a Plane Curve

Sketch the plane curve defined by the parametric equations over the given interval.

$$x = t^2 - 2t \text{ and } y = t - 2 \text{ over the interval } -2 \le t < 3$$

Solution:

In Table 10-2, we have selected arbitrary values of t on the interval $[-2, 3)$ and have computed the corresponding values of x and y. After plotting the points, we draw a curve through the points and place arrows indicating the direction of increasing values of t (Figure 10-47).

Although the value $t = 3$ is not on the interval defining the curve, we evaluate x and y at $t = 3$ to determine the location of the open dot at the upper endpoint.

TABLE 10-2 Values of x and y for Selected Values of t

t	x	y
−2	8	−4
−1	3	−3
0	0	−2
1	−1	−1
2	0	0
3	3	1

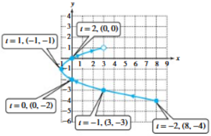


Figure 10-47

Apply the Skills

1. Sketch the plane curve defined by the parametric equations $x = |t| - 2$ and $y = \frac{t}{2}$ over the interval $-4 < t \leq 4$.

In Example 1, the interval for *t* was restricted to $[-2, 3)$. If no interval is given, we will assume that *t* can be any real number for which *x* and *y* are defined. In Example 1, if the restriction of *t* were lifted, the parametric equations would define an entire parabola rather than just the part shown.

Learn: Eliminate the Parameter

Sometimes we can convert parametric equations to an equation in rectangular coordinates by a process called eliminating the parameter. The purpose of eliminating the parameter is to make use of our familiarity with the graphs of many equations and functions expressed in rectangular coordinates.

One way to eliminate a parameter *t* is to solve for *t* or an expression involving *t* in one equation. Then substitute the result in the other equation. However, we must also note restrictions on the domain of the rectangular equation so that the resulting graph is consistent with the restrictions on *x* and *y* in the parametric equations. This is illustrated in Example 2.

Given parametric equations: → Solve for *t* or an expression involving *t* in one equation. → Substitute the result in the other equation. → Identify domain restrictions on the resulting rectangular equation.

Example 2: Eliminating a Parameter

Given $x = \sqrt{t}$ and $y = -t + 4$,

a. Eliminate the parameter and write an equation in rectangular coordinates.

b. Sketch the curve and indicate its orientation.

Solution:

First note that $x = \sqrt{t}$ has the implied restriction that $t \geq 0$. Furthermore, *x* is equated to the principal square root of *t*, which implies that $x \geq 0$. Given the equation $y = -t + 4$, for $t \geq 0$, we have the restriction to $y \leq 4$.

a. From the equation $x = \sqrt{t}$, we have $t = x^2$ for $x \geq 0$. Substituting $t = x^2$ into the equation $y = -t + 4$ gives

$$y = -x^2 + 4 \text{ for } x \geq 0$$

b. The equation $y = -x^2 + 4$ is a parabola opening downward with vertex at $(0, 4)$.

Since $x \geq 0$, we take only the right branch of the parabola.

To determine the orientation of the curve, substitute two values of *t* into the parametric equations. Plot the corresponding points and draw arrows indicating the direction of increasing values of *t*.

When $t = 0$, $x = \sqrt{0} = 0$,

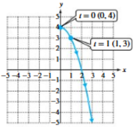
$y = -(0) + 4 = 4$.

The point is $(0, 4)$.

When $t = 1$, $x = \sqrt{1} = 1$,

$y = -(1) + 4 = 3$.

The point is $(1, 3)$.



The curve is oriented downward for increasing values of t.

Apply the Skills

2. Repeat Example 2 with the equations $x = \sqrt{t - 1}$ and $y = t - 1$.

Another method to eliminate a parameter is to combine the equations by using an identity such as $\sin^2\theta + \cos^2\theta = 1$.

Example 3: Eliminating a Parameter

Given $x = -2\sin\theta$ and $y = 2\cos\theta$,

- **a.** Eliminate the parameter and write an equation in rectangular coordinates.
- **b.** Sketch the curve and indicate its orientation.

Solution:

The sine and cosine functions are defined for all real numbers θ. Since $-1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$, the values of x and y are between -2 and 2.

a. Solving for θ in either equation would require an inverse trigonometric function. Instead, we make use of the fact that $\sin^2\theta + \cos^2\theta = 1$. Squaring both sides of the two equations and taking the sum $x^2 + y^2$ gives

$$\begin{aligned} x^2 + y^2 &= (-2\sin\theta)^2 + (2\cos\theta)^2 \\ &= 4\sin^2\theta + 4\cos^2\theta \\ &= 4(\sin^2\theta + \cos^2\theta) \\ &= 4(1) \\ &= 4 \end{aligned}$$

The curve represented in rectangular coordinates is $x^2 + y^2 = 4$.

b. The graph of $x^2 + y^2 = 4$ is a circle centered at the origin with radius 2.

To determine the orientation of the curve, substitute two values of θ into the parametric equations. Plot the corresponding points and draw arrows indicating the direction of increasing values of θ.

When $\theta = 0$, $x = -2\sin(0) = 0$,

$y = 2\cos(0) = 2$.

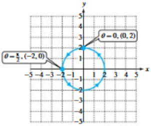
The point is (0, 2).

When $\theta = \frac{\pi}{2}$, $x = -2\sin\left(\frac{\pi}{2}\right) = -2$,

$y = 2\cos\left(\frac{\pi}{2}\right) = 0$.

The point is (−2, 0).

The orientation of the curve is counterclockwise.



Apply the Skills

3. Repeat Example 3 with the equations $x = 4\cos\theta$ and $y = -4\sin\theta$.

In Example 3, the implied domain of the parametric equations is $(-\infty, \infty)$. So, the circle is traced repeatedly for θ values on intervals of 2π. For example:

- For θ on the interval $[0, 2\pi]$, the curve starts at (0, 2) and is traced one time counterclockwise.
- For θ on the interval $[0, 4\pi]$, the curve starts at (0, 2) and is traced two times counterclockwise.
- For θ on the interval $\left[\frac{\pi}{2}, \frac{13\pi}{2}\right]$, the curve starts at (−2, 0) and is traced three times counterclockwise.

It is also important to note that the orientation of a curve defined by parametric equations may change over the interval defining the parameter. In Exercises 21–22, we show a situation where a curve is traversed back and forth in an oscillating manner.

Learn: Write Parametric Equations to Represent a Curve

An equation in x and y in rectangular coordinates defines a set of points (x, y). Suppose that a parameter t is introduced to represent time. Different parametric equations (that is, different parameterizations) can be used to traverse the curve at different speeds and in different directions. This is demonstrated in Example 4.

Example 4: Writing Parametric Equations to Represent an Equation in Rectangular Coordinates

Write parametric equations for the curve defined by $y = x + 2$ with the given definition for x.

a. $x = t$ **b.** $x = 3t$ **c.** $x = -\frac{t}{2}$

Solution:

a. Substituting $x = t$ into the equation $y = x + 2$ yields $y = t + 2$.

The parametric equations are $x = t$ and $y = t + 2$.

b. Substituting $x = 3t$ into the equation $y = x + 2$ yields $y = 3t + 2$.

The parametric equations are $x = 3t$ and $y = 3t + 2$.

c. Substituting $x = -\frac{t}{2}$ into the equation $y = x + 2$ yields $y = -\frac{t}{2} + 2$.

The parametric equations are $x = -\frac{t}{2}$ and $y = -\frac{t}{2} + 2$.

Apply the Skills

4. Write parametric equations for the curve defined by $y = 2x - 3$ with the given definition for x.

a. $x = t$ **b.** $x = -4t$ **c.** $x = \frac{t}{5}$

The graphs of the three parametric representations of the line $y = x + 2$ are shown in Figures 10-48 through 10-50.

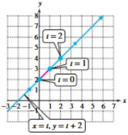


Figure 10-48

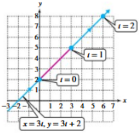


Figure 10-49

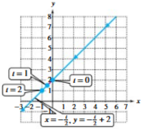


Figure 10-50

For a given interval of t, the different parameterizations traverse different parts of the curve. Notice that for the interval $0 \le t \le 1$, different portions of the line (shown in red) are traversed for each parameterization. Furthermore, if t represents time, then the curve in Figure 10-49 is traversed most quickly because over a 1-sec interval, a longer portion of the curve is represented. Also notice that the curve defined by the parameterization in Figure 10-50 is traversed in the opposite direction than the other two curves.

It is also important to note that a parameterization must be chosen carefully to represent the entire curve defined in rectangular coordinates. For example, the parameterization $x = \sqrt{t}$ and $y = \sqrt{t} + 2$ would not represent the entire line $y = x + 2$. This parameterization would only define the portion of the line from the point (0, 2) to the right.

Learn: Apply Parametric Equations

Parametric equations are particularly well suited for describing objects in motion because curves can be expressed using time as the parameter. Two common applications and the associated parametric equations are summarized as follows.

Modeling Motion Using Parametric Equations

Let $t \geq 0$ represent the time under which an object is in motion along a plane curve, and let (x_0, y_0) represent the initial position of the object at $t = 0$.

- **Uniform linear motion**

$$x = x_0 + at$$
$$y = y_0 + bt$$

The values of a and b are the horizontal and vertical components of velocity, respectively.

- **Projectile motion***

$$x = (v_0 \cos\theta)t + x_0$$
$$y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + y_0$$

where:

θ is the angle of launch relative to the horizontal.

v_0 is the initial speed (magnitude of the initial velocity).

g is the acceleration due to gravity.

($g = 32$ ft/sec² or $g = 9.8$ m/sec² on the surface of the Earth)

*Assuming negligible air resistance.

In Example 5, we investigate an application of motion along a straight path.

Example 5: An Application of Motion Along a Straight Path

An ant walks across a 72-in. by 36-in. picnic table along a straight path as shown in Figure 10-51. The origin of a rectangular coordinate system is placed at the lower left corner of the table.

a. If the ant walks from point $A(72, 24)$ to point $B(0, 12)$ in 24 sec, write parametric equations to represent the ant's path, as a function of the time t (in sec). Assume that the distances in the figure are in inches.

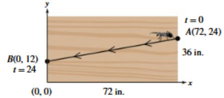


Figure 10-51

b. Determine the ant's location 8 sec after leaving point A.

Solution:

a. For a straight path we use the equations $x = x_0 + at$ and $y = y_0 + bt$.

The initial position (x_0, y_0) of the ant is (72, 24). So, $x_0 = 72$ and $y_0 = 24$.

The ant moves from $x = 72$ to $x = 0$ in 24 sec.

Therefore, the ant's horizontal component of velocity is

$$a = \frac{0 - 72}{24} = -3 \text{ in./sec}$$

Insights

The horizontal component of velocity is negative because the ant is moving from right to left.

The ant moves from $y = 24$ to $y = 12$ in 24 sec.

So, the vertical component of velocity is

$b = \frac{12 - 24}{24} = -\frac{1}{2}$ in./sec

Insights

The vertical component of velocity is negative because the ant is moving downward.

The ant's path is given by

$x = x_0 + at = 72 + (-3)t$ and $y = y_0 + bt = 24 + \left(-\frac{1}{2}\right)t$.

So, $x = 72 - 3t$ and $y = 24 - \frac{1}{2}t$.

b. The ant's location at $t = 8$ sec is given by

$x = 72 - 3(8) = 48$ and $y = 24 - \frac{1}{2}(8) = 20$.

The location is (48, 20).

Apply the Skills

5. Repeat Example 5 with an ant that begins at $A(72, 10)$ and walks to $B(20, 36)$ in 13 sec.

Insights

For an object moving under uniform linear motion from point (x_1, y_1) to point (x_2, y_2) in time Δt, the horizontal and vertical components of velocity are given by

$$a = \frac{x_2 - x_1}{\Delta t} \quad \text{and} \quad b = \frac{y_2 - y_1}{\Delta t}$$

In each case, notice that the pattern is

$$\frac{\begin{pmatrix}\text{final}\\\text{position}\end{pmatrix} - \begin{pmatrix}\text{initial}\\\text{position}\end{pmatrix}}{\Delta t}$$

In Example 6, we use parametric equations to model the path of a projectile.

Example 6: An Application of Projectile Motion

The shot put is a track-and-field event in the Olympics in which competitors throw an iron ball (the shot) as far as they can. The weight of the shot is 7.26 kg (16 lb) for men and 4 kg (8.8 lb) for women. Suppose that a male competitor throws the shot from an initial height of 2 m, at an angle of 38° to the horizontal, with an initial speed of 12 m/sec (Figure 10-52).

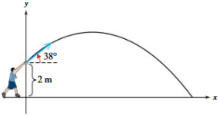


Figure 10-52

a. Write parametric equations to model the path of the shot as a function of the time *t* (in sec).

b. For how many seconds is the shot in the air? Round to the nearest hundredth of a second.

c. Determine the horizontal distance traveled by the shot. Round to the nearest hundredth of a foot.

d. When does the shot reach its maximum height? Find the exact value and round to the nearest hundredth of a second.

e. Determine the maximum height. Round to the nearest hundredth of a foot.

Solution:

a. To begin, we choose a coordinate system with the origin placed at ground level, directly below the point of release. So, $x_0 = 0$ m, $y_0 = 2$ m, $\theta = 38°$, $v_0 = 12$ m/sec, and $g = 9.8$ m/sec^2.

Using the model for projectile motion, we have

$$x = (v_0 \cos\theta)t + x_0 = (12\cos 38°)t + 0$$
$$y = -\tfrac{1}{2}gt^2 + (v_0 \sin\theta)t + y_0 = -\tfrac{1}{2}(9.8)t^2 + (12\sin 38°)t + 2.$$

Simplifying, we have $x = 12\cos 38°t$ and $y = -4.9t^2 + 12\sin 38°t + 2$, where x and y are in meters and t is in seconds.

b. The shot will hit the ground when $y = 0$.

$$-4.9t^2 + 12\sin 38°t + 2 = 0$$
$$t = \frac{-12\sin 38° \pm \sqrt{(12\sin 38°)^2 - 4(-4.9)(2)}}{2(-4.9)}$$

$t \approx 1.74$ sec and $t \approx$ ~~-0.23 sec~~.

Reject the negative solution.

The shot will hit the ground in approximately 1.74 sec.

c. The horizontal distance traveled by the shot is the x value at the time the shot hits the ground. Compute x at $t \approx 1.74$.

$$x \approx 12\cos 38°(1.74) \approx 16.45 \text{ m}$$

The shot travels approximately 16.45 m. This is the length of the throw and score for the competitor.

d. The path of the shot is parabolic. The maximum height is reached at the vertex. To find the time t at which the shot reaches its maximum height, use the vertex formula with $y = -4.9t^2 + 12 \sin 38°t + 2$.

$$t = \frac{-b}{2a} = \frac{-12\sin 38°}{2(-4.9)} \approx 0.75$$

The shot will reach its maximum height approximately 0.75 sec after release.

e. The maximum height is the y value at the vertex. Compute y at $t \approx 0.75$.

$$y = -4.9(0.75)^2 + 12\sin 38°(0.75) + 2 \approx 4.78$$

The maximum height is approximately 4.78 m.

Apply the Skills

6. Repeat Example 6 with a cannonball launched from a cannon at an initial height of 1 m, at an angle of 30° to the horizontal, with an initial speed of 200 m/sec. Round values to 1 decimal place.

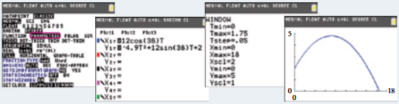
Point of Interest

In Lesson 5.3, we learned that launching a projectile at a 45° angle results in the maximum horizontal distance. However, experiments performed on world-class shot putters show that the initial speed that an athlete can generate throwing the shot decreases with increasing projection angle. Based on empirical data, the optimum projection angle for the shot put is approximately 35°. (*Source*: Journal of Sports Sciences)

TECHNOLOGY CONNECTIONS

Graphing Parametric Equations

We can graph the parametric equations from Example 6 on a graphing utility. First select the "PARAMETRIC" option in the MODE menu. Then enter the pair of parametric equations in the equation editor, set the window parameters, and select the GRAPH button.



```
NORMAL FLOAT AUTO a+bi DEGREE CL
MATHPRINT CLASSIC
NORMAL SCI ENG
FLOAT 0123456789
RADIAN DEGREE
FUNCTION PARAMETRIC POLAR SEQ
THICK DOT-THICK THIN DOT-THIN
SEQUENTIAL SIMUL
REAL a+bi re^(θi)
FULL HORIZONTAL GRAPH-TABLE
FRACTION TYPE: n/d Un/d
ANSWERS: AUTO DEC FRAC-APPROX
GO TO 2ND FORMAT GRAPH: NO YES
STAT DIAGNOSTICS: OFF ON
STAT WIZARDS: ON OFF
SET CLOCK
```

```
NORMAL FLOAT AUTO a+bi RADIAN CL
Plot1 Plot2 Plot3
\X1T=12cos(38)T
 Y1T=-4.9T²+12sin(38)T+2
\X2T=
 Y2T=
\X3T=
 Y3T=
\X4T=
 Y4T=
\X5T=
```

```
NORMAL FLOAT AUTO a+bi DEGREE CL
WINDOW
 Tmin=0
 Tmax=1.75
 Tstep=.05
 Xmin=0
 Xmax=18
 Xscl=2
 Ymin=0
 Ymax=5
 Yscl=1
```

Insights

The value of Tstep in the WINDOW menu affects the speed at which the calculator graphs the curve. Small values of Tstep result in more points being plotted and the curve graphed more slowly. This can be a helpful feature when you want to identify the orientation of a curve.

We conclude this section with a discussion of a special curve with interesting physical properties. Suppose that a circle rolls along a straight path in a plane. The curve traced out by a fixed point P on the circumference of the circle is called a **cycloid** (Figure 10-53).

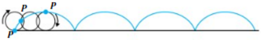


Figure 10-53

An equation of a cycloid in rectangular coordinates is quite cumbersome. Fortunately, the curve is defined nicely with parametric equations using the angle of rotation as the parameter.

Consider a circle with center C and radius r that rolls along the x-axis. Choose the origin to be one of the points where the fixed point $P(x, y)$ on the circumference of the circle touches the x-axis. In Figure 10-54, we show the position of the circle after it has rolled to the right through an angle of $0 < \theta < \frac{\pi}{2}$ radians.

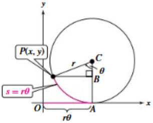


Figure 10-54

The distance $d(O, A)$ that the circle rolls along the x-axis equals the arc length s between A and P. Furthermore, recall that arc length is computed as $s = r\theta$ for θ in radians.

So, $d(O, A) = r\theta$. The x-coordinate of point P is given by

$$x = d(O, A) - d(P, B). \text{ So, } x = r\theta - r\sin\theta.$$

The y-coordinate of P is given by

$$y = d(C, A) - d(C, B). \text{ So, } y = r - r\cos\theta.$$

In summary, the cycloid is represented by $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$.

An inverted cycloid is generated by the same equations with a negative value of r (Figure 10-55). This curve has the interesting physical property that it is a solution to the *brachistochrone* problem posed in 1696 by Swiss mathematician Johann Bernoulli. Suppose that two points A and B lie in a vertical plane with A higher than B, but not directly above B (Figure 10-56).

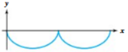


Figure 10-55 **Inverted cycloid**

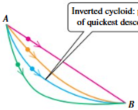


Figure 10-56

The brachistochrone problem asks what curve would minimize the time required for an object to slide from A to B with only the force of gravity to move it? The answer is a half-arch of an inverted cycloid. In Figure 10-56, we show several paths from A to B. Note that although a line is the shortest *distance* between points A and B, the shortest *time* to move from A to B under the influence of gravity occurs along the path of an inverted cycloid.

Insights

The word "brachistochrone" is derived from the Greek (*brachistos*) meaning "shortest" and (*chronos*) meaning "time."

The inverted cycloid also satisfies the *tautochrone* property (property of equal time). Let Q be the lowest point on one arch of an inverted cycloid. If several objects are placed at different points along the cycloid and allowed to slide down the curve under the influence of gravity, they will all reach point Q at the same time (Figure 10-57). This discovery intrigued seventeenth-century clockmakers.

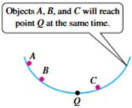

Objects A, B, and C will reach point Q at the same time.

Figure 10-57

Insights

The word "tautochrone" is derived from the Greek (*tauto*) meaning "same" and (*chronos*) meaning "time."

A clock made from a simple pendulum (a bob hanging from a thread swinging back and forth) has slight variations in the period with increasing amplitude of the swing. To address this problem, Dutch mathematician Christiaan Huygens constrained the swing of the pendulum between two metal plates in the shape of an inverted cycloid. As the pendulum swings, the thread wraps and unwraps around the cycloid, causing the bob to follow the path of a cycloid (Figure 10-58). Theoretically, in the absence of friction, this design would ensure equal times for each back-and-forth swing.

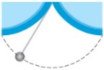


Figure 10-58

Practice 10-6

Practice Exercises

Prerequisite Review

R.1. Solve the equation. $5w^2 + 2w - 1 = 0$

R.2. Find the vertex by using the vertex formula. $T(x) = x^2 - 8x + 7$

Concept Connections

1. If f and g are continuous functions of t over an interval I, then the set of points $(f(t), g(t))$ is called a ____________ curve. The equations $x = f(t)$ and $y = g(t)$ are called ____________ equations, and t is called the ____________.
2. The parametric equations $x = 5\cos\theta$ and $y = -5\sin\theta$ define a circle. In which direction is the curve oriented? (clockwise or counterclockwise)
3. Identify the endpoints of the curve defined by $x = 3 - t$ and $y = 2t + 1$ on the interval $-2 \le t \le 1$.
4. Suppose that an object travels along a straight path in a rectangular coordinate system with units of distance in feet. If the object moves from the point (12, 18) to (36, 16) in 4 sec, the horizontal component of velocity is ____________ and the vertical component of velocity is ____________.

Learn: Graph a Plane Curve by Plotting Points

For Exercises 5–6,

a. Use the parametric equations to compute x and y for the given values of t.

b. Plot the points, sketch the curve, and indicate the orientation of the curve.

5. $x = 2t$ and $y = t + 1$ for $-3 \le t \le 1$

t	−3	−2	−1	0	1
x					
y					

6. $x = -t$ and $y = 3t - 1$ for $0 \le t \le 4$

t	0	1	2	3	4
x					
y					

For Exercises 7–10, sketch the plane curve by plotting points. Indicate the orientation of the curve. (See Example 1, p. 1136)

7. $x = -t^2$ and $y = t + 3$ for $-1 < t \le 3$

8. $x = t^2 - 4$ and $y = t - 1$ for $-2 \le t < 2$

9. $x = t - 4$ and $y = \sqrt{t}$

10. $x = -t - 3$ and $y = |t|$

Learn: Eliminate the Parameter

For Exercises 11–26,

a. Eliminate the parameter and write an equation in rectangular coordinates.

b. Sketch the curve and indicate its orientation. (**See Examples 2–3, pp. 1137–1139**)

11. $x = t + 2$ and $y = t^2 + t$

12. $x = t - 1$ and $y = t^2 - 2t$

13. $x = t - 1$ and $y = \frac{t}{t-1}$

14. $x = t + 2$ and $y = \frac{t}{t+2}$

15. $x = |t + 3|$ and $y = 1 + t$

16. $x = |t - 3|$ and $y = t - 2$

17. $x = e^{-t}$ and $y = e^t$

18. $x = 2e^t$ and $y = e^t - 2$

19. $x = \sqrt{t}$ and $y = \sqrt{16 - t}$

20. $x = -\sqrt{9 - t}$ and $y = \sqrt{t}$

21. $x = \cos t$ and $y = \sin^2 t - 4$ (*Hint*: Use the identity $\sin^2 t + \cos^2 t = 1$.)

22. $x = -\sin t$ and $y = 1 + \cos^2 t$

23. $x = 5\cos\theta$ and $y = -5\sin\theta$

24. $x = 4\sin\theta$ and $y = -4\cos\theta$

25. $x = 1 + 2\cos\theta$ and $y = -2 + 3\sin\theta$ (*Hint*: Solve for $\sin\theta$ and $\cos\theta$. Square the results and apply the identity $\sin^2\theta + \cos^2\theta = 1$.)

26. $x = 5 + 4\sin\theta$ and $y = -3 + 2\cos\theta$

For Exercises 27–30, eliminate the parameter and write an equation in rectangular coordinates to represent the given curve.

27. Circle: $x = h + r\cos\theta$ and $y = k + r\sin\theta$ (*Hint*: Solve for $\cos\theta$ and $\sin\theta$. Then square both equations and use a Pythagorean identity.)

28. Line through the points (x_1, y_1) and (x_2, y_2): $x = x_1 + t(x_2 - x_1)$ and $y = y_1 + t(y_2 - y_1)$

29. Ellipse: $x = h + a\cos\theta$ and $y = k + b\sin\theta$

30. Hyperbola: $x = h + a\sec\theta$ and $y = k + b\tan\theta$

Learn: Write Parametric Equations to Represent a Curve

For Exercises 31–34, write parametric equations for the given curve for the given definitions of *x*. (See Example 4, p. 1139)

31. $y = 2x - 4$

- **a.** $x = t$
- **b.** $x = \frac{t}{3}$
- **c.** $x = -3t$

32. $y = -4x + 1$

- **a.** $x = t$
- **b.** $x = \frac{t}{2}$
- **c.** $x = -4t$

33. $y = x + 4, x > 0$

- **a.** $x = t, t > 0$
- **b.** $x = e^t$

34. $y = -2x, x \geq 0$

- **a.** $x = t, t \geq 0$
- **b.** $x = t^2$

For Exercises 35–42, use the results of Exercises 27–30 and use the parameter *t* to write parametric equations representing the given curve. Answers may vary.

35. Line passing through (0, 2) and (3, 1)

36. Line passing through (0, –1) and (4, 2)

37. Circle with center (4, –3) and radius 5

38. Circle with center $\left(-\frac{1}{2}, 1\right)$ and radius $\frac{1}{2}$

39. Hyperbola with center (0, 0), vertices (0, ±2), and asymptotes $y = \pm\frac{2}{3}x$

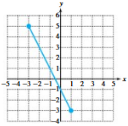
40. Hyperbola with center (1, 3), vertices (–2, 3) and (4, 3), and foci (–4, 3) and (6, 3)

41. Ellipse with center (–2, 1), vertices (1, 1) and (–5, 1), and endpoints of the minor axis (–2, 3) and (–2, –1)

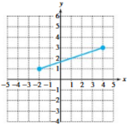
42. Ellipse with center (0, 0), vertices (0, ±5), and foci (0, ±4)

For Exercises 43–46, write parametric equations on a restricted interval for *t* to define the given graph.

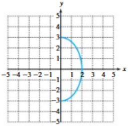
43.



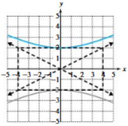
44.



45. Right half of an ellipse



46. Top branch of a hyperbola



Learn: Apply Parametric Equations

For Exercises 47–48, an object undergoes uniform linear motion on a straight path from point A to point B in Δt sec. Write parametric equations over an interval I that describe the motion along the path.

47. $A = (1, 5)$, $B = (7, 3)$, and $\Delta t = 2$ sec

48. $A = (10, 7)$, $B = (-14, 1)$, and $\Delta t = 6$ sec

49. Suppose that the origin on a computer screen is in the lower left corner of the screen. A target moves along a straight line at a constant speed from point $A(670, 450)$ to point $B(250, 210)$ in 4 sec. (**See Example 5, p. 1141**)

- **a.** Write parametric equations to represent the target's path as a function of the time t (in sec) after the target leaves its initial position.
- **b.** Where is the target located 1.4 sec after motion starts?

50. A hospital is located 3 mi west and 4 mi north of the center of town. Suppose that a Medi-Vac helicopter flies at a constant speed from the hospital to the location of an accident 15 mi east and 20 mi south of the center of town in 15 min$\left(\frac{1}{4}\text{hr}\right)$. Choose a coordinate system with the origin at the center of town.

- **a.** Write parametric equations to represent the path of the helicopter as a function of the time t (in hr) after the helicopter leaves the hospital.
- **b.** Where is the helicopter located 10 min after leaving the hospital?

51. An artillery specialist practicing at a firing range on a flat stretch of ground fires a shell from 1 m above the ground, with a muzzle velocity of 96 m/sec at an angle of 16° from the horizontal. Choose a coordinate system with the origin at ground level directly below the launch position. (**See Example 6, p. 1143**)

- **a.** Write parametric equations that model the path of the shell as a function of time t (in sec) after launch.
- **b.** Approximate the time required for the shell to hit the ground. Round to the nearest hundredth of a second.
- **c.** Approximate the horizontal distance that the shell travels before it hits the ground. Round to the nearest meter.
- **d.** When is the shell at its maximum height? Find the exact value and an approximation to the nearest hundredth of a second.
- **e.** Determine the maximum height. Round to the nearest meter.

52. A golfer tees off on level ground and hits the ball with an initial speed of 52 m/sec at an angle of 34° above the horizontal. Choose a coordinate system with the origin at the point where the ball is struck.

- **a.** Write parametric equations that model the path of the ball as a function of time t (in sec).
- **b.** For how long will the ball be in the air before it hits the ground? Round to the nearest hundredth of a second.
- **c.** Approximate the horizontal distance that the ball travels before it hits the ground. Round to the nearest foot.
- **d.** When is the ball at its maximum height? Find the exact value and an approximation to the nearest hundredth of a second.
- **e.** What is the maximum height? Round to the nearest foot.

53. Anna hits a softball at a height of 3 ft from the ground. The softball leaves the bat traveling with an initial speed of 80 ft/sec, at an angle of 30° from the horizontal. Choose a coordinate system with the origin at ground level directly under the point where the ball is struck.

a. Write parametric equations that model the path of the ball as a function of time t (in sec).

b. When is the ball at its maximum height?

c. What is the maximum height? Round to the nearest foot.

d. If an outfielder catches the ball at a height of 5 ft, how long was the ball in the air after being struck? Give the exact answer and the answer rounded to the nearest hundredth of a second.

e. How far is the outfielder from home plate when the ball is caught? Round to the nearest foot.

54. Tony hits a baseball at a height of 3 ft from the ground. The ball leaves the bat traveling with an initial speed of 120 ft/sec at an angle of 45° from the horizontal. Choose a coordinate system with the origin at ground level directly under the point where the ball is struck.

a. Write parametric equations that model the path of the ball as a function of time t (in sec).

b. When is the ball at its maximum height? Give the exact value and round to the nearest hundredth of a second.

c. What is the maximum height?

d. If an outfielder catches the ball at a height of 6 ft, for how long was the ball in the air after being struck? Give the exact answer and the answer rounded to the nearest hundredth of a second.

e. How far is the outfielder from home plate when the ball is caught? Round to the nearest foot.

Mixed Exercises

55. Two cars approach an intersection. Car A is 25 mi west of the intersection traveling 40 mph. Car B is 30 mi north of the intersection traveling 50 mph. Place the origin of a rectangular coordinate system at the intersection.

a. Write parametric equations that model the path of each car as a function of the time $t \geq 0$ (in hr).

b. Determine the times required for each car to reach the intersection. Based on these results, will the cars crash?

c. Write the distance between the cars as a function of the time t.

d. Determine the time at which the two cars are at their closest point. [*Hint*: The function from part (c) is minimized when the radicand is minimized.]

e. How close are the cars at their closest point? Round to the nearest hundredth of a mile.

56. Two planes flying at the same altitude are on a course to fly over a control tower. Plane A is 50 mi east of the tower flying 125 mph. Plane B is 90 mi south of the tower flying 200 mph. Place the origin of a rectangular coordinate system at the intersection.

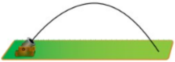
a. Write parametric equations that model the path of each plane as a function of the time $t \geq 0$ (in hr).

b. Determine the times required for each plane to reach a point directly above the tower. Based on these results, will the planes crash?

c. Write the distance between the planes as a function of the time t.

d. How close do the planes pass? Round to the nearest tenth of a mile.

57. Suppose that a projectile is fired from ground level at an angle of 45° with an initial speed of 200 ft/sec. Choose a coordinate system with the origin at the point of launch.



- **a.** Write parametric equations to define the path of the projectile as a function of the time *t* (in sec).
- **b.** What is the range of the projectile? That is, what is the horizontal distance traveled from the point of launch to the point where the projectile lands?
- **c.** What are the coordinates of the projectile at its maximum height?
- **d.** Eliminate the parameter and write an equation in rectangular coordinates to represent the path.
- **e.** If a drone will be at position (848.5, 272.5) at a time 6 sec after the projectile is launched, will the projectile hit the drone? Assume a 1-ft margin of error.

58. A daredevil wants to jump over a canyon on a motorcycle. When the motorcycle leaves the ramp at the edge of the canyon, it travels approximately 88 ft/sec (60 mph) at an angle of 30°. Choose a coordinate system with the origin at the end of the take-off ramp, and assume that the motorcycle lands at the same height.

- **a.** Write parametric equations to define the path of the motorcycle as a function of the time *t* (in sec) after leaving the ramp.
- **b.** If a bird is at a position (90, 26) at a time 1.2 sec after the motorcycle leaves the ramp, will the motorcycle hit the bird? Assume a 1-ft margin of error.
- **c.** What is the horizontal distance across the canyon from the point of take-off to the point of landing? Round to the nearest foot.
- **d.** What are the coordinates (to the nearest foot) of the motorcycle at its maximum height?
- **e.** Eliminate the parameter and write an equation in rectangular coordinates to represent the path.

Recall from Lesson 7.1 that a point (x, y) in rectangular coordinates and a point (r, θ) in polar coordinates are related by the equations $x = r\cos\theta$ and $y = r\sin\theta$. So, the graph of the polar equation $r = f(\theta)$ is the same as the graph of the parametric equations $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. For Exercises 59–60, a polar equation is given.

- **a.** Use a graphing utility in “polar” mode to graph the polar equation.
- **b.** Write parametric equations to represent the graph of the polar equation.
- **c.** Use a graphing utility in “parametric” mode to graph the parametric equations from part (b).

59. $r = 4\sin 2\theta$ for $0 \le \theta \le 2\pi$

60. $r = 3 - 4\cos\theta$ for $0 \le \theta \le 2\pi$

Write About It

For Exercises 61–62, describe the differences in the graphs of C_1 through C_3.

61. C_1: $x = t$ and $y = t^2 - 1$

C_2: $x = e^t$ and $y = e^{2t} - 1$

C_3: $x = \cos t$ and $y = \cos^2 t - 1$

62. C_1: $x = t$ and $y = \sqrt{25 - t^2}$

C_2: $x = e^t$ and $y = \sqrt{25 - e^{2t}}$

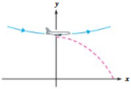
C_3: $x = \cos t$ and $y = \sqrt{25 - \cos^2 t}$

63. Do the parametric equations $x = \sqrt{t}$ and $y = t + 5$ define a parabola? Why or why not?

64. Is the point (16, 12) on the curve defined by $x = 2t$ and $y = t + 1$? Why or why not?

Expanding Your Skills

65. For a short interval of time, a military supply plane flies on a hyperbolic path given by the equation $\frac{y^2}{122.5^2} - \frac{x^2}{100^2} = 1$; $y \geq 0$, where x and y are measured in meters.



a. What are the coordinates of the point on the flight path closest to the ground?

b. When the plane reaches its closest point to the ground, it drops a bag of supplies to people on the ground. Assuming that the plane is traveling 200 m/sec due east at the time of the drop, write parametric equations representing the path of the bag as a function of the time t (in sec) after the drop.

c. Determine the coordinates of the point where the bag hits the ground.

66. Suppose that the position of one particle at time t is given by

$x_1 = 3 + 3\cos t$ and $y_1 = 5\sin t$ for $0 \leq t \leq 2\pi$

Suppose that the position of a second particle is given by

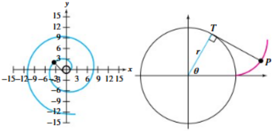
$x_2 = \frac{6}{\pi}t$ and $y_2 = \frac{10}{\pi}t$ for $0 \leq t \leq 2\pi$

a. Use a graphing utility to graph the paths of both particles simultaneously.

b. In how many points do the graphs intersect?

c. Will the particles ever collide? If so, where and at what time?

67. Suppose that an imaginary string of unlimited length is attached to a point on a circle and pulled taut so that it is tangent to the circle. Keeping the string pulled tight, wind (or unwind) the string around the circle. The path traced out by the end of the string as it moves is called an *involute* of the circle (shown in blue).



Given a circle of radius r with parametric equations $x = r\cos\theta$, $y = r\sin\theta$, show that the parametric equations of the involute of the circle are $x = r(\cos\theta + \theta\sin\theta)$ and $y = r(\sin\theta - \theta\cos\theta)$.

Technology Connections

For Exercises 68–70, use a graphing utility to graph the given curve on the recommended viewing window.

68. Witch of Agnesi:

$x = 2\cot t$ and $y = 2\sin^2 t$

t: [0, 2π, 0.1], x: [−6.4, 6.4, 1], y: [−4, 4, 1]

69. Lissajous curve:

$x = 3\cos 3\theta$ and $y = 3\sin 4\theta$

t: $[0, 2\pi, 0.1]$, x: $[-6.4, 6.4, 1]$, y: $[-4, 4, 1]$

70. Involute of a circle:

$x = \cos t + t\sin t$ and $y = \sin t - t\cos t$

t: $[0, 6\pi, 0.1]$, x: $[-32, 32, 4]$, y: $[-20, 20, 4]$

A cycloid is a curve created by rolling a circle along a line. We can also create families of curves by rolling a circle around another *circle*. If a circle of radius r rolls around the *interior* of a larger circle of radius R, a fixed point on the smaller circle traces out a curve called a hypocycloid. For Exercises 71–72, use the following parametric equations for a hypocycloid.

$$x = (R - r)\cos t + r\cos\left[\left(\frac{R-r}{r}\right)t\right] \text{ and}$$
$$y = (R - r)\sin t - r\sin\left[\left(\frac{R-r}{r}\right)t\right]$$

71. a. Write parametric equations for a hypocycloid with $R = 3$ and $r = 1$. The curve defined by these equations is called a *deltoid*.

b. Graph the exterior circle given by $x = 3\cos t$ and $y = 3\sin t$ and the deltoid from part (a). Use $0 \le t \le 2\pi$ and a viewing window of $[-4.8, 4.8, 1]$ by $[-3, 3, 1]$.

c. For what values of t on the interval $[0, 2\pi)$ do the deltoid curve and circle intersect?

72. a. Write parametric equations for a hypocycloid with $R = 4$ and $r = 1$. The curve defined by these equations is called an *astroid* (not to be confused with asteroid).

b. Graph the exterior circle given by $x = 4\cos t$ and $y = 4\sin t$ and the astroid from part (a). Use $0 \le t \le 2\pi$ and a viewing window of $[-6.4, 6.4, 1]$ by $[-4, 4, 1]$.

c. For what values of t on the interval $[0, 2\pi)$ do the astroid curve and circle intersect?

If a circle of radius r rolls around the *exterior* of a circle of radius $R \ge r$, a fixed point on the outer circle traces out a curve called an epicycloid. For Exercises 73–74, use the following parametric equations for an epicycloid.

$$x = (R + r)\cos t - r\cos\left[\left(\frac{R+r}{r}\right)t\right] \text{ and}$$
$$y = (R + r)\sin t - r\sin\left[\left(\frac{R+r}{r}\right)t\right]$$

73. a. Write parametric equations for a epicycloid with $R = 1$ and $r = 1$. The curve defined by these equations is called a *cardioid*.

b. Graph the circle given by $x = \cos t$ and $y = \sin t$ and the *cardioid* from part (a). Use $0 \le t \le 2\pi$ and a viewing window of $[-6.4, 6.4, 1]$ by $[-4, 4, 1]$.

74. a. Write parametric equations for a epicycloid with $R = 2$ and $r = 1$. The curve defined by these equations is called a *nephroid*, meaning “kidney-shaped.”

b. Graph the circle given by $x = 2\cos t$ and $y = 2\sin t$ and the *nephroid* from part (a). Use $0 \le t \le 2\pi$ and a viewing window of $[-6.4, 6.4, 1]$ by $[-4, 4, 1]$.

75. Suppose that a projectile is fired at an angle of inclination θ, where $0 < \theta < \frac{\pi}{2}$, with an initial speed of v_0. The parametric equations representing the path of points (x, y) at time t are given by:

$$x = v_0 t\cos\theta$$
$$y = v_0 t\sin\theta - \frac{gt^2}{2}$$

where g is the acceleration due to gravity.

a. Find the value of t when the projectile hits the ground.

b. If $\theta = \frac{\pi}{6}$, $v_0 = 128$ ft/sec, and $g = 32$ ft/sec^2, how long will it take for the projectile to hit the ground? What is the maximum height of the projectile over its path?

c. Eliminate the parameter t to isolate the y variable.

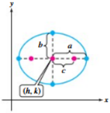
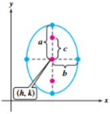
d. Use a graphing utility to graph the path of the projectile with the values of θ, v_0, and g given in part (b).

Go online for more practice problems.

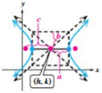
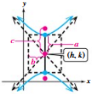
Chapter 10 Review

Chapter 10 Key Concepts

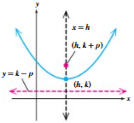
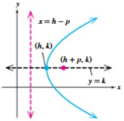
Lesson 10-1 The Ellipse

Key Concepts	Reference
An **ellipse** is the set of all points (x, y) in a plane such that the sum of the distances between (x, y) and two fixed points is a constant. The fixed points are called the **foci** of the ellipse.	p. 1039
Standard forms of an equation of an ellipse: • Major axis horizontal: Equation: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ $a > b > 0$; $c^2 = a^2 - b^2$ $(c > 0)$ Center: (h, k) Vertices: $(h + a, k)$, $(h - a, k)$ Foci: $(h + c, k)$, $(h - c, k)$ Minor axis endpoints: $(h, k + b)$, $(h, k - b)$  • Major axis vertical: Equation: $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ $a > b > 0$; $c^2 = a^2 - b^2$ $(c > 0)$ Center: (h, k) Vertices: $(h, k + a)$, $(h, k - a)$ Foci: $(h, k + c)$, $(h, k - c)$ Minor axis endpoints: $(h + b, k)$, $(h - b, k)$  	p. 1045
$2a$ is the length of the major axis. $2b$ is the length of the minor axis.	p. 1041
The **eccentricity** $e = \frac{c}{a}$ of an ellipse measures the degree of elongation. Note that for an ellipse, $0 < e < 1$.	p. 1050

Lesson 10-2 The Hyperbola

Key Concepts	Reference
A **hyperbola** is the set of all points (x, y) in a plane such that the difference in distances between (x, y) and two fixed points (foci) is a positive constant.	p. 1062
Standard forms of an equation of a hyperbola: - Transverse axis horizontal: Equation: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $a > 0, b > 0; c^2 = a^2 + b^2\ (c > 0)$ Center: (h, k) Vertices: $(h + a, k), (h - a, k)$ Foci: $(h + c, k), (h - c, k)$ Asymptotes: $y - k = \pm\frac{b}{a}(x - h)$  - Transverse axis vertical: Equation: $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ $a > 0, b > 0; c^2 = a^2 + b^2\ (c > 0)$ Center: (h, k) Vertices: $(h, k + a), (h, k - a)$ Foci: $(h, k + c), (h, k - c)$ Asymptotes: $y - k = \pm\frac{a}{b}(x - h)$ 	p. 1064
$2a$ is the length of the transverse axis. $2b$ is the length of the conjugate axis.	p. 1065
The **eccentricity** of a hyperbola is given by $e = \frac{c}{a}$. Note that for a hyperbola, $e > 1$.	p. 1073

Lesson 10-3 The Parabola

Key Concepts	Reference
A **parabola** is the set of all points in a plane that are equidistant from a fixed line (called the **directrix**) and a fixed point (called the **focus**).	p. 1085
Standard forms of an equation of a parabola: - Vertical axis of symmetry: Equation: $(x-h)^2 = 4p(y-k)$ Vertex: (h, k) Focus: $(h, k+p)$ Directrix: $y = k - p$ Axis of symmetry: $x = h$  - If $p > 0$, the parabola opens upward as shown. - If $p < 0$, the parabola opens downward. - Horizontal axis of symmetry: Equation: $(y-k)^2 = 4p(x-h)$ Vertex: (h, k) Focus: $(h+p, k)$ Directrix: $x = h - p$ Axis of symmetry: $y = k$  - If $p > 0$, the parabola opens to the right as shown. - If $p < 0$, the parabola opens to the left.	p. 1092
The value of $\|p\|$ is the focal length (distance between the vertex and focus).	p. 1086
The **latus rectum** is the line segment perpendicular to the axis of symmetry, through the focus, with endpoints on the parabola.	p. 1089
The value of 4 $\|p\|$ is the focal diameter (length of the latus rectum).	p. 1089

Lesson 10-4 Rotation of Axes

Key Concepts	Reference
Rotation of axes: Suppose that the x- and y-axes in a rectangular coordinate system are rotated about the origin through an angle θ to form the x'- and y'-axes. x and y in terms of x' and y' $x = x'\cos\theta - y'\sin\theta$ $y = x'\sin\theta + y'\cos\theta$ x' and y' in terms of x and y $x' = x\cos\theta + y\sin\theta$ $y' = -x\sin\theta + y\cos\theta$	p. 1109
Determine the angle of rotation: To write the general second-degree polynomial equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ with no xy term, rotate the axes through an acute angle θ given by $\cot 2\theta = \frac{A-C}{B}$	p. 1112

Graphing a rotated conic: 1. Determine the angle of rotation. 2. Substitute $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ into the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. The resulting equation will have no xy term. 3. Write the equation from Step 2 in standard form and graph the conic relative to the x'- and y'-axes.	p. 1116
Identify conics by the discriminant of the equation, $B^2 - 4AC$: Excluding degenerate cases, the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ represents An ellipse (or a circle) if $B^2 - 4AC < 0$. A hyperbola if $B^2 - 4AC > 0$. A parabola if $B^2 - 4AC = 0$.	p. 1115

Lesson 10-5 Polar Equations of Conics

Key Concepts	Reference
Alternative definition of conics: Let l be a fixed line (**directrix**) in a plane and let F be a fixed point (**focus**) in the plane, not on the line. A **conic** is the set of points P in the plane such that the ratio of the distance between P and F to the distance from P to l is a constant positive real number e (the **eccentricity**). That is, $\frac{d(P, F)}{d(P, l)} = e$.	p. 1123
Polar equations of conics: A polar equation of the form $r = \frac{ed}{1 \pm e\cos\theta}$ or $r = \frac{ed}{1 \pm e\sin\theta}$ is a conic with focus at the pole and eccentricity $e > 0$. The value $\|d\|$ is the distance between the focus at the pole and the directrix. The conic is 1. A parabola if $e = 1$ (axis of symmetry perpendicular to the directrix). 2. An ellipse if $0 < e < 1$ (major axis perpendicular to the directrix). 3. A hyperbola if $e > 1$ (transverse axis perpendicular to the directrix). The location of the directrix and orientation of the curve for $d > 0$ are summarized in Figure 10-41 on page 1007.	p. 1125
Guidelines to graph a conic in polar form: 1. Write the polar equation in the form $r = \frac{ed}{1 \pm e\cos\theta}$ or $r = \frac{ed}{1 \pm e\sin\theta}$. 2. Determine the value of the eccentricity e and identify the type of conic. 3. Determine the orientation of the conic. 4. Plot points at $\theta = 0, \frac{\pi}{2}, \pi$, and $3\frac{\pi}{2}$. 5. Identify key characteristics of the conic as desired (center, vertices, length of major/minor/transverse/conjugate axes, directrix).	p. 1126

Rotation of conics in polar form: The rotation of a conic about the origin is much more convenient when the conic is expressed in polar versus rectangular form. Consider the graph of $r = f(\theta)$. Replacing θ by $(\theta - \alpha)$ results in • A counterclockwise rotation (rotation through a positive angle) if $\alpha > 0$. • A clockwise rotation (rotation through a negative angle) if $\alpha < 0$.	p. 1131

Lesson 10-6 Plane Curves and Parametric Equations

Key Concepts	Reference
Plane curves: Let f and g be functions of t over an interval I. Then the set of points defined by $(x, y) = (f(t), g(t))$ is called a plane curve. The equations $x = f(t)$ and $y = g(t)$ are called parametric equations of the curve, and t is called the parameter.	p. 1136
Graphing a plane curve: • One method to graph a plane curve defined by parametric equations is to create a table of points. • Another method to graph a plane curve is to eliminate the parameter t, by solving for t or an expression involving t in one equation. Then substitute the result in the other equation. Note any restrictions on the domain of the rectangular equation so that the resulting graph is consistent with the restrictions on x and y in the parametric equations.	p. 1136
Applications of parametric equations: Parametric equations are particularly well suited for describing objects in motion. Let $t \geq 0$ represent the time under which an object is in motion along a plane curve, and let (x_0, y_0) represent the initial position of the object at $t = 0$. **Uniform linear motion** $x = x_0 + at$ $y = y_0 + bt$ The values of a and b are the horizontal and vertical components of velocity, respectively. **Projectile motion*** $x = (v_0 \cos\theta)t + x_0$ $y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + y_0$ *Assuming negligible air resistance. θ is the angle of launch relative to the horizontal, and v_0 is the initial speed (magnitude of the initial velocity). The value of g is the acceleration due to gravity, where $g = 32$ ft/sec² or $g = 9.8$ m/sec² on the surface of the Earth.	p. 1141

Go online for a detailed chapter summary.

Chapter 10 Test

1. How are the graphs of these ellipses similar, and how are they different?

$$\frac{x^2}{36}+\frac{y^2}{144}=1 \qquad \frac{x^2}{144}+\frac{y^2}{36}=1$$

2. Write the standard form of an equation of a circle with center (h, k) and radius r.

3. Write the standard form of an equation of an ellipse with center (h, k) and major axis horizontal.

4. Write the standard form of an equation of an ellipse with center (h, k) and major axis vertical.

5. Write the standard form of an equation of a hyperbola with center (h, k) and transverse axis vertical.

6. Write the standard form of an equation of a hyperbola with center (h, k) and transverse axis horizontal.

7. Write the standard form of an equation of a parabola with vertex (h, k) and vertical axis of symmetry.

8. Write the standard form of an equation of a parabola with vertex (h, k) and horizontal axis of symmetry.

For Exercises 9–16,

a. Identify the equation as representing a circle, an ellipse, a hyperbola, or a parabola.

b. Graph the curve.

c. Identify key features of the graph. That is,

- If the equation represents a circle, identify the center and radius.
- If the equation represents an ellipse, identify the center, vertices, endpoints of the minor axis, foci, and eccentricity.
- If the equation represents a hyperbola, identify the center, vertices, foci, equations of the asymptotes, and eccentricity.
- If the equation represents a parabola, identify the vertex, focus, endpoints of the latus rectum, equation of the directrix, and equation of the axis of symmetry.

9. $\frac{(x-2)^2}{9}+\frac{y^2}{16}=1$

10. $\frac{(x-2)^2}{9}-\frac{y^2}{16}=1$

11. $\frac{(x-2)^2}{16}+y=0$

12. $x^2+y^2-4x-6y+1=0$

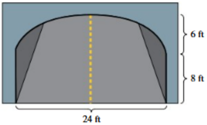
13. $-9x^2+16y^2+64y-512=0$

14. $9x^2+25y^2+72x-50y-731=0$

15. $\frac{(x+4)^2}{4}+\frac{(y-1)^2}{4}=1$

16. $y^2-8y-8x+40=0$

17. The entrance to a tunnel is in the shape of a semiellipse over a 24-ft by 8-ft rectangular opening. The height at the center of the opening is 14 ft.



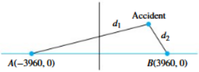
a. Determine the height of the opening at a point 6 ft from the edge of the tunnel. Round to 1 decimal place.

b. Can a 10-ft-high truck pass through the opening if the truck's outer wheel passes at a point 3 ft from the edge of the tunnel?

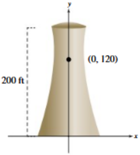
c. Set up a coordinate system with the origin placed on the ground in the center of the roadway. Write a function that represents the height of the opening $h(x)$ (in ft) as a function of the horizontal distance x (in ft) from the center.

18. Neptune orbits the Sun in an elliptical path with the Sun at one focus. At aphelion, Neptune is farthest from the Sun at 4.546×10^9 km. If the eccentricity of the orbit is 0.0113, determine the distance at perihelion (closest point between Neptune and the Sun). Round to the nearest million km.

19. Suppose that two people standing 1.5 mi (7920 ft) apart at points A and B hear a car accident. The time difference between the sound heard at point A and the sound heard at point B is 6 sec. If sound travels 1100 ft/sec, find an equation of the hyperbola (with foci at A and B) at which the accident occurred.



20. A building in the shape of a hyperboloid has hyperbolic cross sections perpendicular to the ground. Placing the origin at the bottom and center of the building, the center of a hyperbolic cross section is $(0, 120)$ with one vertex at $(12, 120)$ and one focus at $(12\sqrt{10}, 120)$. All units are in feet.



a. Write an equation of a hyperbolic cross section through the origin. Assume that there are no restrictions on x or y.

b. Determine the diameter of the building at the base. Round to the nearest tenth of a foot.

c. Determine the diameter of the building at the top. Round to the nearest tenth of a foot.

21. The middle span of the Golden Gate Bridge in San Francisco is supported by cables from two towers 4200 ft apart. The tops of the towers are 500 ft above the roadway. Assuming that the cables are parabolic and are at road level halfway between the towers, find the height of the cables 1000 ft from the center of the bridge. Round to the nearest foot.

22. Write the standard form of an equation of a parabola with focus $(1, 6)$ and directrix $y = -2$.

23. Write the standard form of an equation of an ellipse with vertices $(2, -3)$ and $(6, -3)$ and foci $(3, -3)$ and $(5, -3)$.

24. Write the standard form of an equation of a hyperbola with vertices $(4, -1)$ and $(4, 7)$ and foci $(4, -3)$ and $(4, 9)$.

25. Write the standard form of an equation of a hyperbola with vertices $(-1, -3)$ and $(5, -3)$ and slope of one asymptote $\frac{4}{3}$.

26. If the major axis of an ellipse is 26 units in length and the minor axis is 10 units in length, determine the eccentricity of the ellipse.

27. If the transverse axis of a hyperbola is 20 units and the conjugate axis is 12 units, determine the eccentricity of the hyperbola.

28. If the directrix of a parabola is $x = 2$, and the focus is $(-4, 0)$, determine the focal length and the focal diameter.

29. If the asymptotes of a hyperbola are $y = \frac{3}{5}x + 1$ and $y = -\frac{3}{5}x + 1$, identify the center of the hyperbola.

30. Solve the system of equations.

$$3x^2 - 4y^2 = -13$$

$$5x^2 + 2y^2 = 13$$

31. Describe the graph of the equation.

$$x = -4\sqrt{1 - \frac{y^2}{9}}$$

32. Each equation represents a conic section (nondegenerative case). Identify the type of conic section.

a. $50x^2 - 20xy + 2y^2 - 100x + 8y - 9 = 0$

b. $4x^2 - 16x = -9y^2 + 18y + 4$

c. $x^2 + 4xy = -2y^2 + 2x - 16y + 4$

33. Identify the conic each equation represents and give the eccentricity.

a. $r = \dfrac{6}{2 + 3\sin\theta}$

b. $r = \dfrac{10}{3 - 3\sin\theta}$

c. $r = \dfrac{12}{6 + 3\cos\theta}$

For Exercises 34–35, assume that the *x*- and *y*-axes are rotated through angle θ about the origin to form the *x′*- and *y′*-axes.

34. Given $\theta = 30°$ and $P(8\sqrt{3}, -2\sqrt{3})$ in the *xy*-plane, find the coordinates of *P* in the *x′y′*-plane.

35. Given $\theta = 45°$, write the equation $x^2 + 2xy + y^2 + 3\sqrt{2}x + \sqrt{2}y + 2 = 0$ in *x′y′*-coordinates.

For Exercises 36–37, an equation of a conic (nondegenerative case) is given.

a. Identify the type of conic.

b. Determine an acute angle of rotation to eliminate the *xy* term. Round to the nearest tenth of a degree if necessary.

c. Use a rotation of axes to eliminate the *xy* term in the equation.

d. Sketch the graph.

36. $13x^2 - 10xy + 13y^2 - 24\sqrt{2}x - 24\sqrt{2}y = 0$

37. $8x^2 + 6xy + 9 = 0$

For Exercises 38–39, an equation of a conic is given.

a. Identify the type of conic represented by the equation.

b. Give the eccentricity and write an equation of the directrix in rectangular coordinates.

c. Give the coordinates of the vertex or vertices in polar coordinates.

d. Sketch the graph.

38. $r = \frac{8}{4 + 6\sin\theta}$

39. $r = \frac{14}{6 - 6\cos\theta}$

40. Write a polar equation of a conic with $e = 2$ directrix $y = 3$.

For Exercises 41–42,

a. Eliminate the parameter.

b. Sketch the curve and show the orientation of the curve.

41. $x = 2 - t$ and $y = -2 + 3t$ for $-1 \leq t < 2$

42. $x = 2 - 3\sin\theta$ and $y = 3\cos\theta$

43. Write parametric equations describing the motion of a particle moving under uniform linear motion from point *A*(3, 4) to point *B*(12, −2) in 3 sec.

44. A pyrotechnic rocket is fired from a platform 2 ft high at an angle of 60° from the horizontal with an initial speed of 72 ft/sec. Choose a coordinate system with the origin at ground level directly below the launch position.

a. Write parametric equations that model the path of the shell as a function of the time *t* (in sec) after launch.

b. Approximate the time required for the shell to hit the ground. Round to the nearest hundredth of a second.

c. Approximate the horizontal distance that the shell travels before it hits the ground. Round to the nearest foot.

d. When is the shell at its maximum height? Find the exact value and an approximation to the nearest hundredth of a second.

e. Determine the maximum height.

45. The equation $5x^2 - 10xy + 5y^2 + 10x - 15y - 9 = 0$ represents a conic section (nondegenerative case).

a. Identify the type of conic section.

b. Graph the equation on a graphing utility.

46. The equation $r = \frac{8}{4 - 2\cos\theta}$ represents a conic.

a. Identify the type of conic.

b. Give a new equation if the graph of the conic is rotated clockwise about the origin through the angle $\theta = \frac{5\pi}{6}$.

c. Graph both equations on a graphing utility.

47. Use the parametric mode to graph the curve on a graphing utility.

$x = \sin 4t$ and $y = 3\cos t$, $0 \leq t \leq 2\pi$

GO ONLINE **Go online for more practice problems.**

Chapter 10 Cumulative Review Exercises

1. Use long division to divide. $(3x^4 - 2x^3 - 5x + 1) \div (x^2 + 3)$

2. Is -4 a zero of the polynomial $f(x) = 3x^4 - 6x^3 + 5x - 12$?

3. Is $(x - 3)$ a factor of $2x^4 - 7x^3 + 8x^2 - 17x + 6$?

For Exercises 4–5, simplify completely.

4. $\dfrac{\sqrt[3]{54x^4}}{\sqrt[3]{2x^5}}$

5. $\left(\dfrac{-2x^2y^{-1}}{z^3}\right)^{-4}\left(\dfrac{4x^{-5}}{y^7}\right)^2$

6. Find the midpoint of the line segment with endpoints $(-362, 147)$ and $(118, 24)$.

7. Given $f(x) = \sqrt[3]{x - 7}$, write an equation defining $f^{-1}(x)$.

8. A boat travels 10 mph at a bearing of N30°E in still water. A 2-mph current is directed due west. Determine the actual speed of the boat. Write the exact value and the speed rounded to the nearest tenth of a mph.

9. Determine the asymptotes of the graph of $y = \csc\left(2x - \frac{\pi}{2}\right)$.

For Exercises 10–15, solve the equation or inequality. Write the solution set to the inequalities in interval notation if possible.

10. $e^{3x+2} = 22$

11. $2 - 3\{4 - [x + 2(x - 1) - 3] + 4\} > 0$

12. $21 \le 9 - 2\,|x + 1|$

13. $2x^2 - 8x + 3 \ge 0$

14. $\log_3 x + \log_3(x - 6) = 3$

15. $e^{2x} - 8e^x + 15 = 0$

16. Graph the solution set.
$x^2 + (y - 1)^2 \le 9$

17. Graph $y = 2\sin\left(x - \frac{\pi}{4}\right)$ over one completed period.

18. Find the partial fraction decomposition for $\dfrac{8x - 27}{x^2 - 7x + 12}$

19. Given $f(x) = \dfrac{1}{x - 2}$ and $g(x) = \sqrt{x - 1}$,
    - **a.** Find $(f \circ g)(x)$.
    - **b.** Write the domain of $(f \circ g)(x)$ in interval notation.

20. Given $f(x) = \dfrac{x^2 - 4x}{x + 3}$,
    - **a.** Determine the x-intercept(s) of the graph of f.
    - **b.** Determine the y-intercept.
    - **c.** Find the vertical asymptote(s).
    - **d.** Find the horizontal asymptote.
    - **e.** Find the slant asymptote.

For Exercises 21–24, graph the equations and functions.

21. $\dfrac{x^2}{4} + \dfrac{(y - 1)^2}{9} = 1$

22. $\dfrac{x^2}{4} - \dfrac{(y - 1)^2}{9} = 1$

23. $x^2 = 4(y - 2)$

24. $f(x) = \left(\frac{1}{2}\right)^x + 2$

For Exercises 25–26, solve the system.

25.
$$\begin{aligned} 3x - 4y + 2z &= 5 \\ 5y - 3z &= -12 \\ 7x \qquad + 2z &= 1 \end{aligned}$$

26.
$$\begin{aligned} x^2 - 5y^2 &= 4 \\ 4x^2 + y^2 &= 37 \end{aligned}$$

For Exercises 27–29, refer to the matrices *A* and *B*. Simplify each expression.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 8 \\ 0 & 4 \end{bmatrix}$$

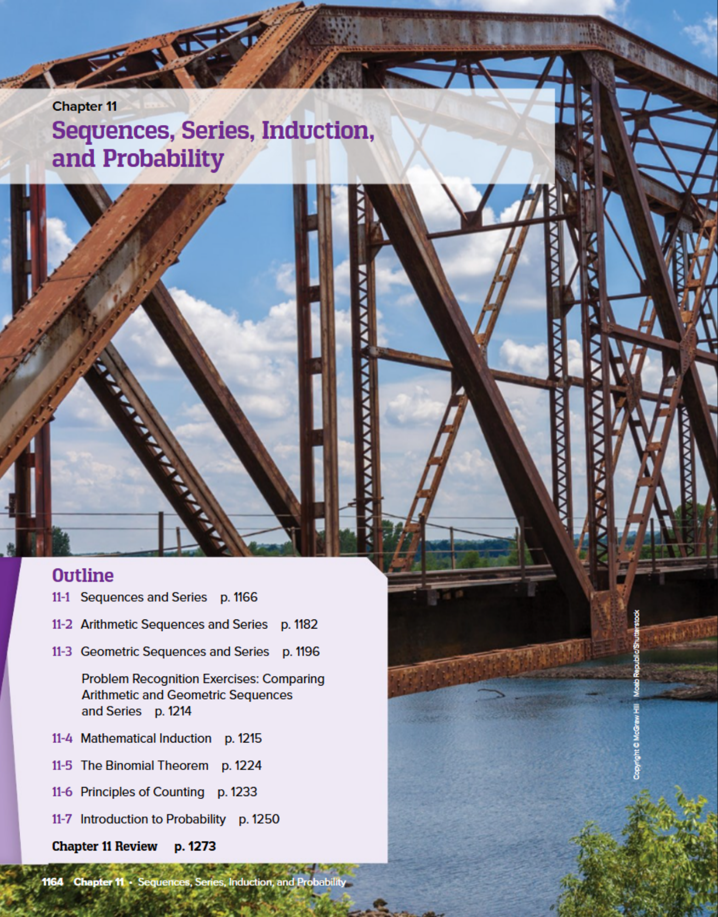
27. $2AB$

28. A^{-1}

29. $|B|$

30. Use Cramer's rule to solve for z.
$$\begin{aligned} 3x - 4y + z &= 3 \\ 2x + 3y \qquad &= 8 \\ 5y + 8z &= 11 \end{aligned}$$

Go online for more practice problems.



Chapter 11

Sequences, Series, Induction, and Probability

Outline

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11-2 Arithmetic Sequences and Series p. 1182

11-3 Geometric Sequences and Series p. 1196

Problem Recognition Exercises: Comparing Arithmetic and Geometric Sequences and Series p. 1214

11-4 Mathematical Induction p. 1215

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11-7 Introduction to Probability p. 1250

Chapter 11 Review p. 1273

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Launch Activity: Repeated Reasoning

Sequences

Hana's math teacher gave an assignment asking the class to go outside and find examples of mathematics happening in the community. While walking through the azalea trail, Hana walked across a bridge that had a really neat design. She took a picture of the bridge then went home to research what the bridge was called. Hana discovered that the type of bridge she walked across was a Warren Truss bridge. When she got back to her math class, the described that the bridge was created by connecting equilateral triangles. Hana even measured the bridge and determined that the length of each rod of the equilateral triangle is 9 feet and the 18-foot bridge she walked across the bridge had 3 equilateral triangles on each side.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer this question: How can you create a formula to determine the total number of rods used in a bridge that is 153 feet long?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

GO ONLINE to answer these questions.

Lesson 11-1

Sequences and Series

What Will You Learn?

After completing this lesson, you should be able to:

- Write Terms of a Sequence from the *n*th Term
- Write Terms of a Sequence Defined Recursively
- Use Factorial Notation
- Use Summation Notation

Learn: Write Terms of a Sequence from the *n*th Term

In day-to-day life, we think of a **sequence** as a progression of elements with some order or pattern. For example, suppose that a new college graduate has to choose between two job offers. Job 1 offers \$75,000 the first year with a \$4000 raise each year thereafter. Job 2 offers \$75,000 with a 5% raise each year. The sequences in Table 11-1 represent the salaries (rounded to the nearest dollar) for each job as a function of the year number.

TABLE 11-1 Job Salaries for Years 1–6

	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6
Job 1	\$75,000	\$79,000	\$83,000	\$87,000	\$91,000	\$95,000
Job 2	\$75,000	\$78,750	\$82,688	\$86,822	\$91,163	\$95,721

By studying the two sequences, we see that by year 5, job 2 has a better salary. If we take the *sum* of the first *n* terms of a sequence, we have a **finite series.** The total salary through the first 5 yr favors job 1 (\$415,000 for job 1 versus \$414,423 for job 2). However, by the end of year 6, job 2 not only pays more each year, but it also has a greater total salary for the 6-yr period (\$510,144 for job 2 versus \$510,000 for job 1).

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From Table 11-1, we can think of a sequence as a function pairing the year number (a positive integer) with the salary. For this reason, it makes sense to define a sequence mathematically as a function whose domain is the set of positive integers.

Finite and Infinite Sequences

An **infinite sequence** is a function whose domain is the set of positive integers. A **finite sequence** is a function whose domain is the set of the first *n* positive integers.

Insights In this text, we address sequences with real-valued terms only.

Although a sequence is defined as a function, we typically do not use function notation to denote the terms of a sequence. Instead, the terms of the sequence are denoted by the letter a with a subscript representing the term number. The sequence of salaries for job 1 would be denoted by

term 1 term 2 term 3

$a_1 = 75{,}000$ $a_2 = 79{,}000$ $a_3 = 83{,}000$ and so on.

The notation $a_1 = 75{,}000$ is used in place of function notation $f(1) = 75{,}000$. Likewise $a_2 = 79{,}000$ is used instead of $f(2) = 79{,}000$ and so on. For a positive integer n, the value a_n is called the ***n*th term** or **general term** of the sequence. The notation $\{a_n\}$ represents the entire sequence.

$$\{a_n\} = a_1, a_2, a_3, \dots, a_n, \dots$$

Insights

Lowercase letters such as a, b, c, and so on are often used to represent the terms of a sequence.

Example 1: Writing Several Terms of a Sequence

Write the first four terms of the sequence defined by the nth term.

a. $a_n = 3n^2 - 4$ **b.** $c_n = 9\left(-\frac{1}{3}\right)^n$

Solution:

a. $a_n = 3n^2 - 4$ To find the first four terms, substitute $n = 1, 2, 3$, and 4.

$a_1 = 3(1)^2 - 4 = -1$

$a_2 = 3(2)^2 - 4 = 8$

$a_3 = 3(3)^2 - 4 = 23$

$a_4 = 3(4)^2 - 4 = 44$ The first four terms are $-1, 8, 23, 44$.

b. $c_n = 9\left(-\frac{1}{3}\right)^n$ To find the first four terms, substitute $n = 1, 2, 3$, and 4.

$c_1 = 9\left(-\frac{1}{3}\right)^1 = -3$

$c_2 = 9\left(-\frac{1}{3}\right)^2 = 1$

$c_3 = 9\left(-\frac{1}{3}\right)^3 = -\frac{1}{3}$

$c_4 = 9\left(-\frac{1}{3}\right)^4 = \frac{1}{9}$ The first four terms are $-3, 1, -\frac{1}{3}, \frac{1}{9}$.

Apply the Skills

1. Write the first four terms of the sequence defined by the nth term.

a. $b_n = 2n + 3$ **b.** $d_n = (-1)^n \cdot n^2$

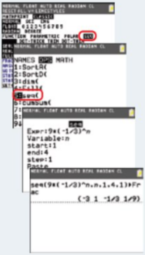
TECHNOLOGY CONNECTIONS

Displaying Terms of a Sequence

If the *n*th term of a sequence is known, a seq function on a graphing utility can display a list of terms. Finding the first four terms of the sequence from Example 1(b) is outlined here.

- Select the MODE key and highlight SEQ.
- Then access the seq function from the catalog or from the LIST menu (OPS submenu) and enter the parameters as prompted.
- Paste the seq command on the home screen and select ENTER.

  seq(9*(–1/3)^*n*, *n*, 1, 4, 1) > Frac


NORMAL FLOAT AUTO REAL RADIAN CL
MATHPRINT CLASSIC
NORMAL SCI ENG
FLOAT 0123456789
RADIAN DEGREE
FUNCTION PARAMETRIC POLAR SEQ
NORMAL FLOAT AUTO REAL RADIAN CL
NAMES OPS MATH
1:SortA(
2:SortD(
3:dim(
4:Fill(
5:seq(
6:cumSum(
7:
8:
9↓
NORMAL FLOAT AUTO REAL RADIAN CL
seq

```
Expr:9*(-1/3)^n
Variable:n
start:1
end:4
step:1
Paste
```

```
seq(9*(-1/3)^n,n,1,4,1)▸Fr
ac
                {-3 1 -1/3 1/9}
```

Notice that the terms of the sequence given in Example 1(b) alternate in signs. Such a sequence is called an **alternating sequence.** The alternation in signs can be represented in the *n*th term by a factor of –1 raised to a variable power that alternates between an even and an odd integer. For example:

$a_n = (-1)^n$ odd-numbered terms, negative; even-numbered terms, positive
$b_n = (-1)^{n+1}$ odd-numbered terms, positive; even-numbered terms, negative

Because a sequence is defined only for positive integers, its graph is a discrete set of points (the points are not connected). Compare the graphs of the sequence defined by $a_n = 8\left(\frac{1}{2}\right)^n$ (see Figure 11-1) versus the function defined by $f(x) = 8\left(\frac{1}{2}\right)^x$ (see Figure 11-2).

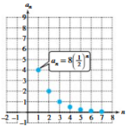


Figure 11-1

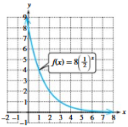


Figure 11-2

The graph of $\{a_n\}$ consists of a discrete set of points that correspond to points on the graph of $f(x) = 8\left(\frac{1}{2}\right)^x$ for positive integers *x*.

Insights

In some cases, we may define a sequence with a domain beginning at zero or some other whole number.

For example, $a_n = \frac{4}{n-1}$ is not defined for $n = 1$.

Therefore, we might restrict the domain to the set of integers greater than or equal to 2.

Learn: Write Terms of a Sequence Defined Recursively

The sequences in Example 1 were defined as a function of the *n*th term. Now we look at sequences defined *recursively*, using a recursive formula. A **recursive formula** defines the *n*th term of a sequence as a function of one or more terms preceding it. This is demonstrated in Example 2.

Example 2: Writing Terms of a Sequence Defined Recursively

Write the first five terms of the sequence defined by $a_1 = 4$ and $a_n = 2a_{n-1} + 1$ for $n > 1$.

Solution:

The first term is given: $a_1 = 4$.

Every term thereafter is defined by $a_n = 2a_{n-1} + 1$

Value of the *n*th term is

two times the value of the preceding term, plus 1.

$a_1 = 4$

$a_2 = 2a_1 + 1 = 2(4) + 1 = 9$ Substitute $a_1 = 4$.

$a_3 = 2a_2 + 1 = 2(9) + 1 = 19$ Substitute $a_2 = 9$.

$a_4 = 2a_3 + 1 = 2(19) + 1 = 39$ Substitute $a_3 = 19$.

$a_5 = 2a_4 + 1 = 2(39) + 1 = 79$ The first five terms are 4, 9, 19, 39, 79.

Apply the Skills

2. Write the first five terms of the sequence defined by:

$c_1 = 5, c_n = 3c_{n-1} - 4$

Perhaps the most famous sequence is the Fibonacci sequence, named after the twelfth-century Italian mathematician Leonardo of Pisa (known as Fibonacci). The Fibonacci sequence is defined recursively as

$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-2} + a_{n-1}$$

The first two terms of the Fibonacci sequence are 1, and each term thereafter is the sum of its two predecessors. We have:

Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Point of Interest

The Fibonacci sequence is of particular interest because it is often observed in nature. For example, male honeybees hatch from eggs that have not been fertilized, so each male honeybee has only 1 parent, a female. A female honeybee hatches from a fertilized egg, so she has two parents: one male and one female. Notice that the number of ancestors for *n* generations for a male honeybee follows the Fibonacci sequence (Figure 11-3).

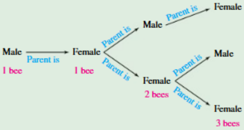


Figure 11-3

Learn: Use Factorial Notation

We now introduce factorial notation to denote the product of the first *n* positive integers.

Definition of *n*!

- Let n be a positive integer. The expression $n!$ (read as "n factorial") is defined as $n! = (n)(n-1)(n-2)\cdots(2)(1)$
- Zero factorial is defined as $0! = 1$.

Examples: $0! = 1$ $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$1! = 1$ $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$2! = 2 \cdot 1 = 2$ $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

$3! = 3 \cdot 2 \cdot 1 = 6$ $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

Example 3: Evaluating Expressions with Factorial Notation

Evaluate.

a. $\dfrac{8!}{3! \cdot 5!}$ **b.** $\dfrac{n!}{(n+2)!}$

Solution:

a. $\dfrac{8!}{3! \cdot 5!} = \dfrac{8 \cdot 7 \cdot \overset{1}{\cancel{6}} \cdot \overset{1}{\cancel{5!}}}{(\cancel{3} \cdot \cancel{2} \cdot 1)(\cancel{5!})} = 56$

b. $\dfrac{n!}{(n+2)!} = \dfrac{\overset{1}{\cancel{n!}}}{(n+2)(n+1)(\cancel{n!})} = \dfrac{1}{(n+2)(n+1)}$

Apply the Skills

3. Evaluate.

a. $\dfrac{9!}{2! \cdot 7!}$ **b.** $\dfrac{(n-1)!}{n!}$

In Example 4, we find a specific term of a sequence in which the expression for the *n*th term contains factorial notation.

Example 4: Finding a Specific Term of a Sequence

Given the sequence defined by $b_n = \frac{n^2}{(n+1)!}$, find b_6.

Solution:

$$b_n = \frac{n^2}{(n+1)!} \xrightarrow[\text{6 for } n.]{\text{Substitute}} b_6 = \frac{(6)^2}{(6+1)!} = \frac{36}{7!} = \frac{\cancel{36}}{7 \cdot \cancel{6} \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = \frac{1}{140}$$

Apply the Skills

4. Given the sequence defined by $c_n = \frac{2^n}{(n+1)!}$, find c_4.

Sometimes we are presented with several terms of a sequence and are asked to find a formula for the *n*th term. In such a case, the goal is to look for a pattern that can be expressed mathematically as a function of the term number.

Example 5: Finding the *n*th Term of a Sequence

Find the *n*th term a_n of a sequence whose first four terms are given.

a. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$ **b.** $\frac{1}{5}, -\frac{4}{25}, \frac{9}{125}, -\frac{16}{625}, \ldots$

Solution:

a. Term number: $n = 1 \quad 2 \quad 3 \quad 4$

$\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$ The numerator is 1 more than the term number: $n + 1$.

$a_n = \frac{n+1}{n+2}$ The denominator is 2 more than the term number: $n + 2$.

b. Term number: $n = 1 \quad 2 \quad 3 \quad 4$

$\frac{1}{5}, -\frac{4}{25}, \frac{9}{125}, -\frac{16}{625}, \ldots$ Terms in the numerator are perfect squares: $1^2, 2^2, 3^2, 4^2, \ldots, n^2$

Terms in the denominator are powers of 5: $5^1, 5^2, 5^3, 5^4, \ldots, 5^n$

$a_n = (-1)^{n+1} \cdot \frac{n^2}{5^n}$ The signs are positive for odd-numbered terms. Therefore, we want a factor of -1 raised to an even exponent on odd-numbered terms and an odd exponent for even-numbered terms: $(-1)^{n+1}$

Apply the Skills

5. Find the nth term a_n of a sequence whose first four terms are given.

a. $-5, 6, -7, 8, \ldots$ **b.** $\frac{1}{3}, \frac{1 \cdot 2}{9}, \frac{1 \cdot 2 \cdot 3}{27}, \frac{1 \cdot 2 \cdot 3 \cdot 4}{81}, \ldots$

Learn: Use Summation Notation

Consider an infinite sequence $\{a_n\} = a_1, a_2, a_3, \ldots$. The sum S of such a sequence is called an **infinite series** and is represented by $S = a_1 + a_2 + a_3 + \cdots$. The sum of the first n terms of the sequence is called the ***n*th partial sum** of the sequence and is denoted by S_n. The nth partial sum of a sequence is a **finite series.**

In mathematics, the Greek letter Σ (sigma) is used to represent summations.

Summation Notation

Given a sequence $a_1, a_2, a_3, \ldots$, the nth partial sum S_n is a finite series and is represented by

n is the upper limit of summation.

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

i is the index of summation.

1 is the lower limit of summation.

The sum of *all* terms in the sequence is an infinite series and is given by

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$$

Any letter such as i, j, k, and n may be used for the index of summation. However, if i is used for the index of summation, do not confuse it with the imaginary number $i = \sqrt{-1}$.

Insights The summation $\sum_{i=1}^{n} a_i$ is read as "the sum of a_i for i equals 1 to n."

Point of Interest

Swiss mathematician and physicist Leonhard Euler (1707–1783) is credited with the use of the modern symbol Σ to represent a sum, i to represent the imaginary unit, and e to represent the base of the natural logarithmic function.

Example 6: Using Summation Notation

Write the terms for each series and evaluate the sum.

a. $\sum_{i=1}^{5}(2i+3)$ **b.** $\sum_{k=3}^{6}(-1)^k\left(\frac{1}{k}\right)$ **c.** $\sum_{n=1}^{6}3$

Solution:

a.
$$\begin{aligned}\sum_{i=1}^{5}(2i+3) &= \overset{i=1}{[2(1)+3]} + \overset{i=2}{[2(2)+3]} + \overset{i=3}{[2(3)+3]} + \overset{i=4}{[2(4)+3]} + \overset{i=5}{[2(5)+3]}\\ &= 5+7+9+11+13\\ &= 45\end{aligned}$$

b.
$$\begin{aligned}\sum_{k=3}^{6}(-1)^k\left(\frac{1}{k}\right) &= \overset{k=3}{\left[(-1)^3\left(\frac{1}{3}\right)\right]} + \overset{k=4}{\left[(-1)^4\left(\frac{1}{4}\right)\right]} + \overset{k=5}{\left[(-1)^5\left(\frac{1}{5}\right)\right]} + \overset{k=6}{\left[(-1)^6\left(\frac{1}{6}\right)\right]}\\ &= -\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}\\ &= -\frac{7}{60}\end{aligned}$$

c.
$$\sum_{n=1}^{6}3 = \underbrace{3+3+3+3+3+3}_{\text{6 terms}} = 18$$

Apply the Skills

6. Evaluate the sum.

a. $\sum_{i=1}^{5}(3i+1)$ **b.** $\sum_{n=2}^{5}(-1)^{n+1}\frac{1}{2^n}$ **c.** $\sum_{k=1}^{5}4$

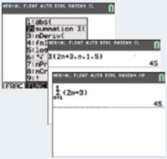
TECHNOLOGY CONNECTIONS

Evaluating a Finite Series

A graphing utility can be used to evaluate a finite series if the *n*th term of the corresponding sequence is known. The Σ symbol can be accessed by selecting ALPHA F2.

- If the calculator is in “Classic” mode, the sum can be evaluated by entering

  Σ (*n*th term, variable, lower limit, upper limit)
- If the calculator is in “Mathprint” mode, then the user is prompted to enter the *n*th term, the variable, and limits of summation in text fields.


NORMAL FLOAT AUTO REAL RADIAN CL
1:abs(
2:summation Σ(
3:nDeriv(
4:fnI
5:log
6:ˣ√
7:nPr
8:nCr
9:!
FRAC
FUNC
NORMAL FLOAT AUTO REAL RADIAN MP

```
Σ(2n+3,n,1,5)
                                 45
```

$$\sum_{n=1}^{5}(2n+3)$$

```
                                 45
```

It is important to note that the lower limit of summation need not be 1. In Example 6(b) for instance, the index of summation ranges from 3 to 6. In fact, sometimes we adjust the limits of summation and the expression being summed to write the summation in a more convenient form. For example, the following three series are all equivalent.

$$\sum_{i=1}^{5}3i = 3(1)+3(2)+3(3)+3(4)+3(5)$$
$$=3+6+9+12+15=45$$

$$\sum_{j=0}^{4}3(j+1) = 3(0+1)+3(1+1)+3(2+1)+3(3+1)+3(4+1)$$
$$=3+6+9+12+15=45$$

$$\sum_{k=2}^{6}3(k-1) = 3(2-1)+3(3-1)+3(4-1)+3(5-1)+3(6-1)$$
$$=3+6+9+12+15=45$$

Example 7: Write a Series Using Summation Notation

Write each sum using summation notation.

a. $\frac{\sqrt{3}}{1!}+\frac{\sqrt{4}}{2!}+\frac{\sqrt{5}}{3!}+\cdots+\frac{\sqrt{n+2}}{n!}$ **b.** $\frac{2}{1}-\frac{3}{4}+\frac{4}{9}-\frac{5}{16}+\frac{6}{25}$

Solution:

a. $\frac{\sqrt{3}}{1!}+\frac{\sqrt{4}}{2!}+\frac{\sqrt{5}}{3!}+\cdots+\frac{\sqrt{n+2}}{n!}$

The series consists of n terms where a formula for the nth term is given.

$=\sum_{i=1}^{n}\frac{\sqrt{i+2}}{i!}$

We have used n as the upper limit of summation. Therefore, use a different variable for the index of summation. We have chosen i.

b.

Look for a relationship between the term number and the values in the numerator and denominator. Choose a variable such as i for the index of summation.

Term: 1 2 3 4 5

$\frac{2}{1}-\frac{3}{4}+\frac{4}{9}-\frac{5}{16}+\frac{6}{25}$

The numerator is 1 more than the term number: $(i+1)$

The denominator is the square of the term number: i^2

$\sum_{i=1}^{5}(-1)^{i+1}\cdot\frac{i+1}{i^2}$

The odd-numbered terms are positive. This can be generated by the factor $(-1)^{i+1}$.

Apply the Skills

7. Write each sum using summation notation.

a. $\frac{2!}{7}+\frac{3!}{14}+\frac{4!}{21}+\cdots+\frac{(n+1)!}{7n}$ **b.** $-\frac{1}{5}+\frac{8}{25}-\frac{27}{125}+\frac{64}{625}$

Several important properties of summation are given below.

Properties of Summation

If $\{a_n\}$ and $\{b_n\}$ are sequences, and c is a real number, then:

1. $\sum_{i=1}^{n}c=cn$ Adding a constant c a total of n times equals cn.
2. $\sum_{i=1}^{n}ca_i=c\sum_{i=1}^{n}a_i$ A constant factor can be factored out of a summation.
3. $\sum_{i=1}^{n}(a_i\pm b_i)=\sum_{i=1}^{n}a_i\pm\sum_{i=1}^{n}b_i$ A single sum or difference can be regrouped as two sums or differences.

The proof of Property 2 follows from the distributive property of real numbers.

$\sum_{i=1}^{n}ca_i=ca_1+ca_2+ca_3+\cdots+ca_n$ Expand the terms in the series.

$=c(a_1+a_2+a_3+\cdots+a_n)$ Apply the distributive property.

$=c\sum_{i=1}^{n}a_i$ Write the sum using summation notation.

The proofs of properties 1 and 3 are examined in Exercises 89 and 90.

Practice 11-1

Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, evaluate the function for the given value.

R.1. $g(x) = \frac{5}{x}$; $g(15)$

R.2. $f(x) = 5$; $f(4)$

R.3. $h(x) = x^2 + 3x$; $h(3)$

For Exercises R.4–R.5, simplify the rational expression.

R.4. $\dfrac{(r-2)^2(2r-1)^4}{(r-2)^4(2r-1)}$

R.5. $\dfrac{p^2(p-6)^5}{p^5(p-6)^2}$

Concept Connections

1. An infinite __________ is a function whose domain is the set of positive integers. A __________ sequence is a function whose domain is the set of the first n positive integers.

2. An __________ sequence is a sequence in which consecutive terms alternate in sign.

3. A __________ formula defines the nth term of a sequence as a function of one or more terms preceding it.

4. For $n \geq 1$, the expression __________ represents the product of the first n positive integers $n(n-1)(n-2) \cdots (2)(1)$. Furthermore, by definition, $0! =$ __________.

5. Given $\sum_{i=1}^{n} a_i$, the variable i is called the __________ of __________. The value 1 is called the __________ limit of summation. The value n is called the upper __________ of summation.

6. One property of summation indicates that $\sum_{i=1}^{n} c =$ __________.

Learn: Write Terms of a Sequence from the nth Term

For Exercises 7–14, the nth term of a sequence is given. Write the first four terms of the sequence. (See Example 1, p. 1167)

7. $a_n = 2n^2 + 3$

8. $b_n = -n^3 + 5$

9. $c_n = 12\left(-\frac{1}{2}\right)^n$

10. $d_n = 64\left(-\frac{1}{4}\right)^n$

11. $a_n = \sqrt{n+3}$

12. $b_n = \sqrt{n-1}$

13. $a_n = e^{3\ln n}$

14. $c_n = \ln e^{2n}$

15. If the nth term of a sequence is $(-1)^n n^2$, which terms are positive and which are negative?

16. If the nth term of a sequence is $(-1)^{n+1}\frac{1}{n}$, which terms are positive and which are negative?

For Exercises 17–20, the nth term of a sequence is given. Find the indicated term.

17. $a_n = 2^n - 1$; find a_{10}

18. $b_n = \frac{5}{n} - 1$; find b_{20}

19. $c_n = 5n - 4$; find c_{157}

20. $d_n = 6n + 7$; find d_{204}

For Exercises 21–24, match the sequence or function with its graph from the choices a–d.

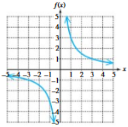
21. $a_n = \sqrt{n}$

22. $f(x) = \sqrt{x}$

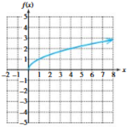
23. $f(x) = \frac{3}{x}$

24. $a_n = \frac{3}{n}$

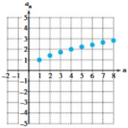
a.



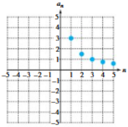
b.



c.



d.



Learn: Write Terms of a Sequence Defined Recursively

For Exercises 25–30, write the first five terms of the sequence defined recursively. (See Example 2, p. 1169)

25. $b_1 = -3$; $b_n = 3b_{n-1} + 4$

26. $a_1 = 11$; $a_n = 4a_{n-1} + 3$

27. $c_1 = 4$; $c_n = \frac{1}{2}c_{n-1} + 4$

28. $d_1 = 30$; $d_n = \frac{1}{3}d_{n-1} - 1$

29. $a_1 = 5$; $a_n = \frac{1}{a_{n-1}}$

30. $b_1 = 10$; $b_n = -\frac{1}{b_{n-1}}$

31. a. The Fibonacci sequence is defined by $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-2} + a_{n-1}$. Write the first 10 terms of the Fibonacci sequence.

b. The formula

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

gives the *n*th term of the Fibonacci sequence. Use a calculator to verify this statement for $n = 1$, $n = 2$, and $n = 3$.

32. The numbers in the sequence defined by $a_1 = 1$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ are referred to as Lucas numbers in honor of French mathematician Edouard Lucas (1842–1891).

a. Find the first eight Lucas numbers.

b. The formula $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$ gives the *n*th Lucas number. Use a calculator to verify this statement for $n = 1$, $n = 2$, and $n = 3$.

Learn: Use Factorial Notation

For Exercises 33–44, evaluate the expression. (See Example 3, p. 1171)

33. 7!

34. 8!

35. 0!

36. 1!

37. $\frac{8!}{6!}$

38. $\frac{12!}{9!}$

39. $\frac{9!}{5! \cdot 4!}$

40. $\frac{10!}{6! \cdot 4!}$

41. $\frac{(n-1)!}{(n+1)!}$

42. $\frac{(n-2)!}{n!}$

43. $\frac{(2n)!}{(2n+1)!}$

44. $\frac{(2n-1)!}{(2n)!}$

For Exercises 45–48, the *n*th term of a sequence is given. Find the indicated term. (See Example 4, p. 1171)

45. $a_n = \frac{2^n}{(n+2)!}$; find a_5

46. $b_n = \frac{4n}{(n+1)!}$; find b_3

47. $c_n = \frac{(3n)!}{5n}$; find c_3

48. $d_n = \frac{(n+3)!}{3^n}$; find d_3

For Exercises 49–56, find the *n*th term a_n of a sequence whose first four terms are given. (See Example 5, p. 1171)

49. 2, 4, 8, 16, ...

50. 3, 9, 27, 81, ...

51. $-\frac{2}{3}, -\frac{3}{6}, -\frac{4}{9}, -\frac{5}{12}, \ldots$

52. $-\frac{8}{11}, -\frac{9}{22}, -\frac{10}{33}, -\frac{11}{44}, \ldots$

53. –1, 4, –9, 16, ...

54. 1, –8, 27, –64, ...

55. $\frac{1 \cdot 2}{2}, \frac{1 \cdot 2 \cdot 3}{4}, \frac{1 \cdot 2 \cdot 3 \cdot 4}{6}, \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{8}, \ldots$

56. $\frac{5}{4}, \frac{10}{9}, \frac{15}{16}, \frac{20}{25}, \ldots$

Learn: Use Summation Notation

For Exercises 57–70, find the sum. (See Example 6, p. 1173)

57. $\sum_{i=1}^{6}(3i-4)$

58. $\sum_{i=1}^{5}(2i+7)$

59. $\sum_{j=1}^{5}(-2j^2)$

60. $\sum_{k=1}^{3}(-3k^3)$

61. $\sum_{j=2}^{5}\left(\frac{1}{2}\right)^j$

62. $\sum_{n=2}^{4}\left(\frac{1}{3}\right)^n$

63. $\sum_{j=1}^{20}6$

64. $\sum_{i=1}^{50}4$

65. $\sum_{k=3}^{7}(-1)^k(6k)$

66. $\sum_{n=3}^{8}(-1)^{n-1}(2n)$

67. $\sum_{m=1}^{3}\frac{m+1}{m}$

68. $\sum_{n=1}^{4}\frac{n-1}{n}$

69. $\sum_{k=1}^{3}(k+2)(k+3)$

70. $\sum_{j=1}^{4}(j+1)(j-1)$

71. **a.** Evaluate $\sum_{i=1}^{n}(-1)^i$ if n is even.

b. Evaluate $\sum_{i=1}^{n}(-1)^i$ if n is odd.

72. **a.** Evaluate $\sum_{i=1}^{n}(-1)^{i+1}$ if n is even.

b. Evaluate $\sum_{i=1}^{n}(-1)^{i+1}$ if n is odd.

73. Given the sequence defined by $a_n = n^2 - 2n$, find the fifth partial sum.

74. Given the sequence defined by $b_n = n^3 - 3n^2$, find the fourth partial sum.

For Exercises 75–86, write the sum using summation notation. There may be multiple representations. Use *i* as the index of summation. (See Example 7, p. 1174)

75. $\frac{1}{2} + \frac{4}{3} + \frac{9}{4} + \frac{16}{5} + \cdots + \frac{n^2}{n+1}$

76. $3 + \frac{1}{2} + \frac{5}{27} + \frac{3}{32} + \cdots + \frac{n+2}{n^3}$

77. $1 + 2 + 3 + 4 + 5$

78. $2 + 4 + 6 + 8 + 10 + 12$

79. $8 + 8 + 8 + 8$

80. $11 + 11 + 11 + 11 + 11$

81. $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81}$

82. $-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}$

83. $c^3 + c^4 + c^5 + \cdots + c^{20}$

84. $a + ar + ar^2 + \cdots + ar^{12}$

85. $\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ (*Hint*: Note the pattern produced by $n!$ for $n = 1$, $n = 2$, and so on.)

86. $\frac{1}{x+1} + \frac{2}{x+2} + \frac{6}{x+3} + \frac{24}{x+4} + \frac{120}{x+5}$

For Exercises 87–88, rewrite each series as an equivalent series with the new index of summation.

87. $\sum_{i=1}^{8}i^2 = \sum_{j=0}^{\square}\square = \sum_{k=2}^{\square}\square$

88. $\sum_{i=1}^{5}(4i) = \sum_{j=0}^{\square}\square = \sum_{k=2}^{\square}\square$

Mixed Exercises

89. Prove that $\sum_{i=1}^{n}c = cn$.

90. Prove that $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n}a_i + \sum_{i=1}^{n}b_i$.

For Exercises 91–94, use the sums $\sum_{i=1}^{50} i^2 = 42{,}925$ and $\sum_{i=1}^{50} i = 1275$ and the properties of summation given on page 1175 to evaluate the given expression.

91. $\sum_{i=1}^{50}(i^2 + 3i)$

92. $\sum_{i=1}^{50}(2i^2 - i)$

93. $\sum_{i=1}^{50}(5i + 4)$

94. $\sum_{i=1}^{50}(6i - 7)$

For Exercises 95–98, determine whether the statement is true or false. If a statement is false, explain why.

95. $\sum_{i=1}^{n}(3i + 7) = 3\sum_{i=1}^{n} i + 7n$

96. $\sum_{i=1}^{n}(i^2 - 4i + 5) = \sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + 5n$

97. $\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$

98. $\sum_{i=1}^{n}\frac{a_i}{b_i} = \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$

99. Expenses for a company for year 1 are \$24,000. Every year thereafter, expenses increase by \$1000 plus 3% of the cost of the prior year. Let a_1 represent the original cost for year 1; that is, $a_1 = 24{,}000$. Use a recursive formula to find the cost a_n in terms of a_{n-1} for each subsequent year, $n \geq 2$.

100. A retirement account initially has \$500,000 and grows by 5% per year. Furthermore, the account owner adds \$12,000 to the account each year after the first. Let a_1 represent the original amount in the account; that is, $a_1 = 500{,}000$. Use a recursive formula to find the amount in the account a_n in terms of a_{n-1} for each subsequent year, $n \geq 2$.

101. In a business meeting, every person at the meeting shakes every other person's hand exactly one time. The total number of handshakes for n people at the meeting is given by $a_n = \frac{1}{2}n(n - 1)$. Evaluate a_{12} and interpret its meaning in the context of this problem.

102. Given a polygon of $n \geq 3$ sides, the sum of the interior angles within the polygon is given by $s_n = 180(n - 2)$. Evaluate s_{10} and interpret its meaning in the context of this problem.

Given a sequence $a_1, a_2, a_3, \ldots, a_n$, the arithmetic mean a is given by $a = \frac{1}{n}\sum_{i=1}^{n} a_i$. Use the arithmetic mean for Exercises 103–104.

103. Consider the sequence defined by $\{a_n\} = 18, 32, 44, 20, 36, 28, 32, 38$. Evaluate $\sum_{i=1}^{8}(a_i - a)^2$.

104. Show that $\sum_{i=1}^{n}(a_i - a) = 0$.

For Exercises 105–108, write the first four terms for each sequence.

105. a. $a_n = 2n + 5$

b. $b_1 = 7, b_n = 2b_{n-1} + 5$

106. a. $a_n = (-1)_n(n + 3)$

b. $b_1 = -4, b_n = (-1)_n(b_{n-1} + 3)$

107. a. $a_n = \frac{n}{n + 1}$

b. $b_n = \frac{n!}{(n + 1)!}$

108. a. $b_n = \frac{3^n}{n + 3}$

b. $b_n = \frac{3^n}{(n + 3)!}$

For Exercises 109–112, write the sum using the given summation notation.

109. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$

a. $\sum_{n=1}^{4}$ **b.** $\sum_{n=0}^{3}$

110. $\frac{3}{2} + 1 + \frac{3}{4} + \frac{3}{5}$

a. $\sum_{n=1}^{4}$ **b.** $\sum_{n=3}^{6}$

111. $1 + (-1) + (-3) + (-5)$

a. $\sum_{n=0}^{3}$ **b.** $\sum_{n=5}^{8}$

112. $3 + 3 + 3 + 3$

a. $\sum_{n=2}^{15}$ **b.** $\sum_{n=23}^{26}$

Write About It

113. Explain the difference between the graph of $a_n = n^2$ and $f(x) = x^2$.

114. What is the difference between a sequence and a series?

115. Given the sequence defined by $a_n = \frac{n}{n-1}$, explain why the domain must be restricted to positive integers $n \geq 2$.

116. The value 1206! is too large to evaluate on most calculators. Explain how you would evaluate $\frac{1206!}{1204!}$ on a calculator.

Expanding Your Skills

117. For $i = \sqrt{-1}$, find the first eight terms of the sequence defined by $a_n = i^n$.

118. For $i = \sqrt{-1}$, find the first eight terms of the sequence defined by $a_n = 1 - i^n$.

119. The terms of the sequence defined by $a_1 = x$ and $a_n = \frac{1}{2}\left(a_{n-1} + \frac{x}{a_{n-1}}\right)$ for $n > 1$ give successively better approximations of $\sqrt{x}$ for $x > 1$. Approximate $\sqrt{2}$ by substituting 2 for x and finding the first four terms of the sequence. Round to 4 decimal places if necessary.

120. Find a formula for the nth term of the sequence. $\sqrt{3}, \sqrt{\sqrt{3}}, \sqrt{\sqrt{\sqrt{3}}}, \ldots$

The On-Line Encyclopedia of Integer Sequences (oeis.org) is a vast repository of named and unnamed sequences. The OEIS not only gives numerical information about a sequence, but also enables you to *listen* to a sequence by converting the numbers to musical notes. After completing exercises 121–122, take a minute to visit the site.

121. The **Look and Say sequence** begins with any single digit from 0 to 9. Subsequent terms are defined by reading the previous term from left to right and "saying" the number of consecutive times a digit occurs in a group followed by the digit. For example, suppose we begin with $a_1 = 2$. To find a_2, we describe the previous term. We would *say* there is "one 2." Therefore, $a_2 = 12$.

To find a_3, describe a_2: One 1 and one 2. Therefore $a_3 = 1112$.

To find a_4, describe a_3: Three 1s and one 2. Therefore $a_4 = 3112$.

So, the Look and Say sequence beginning with 2 is: 2, 12, 1112, 3112, ...

- **a.** Write the first five terms in the Look and Say sequence beginning with 5.
- **b.** Make your own variation of a Look and Say sequence beginning with any positive integer.

122. A **Hailstone sequence** is named for the behavior of the numbers in the sequence. The numbers seem to "rise and fall" like hailstones forming in a cloud. The sequence begins with a positive integer "seed." This is the first term, a_1. Subsequent terms are found as follows: If the previous term is *even*, divide it by 2; if the previous term is *odd*, multiply by 3 and add 1.

For example, begin with $a_1 = 3$. Because 3 is odd, multiply it by 3 and add 1 to obtain $a_2 = 10$. Because 10 is even, divide it by 2 to obtain $a_3 = 5$. So, the Hailstone sequence beginning with 3 is: 3, 10, 5, ...

- **a.** What are the next 8 numbers in the sequence?
- **b.** Use the seed $n = 6$ and write the first 12 terms in the sequence.

An open question in mathematics is whether every Hailstone sequence eventually reaches 1 (cycling 4, 2, 1, 4, 2, 1, ...). This is known as the Collatz conjecture (or the $3n + 1$ problem), introduced in 1937 by Lothar Collatz. As of the year 2020, values up to 2^{68} have been checked by a computer, and all end in 1. But no one has been able to formally prove the conjecture.

Technology Connections

For Exercises 123–124, use a graphing utility to find the first four terms of the sequence.

123. $a_n = 12\left(-\frac{1}{2}\right)^n$ (Exercise 9)

124. $a_n = 64\left(-\frac{1}{4}\right)^n$ (Exercise 10)

For Exercises 125–126, use a graphing utility to find the sum.

125. $\sum_{k=3}^{7}(-1)^k(6k)$ (Exercise 65)

126. $\sum_{n=3}^{8}(-1)^{n-1}(2n)$ (Exercise 66)

127. Using calculus, we can show that the series $\sum_{k=0}^{n}\frac{1}{k!}$ approaches *e* as *n* approaches infinity. Investigate this statement by evaluating the sum for $n = 10$ and $n = 50$.

128. Using calculus, we can show that the series $\sum_{k=1}^{n}(-1)^{k-1}\frac{(0.5)^k}{k}$ approaches ln 1.5 as *n* approaches infinity. Investigate this statement by evaluating the sum for $n = 10$ and $n = 50$.

Point of Interest

Recamán's sequence is named after its inventor, Columbian mathematician Bernardo Recamán Santos, and is defined as follows:

Begin with $a_0 = 0$. Each subsequent term will either be

- The value of the previous term *minus* the term number, provided that this value is positive and not already in the sequence.
- Otherwise, the value of the previous term *plus* the term number.

In summary:

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ a_{n-1} - n & \text{if } a_{n-1} - n > 0 \text{ \textit{and} is not already in the sequence} \\ a_{n-1} + n & \text{otherwise} \end{cases}$$

$$\text{So, } a_1 = \begin{cases} a_0 - 1 & \text{if this is positive \textit{and} is not already in the sequence; } a_1 = 1 \\ a_0 + 1 & \text{otherwise} \end{cases}$$

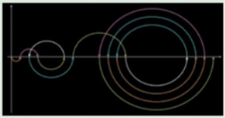
$$a_2 = \begin{cases} a_1 - 2 & \text{if this is positive \textit{and} is not already in the sequence; } a_2 = 3 \\ a_1 + 2 & \text{otherwise} \end{cases}$$

$$a_3 = \begin{cases} a_2 - 3 & \text{if this is positive \textit{and} is not already in the sequence; } a_3 = 6 \\ a_2 + 3 & \text{otherwise} \end{cases}$$

129. Write the next 10 terms in Recamán's sequence.

For Further Study

A visual representation of Recamán's sequence uses the number line and semicircles to illustrate the backward and forward movement of the numbers in the sequence. Visit *The Slightly Spooky Recamán Sequence* (June 2018) video from Numberphile on YouTube.



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Go online for more practice problems.

Lesson 11-2

Arithmetic Sequences and Series

What Will You Learn?

After completing this lesson, you should be able to:

- Identify Specific and General Terms of an Arithmetic Sequence
- Evaluate a Finite Arithmetic Series
- Apply Arithmetic Sequences and Series

Learn: Identify Specific and General Terms of an Arithmetic Sequence

In this lesson and Lesson 11-3, we study two special types of sequences. The first is called an arithmetic sequence. For example, consider the salary plan for a job that pays \$75,000 the first year with a \$4000 raise each year thereafter. The sequence of salaries for the first 5 yr is

Year 1	Year 2	Year 3	Year 4	Year 5
\$75,000	\$79,000	\$83,000	\$87,000	\$91,000

Notice that each term after the first results from adding a fixed constant (\$4000) to its predecessor. This is the characteristic that makes this sequence arithmetic.

Arithmetic Sequence

An **arithmetic sequence** $\{a_n\}$ is a sequence of the form

$$a_1, a_1 + d, a_1 + 2d, a_1 + 3d, a_1 + 4d, \ldots$$

- The value a_1 is the first term, and d is called the **common difference** of the sequence.
- The value of d is the difference of any term after the first and its predecessor. $d = a_{n+1} - a_n$
- The nth term of the sequence is given by $a_n = a_1 + (n - 1)d$.
- The sequence can be defined recursively as a_1, $a_n = a_{n-1} + d$ for $n \geq 2$.

Example 1: Identifying an Arithmetic Sequence and the Common Difference

Determine whether the sequence is arithmetic. If so, identify the common difference.

a. 35, 25, 15, 5, −5, ...

b. 1, 5, 10, 16, 23, ...

Solution:

a. 35, 25, 15, 5, −5, ...

$a_2 - a_1 = 25 - 35 = -10$

$a_3 - a_2 = 15 - 25 = -10$

$a_4 - a_3 = 5 - 15 = -10$

$a_5 - a_4 = -5 - 5 = -10$

The sequence is arithmetic because the difference between each term and its predecessor is the same constant.

The common difference is $d = -10$.

b. 1, 5, 10, 16, 23, ...

$a_2 - a_1 = 5 - 1 = 4$

$a_3 - a_2 = 10 - 5 = 5$

The sequence is *not* arithmetic because the difference between a_2 and a_1 is different than the difference between a_3 and a_2. That is, the difference between consecutive terms is not the same.

Apply the Skills

1. Determine whether the sequence is arithmetic. If so, identify the common difference.

a. 12, 5, −2, −9, −16, ...

b. 1, 4, 9, 16, 25, ...

Example 2: Writing the Terms of a Sequence

a. Write the first five terms of an arithmetic sequence with first term −5 and common difference 4.

b. Write a recursive formula to define the sequence.

Solution:

a. $$-5,\ \underbrace{\qquad}_{+4}\ -1,\ \underbrace{\qquad}_{+4}\ 3,\ \underbrace{\qquad}_{+4}\ 7,\ \underbrace{\qquad}_{+4}\ 11$$

Because the common difference is 4, each term after the first must be 4 *more than* its predecessor.

b. The recursive formula for an arithmetic sequence is $a_1, a_n = a_{n-1} + d$ for $n \geq 2$.

The sequence is defined by $a_1 = -5$ and $a_n = a_{n-1} + 4$ for $n \geq 2$.

Apply the Skills

2. **a.** Write the first four terms of an arithmetic sequence with first term 8 and common difference −3.

b. Write a recursive formula to define the sequence.

Example 3: Applying an Arithmetic Sequence

A park ranger in Everglades National Park measures the water level in one region of the park for a 5-day period during a drought.

Day number	1	2	3	4	5
Water level (in.)	54.0	53.2	52.4	51.6	50.8

a. Based on the given data, does the water level follow an arithmetic progression?

b. Write an expression for the *n*th term of the sequence where *n* represents the day number.

c. Predict the water level on day 30 if this trend continues.

Solution:

a. $a_2 - a_1 = 53.2 - 54.0 = -0.8$ The sequence is arithmetic because the difference between each term and its predecessor is the same constant.

$a_3 - a_2 = 52.4 - 53.2 = -0.8$

$a_4 - a_3 = 51.6 - 52.4 = -0.8$

$a_5 - a_4 = 50.8 - 51.6 = -0.8$ The common difference is $d = -0.8$ in.

b. $a_n = a_1 + (n-1)d$ Formula for the *n*th term of an arithmetic sequence. Substitute 54 for a_1 and -0.8 for *d*.

$a_n = 54 + (n-1)(-0.8)$

c. $a_{30} = 54 + (30-1)(-0.8)$ To find a_{30}, substitute 30 for *n*.

$= 30.8$

The water level on day 30 will be 30.8 in. if this trend continues.

Apply the Skills

3. A homeowner has kept records of the average monthly electric bill for 4 yr.

Year	1	2	3	4
Amount ($)	102.60	108.00	113.40	118.80

a. Do the average monthly electric bills follow an arithmetic progression?

b. Write an expression for the *n*th term of the sequence.

c. Predict the average monthly bill for year 6 if this trend continues.

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The *n*th term of the sequence from Example 3 can be written in several equivalent algebraic forms.

$a_n = 54 + (n - 1)(-0.8)$

$a_n = 54 - 0.8n + 0.8$ Apply the distributive property.

$a_n = -0.8n + 54.8$

The expression $a_n = -0.8n + 54.8$ resembles the slope-intercept form of a linear function $f(x) = mx + b$. In fact, an arithmetic sequence is a linear function whose domain is the set of positive integers. The graph of the sequence from Example 3 is shown in Figure 11-4. The value of the common difference is negative ($d = -0.8$), indicating that the progression of points slopes downward.

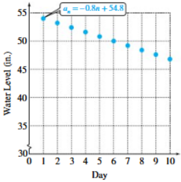


Figure 11-4

In Example 4, we find specific terms of an arithmetic sequence given information about the sequence.

Example 4: Finding a Specific Term of an Arithmetic Sequence

Find the ninth term of the arithmetic sequence in which $a_1 = -4$ and $a_{22} = 164$.

Solution:

To find the value of a_9, we need to determine the common difference *d*. To find *d*, substitute $a_1 = -4$, $n = 22$, and $a_{22} = 164$ into the formula for the *n*th term.

$$a_n = a_1 + (n - 1)d$$

$$164 = -4 + (22 - 1)d$$

$$164 = -4 + 21d$$

$$d = 8$$

Therefore, $a_n = -4 + (n - 1)(8)$. With the values of a_1 and *d* known, the *n*th term is represented by $a_n = -4 + (n - 1)(8)$.

$a_n = -4 + 8n - 8$ Simplify.

$a_n = 8n - 12$

$a_9 = 8(9) - 12$ To find a_9 substitute $n = 9$.

$= 60$

Apply the Skills

4. Find the tenth term of the arithmetic sequence in which $a_1 = 12$ and $a_{30} = 128$.

In Example 5, we use the general formula for the *n*th term of an arithmetic sequence to find the number of terms in an arithmetic sequence.

Example 5: Finding the Number of Terms in an Arithmetic Sequence

Find the number of terms of the finite arithmetic sequence 7, 3, –1, –5, … , –113.

Solution:

$a_n = a_1 + (n - 1)d$ The first term of the sequence is 7. The *n*th term is –113. The common difference is $d = -4$.

$-113 = 7 + (n - 1)(-4)$ To find the number of terms *n*, substitute $a_1 = 7$, $d = -4$, and $a_n = -113$ into the formula for the *n*th term.

$$-113 = 7 - 4n + 4$$

$$-113 = 11 - 4n$$

$$4n = 124$$

$n = 31$ There are 31 terms.

Apply the Skills

5. Find the number of terms of the finite arithmetic sequence 16, 11, 6, 1, … , –239.

Example 6: Finding a Specific Term of an Arithmetic Sequence

For an arithmetic sequence, $a_{15} = 49$ and $a_{27} = 85$. Find the 500th term.

Solution:

Substituting $a_{15} = 49$ and $a_{27} = 85$ into the formula $a_n = a_1 + (n - 1)d$, we can set up a system of linear equations to solve for a_1 and d.

$$\begin{array}{ll} a_n = a_1 + (n-1)d & a_n = a_1 + (n-1)d \\ 49 = a_1 + (15-1)d & 85 = a_1 + (27-1)d \\ 49 = a_1 + 14d & 85 = a_1 + 26d \end{array}$$

Substitute $a_{15} = 49$ and $a_{27} = 85$ into the formula for the *n*th term.

The two equations form a system of linear equations in two variables.

$$\begin{array}{lcrl} 49 = a_1 + 14d & \xrightarrow{\text{Multiply by } -1} & -49 = -a_1 - 14d & 49 = a_1 + 14d \\ 85 = a_1 + 26d & & \underline{85 = a_1 + 26d} & \rightarrow 49 = a_1 + 14(3) \\ & & 36 = 12d & 7 = a_1 \\ & & 3 = d & \end{array}$$

With $d = 3$ and $a_1 = 7$, we have an nth term of $a_n = 7 + (n - 1)3$. Therefore, the 500th term is given by $a_{500} = 7 + (500 - 1)3 = 1504$.

Insights

In Example 6, we can also find the value of d by dividing the difference of a_{27} and a_{15} by the number of terms between a_{27} and a_{15}. That is,

$$d = \frac{85 - 49}{27 - 15} = 3$$

Apply the Skills

6. For an arithmetic sequence, $a_{11} = 65$ and $a_{25} = 149$. Find the 400th term.

Learn: Evaluate a Finite Arithmetic Series

Consider the finite arithmetic sequence 1, 4, 7, 10, 13. The sum of the first n terms of the sequence is called the ***n*th partial sum** and is denoted by S_n. In this case, we have

$$\begin{aligned} S_1 &= 1 && = 1 \\ S_2 &= 1 + 4 && = 5 \\ S_3 &= 1 + 4 + 7 && = 12 \\ S_4 &= 1 + 4 + 7 + 10 && = 22 \\ S_5 &= 1 + 4 + 7 + 10 + 13 && = 35 \end{aligned}$$

Insights

The sum of the first n terms of an arithmetic sequence is called a **finite arithmetic series.**

For a large number of terms, adding the terms individually would be a cumbersome process, so instead, we observe the following pattern. We write the sum of the first five terms of the sequence in both ascending and descending order.

Ascending order: $S_5 = 1 + 4 + 7 + 10 + 13$

Descending order: $\underline{S_5 = 13 + 10 + 7 + 4 + 1}$

$2S_5 = 14 + 14 + 14 + 14 + 14$ Now add to get $2S_5$.

$$2S_5 = 5(14)$$

$$\frac{2S_5}{2} = \frac{5(14)}{2}$$

$$S_5 = 35 \checkmark$$

By adding the terms in ascending and descending order, we double the sum but create a pattern that is easily added. We can use a similar process to find the sum S_n of the first n terms of an arithmetic sequence: $a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$.

$$\begin{aligned}
S_n &= a_1 \quad + (a_1 + d) + (a_1 + 2d) + \cdots + a_n && \text{Ascending order}\\
\underline{S_n} &\underline{= a_n \quad + (a_n - d) + (a_n - 2d) + \cdots + a_1} && \text{Descending order}\\
2S_n &= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n)\\
2S_n &= n(a_1 + a_n)\\
S_n &= \frac{n}{2}(a_1 + a_n)
\end{aligned}$$

nth Partial Sum of an Arithmetic Sequence

The sum S_n of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}(a_1 + a_n)$$

where a_1 is the first term of the sequence, and a_n is the nth term of the sequence.

In Example 7, we use the formula for the nth partial sum of an arithmetic sequence to find the sum of the first 50 positive even integers.

Example 7: Finding an *n*th Partial Sum of an Arithmetic Sequence

Find the sum of the first 50 terms of the sequence. 2, 4, 6, 8, 10, ...

Solution:

The sequence is arithmetic because each term is 2 more than its predecessor. The common difference is 2. To find the sum of the first 50 terms, we need to know the values of a_1 and a_{50}. To find a_{50}, we write a formula for the nth term and then evaluate the expression for $n = 50$.

$$\begin{aligned}
a_n &= a_1 + (n-1)d && \text{General expression for the } n\text{th term}\\
a_n &= 2 + (n-1)(2) && \text{Substitute } a_1 = 2 \text{ and } d = 2\\
a_n &= 2n\\
a_{50} &= 2(50) && \text{To find } a_{50}\text{, substitute 50 for } n.\\
a_{50} &= 100\\
S_n &= \frac{n}{2}(a_1 + a_n) && \text{Now find the sum of the first 50 terms.}\\
S_{50} &= \frac{50}{2}(2 + 100) && \text{Substitute } n = 50,\ a_1 = 2, \text{ and } a_{50} = 100.\\
S_{50} &= 2550 && \text{The sum of the first 50 positive even integers is 2550.}
\end{aligned}$$

Apply the Skills

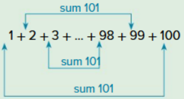
7. Find the sum of the first 50 terms of the sequence. 1, 3, 5, 7, 9, ...

Point of Interest

An old story suggests that at the age of 7, German mathematician Carl Friedrich Gauss (1777–1855) amazed his teachers by quickly computing the sum of the first 100 positive integers. Gauss realized that the sum of the integers from 1 to 100 was the same as 50 pairs of numbers each summing to 101. As the story goes, Gauss returned the correct sum of 5050 within minutes by taking the product of 50(101).

Sum of the first 100 positive integers:

50 pairs, each summing to 101.



In Example 8, we evaluate an arithmetic series written in summation notation.

Example 8: Evaluating a Finite Arithmetic Series

Find the sum. $\sum_{i=1}^{60}(3i+5)$

Solution:

The expression $3i + 5$ is linear in the variable i. This tells us that the terms of the series form an arithmetic progression. To verify, we can write several terms of the sum.

$$\begin{aligned}\sum_{i=1}^{60}(3i+5) &= [3(1)+5]+[3(2)+5]+[3(3)+5]+\cdots+[3(60)+5]\\ &= 8+11+14+\cdots+185\end{aligned}$$

The individual terms in the series 8, 11, 14, ... , 185 form an arithmetic sequence $\{a_n\}$ with a common difference of 3. The first term is $a_1 = 8$, and the 60th term is $a_{60} = 185$. The value of the series is equal to the 60th partial sum of the sequence of terms.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{60} = \frac{60}{2}(8 + 185) \qquad \text{Substitute } n = 60,\ a_1 = 8, \text{ and } a_{60} = 185.$$

$$S_{60} = 5790$$

Check Your Work

When we apply the formula $S_n = \frac{n}{2}(a_1 + a_n)$ to find the sum $\sum_{i=1}^{n} a_n$, the index of summation, i, must begin at 1.

Apply the Skills

8. Find the sum $\sum_{i=1}^{80}(4i+3)$

Learn: Apply Arithmetic Sequences and Series

In Example 9, we use an arithmetic sequence and series in an application.

Example 9: Applying an Arithmetic Sequence and Series

Suppose that a job offers a starting salary of \$75,000 with a raise of \$4000 every year thereafter.

a. Write an expression for the nth term of an arithmetic sequence that represents the salary as a function of the number of years of employment, n.

b. Find the total income for an employee who works at the job for 20 yr.

Solution:

a. $a_n = a_1 + (n-1)d$

$a_n = 75{,}000 + (n-1)(4000)$ Substitute $a_1 = 75{,}000$ and $d = 4000$.

$a_n = 75{,}000 + 4000n - 4000$ Simplify.

$a_n = 4000n + 71{,}000$

b. $a_{20} = 75{,}000 + (20-1)(4000)$ To find the total income over 20 yr, we need to know a_1 and a_{20}. Use the nth term to find a_{20}.

$a_{20} = 151{,}000$

$S_n = \frac{n}{2}(a_1 + a_n)$

$S_{20} = \frac{20}{2}(75{,}000 + 151{,}000)$ Substitute $n = 20$, $a_1 = 75{,}000$, and $a_{20} = 151{,}000$.

$S_{20} = 2{,}260{,}000$

Apply the Skills

9. A teaching position has a starting salary of \$60,000 with a raise of \$3000 every year thereafter.

a. Write an expression for the nth term of an arithmetic sequence that represents the salary as a function of the number of years of employment, n.

b. Find the total income for an employee who works at the job for 30 yr.

Practice 11-2

Practice Exercises

Prerequisite Review

R.1. Given $f(x) = 7x - 9$, evaluate $f(2)$.

R.2. Determine the slope and y-intercept of the line $y = -\frac{1}{4}x + 8$.

For Exercises R.3–R.4, solve the system.

R.3. $6x - 2y = 10$

$2x - 10y = 22$

R.4. $0.3x - 0.4y = -1.6$

$0.9x + 0.1y = -3.5$

Concept Connections

1. A(n) ______________ sequence is a sequence in which each term after the first is found by adding a fixed constant to its predecessor.

2. The difference between any term after the first and its predecessor in an arithmetic sequence is called the ______________ ______________ and is denoted by d.

3. Given an arithmetic sequence with first term a_1 and common difference d, the nth term is represented by the formula $a_n =$ ______________ or by the recursive formula $a_n =$ ______________ for $n \geq 2$.

4. An arithmetic sequence is a linear function whose domain is the set of ______________ integers.

5. The sum of the first n terms of a sequence is called the nth ______________ sum and is denoted by S_n.

6. Given an arithmetic sequence with first term a_1 and nth term a_n, the nth partial sum is given by the formula $S_n =$ ____________.

Learn: Identify Specific and General Terms of an Arithmetic Sequence

For Exercises 7–14, determine whether the sequence is arithmetic. If so, find the common difference. (See Example 1, p. 1182)

7. 15, 19, 23, 27, ...

8. 256, 268, 280, 292, ...

9. 9, −2, −13, −24, ...

10. 8, 0, −8, −16, ...

11. 18, 22, 27, 33, ...

12. 2, 4, 8, 16, ...

13. $4, \frac{14}{3}, \frac{16}{3}, 6, \ldots$

14. $3, \frac{15}{4}, \frac{9}{2}, \frac{21}{4}, \ldots$

For Exercises 15–18,

a. Write the first five terms of an arithmetic sequence with the given first term and common difference.

b. Write a recursive formula to define the sequence. (**See Example 2, p. 1183**)

15. $a_1 = 3, d = 10$

16. $a_1 = 6, d = 5$

17. $a_1 = 4, d = -2$

18. $a_1 = 5, d = -3$

For Exercises 19–24,

a. Write a nonrecursive formula for the nth term of the arithmetic sequence $\{a_n\}$ based on the given information.

b. Find the indicated term.

19. a. $a_1 = 7, d = 10$

b. Find a_{22}.

20. a. $a_1 = 102, d = 4$

b. Find a_{43}.

21. a. $a_1 = -12, d = 5$

b. Find a_{20}.

22. a. $a_1 = -4, d = 6$

b. Find a_{18}.

23. a. $a_1 = \frac{1}{2}, d = \frac{1}{3}$

b. Find a_{10}.

24. a. $a_1 = \frac{2}{3}, d = \frac{1}{2}$

b. Find a_8.

25. K'yeon has 8 unread emails in his inbox before going on vacation. While on vacation, K'yeon does not read email. If he receives an average of 22 emails each day, write the *n*th term of a sequence defining the number of unread emails in his box at the end of day *n* of K'yeon's vacation.

26. Lita has a personal trainer who encourages cardiovascular exercise. In the first week of training, Lita walks for 10 min on a treadmill every day. Each week thereafter, Lita increases the time on the treadmill by 5 min. Write the *n*th term of a sequence defining the number of minutes spent on the treadmill per day for Lita's *n*th week at the gym.

27. A new drug and alcohol rehabilitation program performs outreach for members of the community. The number of participants for a 4-week period is given in the table. **(See Example 3, p. 1184)**

Week number	1	2	3	4
Number of participants	34	50	66	82

a. Based on the data given, does the number of participants follow an arithmetic progression?

b. Write an expression for the *n*th term of the sequence representing the number of participants, where *n* represents the week number.

c. Predict the number of participants in week 10 if this trend continues.

28. A student studying to be a veterinarian's assistant keeps track of a kitten's weight each week for a 5-week period after birth.

Week number	1	2	3	4	5
Weight (lb)	0.6	0.88	1.16	1.44	1.72

a. Based on the data given, does the weight of the kitten follow an arithmetic progression?

b. Write an expression for the *n*th term of the sequence representing the kitten's weight, *n* weeks after birth.

c. If the weight of the kitten continues to increase linearly for 3 months, predict the kitten's weight 12 weeks after birth.

29. Suppose that an object starts with an initial velocity of v_0 (in ft/sec) and moves under a constant acceleration a (in ft/sec²). Then the velocity v_n (in ft/sec) after n seconds is given by $v_n = v_0 + a_n$. Show that this sequence is arithmetic. (*Hint*: Show that $v_{n+1} - v_n$ is constant.)

30. Suppose that an object is dropped from rest from a helicopter. Assuming negligible air resistance, the vertical acceleration is 32 ft/sec². Using the formula for v_n given in Exercise 29,

a. Write the *n*th term v_n of the sequence representing the velocity in the downward direction after n seconds.

b. Determine the vertical velocity 1 sec after release and 5 sec after release.

31. Find the 8th term of an arithmetic sequence with $a_1 = -2$ and $a_{15} = 68$. **(See Example 4, p. 1185)**

32. Find the 19th term of an arithmetic sequence with $a_1 = -11$ and $a_{30} = 163$.

33. Find the 35th term of an arithmetic sequence with $a_1 = 50$ and $a_{22} = -265$.

34. Find the 46th term of an arithmetic sequence with $a_1 = 210$ and $a_{60} = -262$.

For Exercises 35–38, find the number of terms of the finite arithmetic sequence. (See Example 5, p. 1186)

35. 8, 14, 20, 26, … , 320

36. 7, 16, 25, 34, … , 574

37. 11, 10.7, 10.4, 10.1, … , −3.4

38. 9, 8.4, 7.8, 7.2, … , −39

39. Given an arithmetic sequence with $a_{14} = 148$ and $a_{35} = 316$, find a_1 and d.

40. Given an arithmetic sequence with $a_{12} = -76$ and $a_{51} = -193$, find a_1 and d.

For Exercises 41–46, two terms of an arithmetic sequence are given. Find the indicated term. (See Example 6, p. 1186)

41. $a_{15} = 86$, $a_{34} = 200$; Find a_{150}.

42. $a_{12} = 52$, $a_{51} = 208$; Find a_{172}.

43. $b_{32} = -303$, $b_{54} = -567$; Find b_{214}.

44. $b_{64} = -456$, $b_{81} = -575$; Find b_{105}.

45. $c_{14} = 7.5$, $c_{101} = 29.25$; Find c_{400}.

46. $c_{21} = -11.16$, $c_{116} = -7.36$; Find c_{505}.

47. If the third and fourth terms of an arithmetic sequence are −6 and −9, what are the first and second terms?

48. If the third and fourth terms of an arithmetic sequence are 12 and 16, what are the first and second terms?

Learn: Evaluate a Finite Arithmetic Series

For Exercises 49–50, find the sum.

49. $4 + 7 + 10 + 13 + 16 + 19 + 22 + 25 + 28 + 31 + 34 + 37$

50. $8 + 12 + 16 + 20 + 24 + 28 + 32 + 36 + 40 + 44$

51. Find the sum of the first 40 terms of the sequence. {1, 6, 11, 16, …} (**See Example 7, p. 1188**)

52. Find the sum of the first 60 terms of the sequence. {2, 10, 18, 26, …}

53. Find the sum. $5 + 4.5 + 4 + 3.5 + \cdots + (-30.5)$

54. Find the sum. $8 + 7.2 + 6.4 + 5.6 + \cdots + (-45.6)$

For Exercises 55–64, find the sum. (See Example 8, p. 1189)

55. $\sum_{j=1}^{18}(j + 6)$

56. $\sum_{j=1}^{15}(j - 3)$

57. $\sum_{i=1}^{50}(2i + 6)$

58. $\sum_{i=1}^{40}(3i - 7)$

59. $\sum_{k=1}^{162}\left(3 - \frac{1}{2}k\right)$

60. $\sum_{k=1}^{141}\left(4 - \frac{1}{4}k\right)$

61. $-1 + 4 + 9 + \cdots + 49$

62. $12 + 16 + 20 + \cdots + 84$

63. $-7 + (-11) + (-15) + \cdots + (-39)$

64. $-18 + (-23) + (-28) + \cdots + (-183)$

Learn: Apply Arithmetic Sequences and Series

65. Suzanne must choose between two job offers. The first job offers $64,000 for the first year. Each year thereafter, she would receive a $3200 raise. The second job offers $60,000 in the first year. Each year thereafter, she would receive a $5000 raise. (**See Example 9, p. 1190**)

- **a.** If she anticipates working for the company for 5 yr, find the total amount she would earn from each job.
- **b.** If she anticipates working for the company for 10 yr, find the total amount she would earn from each job.

66. José must choose between two job offers. The first job pays \$50,000 the first year. Each year thereafter, he would receive a raise of \$2400. A second job offers \$54,000 the first year with a raise of \$2000 each year thereafter. However, with the second job, José would have to pay \$100 per month out of his paycheck for health insurance.

a. If José anticipates working for the company for 6 yr, find the total amount he would earn from the first job and the total amount he would earn from the second job.

b. If he anticipates working for the company for 12 yr, find the total amount he would earn from the first job and the total amount he would earn from the second job.

67. An object in free fall is dropped from a tall cliff. It falls 16 ft in the first second, 48 ft in the second second, 80 ft in the third second, and so on.

a. Write a formula for the *n*th term of an arithmetic sequence that represents the distance d_n (in ft) that the object will fall in the *n*th second.

b. How far will the object fall in the 8th second?

c. What is the total distance that the object will fall in 8 sec?

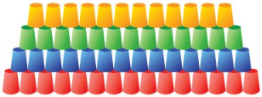
68. A ball rolling down an inclined plane rolls 4 in. in the first second, 8 in. in the second second, 12 inches in the third second, and so on.

a. Write a formula for the *n*th term of an arithmetic sequence that represents the distance d_n (in inches) that the ball will roll in the *n*th second.

b. How far will the ball roll in the 10th second?

c. What is the total distance that the ball will travel in 10 sec?

69. The students at Prairiewood Elementary plan to make a pyramid out of plastic cups. The bottom row has 15 cups. Moving up the pyramid, the number of cups in each row decreases by 1.



a. If the students build the pyramid so that there are 12 rows, how many cups will be at the top?

b. How many total cups will be required?

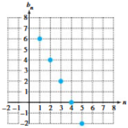
70. A theater has 32 rows. The first row has 18 seats, and each row that follows has three more seats than the row in front.

a. Determine the number of seats in row 32.

b. Determine the total number of seats in the theater.

Mixed Exercises

71. Refer to the graph of the sequence $\{b_n\}$.



a. Estimate the first four terms of the sequence.

b. Write a formula for the *n*th term of the sequence.

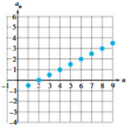
c. Find the 30th term.

d. Find the sum of the first 30 terms.

e. If $b_n = -180$, what is *n*?

f. Find the difference between b_{88} and b_{20}.

72. Refer to the graph of the sequence $\{a_n\}$.



- **a.** Estimate the first four terms of the sequence.
- **b.** Write a formula for the nth term of the sequence.
- **c.** Find the 20th term.
- **d.** Find the sum of the first 20 terms.
- **e.** If $a_n = 256$, what is n?
- **f.** Find the difference between a_{50} and a_{32}.

73. Find the sum of the integers from –20 to 256.

74. Find the sum of the integers from –102 to 57.

75. Compute the sum of the first 50 positive integers that are exactly divisible by 5.

76. Compute the sum of the first 60 positive integers that are exactly divisible by 4.

77. Compute the sum of all integers between 40 and 100 that are exactly divisible by 3.

78. Compute the sum of all integers between 60 and 150 that are exactly divisible by 8.

79. a. Use the formula $S_n = \frac{n}{2}(a_1 + a_n)$ to show that the sum of the first n positive integers is $S_n = \frac{n}{2}(1 + n)$.

- **b.** Find the sum of the first 100 positive integers.
- **c.** Find the sum of the first 1000 positive integers.

80. a. Use the formula $S_n = \frac{n}{2}(a_1 + a_n)$ to show that the sum 1, 3, 5, ... $(2n - 1) = n^2$.

- **b.** Find the sum of the first 100 positive odd integers.

For Exercises 81–86, a description of a sequence is given. Find

- **a.** The 10th term
- **b.** The sum of the first eight terms

81. The first term is 8, and the common difference is 5.

82. The first term is 0.04, and the common difference is 9.

83. The second term is 8, and the 22nd term is 128.

84. The fourth term is 1.75, and the 30th term is 14.75.

85. The nth term is $3n - 1$.

86. The nth term is $\frac{2n}{15}$.

The arithmetic mean (average) of two numbers c and d is given by $x = \frac{c+d}{2}$. The value x is equidistant between c and d, so the sequence c, x, d is an arithmetic sequence. Inserting k equally spaced values between c and d yields the arithmetic sequence $c, x_1, x_2, x_3, x_4, \ldots, x_k, d$. Use this information for Exercises 87–88.

87. Insert three arithmetic means between 4 and 28. (*Hint*: There will be a total of five terms. Write the nth term of an arithmetic sequence with $a_1 = 4$ and $a_5 = 28$, and then find a_2, a_3, and a_4.)

88. Insert four arithmetic means between 19 and 64.

Write About It

89. Suppose that $\{a_1, a_2, a_3, \ldots\}$ is an arithmetic sequence with common difference d. Explain why $\{a_1, a_3, a_5, \ldots\}$ is also an arithmetic sequence.

90. Suppose you are helping a friend with the homework for this lesson. Explain how to construct an arithmetic sequence.

Expanding Your Skills

91. Determine the nth term of the arithmetic sequence whose nth partial sum is $n^2 + 2n$. [*Hint*: The nth term of the sequence is the difference between the sum of the first n terms and the first $(n - 1)$ terms.]

92. Determine the nth term of the arithmetic sequence whose nth partial sum is $2n^2$.

GO ONLINE **Go online for more practice problems.**

Lesson 11-3

Geometric Sequences and Series

What Will You Learn?

After completing this lesson, you should be able to:

- Identify Specific and General Terms of a Geometric Sequence
- Evaluate Finite Geometric Series
- Evaluate Infinite Geometric Series
- Find the Value of an Annuity

Learn: Identify Specific and General Terms of a Geometric Sequence

The sequence 2, 4, 8, 16, 32, ... is not an arithmetic sequence because the difference between consecutive terms is not the same constant. However, a different pattern exists. Notice that each term after the first is 2 times the preceding term. This sequence is called a geometric sequence.

Geometric Sequence

A **geometric sequence** $\{a_n\}$ is a sequence of the form

$$a_1, a_1r, a_1r^2, a_1r^3, a_1r^4, \ldots$$

- The value a_1 is the first term, and r is called the **common ratio** of the sequence.
- The value of r is the quotient of any term after the first and its predecessor.

$$r = \frac{a_{n+1}}{a_n}$$

- The nth term of the sequence is given by $a_n = a_1r^{n-1}$.
- The sequence can be defined recursively as a_1, $a_n = a_{n-1}r$ for $n \geq 2$.

A geometric sequence is recognized by dividing any term after the first by its predecessor. In all cases the quotient is the same ratio r.

Example 1: Identifying a Geometric Sequence and the Common Ratio

Determine whether the sequence is geometric. If so, identify the common ratio.

a. $18, 6, 2, \frac{2}{3}, \ldots$ **b.** $3, 12, 48, 240, \ldots$

Solution:

a. $18, 6, 2, \frac{2}{3}, \ldots$

$\frac{a_2}{a_1} = \frac{6}{18} = \frac{1}{3}, \quad \frac{a_3}{a_2} = \frac{2}{6} = \frac{1}{3},$

$\frac{a_4}{a_3} = \frac{\frac{2}{3}}{2} = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$

The sequence is geometric because the ratio between each term and its predecessor is the same constant. The common ratio is $r = \frac{1}{3}$.

b. $3, 12, 48, 240, \ldots$

$\frac{a_2}{a_1} = \frac{12}{3} = 4, \quad \frac{a_3}{a_2} = \frac{48}{12} = 4,$

$\frac{a_4}{a_3} = \frac{240}{48} = 5$

The sequence is *not* geometric because the ratio between a_4 and a_3 is different than the ratio of other pairs of consecutive terms.

Apply the Skills

1. Determine whether the sequence is geometric. If so, identify the common ratio.

a. $80, 20, 5, \frac{5}{4}, \ldots$ **b.** $1, 8, 64, 640, \ldots$

Example 2 illustrates the fundamental characteristic of a geometric sequence; that is, each term of a geometric sequence is a constant multiple of the preceding term.

Example 2: Writing Several Terms of a Geometric Sequence

Write the first five terms of a geometric sequence with $a_1 = 5$ and $r = -2$.

Solution:

By definition, a geometric sequence follows the pattern $a_1, a_1r, a_1r^2, a_1r^3, \ldots$. The first five terms are

$$\begin{aligned} a_1 &= 5 \\ a_2 &= 5(-2) = -10 \\ a_3 &= 5(-2)^2 = 20 \\ a_4 &= 5(-2)^3 = -40 \\ a_5 &= 5(-2)^4 = 80 \end{aligned}$$

Insights

Alternatively, we can use a recursive formula: $a_1 = 5$ and $a_n = a_{n-1} \cdot r$ for $n \geq 2$.

$$\begin{aligned} a_2 &= a_1 \cdot (-2) = 5(-2) = -10 \\ a_3 &= a_2 \cdot (-2) = -10(-2) = 20 \\ a_4 &= a_3 \cdot (-2) = 20(-2) = -40 \\ a_5 &= a_4 \cdot (-2) = -40(-2) = 80 \end{aligned}$$

Apply the Skills

2. Write the first five terms of the geometric sequence with $a_1 = 2$ and $r = -3$.

In Example 3, we write an expression for the *n*th term of a geometric sequence given the first four terms.

Example 3: Writing the *n*th Term of a Geometric Sequence

Write a formula for the *n*th term of the geometric sequence. $2, 3, \frac{9}{2}, \frac{27}{4}, \ldots$

Solution:

$r = \frac{a_2}{a_1} = \frac{3}{2}$ Dividing any term by its predecessor, we have a common ratio of $\frac{3}{2}$.

$a_n = a_1 r^{n-1}$ Begin with the formula for the *n*th term of a geometric sequence.

$a_n = 2\left(\frac{3}{2}\right)^{n-1}$ Substitute $a_1 = 2$ and $r = \frac{3}{2}$.

Apply the Skills

3. Write the *n*th term of the geometric sequence.

$$4, 5, \frac{25}{4}, \frac{125}{16}, \ldots$$

A graph of several terms of the geometric sequence from Example 3 is shown in Figure 11-5. The points representing the sequence coincide with points on the graph of the exponential function $f(x) = 2\left(\frac{3}{2}\right)^{x-1}$ for positive integer values of x (Figure 11-6). In fact, a geometric sequence with $r > 0$ and $r \neq 1$ is an exponential function whose domain is restricted to the set of positive integers.

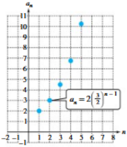


Figure 11-5

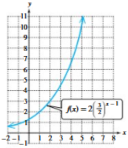


Figure 11-6

Example 4: Finding a Specified Term of a Geometric Sequence

Find the fifth term of a geometric sequence $\{a_n\}$ given that $a_1 = 15$ and $a_2 = -9$.

Solution:

$r = \frac{a_2}{a_1} = \frac{-9}{15} = -\frac{3}{5}$ Divide a_2 by a_1 to obtain the common ratio $-\frac{3}{5}$.

$a_n = a_1 r^{n-1}$ Use the formula for the nth term of a geometric sequence.

$a_n = 15\left(-\frac{3}{5}\right)^{n-1}$ Substitute $a_1 = 15$ and $r = -\frac{3}{5}$ to get the nth term for this sequence.

$a_5 = 15\left(-\frac{3}{5}\right)^{5-1}$ To find a_5, substitute 5 for n.

$a_5 = \frac{243}{125}$

Apply the Skills

4. Find the sixth term of a geometric sequence $\{a_n\}$ given that $a_1 = 64$ and $a_2 = -16$.

Example 4 illustrates that a geometric sequence with a negative common ratio is an alternating sequence. The graph of the first 10 terms of the sequence is shown in Figure 11-7.

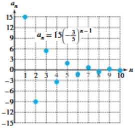


Figure 11-7

Example 5: Writing the *n*th Term of a Geometric Sequence

Given the terms $a_2 = 80$ and $a_5 = 40.96$ of a geometric sequence, find r, a_1, and a_n.

Solution:

In this example, we need to find two unknown quantities. One strategy is to begin with the formula for the *n*th term of the geometric sequence and substitute the known values for terms 2 and 5 of the sequence. Then solve the resulting system of nonlinear equations.

$$a_n = a_1r^{n-1}$$

$$80 = a_1r^{(2-1)} \longrightarrow 80 = a_1r \qquad \text{Substitute } a_2 = 80.$$

$$40.96 = a_1r^{(5-1)} \longrightarrow 40.96 = a_1r^4 \qquad \text{Substitute } a_5 = 40.96.$$

Substitute $a_1 = \frac{80}{r}$ from the first equation into the second equation: $40.96 = a_1r^4$.

$$40.96 = \frac{80}{r}r^4$$

$$40.96 = 80r^3$$

$$0.512 = r^3 \qquad \text{Divide by 80.}$$

$$r = 0.8 \qquad \text{Take the cube root of both sides.}$$

$$a_1 = \frac{80}{r} = \frac{80}{0.8} = 100 \qquad \text{To solve for } a_1\text{, substitute } r = 0.8 \text{ into the equation } a_1 = \frac{80}{r}.$$

The value of r is 0.8, the value of a_1 is 100, and the *n*th term is $a_n = 100(0.8)^{n-1}$.

Apply the Skills

5. Given the terms $a_2 = 54$ and $a_5 = 182.25$ of a geometric sequence, find r, a_1, and a_n.

Learn: Evaluate Finite Geometric Series

The *n*th partial sum S_n of the first n terms of a geometric sequence is a **finite geometric series**. Consider the geometric series:

$$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1}$$

Suppose that we subtract rS_n from S_n.

$$\begin{array}{rlllll} S_n = & a_1 & + a_1r & + a_1r^2 & + \cdots + a_1r^{n-1} & \\ rS_n = & a_1r & + a_1r^2 & + a_1r^3 & + \cdots + a_1r^n & \\ \hline \end{array}$$

$$S_n - rS_n = (a_1 - a_1r) + (a_1r - a_1r^2) + (a_1r^2 - a_1r^3) + \cdots + (a_1r^{n-1} - a_1r^n)$$

$$S_n - rS_n = a_1 - a_1r^n \qquad \text{The terms in red form a sum of zero.}$$

$$S_n(1 - r) = a_1(1 - r^n) \qquad \text{Factor each side of the equation.}$$

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \qquad \text{Divide by } (1 - r).$$

nth Partial Sum of a Geometric Sequence

The sum S_n of the first n terms of a geometric sequence is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

where a_1 is the first term of the sequence and r is the common ratio, $r \neq 1$.

Example 6: Evaluating a Finite Geometric Series

Find the sum. $\sum_{i=1}^{6} 4\left(\frac{1}{2}\right)^{i-1}$

Solution:

$$\sum_{i=1}^{6} 4\left(\frac{1}{2}\right)^{i-1} = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

The individual terms in the series form a geometric sequence with $a_1 = 4$ and $r = \frac{1}{2}$. The value of the series is the sixth partial sum of the sequence of terms.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{4\left[1 - \left(\frac{1}{2}\right)^6\right]}{1 - \frac{1}{2}} = \frac{4\left(1 - \frac{1}{64}\right)}{\frac{1}{2}} = \frac{63}{8}$$

Apply the formula for the nth partial sum with $n = 6$.

Apply the Skills

6. Find the sum. $\sum_{i=1}^{7} 9\left(\frac{1}{3}\right)^{i-1}$

Example 7: Evaluating a Finite Geometric Series

Find the sum of the finite geometric series. $5 + 10 + 20 + \cdots + 5120$

Solution:

The common ratio is 2 and $a_1 = 5$. The nth term of the sequence of terms can be written as $a_n = 5(2)^{n-1}$. To find the number of terms n, substitute 5120 for a_n.

$5120 = 5(2)^{n-1}$

$1024 = 2^{n-1}$ Divide both sides by 5.

$2^{10} = 2^{n-1}$ Apply the equivalence property of exponential expressions.

$10 = n - 1$

$n = 11$ The sequence has 11 terms.

With $a_1 = 5$, $r = 2$, and $n = 11$, we have

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{5(1 - 2^{11})}{1 - 2} = 10{,}235$$

Insights

The equivalence property of exponential expressions indicates that if $b^n = b^m$, then $n = m$. That is, if two exponential expressions with the same base are equal, then their exponents are equal.

Apply the Skills

7. Find the sum of the finite geometric series.

$$3 + 6 + 12 + \cdots + 768$$

Learn: Evaluate Infinite Geometric Series

For a positive integer n, the series $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$ is called a **finite series** because there are a finite number of terms. The series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$ is called an **infinite series** because there are an infinite number of terms. Although it is impossible to add an infinite number of terms on a term-by-term basis, we might ask if the sum approaches a limiting value. If the nth partial sum S_n of an infinite series approaches a number L as $n \to \infty$, we say that the series **converges** and we call L the sum of the series. If a series does not converge, we say that the series **diverges** (the sum does not exist).

Consider the geometric sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \left(\frac{1}{2}\right)^n, \ldots$ and the corresponding infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \left(\frac{1}{2}\right)^n + \cdots$. To visualize the infinite series, suppose that we add $\frac{1}{2}$ of a pie plus $\frac{1}{4}$ of a pie plus $\frac{1}{8}$ of a pie and so on. Will we approach some finite amount of pie? From Figure 11-8, it appears that the sum will be 1 whole unit.

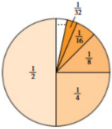


Figure 11-8

For an infinite geometric sequence with $-1 < r < 1$ (equivalently $|r| < 1$), the value of r^n will become smaller as n gets larger. For example:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \overset{\approx 0.0009766}{\left(\frac{1}{2}\right)^{10}}, \ldots, \overset{\approx 0.0000305}{\left(\frac{1}{2}\right)^{15}}, \ldots$$

In fact, for $|r| < 1$, as $n \to \infty$, $r^n \to 0$. Therefore, as n approaches infinity

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \text{ approaches } \frac{a_1(1 - 0)}{1 - r} = \frac{a_1}{1 - r}.$$

Sum of an Infinite Geometric Series

Given an infinite geometric series $a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots$, with $|r| < 1$, the sum S of all terms in the series is given by

$$S = \frac{a_1}{1-r}$$

Note: If $|r| \geq 1$, then the sum does not exist.

Example 8: Evaluating an Infinite Geometric Series

Find the sum if possible. **a.** $\sum_{i=1}^{\infty} 5\left(-\frac{1}{3}\right)^{i-1}$ **b.** $2 + \frac{8}{3} + \frac{32}{9} + \frac{128}{27} + \cdots$

Solution:

a. $\sum_{i=1}^{\infty} 5\left(-\frac{1}{3}\right)^{i-1} = 5 - \frac{5}{3} + \frac{5}{9} - \frac{5}{27} + \frac{5}{81} + \cdots$

The sum is an infinite geometric series with $a_1 = 5$ and $r = -\frac{1}{3}$.

Because $|r| = \left|-\frac{1}{3}\right| < 1$, we have $S = \frac{a_1}{1-r} = \frac{5}{1-\left(-\frac{1}{3}\right)} = \frac{15}{4}$.

The sum is $\frac{15}{4}$.

b. $2 + \frac{8}{3} + \frac{32}{9} + \frac{128}{27} + \cdots$ is a geometric series with $a_1 = 2$ and $r = \frac{4}{3}$.

Because $|r| = \left|\frac{4}{3}\right| \geq 1$, the sum does not exist.

Apply the Skills

8. Find the sum if possible.

a. $\sum_{i=1}^{\infty} 4\left(\frac{3}{4}\right)^{i-1}$ **b.** $2 + \frac{5}{2} + \frac{25}{8} + \frac{125}{32} + \cdots$

In Example 9, we use the formula for the sum of an infinite geometric series to write a repeating decimal as a fraction.

Example 9: Writing a Repeating Decimal as a Fraction

a. Write 0.5 as a fraction. **b.** Write 1.975 as a fraction.

Solution:

a. $0.5 = 0.5555... = \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \frac{5}{10{,}000} + \cdots$

$$= \underbrace{\frac{5}{10} + \frac{5}{10}\left(\frac{1}{10}\right) + \frac{5}{10}\left(\frac{1}{100}\right) + \frac{5}{10}\left(\frac{1}{1000}\right) + \cdots}_{\text{Infinite geometric series with } a_1 = \frac{5}{10} \text{ and } r = \frac{1}{10}} \longrightarrow S = \frac{a_1}{1-r} = \frac{\frac{5}{10}}{1-\frac{1}{10}} = \frac{\frac{5}{10}}{\frac{9}{10}} = \frac{5}{9}$$

$0.5 = \frac{5}{9}$

b. $$1.975 = 1.9757575... = \frac{19}{10} + \left[\frac{75}{1000} + \frac{75}{100{,}000} + \cdots\right]$$

$$= \frac{19}{10} + \underbrace{\left[\frac{75}{1000} + \frac{75}{1000}\left(\frac{1}{100}\right) + \cdots\right]}$$

Infinite geometric series
with $a_1 = \frac{75}{1000}$ and $r = \frac{1}{100} \longrightarrow S = \frac{a_1}{1-r} = \frac{\frac{75}{1000}}{1 - \frac{1}{100}} = \frac{\frac{75}{1000}}{\frac{99}{100}} = \frac{75}{990} = \frac{5}{66}$

$$1.9\overline{75} = \frac{19}{10} + \frac{5}{66} = \frac{326}{165}$$

Apply the Skills

9. **a.** Write 0.7 as a fraction. **b.** Write $0.\overline{34}$ as a fraction.

When money is infused into the economy, a percentage of the money received by individuals or businesses is often respent over and over again. Economists call this the **multiplier effect.**

Example 10: Investigating an Application of an Infinite Geometric Series—The Multiplier Effect

Suppose that \$200 million is spent annually by tourists in a certain state. Further suppose that 75% of the money is respent in the state and then respent over and over again, each time at a rate of 75%. Determine the theoretical total amount spent from the initial \$200 million if the money can be respent an infinite number of times.

Solution:

The total amount spent can be represented by the infinite geometric series where all values are in \$ millions.

$$200 + \overset{\text{75\% of 200}}{150} + \overset{\text{75\% of 150}}{112.5} + \cdots = 200 + 200(0.75) + 150(0.75) + \cdots$$

$$= 200 + 200(0.75) + 200(0.75)^2 + \cdots$$

We have $a_1 = 200$ and $r = 0.75$. Because $|r| = |0.75| < 1$, we have

$$S = \frac{a_1}{1-r} = \frac{200}{1-0.75} = 800.$$

Theoretically, the total amount spent is \$800 million.

Apply the Skills

10. Suppose that after a tax rebate an individual spends \$210. The money is then respent over and over again, each time at a rate of 70%. Determine the total amount spent. Assume that the money can be respent an infinite number of times.

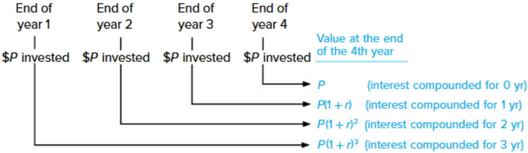
Learn: Find the Value of an Annuity

In Lesson 3-2, we studied applications of exponential functions involving compound interest. We learned that if P dollars is invested in an account at interest rate r, compounded annually for t years, then the amount A in the account is given by

$$A = P(1 + r)^t$$

However, rather than making a one-time lump sum payment of P dollars, many individuals will invest smaller amounts at regular and more frequent intervals. Such a sequence of fixed payments made (or received) by an individual over a fixed period of time is called an **annuity.**

Suppose that an individual invests P dollars at the *end* of each year for 4 yr at interest rate r. The money deposited at the end of the first year will have 3 yr in which to earn interest. The money invested at the end of the second year will earn interest for 2 yr, and the money invested at the end of the third year will earn interest for 1 yr. However, the money invested at the end of the fourth year will not earn any interest.


End of year 1
End of year 2
End of year 3
End of year 4
$P invested
$P invested
$P invested
$P invested
Value at the end of the 4th year
P (interest compounded for 0 yr)
$P(1 + r)$ (interest compounded for 1 yr)
$P(1 + r)^2$ (interest compounded for 2 yr)
$P(1 + r)^3$ (interest compounded for 3 yr)

The total amount A, also called the **future value** of the annuity, is given by

$$A = P + P(1 + r) + P(1 + r)^2 + P(1 + r)^3$$

This is a finite geometric series with $a_1 = P$ and common ratio of $(1 + r)$. Using the relationship $S_n = \frac{a_1(1 - r^n)}{1 - r}$, the total amount invested for t years is given by

$$A = \frac{P[1 - (1 + r)^t]}{1 - (1 + r)} = \frac{P[1 - (1 + r)^t]}{-r} \text{ or simply } A = \frac{P[(1 + r)^t - 1]}{r}$$

If deposits are made n times per year, then the interest rate per compounding period is $\frac{r}{n}$, and the total number of times the money is compounded is nt.

Insights

In this lesson we study **ordinary annuities.** These are annuities in which money is invested at the end of the compounding periods. If the money is invested at the beginning of the compounding periods, such an investment is called an **annuity due** and the future value is computed with a different formula.

Future Value of an Ordinary Annuity

Suppose that P dollars is invested at the end of each compounding period n times per year at interest rate r. Then the value A (in \$) of the annuity after t years is given by

$$A = \frac{P\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}$$

Example 11: Finding the Value of an Annuity

Suppose that an employee working for a state college puts aside \$150 at the end of each month in a tax-sheltered annuity. The annuity pays 6% annual interest compounded monthly. Suppose that the employee begins contributing at age 28.

a. How much will the annuity be worth by the time the employee reaches age 62?

b. How much interest will have been earned?

Solution:

a. $A = \dfrac{P\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}$

$A = \dfrac{150\left[\left(1+\frac{0.06}{12}\right)^{(12)(34)}-1\right]}{\frac{0.06}{12}}$

$P = \$150$
$r = 0.06$
$n = 12$ (the money is invested monthly)
$t = 62 - 28 = 34$ yr

$A = 199{,}548.50$

The annuity will be worth \$199,548.50 when the employee reaches age 62.

b. \$150 is invested 12 times per year for 34 yr.

Principal invested: (\$150)(12)(34) = \$61,200

The amount of interest is \$199,548.50 − \$61,200 = \$138,348.50.

Apply the Skills

11. Suppose that an employee contributes \$100 to an annuity at the end of each month for 25 yr. If the annuity earns 7%,

a. Determine the value of the annuity at the end of the 25-yr period.

b. How much interest will be earned?

Point of Interest

The contributions made to the investment in Example 11 are **tax deferred**. This means that the principal invested each month has not yet been taxed. Because the principal is invested pretax, the individual potentially has more money available to invest. Furthermore, no taxes are paid on either the principal or interest until the money is withdrawn. This type of plan is meant as a long-term investment, and withdrawals are typically taken after age 59½ without penalty.

Practice 11-3

Practice Exercises

Prerequisite Review

R.1. Simplify. $-\frac{5}{6} \div \frac{2}{3}$

R.2. Solve for x. $3^x = 81$

R.3. For $g(x) = \left(\frac{1}{4}\right)^x$ find $g(0)$, $g(1)$, $g(2)$, $g(-1)$, and $g(-2)$.

R.4. For $h(x) = 16^x$ find $h(0)$, $h(1)$, and $h(-1)$

R.5. Suppose that an investor deposits $14,000 in a savings account for 10 yr at 2% interest. Use the model $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ for P dollars in principal invested at an interest rate r compounded n times per year for t years for the following compounding options. Round to the nearest dollar.

- **a.** Interest compounded annually
- **b.** Interest compounded quarterly
- **c.** Interest compounded monthly

Concept Connections

1. A ________________ sequence is a sequence in which each term after the first is the product of the preceding term and a fixed nonzero real number, called the common ______________, r.

2. The nth term of a geometric sequence with first term a_1 and common ratio r is given by $a_n =$ ________________.

3. The nth partial sum S_n of the first n terms of a geometric sequence is a (finite/infinite) geometric series.

4. The sum S_n of the first n terms of a geometric sequence with first term a_1 and common ratio r is given by the formula $S_n =$ ________________.

5. Given a geometric sequence with $|r| < 1$, the value of $r^n \to$ __________ as $n \to \infty$.

6. Given an infinite geometric series with first term a_1 and common ratio r, if $|r| < 1$, then the sum S is given by the formula $S =$ __________. If $|r| \geq 1$, then the sum (does/does not) exist.

7. A sequence of payments made at equal intervals over a fixed period of time is called an ________________.

8. Suppose that an infinite series $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ approaches a value L as $n \to \infty$. Then the series ________________. Otherwise, the series ________________.

Learn: Identify Specific and General Terms of a Geometric Sequence

For Exercises 9–18, determine whether the sequence is geometric. If so, find the value of *r*. (See Example 1, p. 1197)

9. 6, 18, 54, 162, ...

10. 4, 20, 100, 500, ...

11. $-7, \frac{7}{2}, -\frac{7}{4}, \frac{7}{8}, \ldots$

12. $5, -\frac{5}{3}, \frac{5}{9}, -\frac{5}{27}, \ldots$

13. 3, 12, 60, 360, ...

14. 7, 14, 42, 88, ...

15. $\sqrt{5}, 5, 5\sqrt{5}, 25, \ldots$

16. $\sqrt[3]{7}, \sqrt[3]{49}, 7, 7\sqrt[3]{7}, \ldots$

17. $2, \frac{4}{t}, \frac{8}{t^2}, \frac{16}{t^3}, \ldots$

18. $\frac{5}{a^2}, \frac{15}{a^4}, \frac{45}{a^6}, \frac{135}{a^8}, \ldots$

For Exercises 19–24, write the first five terms of a geometric sequence $\{a_n\}$ based on the given information about the sequence. (See Example 2, p. 1197)

19. $a_1 = 7$ and $r = 2$

20. $a_1 = 2$ and $r = 3$

21. $a_1 = 24$ and $r = -\frac{2}{3}$

22. $a_1 = 80$ and $r = -\frac{4}{5}$

23. $a_1 = 36$ and $a_n = \frac{1}{2}a_{n-1}$ for $n \geq 2$

24. $a_1 = 27, a_n = \frac{1}{3}a_{n-1}$ for $n \geq 2$

For Exercises 25–30, write a formula for the *n*th term of the geometric sequence. (See Example 3, p. 1198)

25. 5, 10, 20, 40, ...

26. 3, 6, 12, 24, ...

27. $-2, -1, -\frac{1}{2}, -\frac{1}{4}, \ldots$

28. $-8, -2, -\frac{1}{2}, -\frac{1}{8}, \ldots$

29. $\frac{16}{3}, -4, 3, -\frac{9}{4}, \ldots$

30. $\frac{18}{5}, -\frac{6}{5}, \frac{2}{5}, -\frac{2}{15}, \ldots$

31. A farmer depreciates a \$100,000 tractor. The resale value of the tractor n years after purchase is 85% of its value from the previous year.

a. Write a formula for the nth term of a sequence that represents the resale value of the tractor n years after purchase.

b. What will the resale value be 5 yr after purchase? Round to the nearest \$1000.

32. A Coulter Counter is a device used to count the number of microscopic particles in a fluid, most notably, cells in blood. A hospital depreciates a \$9000 Coulter Counter at a rate of 75% per year after purchase.

a. Write a formula for the nth term of a sequence that represents the resale value of the device n years after purchase.

b. What will the resale value be 4 yr after purchase? Round to the nearest \$100.

33. Doctors in a certain city report 24 confirmed cases of the flu to the health department. At that time, the health department declares a flu epidemic. If the number of reported cases increases by roughly 30% each week thereafter, find the number of cases 10 weeks after the initial report. Round to the nearest whole unit.

34. A homeowner buys a house for \$240,000 and finds that the value of the property increases by 3% per year thereafter. Assuming that the trend continues, find the value of the home 15 yr later. Round to the nearest \$1000.

For Exercises 35–42, find the indicated term of a geometric sequence from the given information. (See Example 4, p. 1199)

35. $a_1 = 12$ and $a_2 = -8$. Find the sixth term.

36. $a_1 = 16$ and $a_2 = -12$. Find the fifth term.

37. $a_1 = 2$ and $a_4 = 16$. Find a_{12}.

38. $a_1 = 4$ and $a_4 = 108$. Find a_{10}.

39. $a_2 = -6$ and $r = \frac{1}{2}$. Find a_7.

40. $a_2 = -15$ and $r = \frac{1}{3}$. Find a_8.

41. $a_5 = -\frac{16}{9}$ and $r = -\frac{2}{3}$. Find a_1.

42. $a_6 = \frac{5}{16}$ and $r = -\frac{1}{2}$. Find a_1.

43. If the second and third terms of a geometric sequence are 15 and 75, what is the first term?

44. If the second and third terms of a geometric sequence are 4 and 1, what is the first term?

For Exercises 45–48, find a_1 and r for a geometric sequence $\{a_n\}$ from the given information. (See Example 5, p. 1200)

45. $a_2 = 18$ and $a_5 = 144$

46. $a_2 = 21$ and $a_7 = 5103$

47. $a_3 = 72$ and $a_6 = -\frac{243}{8}$

48. $a_3 = 45$ and $a_6 = -\frac{243}{25}$

Learn: Evaluate Finite and Infinite Geometric Series

For Exercises 49–72, find the sum of the geometric series, if possible. (See Examples 6–8, pp. 1201–1203)

49. $\sum_{n=1}^{10} 3(2)^{n-1}$

50. $\sum_{n=1}^{8} 4(3)^{n-1}$

51. $\sum_{k=1}^{7} 6\left(\frac{2}{3}\right)^{k-1}$

52. $\sum_{j=1}^{7} 2\left(\frac{3}{4}\right)^{j-1}$

53. $15 + 5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \frac{5}{81}$

54. $50 + 10 + 2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125}$

55. $2 + 6 + 18 + \cdots + 13{,}122$

56. $4 + 12 + 36 + \cdots + 78{,}732$

57. $1 + \frac{2}{3} + \frac{4}{9} + \cdots + \frac{32}{243}$

58. $\frac{8}{3} + 2 + \frac{3}{2} + \cdots + \frac{243}{512}$

59. $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$

60. $1 + \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \cdots$

61. $-2 - \frac{1}{2} - \frac{1}{8} - \frac{1}{32} - \cdots$

62. $-5 - 1 - \frac{1}{5} - \frac{1}{25} - \cdots$

63. $2 + 8 + 32 + 128 + \cdots$

64. $1 + 6 + 36 + 216 + \cdots$

65. $\sum_{j=1}^{\infty} \left(\frac{2}{3}\right)^{j-1}$

66. $\sum_{i=1}^{\infty} \left(\frac{3}{4}\right)^{i-1}$

67. $\sum_{k=1}^{\infty} \left(\frac{3}{2}\right)^{k-1}$

68. $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^{n-1}$

69. $\sum_{i=1}^{12} 4(2)^i$ (*Hint*: Rewrite the expression within the summation so that the base of 2 appears to the $(i - 1)$ power.]

70. $\sum_{j=1}^{10} 5(2)^j$

71. $\sum_{n=3}^{\infty} 4\left(\frac{1}{2}\right)^{n-1}$ [*Hint*: Evaluate the infinite sum from $n = 1$ to infinity. Then subtract the terms corresponding to $n = 1$ and $n = 2$.)

72. $\sum_{n=4}^{\infty} 18\left(\frac{1}{3}\right)^{n-1}$

For Exercises 73–80, write the repeating decimal as a fraction. (See Example 9, p. 1203)

73. 0.8

74. 0.2

75. 0.64

76. 0.78

77. $0.\overline{81}$

78. $0.\overline{72}$

79. $3.4\overline{25}$

80. $4.1\overline{62}$

81. Bike Week in Daytona Beach brings an estimated 500,000 people to the town. Suppose that each person spends an average of $300.

- **a.** How much money is infused into the local economy during Bike Week?
- **b.** If the money is respent in the community at a rate of 68%, determine the total amount spent. Assume that the money is respent an infinite number of times. (**See Example 10, p. 1204**)

82. An individual with questionable integrity prints and spends $12,000 in counterfeit money. If the "money" is respent over and over again each time at a rate of 76%, determine the total amount spent. Assume that the "money" is respent an infinite number of times without being detected.

83. Rafael received an inheritance of $18,000. He saves $6480 and then spends $11,520 of the money on college tuition, books, and living expenses for school. If the money is respent over and over again in the community an infinite number of times, at a rate of 64%, determine the total amount spent.

84. A tax rebate returns $100 million to individuals in the community. Suppose that $25,000,000 is put into savings, and that $75,000,000 is spent. If the money is spent over and over again an infinite number of times, each time at a rate of 75%, determine the total amount spent.

Learn: Find the Value of an Annuity

For Exercises 85–86, find the value of an ordinary annuity in which regular payments of *P* dollars are made at the end of each compounding period, *n* times per year, at an interest rate *r* for *t* years.

85. $P = \$200$, $n = 12$, $r = 5\%$, $t = 30$ yr

86. $P = \$100$, $n = 24$, $r = 5.5\%$, $t = 28$ yr

87.
- **a.** An employee invests $100 per month in an ordinary annuity. If the interest rate is 6%, find the value of the annuity after 20 yr. (**See Example 11, p. 1206**)
- **b.** If the employee invests $200 instead of $100 at 6%, find the value of the annuity after 20 yr. Compare the result to part (a).
- **c.** If the employee invests $100 per month in the annuity at 6% interest, find the value after 40 yr. Compare the result to part (a).

88.
- **a.** An employee invests $500 per month in an ordinary annuity. If the interest rate is 5%, find the value of the annuity after 18 yr.
- **b.** If the employee invests $1000 per month in the annuity instead of $500 at 5% interest, find the value of the annuity after 18 yr. Compare the result to part (a).
- **c.** If the employee invests $500 per month in the annuity at 5% interest, find the value of the annuity after 36 yr. Compare the result to part (a).

Mixed Exercises

89. **a.** Given a geometric sequence whose nth term is $a_n = 6(0.4)^n$, are the terms of this sequence increasing or decreasing?

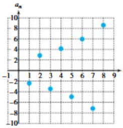
b. Given a geometric sequence whose nth term is $a_n = 3(1.4)^n$, are the terms of this sequence increasing or decreasing?

90. **a.** Given a geometric sequence whose nth term is $a_n = -8(0.2)^n$, are the terms of this sequence increasing or decreasing?

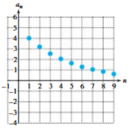
b. Given a geometric sequence whose nth term is $a_n = -5(1.6)^n$, are the terms of this sequence increasing or decreasing?

For Exercises 91–94, match the sequence with its graph from the choices a–d.

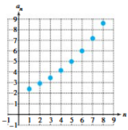
a.



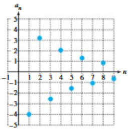
b.



c.



d.



91. $a_n = 5(0.8)^n$

92. $a_n = 2(1.2)^n$

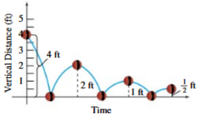
93. $a_n = 5(-0.8)^n$

94. $a_n = 2(-1.2)^n$

95. The initial swing (one way) of a pendulum makes an arc of 24 in. Each swing (one way) thereafter makes an arc of 98% of the length of the previous swing. What is the total arc length that the pendulum travels?

96. The initial swing (one way) of a pendulum makes an arc of 4 ft. Each swing (one way) thereafter makes an arc of 90% of the length of the previous swing. What is the total arc length that the pendulum travels?

97. A child drops a ball from a height of 4 ft. With each bounce, the ball rebounds to 50% of its original height. After the ball falls from its initial height of 4 ft, the vertical distance traveled for every bounce thereafter is doubled (the ball travels up and down).



The total vertical distance traveled by the ball is given by

$$4 + 2\cdot(2) + 2(1) + 2\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{4}\right)\cdots$$

Vertical distance for each bounce is doubled

$$= \underbrace{4}_{\text{Initial height}} + \underbrace{4 + 2 + 1 + \tfrac{1}{2} +}_{\text{Infinite geometric series}} \cdots$$

Determine the total vertical distance traveled by the ball.

98. A ball is dropped from a height of 12 ft. With each bounce, the ball rebounds to $\frac{3}{4}$ of its height. Determine the total vertical distance traveled by the ball.

99. Suppose that an individual is paid \$0.01 on day 1 and every day thereafter, the payment is doubled.

a. Write a formula for the nth term of a sequence that gives the payment (in \$) on day n.

b. How much will the individual earn on day 10? day 20? and day 30?

c. What is the total amount earned in 30 days?

100. The vibration of sound is measured in cycles per second, also called hertz (Hz). The frequency for middle C on a piano is 256 Hz. The C above middle C (one octave above) is 512 Hz. The frequencies of musical notes follow a geometric progression.

a. Find the frequency for C two octaves above middle C.

b. Find the frequency for C one octave *below* middle C.

101. The yearly salary for job A is \$60,000 initially with an annual raise of \$3000 every year thereafter. The yearly salary for job B is \$56,000 for year 1 with an annual raise of 6%.

a. Consider a sequence representing the salary for job A for year *n*. Is this an arithmetic or geometric sequence? Find the total earnings for job A over 20 yr.

b. Consider a sequence representing the salary for job B for year *n*. Is this an arithmetic or geometric sequence? Find the total earnings for job B over 20 yr. Round to the nearest dollar.

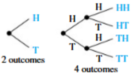
c. What is the difference in total salary between the two jobs over 20 yr?

102. **a.** Jacob has a job that pays \$48,000 the first year with a 4% raise each year thereafter. Find the sum of Jacob's yearly salaries over a 20-yr period. Round to the nearest dollar.

b. Cherise has a job that pays \$48,000 the first year with a 4.5% raise each year thereafter. Find the sum of Cherise's yearly salaries over a 20-yr period. Round to the nearest dollar.

c. How much more will Cherise earn than Jacob over the 20-yr period?

103. If a fair coin is flipped *n* times, the number of head/tail arrangements follows a geometric sequence. In the figure, if the coin is flipped 1 time, there are two possible outcomes, H or T. If the coin is flipped 2 times, then there are four possible outcomes: HH, HT, TH, and TT.



a. Write a formula for the *n*th term of a sequence representing the number of outcomes if a fair coin is flipped *n* times.

b. How many outcomes are there if a fair coin is flipped 10 times?

104. An ancestor is a person from whom an individual is descended (a parent, a grandparent, a great-grandparent, and so on). Every individual has two biological parents, a mother and a father. The mother and father in turn each have two biological parents, and so on.

1st Generation	2nd Generation	3rd Generation
2 parents	4 grandparents	8 great-grandparents

The terms of the sequence 2, 4, 8, 16, ... give the additional number of ancestors for each generation back in time. Determine the total number of ancestors that an individual has going back 12 generations.

Write About It

105. Explain the difference between an arithmetic sequence and a geometric sequence.

106. Explain why two terms are not sufficient to determine whether an infinite sequence is arithmetic or geometric. For example, explain why 4, 16, ... can be arithmetic, geometric, or neither.

107. A geometric sequence $a_1, a_2, a_3, \ldots$ has a common ratio *r*. Explain why the sequence $a_1, a_3, a_5, \ldots$ is also geometric and determine the common ratio.

Expanding Your Skills

108. Show that $x, x + 2, x + 4, x + 6, \ldots$ is *not* a geometric sequence.

109. If $a_1, a_2, a_3, a_4, \ldots$ is a geometric sequence with common ratio *r*, show that $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4} \cdots$ is also a geometric sequence and determine the value of *r*.

110. Suppose that $a_1, a_2, a_3, \ldots$ is an arithmetic sequence with common difference *d*. Show that $10^{a_1}, 10^{a_2}, 10^{a_3}, 10^{a_4}, \ldots$ is a geometric sequence and find the common ratio *r*.

111. Suppose that $a_1, a_2, a_3, \ldots$ is a geometric sequence with $r > 0$ and $a_1 > 0$. Show that the sequence $\log a_1, \log a_2, \log a_3, \ldots$ is arithmetic and find the common difference *d*.

112. Given the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$,

a. Find the sum.

b. How many terms must be taken so that the *n*th partial sum is within $\frac{1}{100}$ of the actual sum?

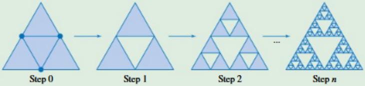
113. Determine whether the sequence ln 1, ln 2, ln 4, ln 8, ... is arithmetic or geometric. If the sequence is arithmetic, find *d*. If the sequence is geometric, find *r*.

Point of Interest

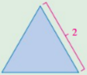
The mathematician Benoit Mandelbrot wrote (*New Scientist magazine*, 14 September, 1990) "I coined the word fractal in 1975 from the Latin 'fractus' which describes a broken stone—broken up and irregular. Fractals are geometrical shapes that, contrary to those of Euclid, are not regular at all. First, they are irregular all over. Second, they have the same degree of irregularity on all scales. A fractal object looks the same when examined from far away or nearby—it is self-similar."

The fractal Sierpinski triangle can be constructed by the following steps.

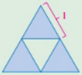
Begin with a solid, filled-in equilateral triangle and identify the midpoints of each side. Connect the midpointst to form a new smaller, inverted equilateral triangle which we remove. Repeat the same process in each of the 3 blue triangles. The Sierpinski triangle is what remains behind when this process continues forever.



114. Assume that each side of the original triangle measures 2 units. For an equilateral triangle of side *s*, the area is given by $A=\frac{\sqrt{3}}{4}s^2$. At each step, we can calculate the area of the portion of the triangle remaining. (Assume the answers are in square units.)



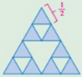
$A_0=\frac{\sqrt{3}}{4}(2)^2=\sqrt{3}$



One blue triangle:

$A=\frac{\sqrt{3}}{4}(1)^2=\frac{\sqrt{3}}{4}$

3 blue triangles: $A_1=3\cdot\frac{\sqrt{3}}{4}=\frac{3}{4}\sqrt{3}$



One blue triangle:

$A=\frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2=\frac{\sqrt{3}}{16}$

9 blue triangles: $A_2=9\cdot\frac{\sqrt{3}}{16}=\frac{9}{16}\sqrt{3}$

a. The area of the blue triangles at each stage is given by the sequence $\sqrt{3}, \frac{3}{4}\sqrt{3}, \frac{9}{16}\sqrt{3}, \ldots$. Write the common ratio and a formula for the *n*th term of this geometric sequence. Is this sequence decreasing or increasing?

b. What is the area of this Sierpinski triangle at step 5?

c. If this process continues indefinitely, what will the area of the triangle become?

Problem Recognition Exercises

Comparing Arithmetic and Geometric Sequences and Series

For Exercises 1–10, determine if the *n*th term of the sequence defines an arithmetic sequence, a geometric sequence, or neither. If the sequence is arithmetic, find the common difference *d*. If the sequence is geometric, find the common ratio *r*.

1. $a_n = \frac{2n}{3} + \frac{1}{4}$

2. $a_n = -\frac{3}{5}n - \frac{1}{3}$

3. $a_n = (-1)^{n-1}$

4. $a_n = (-2)^n$

5. $a_n = \frac{n-1}{n+1}$

6. $a_n = \frac{3n-4}{2n+1}$

7. $a_n = 5(\sqrt{5})^{n-1}$

8. $a_n = 6(\sqrt{2})^{n+1}$

9. $a_n = 5 + \sqrt{2}n$

10. $a_n = \frac{\sqrt{3}}{2}n + 4$

For Exercises 11–28, evaluate the sum if possible.

11. $\sum_{i=1}^{1506} 5$

12. $\sum_{i=1}^{417} (-3)$

13. $\sum_{i=1}^{716} (-1)^{i+1}$

14. $\sum_{j=1}^{2651} (-1)^j$

15. $\sum_{n=1}^{3} \frac{n-2}{n+1}$

16. $\sum_{j=1}^{4} (j^4 - 2j^2)$

17. $\sum_{n=1}^{4} -3\left(\frac{1}{3}\right)^{n-1}$

18. $\sum_{n=1}^{5} -2\left(\frac{3}{4}\right)^{n-1}$

19. $\sum_{n=1}^{\infty} 6\left(\frac{5}{3}\right)^{n-1}$

20. $\sum_{i=1}^{\infty} 7\left(\frac{4}{3}\right)^{i-1}$

21. $\sum_{i=1}^{27} (-6i - 4)$

22. $\sum_{i=1}^{41} (-5i - 3)$

23. $\sum_{n=1}^{\infty} 8\left(\frac{1}{2}\right)^n$

24. $\sum_{n=1}^{\infty} 12\left(\frac{1}{3}\right)^n$

25. $36 + 30 + 25 + \frac{125}{6} + \cdots$

26. $54 + 36 + 24 + 16 + \cdots$

27. $3 + 11 + 19 + 27 + \cdots + 363$

28. $36 + 29 + 22 + 15 + \cdots + (-419)$

GO ONLINE Go online for more practice problems.

Lesson 11-4

Mathematical Induction

What Will You Learn?

After completing this lesson, you should be able to:

- Prove a Statement Using Mathematical Induction
- Prove a Statement Using the Extended Principle of Mathematical Induction

Learn: Prove a Statement Using Mathematical Induction

In this lesson, we present a technique of mathematical proof that enables us to prove the validity of a statement that is true over the set of positive integers. This technique is called mathematical induction.

Consider the following sum:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{n(n+1)}$$

The sequence of terms that make up the sum is neither arithmetic nor geometric. Therefore, we have no formula readily available to evaluate the sum of the first n terms.

We might consider evaluating the nth partial sum for several positive integers n to determine if a pattern exists. For each value of n, we have a corresponding statement P_n involving the value of the partial sum.

$n=1$ $\quad \frac{1}{1\cdot 2}=\frac{1}{2}$ Call this statement P_1.

$n=2$ $\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$ Call this statement P_2.

$n=3$ $\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{3}{4}$ Call this statement P_3.

$n=4$ $\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}=\frac{4}{5}$ Call this statement P_4.

For the first four terms, the numerator is the same as the term number n and the denominator is one more than the term number. From this observation, we might make the following hypothesis, which we call P_n.

$$P_n: =\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Insights

The proof of the statement $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$ is given in Example 2.

In words, the hypothesis P_n suggests that the sum of the first n terms of the sequence $\left\{\frac{1}{n(n+1)}\right\}$ is $\frac{n}{n+1}$. Equivalently, $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

We have shown that the statement P_n is true for $n = 1, 2, 3$, and 4, but what about values of n thereafter? Because there are infinitely many positive integers, there are actually infinitely many statements to prove, and we cannot approach them on a case-by-case basis. Instead we will use the principle of mathematical induction.

Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n, and let k be an arbitrary positive integer. Then P_n is true for all positive integers n if

1. P_1 is true, and
2. The truth of P_k implies the truth of P_{k+1}.

P_n represents the statements, P_1, P_2, P_3, and so on. Mathematical induction is a two-part process to prove all the statements in the sequence. We first need to show that the statement is true for $n = 1$. That is, prove that P_1 is true. The second part involves proving that if any statement P_k in the sequence is true, then the statement that follows is also true (P_k implies P_{k+1}).

Intuitively, mathematical induction is similar to the sequential effect of falling dominos. Pushing the first domino down is analogous to proving P_1. Then showing that any falling domino in the sequence will cause the next domino to fall is the idea behind part 2 in mathematical induction. That is, if domino 1 falls, and any falling domino makes the next domino in line fall, then all the dominos will fall.

In Examples 1 and 2, we use mathematical induction to prove a statement involving a summation.

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Example 1: Using Mathematical Induction

Use mathematical induction to prove that $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$.

Solution:

Let P_n denote the statement $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$.

1. Show that P_1 is true.

For $n = 1$, the sum is 1 which equals $(1)^2$.

Therefore, P_1 is true.

We need to show that P_1 is true. That is, we need to show that the left and right sides of the statement are equal for $n = 1$.

2. Next, assume that P_k is true and show that P_{k+1} is true.

The statement that P_k is true is called the **inductive hypothesis.**

Assume that P_k is true.

Assume that $1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$.

We must show that P_{k+1} is true.

We must show that $1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2$. (1)

The left side of statement P_{k+1} can be written as

$$\overbrace{[1 + 3 + 5 + 7 + \cdots + (2k - 1)]}^{\text{Sum of first } k \text{ terms}} + \overbrace{[2(k + 1) - 1]}^{(k + 1)\text{th term}}$$

$$= \quad k^2 \quad + [2(k + 1) - 1]$$

By the inductive hypothesis, we can replace the sum of the first k terms by k^2.

$$= k^2 + (2k + 2 - 1)$$

$$= k^2 + 2k + 1$$

$= (k + 1)^2$ as desired. This matches equation (1).

We have shown that P_1 is true, and that if P_k is true, then P_{k+1} is true. By mathematical induction, we conclude that the statement is true for all positive integers.

Check Your Work

It is important to use parentheses when substituting $(k + 1)$ into a statement for n.

Apply the Skills

1. Use mathematical induction to prove P_n: $2 + 4 + 6 + 8 + \cdots + 2n = n(n + 1)$.

In Example 2, we prove the statement introduced at the beginning of this lesson.

Example 2: Using Mathematical Induction

Use mathematical induction to prove that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

Solution:

Let P_n denote the statement $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

1. Show that P_1 is true.

For $n=1$, the sum equals $\frac{1}{1\cdot 2}=\frac{1}{2}$ which is the same as $\frac{n}{n+1}=\frac{1}{1+1}=\frac{1}{2}$.

left side of statement

right side of statement

2. Next, assume that P_k is true.

Assume that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$.

Assume that P_k is true. This is the inductive hypothesis.

We must show that P_{k+1} is true.

We must show that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)\,[(k+1)+1]}=\frac{k+1}{(k+1)+1}=\frac{k+1}{k+2}. \qquad (1)$$

The left side of statement P_{k+1} can be written as

$$\overbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}}^{\text{Sum of first } k \text{ terms}}+\overbrace{\frac{1}{(k+1)[(k+1)+1]}}^{(k+1)\text{th term}}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)[(k+1)+1]}$$

By the inductive hypothesis, replace the sum of the first k terms by $\frac{k}{k+1}$.

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

Simplify.

$$=\frac{k}{(k+1)}\cdot\frac{(k+2)}{(k+2)}+\frac{1}{(k+1)(k+2)}$$

Write terms with a common denominator.

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

Add the fractions.

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

Factor the numerator.

$=\frac{k+1}{k+2}$ as desired. This matches equation (1).

We have shown that P_1 is true, and that if P_k is true, then P_{k+1} is true. By mathematical induction, we conclude that the statement is true for all positive integers.

Apply the Skills

2. Use mathematical induction to prove that

P_n: $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$.

Mathematical induction can be used to prove the following useful summation formulas involving powers of the first n positive integers. The proofs of these formulas are addressed in Exercises 17–20.

Sums of Powers

Let n represent a positive integer. Then,

1. $\sum_{i=1}^{n} 1 = n$
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

In Examples 3 and 4, we use mathematical induction to prove statements that do not involve a sum.

Example 3: Using Mathematical Induction

Use mathematical induction to prove that 4 is a factor of $9^n - 1$.

Solution:

Let P_n be the statement: 4 is a factor of $9^n - 1$.

To show that 4 is a factor of any expression, we need to write the expression as the product of 4 and some positive integer that we call a.

1. Show that P_n is true for $n = 1$.

   P_1 reads: 4 is a factor of $9^1 - 1 = 8$.

   P_1 is true because $8 = 4 \cdot 2$, indicating that 4 is indeed a factor of 8.

2. Next, assume that P_k is true and show that P_{k+1} is true:

   Assume that 4 is a factor of $9^k - 1$.

   This implies that $9^k - 1 = 4a$ or equivalently $9^k = 4a + 1$. Inductive hypothesis

   We must show that 4 is a factor of $9^{k+1} - 1$.

   $9^{k+1} - 1 = 9 \cdot 9^k - 1$ From the inductive hypothesis, replace 9^k by $4a + 1$.

   $= 9 \cdot (4a + 1) - 1$

   $= 36a + 9 - 1$ Apply the distributive property.

   $= 36a + 8$

   $= 4(9a + 2)$ Factor. The value 4 is a factor of the expression $4(9a + 2)$. Therefore, 4 is a factor of $9^{k+1} - 1$.

We have shown that P_1 is true, and that if P_k is true, then P_{k+1} is true. By mathematical induction, we conclude that the statement is true for all positive integers.

Apply the Skills

3. Use mathematical induction to prove that 2 is a factor of $5^n + 1$.

Learn: Prove a Statement Using the Extended Principle of Mathematical Induction

Mathematical induction can be extended to prove statements that might hold true only for integers greater than or equal to some positive integer j. In such a case, we use the extended principle of mathematical induction.

Extended Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n. Then P_n is true for all positive integers $n \geq j$ if

1. P_j is true, and
2. For an integer k, if $k \geq j$, the truth of P_k implies the truth of P_{k+1}.

Example 4: Using the Extended Principle of Mathematical Induction

Use mathematical induction to prove that $n! > 2^n$ for positive integers $n \geq 4$.

Solution:

Let P_n be the statement: $n! > 2^n$.

1. First show that P_n is true for $n = 4$.

   To prove that P_4 is true, show that $4! > 2^4$.

   $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ and $2^4 = 16$. Therefore, $4! > 2^4$, indicating that P_4 is true.

2. Next, assume that P_k is true for $k \geq 4$, and show that P_{k+1} is true.

   Assume that $k! > 2^k$ and show that $(k + 1)! > 2^{k+1}$ where $k \geq 4$.

   $(k + 1)! = (k + 1) \cdot k!$

   $> (k + 1)(2^k)$ By the inductive hypothesis, $k! > 2^k$. Therefore, we can replace $k!$ by 2^k and replace the $=$ sign with $>$.

   $> 2 \cdot 2^k$ For $k \geq 4$, the expression $(k + 1) > 2$. Therefore, we can replace $(k + 1)$ by 2 and maintain the $>$ symbol.

   $= 2^{k+1}$

   From the string of inequalities we have shown that $(k + 1)! > 2^{k+1}$.

We have shown that P_4 is true, and that if P_k is true, then P_{k+1} is true. By mathematical induction, we conclude that $n! > 2^n$ is true for all integers, $n \geq 4$.

Insights

In part 2 of Example 4, we manipulate the expression $(k + 1)!$ on the left side to show that it is greater than the expression on the right side for $k \geq 4$.

Apply the Skills

4. Use mathematical induction to prove that $\left(\frac{3}{2}\right)^n > 2n$ for $n \geq 7$.

Point of Interest

Mathematical induction is one type of mathematical proof, but there are many other techniques, including proof by contradiction. This technique uses the premise that a statement's being false would imply a contradiction. This approach was used in the complex proof of Fermat's last theorem.

The equation $x^2 + y^2 = z^2$ has infinitely many positive integer solutions for a, b, and c. Such solutions are called Pythagorean triples, such as $a = 5$, $b = 12$, and $c = 13$. However, French mathematician Pierre de Fermat (1601–1665) posed the statement that the equation $x^n + y^n = z^n$ has *no* such solutions for positive integers $n > 2$. Fermat made this claim, now famously called Fermat's last theorem, while annotating a copy of Diophantus' *Arithmetika*. Fermat included a comment that he had "found a remarkable proof of this fact, but there is not enough space in the margin to write it."

Proof of this theorem eluded mathematicians for over 350 yr until Andrew Wiles of Princeton University announced in June of 1993 that he had solved the problem. Wiles's proof is extremely complex and was revised to correct a slight but critical flaw shortly after publication. The proof involves the use of mathematics that did not exist in Fermat's time, and for this reason mathematicians believe that Fermat did not have a simpler proof.

Practice 11-4

Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, factor completely.

R.1. $x^4 + x^2$

R.2. $-15r^4 - 60r^3$

R.3. $w^2 + 4w + 4$

R.4. $v^2 - 9v + 18$

For Exercises R.5–R.6, simplify.

R.5. $t - 4 + \frac{1}{t + 4}$ **R.6.** $\frac{9}{a(a - 3)} + \frac{3}{a}$

Concept Connections

1. Let P_n be a statement involving the positive integer n, and let k be an arbitrary positive integer. Proof by mathematical ______________ indicates that P_n is true for all positive integers n if (1) ___________ is true, and (2) the truth of P_k implies the truth of ___________.

2. The statement that P_k is true is called the ______________ hypothesis.

Learn: Prove a Statement Using Mathematical Induction

For Exercises 3–16, use mathematical induction to prove the given statement for all positive integers *n*. (See Examples 1–2, pp. 1217–1218)

3. $2 + 6 + 10 + \cdots + (4n - 2) = 2n^2$

4. $2 + 8 + 14 + \cdots + (6n - 4) = n(3n - 1)$

5. $5 + 8 + 11 + \cdots + (3n + 2) = \frac{n}{2}(3n + 7)$

6. $4 + 9 + 14 + \cdots + (5n - 1) = \frac{n}{2}(5n + 3)$

7. $8 + 4 + 0 + \cdots + (-4n + 12) = -2n(n - 5)$

8. $12 + 6 + 0 + \cdots + (-6n + 18) = -3n(n - 5)$

9. $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$

10. $1 + 3 + 3^2 + 3^3 + \cdots + 3^{n-1} = \frac{1}{2}(3^n - 1)$

11. $\frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \cdots + \frac{3}{4^n} = 1 - \left(\frac{1}{4}\right)^n$

12. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \left(\frac{1}{2}\right)^n$

13. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$

14. $1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + \cdots + n(2n + 1) = \frac{n(n + 1)(4n + 5)}{6}$

15. $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{n + 1}\right) = \frac{1}{n + 1}$

16. $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{(n + 1)^2}\right) = \frac{n + 2}{2(n + 1)}$

For Exercises 17–24, use mathematical induction to prove the given statement for all positive integers *n*. (See Example 3, p. 1219)

17. $\sum_{i=1}^{n} 1 = n$

18. $\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$

19. $\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$

20. $\sum_{i=1}^{n} i^3 = \frac{n^2(n + 1)^2}{4}$

21. 2 is a factor of $5^n - 3$.

22. 2 is a factor of $7^n - 3$.

23. $4^n - 1$ is divisible by 3.

24. $5^n - 1$ is divisible by 4.

Learn: Prove a Statement Using the Extended Principle of Mathematical Induction

For Exercises 25–28, use trial-and-error to determine the smallest positive integer *n* for which the given statement is true.

25. $n! > 3^n$

26. $(n + 1)! > 4^n$

27. $3n < 2^n$

28. $5n < 3^n$

For Exercises 29–32, use mathematical induction to prove the given statement. (See Example 4, p. 1220)

29. $n! > 3^n$ for positive integers $n \geq 7$.

30. $(n + 1)! > 4^n$ for positive integers $n \geq 6$.

31. $3n < 2^n$ for positive integers $n \geq 4$.

32. $5n < 3^n$ for positive integers $n \geq 3$.

Mixed Exercises

For Exercises 33–36, use mathematical induction to prove the given statement for all positive integers *n* and real numbers *x* and *y*.

33. $(xy)^n = x^n y^n$

34. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ provided that $y \neq 0$.

35. If $x > 1$, then $x^n > x^{n-1}$.

36. If $0 < x < 1$, then $x^n < x^{n-1}$.

We have used mathematical induction to prove that a statement is true for all positive integers *n*. To show that a statement is *not* true, all we need is one case in which the statement is false. This is called a counterexample. For Exercises 37–38, find a counterexample to show that the given statement is *not* true.

37. The expression $n^2 - n + 11$ is prime for all positive integers n.

38. The inequality $5^n > n!$ is true for all positive integers n.

Write About It

39. Explain the difference between the principle of mathematical induction and the extended principle of mathematical induction.

40. Suppose that a fellow student showed that the expression $n^2 + n + 1$ is prime for $n = 1$, 2, and 3. Explain why this is not a sufficient proof that the expression is prime for all positive integers n.

Expanding Your Skills

41. Show that $n^2 - n$ is even for all positive integers n.

42. Show that $n^2 - n + 1$ is odd for all positive integers n.

For Exercises 43–44, use the Fibonacci sequence $\{F_n\} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$. Recall that the Fibonacci sequence can be defined recursively as $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

43. Prove that $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ for positive integers $n \geq 3$.

44. Prove that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$ for all positive integers n.

Go online for more practice problems.

Lesson 11-5

The Binomial Theorem

What Will You Learn?

After completing this lesson, you should be able to:

- Determine Binomial Coefficients
- Apply the Binomial Theorem
- Find a Specific Term in a Binomial Expansion

Learn: Determine Binomial Coefficients

In Lesson R-3 we learned how to square a binomial.

$$(a+b)^2 = a^2 + 2ab + b^2$$

The expression $a^2 + 2ab + b^2$ is called the **binomial expansion** of $(a+b)^2$. To expand $(a+b)^3$, we can find the product $(a+b)(a+b)^2$.

$$\begin{aligned}(a+b)(a+b)^2 &= (a+b)(a^2+2ab+b^2)\\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

Similarly, to expand $(a+b)^4$, we can multiply $(a+b)$ by $(a+b)^3$. Using this method, we can expand several powers of $(a+b)$.

$(a+b)^0 = 1$

$(a+b)^1 = 1a + 1b$

$(a+b)^2 = 1a^2 + 2ab + 1b^2$

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

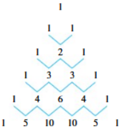


Figure 11-9

From the expansion of $(a+b)^n$, we note the following patterns.

- The exponent on a decreases from n to 0 on sequential terms from left to right.
- The exponent on b increases from 0 to n on sequential terms from left to right.
- The sum of the exponents on each term (that is, the degree of each term) is n.
- The number of terms in the expansion is $(n+1)$. For example, the expansion of $(a+b)^4$ has five terms.

The coefficients for the expansion of $(a+b)^n$ follow a triangular array of numbers called **Pascal's triangle** (Figure 11-9), named after the French mathematician Blaise Pascal (1623–1662). Each row begins and ends with a 1, and each entry in between is the sum of the two diagonal entries from the row above. For example, in the last row of Figure 11-9, we have

$$1, \quad 1+4=5, \quad 4+6=10, \quad 6+4=10, \quad 4+1=5, \text{ and } 1.$$

Example 1: Expanding a Binomial

Expand. $(a + b)^6$

Solution:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

The expansion of $(a + b)^6$ will have 7 terms (one more than the exponent of 6). Using the entries of the seventh row of Pascal's triangle as the coefficients, the expansion of $(a + b)^6$ is:

$$(a + b)^6 = 1a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + 1b^6$$

Apply the Skills

1. Expand. $(a + b)^7$

Determining the coefficients of a binomial expansion using Pascal's triangle would be cumbersome for expansions of higher degree. Instead we can compute the coefficients of a binomial expansion using the following formula.

Coefficients of a Binomial Expansion

Let n and r be nonnegative integers with $n \geq r$. The expression $\binom{n}{r}$ (read as "n choose r") is called a **binomial coefficient** and is defined by

$$\binom{n}{r} = \frac{n!}{r!(n - r)!}$$

Note: The notation ${}_nC_r$ is often used in place of $\binom{n}{r}$.

Insights

The expression $\binom{n}{r}$ will also be used in Lesson 11-6 when we study counting principles. $\binom{n}{r}$ represents the number of ways we can *choose* a group of r items in any order from a group of n items.

Example 2: Computing Binomial Coefficients

Evaluate.

a. $\binom{6}{0}$ **b.** $\binom{6}{1}$ **c.** $\binom{6}{2}$ **d.** $\binom{6}{6}$

Solution:

For parts (a)–(d), apply $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

a. $\binom{6}{0} = \frac{6!}{0! \cdot (6-0)!} = \frac{6!}{0! \cdot 6!} = \frac{6!}{1 \cdot 6!} = 1$ — $n = 6$ and $r = 0$ Recall that $0! = 1$.

b. $\binom{6}{1} = \frac{6!}{1! \cdot (6-1)!} = \frac{6!}{1! \cdot 5!} = \frac{6 \cdot 5!}{1 \cdot 5!} = 6$ — $n = 6$ and $r = 1$

c. $\binom{6}{2} = \frac{6!}{2! \cdot (6-2)!} = \frac{6!}{2! \cdot 4!} = \frac{6 \cdot 5 \cdot 4!}{2 \cdot 1 \cdot 4!} = 15$ — $n = 6$ and $r = 2$

d. $\binom{6}{6} = \frac{6!}{6! \cdot (6-6)!} = \frac{6!}{6! \cdot 0!} = \frac{6!}{6! \cdot 1} = 1$ — $n = 6$ and $r = 6$

Apply the Skills

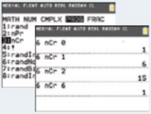
2. Evaluate.

a. $\binom{5}{0}$ **b.** $\binom{5}{1}$ **c.** $\binom{5}{2}$ **d.** $\binom{5}{5}$

TECHNOLOGY CONNECTIONS

Evaluating Binomial Coefficients

To evaluate binomial coefficients on a graphing utility, use the ${}_nC_r$ function found in the MATH menu under PRB.



```
6 nCr 0
                   1
6 nCr 1
                   6
6 nCr 2
                  15
6 nCr 6
                   1
```

Learn: Apply the Binomial Theorem

From Example 2, notice that $\binom{6}{0} = 1$, $\binom{6}{1} = 6$, and $\binom{6}{2} = 15$ are the coefficients for the first three terms of the expansion of $(a + b)^6$. The value $\binom{6}{6} = 1$ is the coefficient of the last term of the expansion. Using the values of $\binom{n}{r}$ for the coefficients, we have a formula for the expansion of the binomial $(a + b)^n$. This is called the **binomial theorem**.

The Binomial Theorem

Let n be a positive integer. The expansion of $(a + b)^n$ is given by

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r\end{aligned}$$

Example 3: Applying the Binomial Theorem

Expand by using the binomial theorem. $(2x + 3)^4$

Solution:

The expression $(2x + 3)^4$ is in the form $(a + b)^4$ with $a = 2x$ and $b = 3$.

$$(a + b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

$$\begin{aligned}(2x + 3)^4 &= \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(3) + \binom{4}{2}(2x)^2(3)^2 + \binom{4}{3}(2x)(3)^3 + \binom{4}{4}(3)^4 \\ &= 1(16x^4) + 4(8x^3)(3) + 6(4x^2)(9) + 4(2x)(27) + 1(81) \\ &= 16x^4 + 96x^3 + 216x^2 + 216x + 81\end{aligned}$$

Apply the Skills

3. Expand by using the binomial theorem. $(4y + 5)^4$

In Example 4, we expand a difference of terms.

Example 4: Applying the Binomial Theorem

Expand by using the binomial theorem. $(3x^2 - 5y)^3$

Solution:

The expression $(3x^2 - 5y)^3 = [3x^2 + (-5y)]^3$ is in the form $(a + b)^3$ with $a = 3x^2$ and $b = -5y$.

$$(a + b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$$

$$\begin{aligned}[3x^2 + (-5y)]^3 &= \binom{3}{0}(3x^2)^3 + \binom{3}{1}(3x^2)^2(-5y) + \binom{3}{2}(3x^2)(-5y)^2 + \binom{3}{3}(-5y)^3 \\ &= 1(27x^6) + 3(9x^4)(-5y) + 3(3x^2)(25y^2) + 1(-125y^3) \\ &= 27x^6 - 135x^4y + 225x^2y^2 - 125y^3\end{aligned}$$

Apply the Skills

4. Expand by using the binomial theorem. $(2t^4 - 3v)^3$

Learn: Find a Specific Term in a Binomial Expansion

Consider the first four terms of the binomial expansion of $(a+b)^n$.

The exponent on b is one less than the term number.

$$\underset{\uparrow}{\binom{n}{0}}a^n\overset{\downarrow}{b^0} + \underset{\uparrow}{\binom{n}{1}}a^{n-1}\overset{\downarrow}{b^1} + \underset{\uparrow}{\binom{n}{2}}a^{n-2}\overset{\downarrow}{b^2} + \underset{\uparrow}{\binom{n}{3}}a^{n-3}\overset{\downarrow}{b^3}$$

These values match the exponent on b.

From the observed pattern, the kth term has a factor of b raised to the $k-1$ power. That is, the exponent on b is one less than the term number. Furthermore, the sum of the exponents must equal n. Therefore, the factor a is raised to the $n-(k-1)$ power. That is, the exponent on a is n minus the exponent on b.

kth Term of a Binomial Expansion

Let n and k be positive integers with $k \le n+1$. The kth term of $(a+b)^n$ is

$$\binom{n}{k-1}a^{n-(k-1)}b^{k-1}$$

Example 5: Finding a Specific Term of a Binomial Expansion

Find the eighth term of $(2x+y^4)^{10}$.

Solution:

To find the eighth term of $(a+b)^{10}$, we require the exponent on b to be 7 (one less than the term number). Therefore, the exponent on a must be 3 so that the sum of the exponents is 10.

Sum of exponents must be 10.

1 less than the term number

Therefore, the eighth term of $(a+b)^{10}$ is $\binom{10}{7}a^3b^7$.

same as exponent on b

The eighth term of $(2x+y^4)^{10}$ is $\binom{10}{7}(2x)^3(y^4)^7 = 120(8x^3)(y^{28}) = 960x^3y^{28}$.

Alternatively, we can apply the formula for the kth term of a binomial expansion with $n=10$, $k=8$, $a=2x$, and $b=y^4$.

$$\binom{n}{k-1}a^{n-(k-1)}b^{k-1} = \binom{10}{8-1}(2x)^{10-(8-1)}(y^4)^{8-1} = \binom{10}{7}(2x)^3(y^4)^7 = 960x^3y^{28}$$

The eighth term of the expansion is $960x^3y^{28}$.

Apply the Skills

5. Find the seventh term of $(3c+d^5)^9$.

Practice 11-5

Practice Exercises

Prerequisite Review

For Exercises R.1–R.5, multiply and simplify.

R.1. $(5v + 2)^2$

R.2. $(8a^2 - 9b^3)^2$

R.3. $(4x + \sqrt{15})^2$

R.4. $\left(\frac{1}{5}m + 2\right)^2$

R.5. $(4a - 3b)^3$

Concept Connections

1. The expression $a^3 + 3a^2b + 3ab^2 + b^3$ is called the ____________ expansion of $(a + b)^3$.

2. Consider $(a + b)^n$, where n is a whole number. How many terms are in the binomial expansion?

3. Consider $(a + b)^n$, where n is a whole number. What is the degree of each term in the expansion?

4. Consider $(a + b)^n$, where n is a whole number. The coefficients of the terms in the expansion can be found by using ____________ triangle or by using $\binom{n}{r}$.

5. For positive integers n and k ($k \leq n + 1$), the kth term of $(a + b)^n$ is given by $\left(\;\right)a^{\square}b^{\square}$.

6. Given $(a + b)^{17}$, the 12th term is given by $\left(\;\right)a^{\square}b^{\square}$.

Learn: Determine Binomial Coefficients

7. **a.** Write the first five rows of Pascal's triangle.

b. Write the expansion of $(x - y)^4$. (**See Example 1, p. 1225**)

8. **a.** Write the first four rows of Pascal's triangle.

b. Write the expansion of $(c - d)^3$.

For Exercises 9–10, evaluate the given expressions. Compare the results to the fifth and sixth rows of Pascal's triangle. (See Example 2, p. 1225)

9. **a.** $\binom{4}{0}$ **b.** $\binom{4}{1}$ **c.** $\binom{4}{2}$

d. $\binom{4}{3}$ **e.** $\binom{4}{4}$

10. **a.** $\binom{5}{0}$ **b.** $\binom{5}{1}$ **c.** $\binom{5}{2}$

d. $\binom{5}{3}$ **e.** $\binom{5}{4}$ **f.** $\binom{5}{5}$

For Exercises 11–14, evaluate the expression.

11. $\binom{13}{3}$

12. $\binom{17}{15}$

13. $\binom{11}{5}$

14. $\binom{9}{4}$

Learn: Apply the Binomial Theorem

For Exercises 15–28, expand the binomial by using the binomial theorem. (See Examples 3–4, p. 1227)

15. $(3x + 1)^5$

16. $(5x + 3)^5$

17. $(7x + 3)^3$

18. $(6x + 5)^3$

19. $(2x - 5)^4$

20. $(4x - 1)^4$

21. $(2x^3 - y)^5$

22. $(3y^2 - z)^5$

23. $(p^2 - w^4)^6$

24. $(t^3 - v^5)^6$

25. $(0.2 + 0.1k)^4$

26. $(0.1 + 0.3m)^4$

27. $\left(\frac{c}{2} - d\right)^3$

28. $\left(\frac{x}{3} - n\right)^3$

Learn: Find a Specific Term in a Binomial Expansion

For Exercises 29–40, find the indicated term of the binomial expansion. (See Example 5, p. 1228)

29. $(m + n)^{10}$; seventh term

30. $(p + q)^{11}$; ninth term

31. $(c - d)^8$; fourth term

32. $(x - y)^9$; sixth term

33. $(u^2 + 2v^4)^{15}$; tenth term

34. $(y^3 + 2z^2)^{14}$; tenth term

35. $\left(\sqrt{3x^2}+y^3\right)^9$; fourth term

36. $\left(\sqrt{2u^3}+v^4\right)^8$; third term

37. $(h^4-1)^{12}$; middle term

38. $(k^3-1)^{10}$; middle term

39. $(p^4+3q)^8$; term containing p^{12}

40. $(a^3+4b)^6$; term containing a^9

Mixed Exercises

41. Expand $(e^x-e^{-x})^4$.

42. Expand $(e^x+e^{-x})^3$.

43. Use the binomial theorem to expand $(x+y-z)^3$.

44. Use the binomial theorem to expand $(a+b-2)^3$.

45. Use the binomial theorem to find $(1.01)^4$. [*Hint*: Write the expression as $(1+0.01)^4$.]

46. Use the binomial theorem to find $(1.1)^5$.

47. Simplify. $(x+y)^3-(x-y)^3$

48. Simplify. $(x+1)^3-(x-1)^3$

For Exercises 49–52, simplify the difference quotient: $\frac{f(x+h)-f(x)}{h}$

49. $f(x)=2x^3+4$

50. $f(x)=4x^3+3$

51. $f(x)=x^4-5x^2+1$

52. $f(x)=x^4-6x^2-4$

For Exercises 53–56, use the binomial theorem to find the value of the complex number raised to the given power. Recall that $i=\sqrt{-1}$.

53. $(2+i)^3$

54. $(3+i)^3$

55. $(5-2i)^4$

56. $(6-5i)^4$

57. Show that $\binom{n}{r}$ and $\binom{n}{n-r}$ are equivalent.

58. Show that $\binom{n}{0}=1$ and $\binom{n}{n}=1$.

Write About It

59. Explain why $\binom{-13}{3}$ is undefined.

60. Explain why $\binom{3}{5}$ is undefined.

61. Explain why many graphing utilities give an error message for the expression $\frac{120!}{118!}$. How can this expression be evaluated by hand?

Expanding Your Skills

Stirling's formula (named after Scottish mathematician, James Stirling: 1692–1770) is used to approximate large values of $n!$. Stirling's formula is $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. For Exercises 62–63,

a. Use Stirling's formula to approximate the given expression. Round to the nearest whole unit.

b. Compute the actual value of the expression.

c. Determine the percent difference between the approximate value and the actual value. Round to the nearest tenth of a percent.

62. 8!

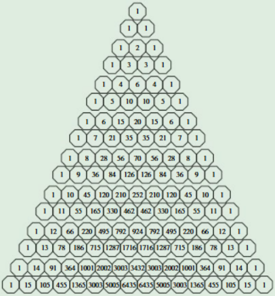
63. 12!

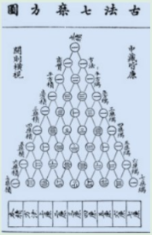
Using techniques from calculus, we can show that $(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$, for $|x|<1$. This formula can be used to evaluate binomial expressions raised to noninteger exponents. For Exercises 64–65, use the first four terms of this infinite series to approximate the given expression. Round to 3 decimal places if necessary.

64. $(1.4)^{3/2}$ [*Hint*: $(1.4)^{3/2}=(1+0.4)^{3/2}$]

65. $\sqrt[4]{(1.3)^3}$

Point of Interest





The array of numbers we call Pascal's triangle was also known in India, China, and Iran hundreds of years earlier. The 10th century Indian mathematician Halayudha gave a description of how to fit binomial coefficients into a triangular array. The Persian poet and mathematician Omar Kayyam also described a triangle of values. One of the earliest illustrations of the triangular array appears in 1261 in the mathematical text *Xiangjie Jiuzhang Suanfa*.

For Exercises 66–69, we examine some of the interesting patterns appearing in the rows, columns, and diagonals of Pascal's triangle.

66. a. Write the sequence corresponding to the sum of the numbers in each row of Pascal's triangle for the firstnine rows.

b. Let n represent the row number in Pascal's triangle, beginning with $n = 0$. Write a formula for the nth term of the sequence representing the sum of the numbers in row n.

c. Use the binomial theorem to prove the result in part (b). (*Hint*: For $(a + b)^n$, let $a = 1$ and $b = 1$.)

67. Examine the rows in Pascal's triangle that begin with a prime number after the leftmost entry of 1. What do you observe about the relationship between the first prime and the other entries in the row (not counting the last entry of 1)?

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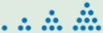
68. A triangular number represents objects stacked in the shape of an equilateral triangle. The first triangular number is $T_1 = 1$, followed by $T_2 = 3$, $T_3 = 6$, and so on.



a. What is the fifth triangular number?

b. Write the first six terms of the sequence of triangular numbers by starting at the beginning or end of the third row of Pascal's triangle and following the diagonal downward.

c. The triangular numbers may also be represented by $T_n = \binom{n+1}{2}$. What is the 20th triangular number?

d. Following the diagonal from either end of the fourth row of Pascal's triangle will give **tetrahedral numbers** (triangular pyramid numbers). The first tetrahedral number is 1. The second tetrahedral number is 4, (geometrically, we envision three round objects touching, then place a fourth on top.) The third tetrahedral number is 10. How would you arrange 10 round objects in a stack to form a tetrahedron? What is the fifth tetrahedral number?

e. The tetrahedral numbers may also be represented as follows:

$$n\text{th tetrahedral number} = \binom{n+2}{3}.$$

What is the eighth tetrahedral number?

69. If we darken out all the odd-numbered entries in Pascal's triangle and leave the even-numbered entries light, what pattern seems to appear? (*Hint*: Refer to Lesson 11-3, Exercise 114.)

Go online for more practice problems.

Lesson 11-6

Principles of Counting

What Will You Learn?

After completing this lesson, you should be able to:

- Apply the Fundamental Principle of Counting
- Count Permutations
- Count Combinations

Learn: Apply the Fundamental Principle of Counting

Many states have lotteries as a means to raise income. One such game pays a grand prize to players who choose six distinct numbers from 1 to 53 (in any order) that match the same group of six numbers in the drawing. The number of such six-number combinations is 22,957,480 (see Example 8). If a player chooses one group of six numbers, then the probability (likelihood) of winning the grand prize is $\frac{1}{22,957,480}$. This value (roughly 1 in 23 million) means that it is highly unlikely that anyone will win the grand prize. To put this in perspective, a more likely event is flipping a coin 24 times and getting all heads, which has a probability of $\frac{1}{16,777,216}$.

In this lesson and in Lesson 11-7, we present basic principles of counting and how these principles apply to probability. We begin in Example 1, by using a figure called a tree diagram to organize different outcomes of a sequence of events.

Example 1: Counting the Outcomes of a Sequence of Events

Suppose that a frozen custard shop offers 4 flavors of custard, 2 types of syrup, and 2 toppings. If a customer selects 1 item from each group for a sundae, how many different sundaes can be made?

Custard Flavors	Syrups	Toppings
Vanilla (V)	Hot fudge (H)	Nuts (N)
Chocolate (C)	Butterscotch (B)	Granola (G)
Mint chip (M)		
Peanut butter (P)		

Solution:

We can depict the 4 flavors of custard by the leftmost branches in Figure 11-10. Then for each branch of custard, there are 2 branches representing the type of syrup. Finally, for each custard-syrup arrangement, there are 2 toppings.

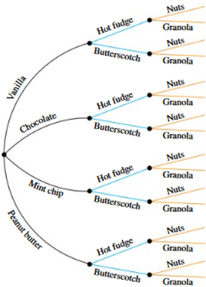

Vanilla
Chocolate
Mint chip
Peanut butter
Hot fudge
Butterscotch
Hot fudge
Butterscotch
Hot fudge
Butterscotch
Hot fudge
Butterscotch
Nuts
Granola
Nuts
Granola
Nuts
Granola
Nuts
Granola
Nuts
Granola
Nuts
Granola
Nuts
Granola
Nuts
Granola

Figure 11-10

The tree diagram shows 16 different sundaes:

VHN, VHG, VBN, VBG, CHN, CHG, CBN, CBG, MHN, MHG, MBN, MBG, PHN, PHG, PBN, and PBG.

Notice that the number of outcomes is easily found by multiplying the number of choices from each group:

number of custards

number of syrups

number of toppings

Total outcomes: $(4)(2)(2) = 16$

Apply the Skills

1. A single-topping pizza can be made from either thick or thin crust, and a choice of 6 toppings. How many different types of single-topping pizzas can be made?

The result from Example 1 suggests that the number of outcomes for the sequence of events is the product of the number of outcomes for each individual event. This is generalized as the **fundamental principle of counting.**

Fundamental Principle of Counting

If one event can occur in m different ways and a second event can occur in n different ways, then the number of ways that the two events can occur in sequence is $m \cdot n$.

In Example 2, we demonstrate that the fundamental principle of counting can be applied for a sequence of many events.

Example 2: Applying the Fundamental Principle of Counting

A computer password must have three letters (either upper- or lowercase), followed by two digits. How many passwords can be made if

a. There are no restrictions on the letters or digits?

b. No letter or digit may be used more than once?

Solution:

a. The computer password is a sequence of five characters. The number of arrangements of letters and digits is given by the product of the number of outcomes for each character. There are 26 letters of the alphabet from which to choose, but these may be either upper- or lowercase. So, there are 52 choices of letters. The digits in the password may be selected from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Therefore, the total number of passwords is given by

$$\overbrace{\underline{52}\cdot\underline{52}\cdot\underline{52}}^{\text{There are 52 choices for each letter.}}\cdot\overbrace{\underline{10}\cdot\underline{10}}^{\text{There are 10 choices for each digit.}}=14{,}060{,}800$$

b. If no letter may be repeated, then the number of letters from which to choose must be decreased by one when the second letter is selected. Likewise, the number of letters must be decreased by one again when the third letter is selected. The same logic is used when selecting the digits. Once the first digit is selected in the password, there are only nine remaining digits available for the second choice.

$$\overbrace{\underline{52}\cdot\underline{51}\cdot\underline{50}}^{\text{The number of letters is decreased by one for each letter in the sequence.}}\cdot\overbrace{\underline{10}\cdot\underline{9}}^{\text{There are 10 choices for the first digit, but only 9 remaining for the second.}}=11{,}934{,}000$$

Apply the Skills

2. A code for an alarm system is made up of two uppercase letters, followed by four digits. How many codes can be made if

a. There are no restrictions on the letters or digits?

b. No letter or digit may be used more than once?

Example 3: Applying the Fundamental Principle of Counting

A quiz has five true/false questions and five multiple-choice questions. The multiple-choice questions each have four possible answers, of which only one is correct. If a student forgets to study and must guess on each question, in how many ways can the student answer the questions on the test?

Solution:

The quiz questions form a 10-stage event. Each true/false question has two possible choices (true or false). Each multiple-choice question has four possible choices (for example: a, b, c, or d). Therefore, the number of ways that the questions can be answered on the test is given by

$$\overbrace{\underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\underline{2}}^{\text{Each true/false question has 2 choices}}\cdot\overbrace{\underline{4}\cdot\underline{4}\cdot\underline{4}\cdot\underline{4}\cdot\underline{4}}^{\text{Each multiple-choice question has 4 choices}} = 32{,}768$$

Apply the Skills

3. A quiz has four true/false questions and six multiple-choice questions. The multiple-choice questions each have five possible answers, of which only one is correct. In how many ways can the questions on the test be answered?

Learn: Count Permutations

We now look at a situation in which *n* items are to be arranged in order. Such an arrangement is called a **permutation.** For example, consider the different arrangements of the letters in the word FIVE.

$$\left.\begin{array}{llllll} \text{FIVE} & \text{FIEV} & \text{FVIE} & \text{FVEI} & \text{FEIV} & \text{FEVI} \\ \text{IFVE} & \text{IFEV} & \text{IVEF} & \text{IVFE} & \text{IEVF} & \text{IEFV} \\ \text{VFIE} & \text{VFEI} & \text{VEIF} & \text{VEFI} & \text{VIEF} & \text{VIFE} \\ \text{EFIV} & \text{EFVI} & \text{EIFV} & \text{EIVF} & \text{EVIF} & \text{EVFI} \end{array}\right\} \text{24 arrangements}$$

There are 24 such arrangements. We can arrive at the same conclusion by applying the fundamental principle of counting. There are 4 letters available for the first choice in the arrangement. That leaves 3 letters remaining for the second choice, followed by 2 letters for the third choice, and only 1 letter for the last choice. Therefore, the number of permutations of 4 distinct letters is given by

$$4\cdot 3\cdot 2\cdot 1 = 24$$

We generalize this as follows:

Number of Permutations of *n* Distinguishable Elements

The number of permutations of *n* distinct elements is *n*!.

Note: This means that there are *n*! ways to arrange *n* distinguishable items in various orders.

Now suppose that we wanted to arrange the letters in the word NINE. Notice that if we were to switch the first and third letters (both "N's"), we would get the same arrangement. This is because the two N's are indistinguishable. This means that when counting the number of permutations we must "divide out" the number of ways that the two N's can be arranged. The two N's can be arranged in 2! different ways. This leads to the following result.

Number of Permutations of *n* Elements, Some Indistinguishable

Consider a set of n elements with r_1 duplicates of one kind, r_2 duplicates of a second kind, ... , r_k duplicates of a kth kind. Then the number of distinguishable permutations of the n elements of the set is

$$\frac{n!}{r_1! \cdot r_2! \cdot \cdots \cdot r_k!}$$

Note: The factors of $r_1!, r_2!, \ldots, r_k!$ in the denominator remove the repetition of arrangements that arise from the indistinguishable elements.

Example 4: Counting Permutations

a. Determine the number of ways that 6 people can be arranged in line at a ticket counter.

b. Determine the number of ways that the letters in the word MICROSCOPIC can be arranged.

Solution:

a. No two people can be the same. Therefore, we are arranging 6 different (distinguishable) items in various orders. The number of such permutations is given by

$6! = 720$ There are 720 ways in which 6 people can be arranged in line.

b. The word MICROSCOPIC has 11 letters. However, the letter I appears twice, the letter O appears twice, and the letter C appears 3 times. The number of unique arrangements of these letters is

$$\frac{11!}{2! \cdot 2! \cdot 3!} = 1,663,200$$

Apply the Skills

4. a. In how many ways can 7 different books be arranged on a bookshelf?

b. Determine the number of ways that the letters in the word BOOKKEEPING can be arranged.

In Example 5, we determine the number of permutations from a group of n items in which fewer than n items are selected at one time.

Example 5: Finding Permutations of *n* Items Taken *r* at a Time

Suppose that 5 students (Alberto, Beth, Carol, Dennis, and Erik) submit applications for scholarships. There are two scholarships available. The first is for $1000 and the second is for $500. In how many ways can 2 students from the 5 be selected to receive the scholarships?

Solution:

We begin by labeling the students as A, B, C, D, and E. We can construct a list of the possible outcomes.

AB	AC	AD	AE	BC	BD	BE	CD	CE	DE
BA	CA	DA	EA	CB	DB	EB	DC	EC	ED

Each outcome in the second row involves the same 2 people as the outcome directly above in the first row, but in the reverse order. The order of selection is important. For example:

- AB means that Alberto gets $1000 and Beth gets $500, whereas
- BA means that Beth gets $1000 and Alberto gets $500.

There are 20 possible ways to select 2 people from 5 people in which the order of selection is relevant. That is, there are 20 permutations of 5 people taken 2 at a time.

Apply the Skills

5. Determine the number of ways that 2 students from a group of 4 students can be selected to hold the positions of president and vice president of student government.

The number of permutations of 5 people taken 2 at a time can also be computed by using the fundamental principle of counting. There are 5 candidates for the $1000 scholarship, which leaves 4 students left over for the $500 scholarship. The product is $5 \cdot 4 = 20$.

This result can also be obtained by using factorial notation:

Number of people in the group (5).

$$\frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \cdot 4 \cdot \overset{1}{\cancel{(3 \cdot 2 \cdot 1)}}}{\cancel{(3 \cdot 2 \cdot 1)}} = 20$$

Number of people to be selected (2).

Suppose that *n* represents the number of distinct elements in a group from which *r* elements will be chosen in a particular order. We call each arrangement a **permutation** of *n* elements taken *r* at a time and denote the number of such permutations as ${}_nP_r$.

Number of Permutations of *n* Elements Taken *r* at a Time

The number of permutations of *n* elements taken *r* at a time is given by

$${}_nP_r = \frac{n!}{(n-r)!}, \quad \text{or equivalently,} \quad {}_nP_r = \overbrace{n(n-1)(n-2)\cdots(n-r=1)}^{r_{\text{factors}}}$$

Note: ${}_nP_r$ counts the number of permutations of *n* items taken *r* at a time under the assumption that no item can be selected more than once, and that each of the *n* items is distinguishable.

Example 6: Counting Permutations in an Application

If 8 horses enter a race, in how many ways can the horses finish first, second, and third?

Solution:

We must find the number of ways that 3 horses can be selected from 8 horses in a prescribed order (first, second, third). This is given by ${}_8P_3$.

$${}_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8\cdot 7\cdot 6\cdot \overset{1}{\cancel{5!}}}{\cancel{5!}} = 8\cdot 7\cdot 6 = 336$$

Alternatively: $${}_8P_3 = \underbrace{\overset{n=8}{8}\cdot 7\cdot 6}_{3\text{ factors}} = 336$$

There are 336 possible first-, second-, and third-place arrangements.

Insights

The alternative formula for ${}_nP_r$ indicates that we multiply *n* times the consecutive integers less than *n* until a total of *r* factors is reached.

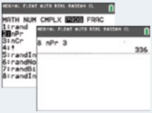
Apply the Skills

6. A judge at the County Fair must give blue, red, and white ribbons for first-, second-, and third-place entries in a poetry contest. If there are 12 contestants, in how many ways can the judge award the ribbons?

TECHNOLOGY CONNECTIONS

Evaluating a Number of Permutations, ${}_nP_r$

Most graphing utilities can evaluate the number of permutations of *n* elements taken *r* at a time. For example, use the ${}_nP_r$ function found in the MATH menu under PRB.



```
8 nPr 3
                    336
```

Learn: Count Combinations

Consider the situation in Example 5 in which 5 students are selected for 2 scholarships. If the scholarships were for *equal* amounts (say for $500 each), then the order in which the 2 students are selected does not matter. This is because the "prizes" are indistinguishable; that is, the outcomes AB and BA are the same because student A and student B would each receive $500.

In this scenario, we want to select a set of 2 people from a set of 5 people without regard to order. In such a case, the outcomes are called *combinations* (rather than permutations). From the list of permutations given in Example 5, we can strike through the redundant cases that arise from the order of the two individuals in the group. As a result, there are 10 combinations of 5 people taken 2 at a time.

AB	AC	AD	AE	BC	BD	BE	CD	CE	DE
~~BA~~	~~CA~~	~~DA~~	~~EA~~	~~CB~~	~~DB~~	~~EB~~	~~DC~~	~~EC~~	~~ED~~

To find the number of combinations, we divide the number of permutations (in this case 20) by 2! because there are 2! ways in which two letters can be arranged in order.

$$\text{Number of combinations} = \frac{{}_5P_2}{2!} = \frac{20}{2} = 10$$

To generalize, suppose that *n* represents the number of distinct elements in a group from which *r* elements will be chosen in *no particular order*. We call each selection a **combination** of *n* elements taken *r* at a time and denote the number of such combinations as ${}_nC_r$.

Number of Combinations of *n* Elements Taken *r* at a Time

The number of combinations of *n* elements taken *r* at a time is given by

$${}_nC_r = \frac{n!}{r! \cdot (n-r)!}, \quad \text{or equivalently,} \quad {}_nC_r = \frac{{}_nP_r}{r!}$$

Insights

The number of combinations ${}_nC_r$ is equal to the number of permutations ${}_nP_r$ divided by $r!$. The value $r!$ in the denominator "divides out" the redundant cases involving different arrangements of r elements within the same group.

Example 7: Comparing Combinations and Permutations

a. In how many ways can 3 students from a group of 15 students be selected to serve on a committee?

b. In how many ways can 3 students from a group of 15 students be selected to serve on a committee, if the students will hold the offices of president, vice president, and treasurer?

Solution:

a. In this situation, the students will serve indistinguishable roles on the committee. Therefore, the order in which the 3 students are selected is not relevant. The number of such committees is the number of *combinations* of $n = 15$ students taken $r = 3$ at a time.

$${}_{15}C_3 = \frac{15!}{3! \cdot (15-3)!} = \frac{15!}{3! \cdot 12!} = \frac{\overset{5}{\cancel{15}} \cdot \overset{7}{\cancel{14}} \cdot 13 \cdot \overset{1}{\cancel{12!}}}{\cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{12!}} = 455$$

b. Now the students will be selected to a committee and assigned different roles (president, vice president, and treasurer). In this case, the order in which the students are selected matters. We need to count the number of *permutations* of $n = 15$ students taken $r = 3$ at a time.

$${}_{15}P_3 = \frac{15!}{(15-3)!} = \frac{15!}{12!} = \frac{15 \cdot 14 \cdot 13 \cdot \overset{1}{\cancel{12!}}}{\cancel{12!}} = 2730$$

Apply the Skills

7. Suppose that 20 people enter a raffle.

a. In how many ways can four different people among the 20 be selected to receive prizes of \$50 each?

b. In how many ways can four different people among the 20 be selected to receive prizes of \$50, \$25, \$10, and \$5?

In Example 7, notice that there are $3! = 6$ times as many permutations as combinations. The reason is that for each combination of 3 items, such as A, B, and C, there are $3! = 6$ permutations: ABC, ACB, BAC, BCA, CAB, CBA.

Example 8: Counting Combinations in an Application

In a particular state lottery game, a player must select a group of 6 numbers (without regard to order) from the numbers 1 to 53. If the 6 numbers match the numbers in the drawing, then the player wins the grand prize. How many groups of 6 numbers are possible?

Solution:

In this situation, the order in which the group of 6 numbers is selected does not matter. Therefore, we need to compute the number of combinations of $n = 53$ numbers taken $r = 6$ at a time.

$$_{53}C_6 = \frac{53!}{6! \cdot (53-6)!} = \frac{53!}{6! \cdot 47!} = 22,957,480$$

There are 22,957,480 different groups of 6 numbers taken from 53 numbers.

Point of Interest

As mentioned in the description in the lesson opener, the probability of winning the grand prize in this state lottery game for a player who plays 1 combination of 6 numbers is $\frac{1}{22,957,480}$.

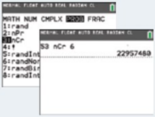
Apply the Skills

8. One California lottery game offers a grand prize to a player who selects the correct group of 5 numbers (in any order) from the numbers 1 to 39. How many groups of 5 numbers are possible?

TECHNOLOGY CONNECTIONS

Evaluating a Number of Combinations, $_nC_r$

Most graphing utilities can evaluate the number of combinations of n elements taken r at a time. For example, use the $_nC_r$ function found in the MATH menu under PRB.



```
53 nCr 6
                    22957480
```

In Example 9 we encounter a situation in which more than one technique of counting must be used.

Example 9: Applying Multiple Principles of Counting

Suppose that a group of 3 full-time employees and 2 part-time employees is to be selected for a special project from a group of 8 full-time employees and 7 part-time employees. In how many ways can such a group be formed?

Solution:

This situation can be thought of as a sequence of two events in which we apply the fundamental principle of counting.

$$\begin{pmatrix}\text{Total number}\\ \text{of groups}\end{pmatrix} = \begin{pmatrix}\text{Number of ways to select}\\ \text{3 full-time employees from}\\ \text{8 full-time employees}\\ \text{in any order}\end{pmatrix} \cdot \begin{pmatrix}\text{Number of ways to select}\\ \text{2 part-time employees from}\\ \text{7 part-time employees}\\ \text{in any order}\end{pmatrix}$$

$$\begin{aligned} &= {}_8C_3 \cdot {}_7C_2 \\ &= 56 \cdot 21 \\ &= 1176 \end{aligned}$$

Check Your Work

The roles of the employees in the group are indistinguishable. Therefore, count the number of combinations rather than permutations.

Apply the Skills

9. The coach of a co-ed softball team must select 4 women and 5 men from a group of 7 women and 10 men to play in a game. In how many ways can such a team be formed?

Practice 11-6

Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, evaluate the expression.

R.1. 9! **R.2.** 0! **R.3.** $\frac{7!}{5!}$ **R.4.** $\frac{10!}{4!6!}$

Concept Connections

1. The fundamental __________ of __________ indicates that if one event can occur in m different ways, and a second event can occur in n different ways, then the two events can occur in sequence in __________ different ways.
2. If n items are arranged in order, then each arrangement is called a __________ of n items.
3. The number of ways that n distinguishable items can be arranged in various orders is __________.
4. Consider a set of n elements of which one element is repeated r times. Then the number of permutations of the elements of the set is given by $\frac{\square}{\square}$.
5. Suppose that n represents the number of distinct elements in a group from which r elements will be chosen in a particular order. Then each arrangement is called a __________ of n items taken __________ at a time.
6. Suppose that n represents the number of elements in a group from which r elements will be selected in no particular order. Then each group selected is called a __________ of n elements taken __________ at a time.
7. The number of permutations of n elements taken r at a time is denoted by ${}_nP_r$ and is computed by ${}_nP_r = \frac{\square}{\square}$ or ${}_nP_r = n(n-1)(n-2) \cdots (n-r+1)$.
8. The number of combinations of n elements taken r at a time is denoted by ${}_nC_r$ and is computed by __________.

Learn: Apply the Fundamental Principle of Counting

For Exercises 9–14, consider the set of integers from 1 to 20, inclusive. If one number is selected, in how many ways can we obtain

9. an even number?
10. an odd number?
11. a prime number?
12. a number that is a multiple of 5?
13. a number that is a multiple of 10?
14. a number that is divisible by 4?
15. At a hospital, the dinner menu consists of 4 choices of entrée, 3 choices of salad, 6 choices of beverage, and 4 choices of dessert. How many different meals can be formed if a patient chooses one item from each category? (**See Example 1, p. 1233**)
16. Casey travels several times a year for work, taking 2 pairs of slacks, 6 shirts, and 4 scarves, all of different colors. Assuming that the items all match together well, how many different outfits does Casey have if one item is selected from each category?
17. A license plate has 3 capital letters followed by 3 digits.
    - **a.** How many license plates can be made if there are no restrictions on the letters or digits? (**See Example 2, p. 1235**)
    - **b.** How many license plates can be made if no digit or letter may be repeated?
18. A security company requires its employees to have a 7-character computer password that must consist of 5 lowercase letters and 2 digits.
    - **a.** How many passwords can be made if there are no restrictions on the letters or digits?
    - **b.** How many passwords can be made if no digit or letter may be repeated?

19. The call letters for a radio station consist of four capital letters, starting with either K or W.

a. How many 4-letter arrangements are possible, assuming that letters may be repeated?

b. How many 4-letter arrangements are possible, assuming that the letters may not be repeated?

20. An employee identification code for a hospital consists of 2 letters from the set {A, B, C, D} followed by 4 digits.

a. How many identification codes are possible if both letters and digits may be repeated?

b. How many identification codes are possible if letters and digits may not be repeated?

21. An online survey is used to monitor customer service. The survey consists of 14 questions. Ten questions have 5 possible responses (strongly agree, agree, neutral, disagree, and strongly disagree). The remaining 4 questions are yes/no questions. In how many different ways can the survey be filled in? (**See Example 3, p. 1235**)

22. A test consists of 3 multiple-choice questions, each with four possible responses, and 7 true/false questions. In how many ways can a student answer the questions on the test?

23. On a computer, 1 *bit* is a single binary digit and has two possible outcomes: either 1 or 0. One *byte* is 8 bits. That is, a byte is a sequence of 8 binary digits. How many arrangements of 1's and 0's can be made with one byte?

24. Older models of garage door remote controls have a sequence of 10 switches that are individually placed in an up or down position. The remote control can "talk to" the overhead door unit if the 10 corresponding switches in the unit are in the same up/down sequence. How many up/down sequences are possible in an arrangement of 10 switches?

Learn: Count Permutations and Combinations

25. a. In how many ways can the letters in the word FLORIDA be arranged? (**See Example 4, p. 1237**)

b. In how many ways can the letters in the word MISSISSIPPI be arranged?

26. a. In how many ways can the letters in the word XRAY be arranged?

b. In how many ways can the letters in the word MAMMOGRAM be arranged?

27. In how many ways can the letters in the word WRONG be arranged and misspelled?

28. In how many ways can the letters in the word EXACTLY be arranged and misspelled?

29. A delivery truck must make 4 stops at locations *A*, *B*, *C*, and *D*. Over several weeks, management asks the driver to drive each possible route and record the time required to complete the route. This is to determine the most time-efficient route. How many possible routes are there?

30. In how many ways can 6 people in a family be lined up for a photograph?

For Exercises 31–36, evaluate ${}_{n}P_{r}$.

31. ${}_{6}P_{4}$ **32.** ${}_{8}P_{5}$

33. ${}_{12}P_{2}$ **34.** ${}_{11}P_{3}$

35. ${}_{9}P_{9}$ **36.** ${}_{5}P_{5}$

37. Evaluate ${}_{20}P_{3}$ and interpret its meaning.

38. Evaluate ${}_{15}P_{6}$ and interpret its meaning.

For Exercises 39–44, evaluate ${}_{n}C_{r}$.

39. ${}_{6}C_{4}$ **40.** ${}_{8}C_{5}$

41. ${}_{12}C_{2}$ **42.** ${}_{11}C_{3}$

43. ${}_{9}C_{9}$ **44.** ${}_{5}C_{5}$

45. Evaluate ${}_{20}C_{3}$ and interpret its meaning.

46. Evaluate ${}_{15}C_{6}$ and interpret its meaning.

47. Given {A, B, C},

- **a.** List all the permutations of two elements from the set. (**See Example 5, p. 1238**)
- **b.** List all the combinations of two elements from the set.

48. Given {W, X, Y, Z},

- **a.** List all the permutations of three elements from the set.
- **b.** List all the combinations of three elements from the set.

49. Suppose that 9 horses run a race. How many first-, second-, and third-place finishes are possible? (**See Example 6, p. 1239**)

50. In how many ways can a judge award blue, red, and yellow ribbons to 3 films at a film festival if there are 10 films entered?

51. **a.** In a drama class, 5 students are to be selected from 24 students to perform a synchronized dance. In how many ways can 5 students be selected from the 24? (**See Example 7, p. 1241**)

- **b.** Determine the number of ways that 5 students can be selected from the class of 24 to play 5 different roles in a short play.

52. **a.** There are 100 members of the U.S. Senate. Suppose that 4 senators currently serve on a committee. In how many ways can 4 more senators be selected to serve on the committee?

- **b.** In how many ways can a group of 3 U.S. senators be selected from a group of 7 senators to fill the positions of chair, vice-chair, and secretary for the Ethics Committee?

53. A chess tournament has 16 players. Every player plays every other player exactly once. How many chess matches will be played?

54. Suppose that a tennis tournament has 64 players. In how many ways can a pairing for a first-round match be made between 2 players among the 64 players? Assume that each player can play any other player without regard to seeding.

55. In the Minnesota lottery game "Gopher 5," a player wins the grand prize by choosing the same group of 5 numbers from 1 through 47 as is chosen by the computer. How many 5-number groups are possible? (**See Example 8, p. 1242**)

56. In a New York state lottery game, a player wins the grand prize by choosing the same group of 6 numbers from 1 through 59 as is chosen by the computer. How many 6-number groups are possible?

57. In ballroom dancing, one partner is designated as the "lead" and the other is designated as the "follow." A ballroom dance instructor wants to choose 3 leads and 3 follows from a group of 7 leads and 9 follows to demonstrate a new dance pattern. How many different groups of 3 leads and 3 follows can be formed? (**See Example 9, p. 1243**)

58. A committee of 4 juniors and 4 seniors is to be made up from a group of students containing 12 juniors and 9 seniors. In how many ways can such a committee be formed?

Mixed Exercises

59. In a "Pick-4" game, a player wins a prize for matching a 4-digit number from 0000 to 9999 with the number randomly selected during the drawing. How many 4-digit numbers can a player choose? Assume that a number can start with a zero or zeros such as 0001.

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60. In a "Numbers" game, a player wins a prize for matching a 3-digit number from 000 to 999 with the number randomly selected during the drawing. How many 3-digit numbers can a player choose? Assume that a number can start with a zero or zeros such as 001.

61. Liza is a basketball coach and must select 5 players out of 12 players to start a game. In how many ways can the 5 players be selected if each player is equally qualified to play each position?

62. Neeja has a list of 8 books that must be read for an upcoming class. To get a head start, Neeja wants to do some of the reading over the summer but only has time to read 5 of the 8 books. In how many ways can 5 books be selected from 8 books?

63. In how many ways can a manager assign 5 employees at a coffee shop to 5 different tasks? Assume that each employee is assigned to exactly one task.

64. In how many ways can a class of 12 kindergarten children line up at the cafeteria?

65. Twenty batteries have been sitting in a drawer for 2 yr. There are 4 dead batteries among the 20. If three batteries are selected at random, determine the number of ways in which

a. 3 dead batteries can be selected.

b. 3 good batteries can be selected.

c. 2 good batteries and 1 dead battery can be selected.

66. There are 30 seeds in a package. Five seeds are defective (will not germinate). If four seeds are selected at random, determine the number of ways in which

a. 4 defective seeds can be selected.

b. 4 good seeds can be selected.

c. 2 good seeds and 2 defective seeds can be selected.

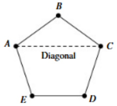
67. A "combination" lock is opened by correctly "dialing" 3 numbers from 0 to 39, inclusive. The user who knows the code turns the dial to the right to the first number in the code, then to the left to find the second number in the code, and then back to the right for the third number in the code. If someone does not know the code and tries to guess, how many guesses are possible?

68. A palindrome is an arrangement of letters that reads the same way forward and backward. For example, one five-letter palindrome is: ABCBA.

a. How many 5-letter palindromes are possible from a 26-letter alphabet?

b. How many 4-letter palindromes are possible from a 26-letter alphabet?

A line segment connecting any two nonadjacent vertices of a polygon is called a *diagonal* of the polygon. For Exercises 69–72, determine the number of diagonals for the given polygon.



69. quadrilateral (4 sides)

70. pentagon (5 sides)

71. hexagon (6 sides)

72. octagon (8 sides)

For Exercises 73–78, consider the set of numbers $\{0, 1, 2, 3, 4, 5\}$. How many 3-digit codes can be formed with the given restrictions?

73. The code has no restrictions.

74. The code may not contain repeated digits.

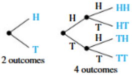
75. The code must represent a 3-digit number. (*Hint*: This means that the first digit cannot be zero.)

76. The code must represent a 3-digit number that is a multiple of 5.

77. The code must represent an even 3-digit number.

78. All three digits in the code must be the same.

79. In Exercise 103 from Lesson 11-3, we learned that if a fair coin is flipped n times, the number of head/tail arrangements follows a geometric sequence.



a. Determine the number of head/tail arrangements if a fair coin is flipped 3 times, 4 times, and 5 times.

b. If a cat has three kittens, how many male/female sequences are possible for the three births? List the outcomes using “M” for male and “F” for female.

c. If a cat has 6 kittens, how many male/female sequences are possible for the six births?

80. Social Security numbers assigned in the United States are comprised of 9 digits of the form ___-__-____. Would there be enough Social Security numbers for the population of China if the Chinese used the same system? (*Hint*: The population of China is approximately 1.5 billion.)

81. Airlines often oversell seats on an airplane. This is done so that the seats for the few passengers that are no-shows will still have been sold. Sometimes, however, all passengers show up and there are more ticketed passengers than seats. Suppose that one flight has 160 passengers and only 156 seats. Determine the number of ways that the airline can select 4 people at random to place on a different flight.

82. In how many ways can a platoon leader select 4 soldiers from among 15 soldiers to secure a building?

83. A car can comfortably hold a family of five. If two people among the five can drive, how many different seating arrangements are possible?

84. A television station must play twelve 30-sec commercials during a half-hour show. In how many ways can the commercials be aired?

Write About It

85. Consider a horse race with 8 horses. Explain how the fundamental principle of counting or the permutation rule can be used to determine the number of first-, second-, and third-place arrangements.

86. Explain why the number of combinations of n items taken r at a time can be computed by $\frac{_nP_r}{r!}$.

Expanding Your Skills

87. Three biology books, 4 math books, and 2 physics books are to be placed on a bookshelf where the books in each discipline are grouped together. In how many ways can the books be arranged on the bookshelf?

88. A softball team has 9 players consisting of 3 adults and 6 children. In how many ways can the coach arrange the batting order if the adults must bat consecutively and the children must bat consecutively?

89. Given that ${}_nC_{n-2} = 15$, solve for n. $n = 6$

90. Given that ${}_nP_{n-2} = 60$, solve for n. $n = 5$

91. **a.** Show that ${}_nC_r = {}_nC_{n-r}$.

b. Given that ${}_nC_8 = {}_nC_3$, solve for n.

92. Given that ${}_nP_r = 336$ and ${}_nC_r = 56$, find r.

93. Suppose that six different symbols are to be used to make a password. The password can be of any length (e.g., a one-symbol password, a two-symbol password, a three-symbol password, and so on). If no symbol can be used more than once, how many passwords are possible?

94. Two parents have 10 different photos of their new baby and 3 different photos of their wedding. They want to arrange 4 photos of the baby and 2 wedding photos on the shelf in such a way that the wedding photos are side-by-side. How many arrangements of photographs can they make?

95. Suppose that a four-digit number is to be formed using the digits 2, 4, 6, and 8, and that each digit is to be used exactly once in the number. Find the *sum* of all such four-digit numbers.

96. **a.** In how many ways can 6 people be seated at a round table?

b. In how many ways can 7 people be seated at a round table if a married couple must sit together?

97. In how many ways can 8 different cookies be distributed among 4 children so that each child receives 2 cookies?

98. In how many ways can a student choose one or more books to read from a group of 5 books?

Go online for more practice problems.

Lesson 11-7

Introduction to Probability

What Will You Learn?

After completing this lesson, you should be able to:

- Determine Theoretical Probabilities
- Determine Empirical Probabilities
- Find the Probability of the Union of Two Events
- Find the Probability of Sequential Independent Events

Learn: Determine Theoretical Probabilities

The Centers for Disease Control publishes National Vital Statistics Reports every year that provide data for birth rates and mortality rates based on gender, race, age, and other factors. The data in Table 11-4 give the 1-yr survival rates for people in the United States for selected ages.

TABLE 11-4 Survival Rates for Selected Ages

Age (yr)	Probability of 1-yr Survival
10	0.9999
20	0.9991
30	0.9990
40	0.9981
50	0.9959
60	0.9907
70	0.9791
80	0.9434
90	0.8543

From the table, the probability that a 20-yr-old will live to age 21 is 0.9991. Put another way, this means that approximately 99.91% of 20-yr-olds will live to age 21 (see Example 5).

A **probability** is a value assigned to an event that quantifies the likelihood of the event to occur. To compute the probability of an event, we first need to define several key terms. An **experiment** is a test with an uncertain outcome. The set of all possible outcomes of an experiment is called the **sample space** of the experiment. For example:

Experiment:	**Sample Space:**
Flip a coin	{head, tail}
Roll a single 6-sided die	{1, 2, 3, 4, 5, 6}

An **event** is a subset of the sample space and is often denoted by E or some other capital letter.

Description of Event:	**Set Representing the Event:**
Flip a coin and the outcome is "head"	{head}
Roll a six-sided die and an even number comes up	{2, 4, 6}

The number of elements in an event E is often denoted by $n(E)$. For example, given the event $E = \{2, 4, 6\}$, $n(E) = 3$.

Insights

The word "die" is the singular of the word "dice." That is, we roll one die but we roll two dice.

If we assume that all elements in a sample space are equally likely to occur, then we define the theoretical probability of an event as follows:

Theoretical Probability of Event E

Let S represent a sample space with equally likely outcomes, and let E be a subset of S. Then the probability of event E, denoted by $P(E)$, is given by

$$P(E) = \frac{n(E)}{n(S)} \begin{array}{l} \longleftarrow \text{Number of elements in the event} \\ \longleftarrow \text{Number of elements in the sample space} \end{array}$$

The probability of an event is the relative frequency of the event compared to the sample space. For example, if E is the event of rolling an even number on a die, then $E = \{2, 4, 6\}$ and $S = \{1, 2, 3, 4, 5, 6\}$. Then,

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

The value of a probability can be written as a fraction, as a decimal, or as a percent. Therefore, $P(E) = \frac{1}{2}$, or 0.5, or 50%.

Check Your Work

The probability $\frac{1}{2}$ does not mean that exactly 1 out of every 2 rolls of a die will result in an even number. Instead, it is a ratio that we expect experimental observations to approach after the experiment is performed a large number of times.

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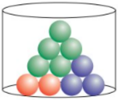
Example 1: Finding the Probability of an Event

Suppose that a box contains two red, three blue, and five green marbles. If one marble is selected at random, find the probability of the event.



a. E_1: A red marble is selected.

b. E_2: A blue marble is selected.

c. E_3: A white marble is selected.

Solution:

Each of the 10 marbles in the box is equally likely to be selected. Denote the sample space as $S = \{R_1, R_2, B_1, B_2, B_3, G_1, G_2, G_3, G_4, G_5\}$.

a. $E_1 = \{R_1, R_2\}$ The number of red marbles is 2.

Therefore, $P(E_1) = \frac{n(E_1)}{n(S)} = \frac{2}{10} = \frac{1}{5}$. The probability of selecting a red marble is $\frac{1}{5}$.

b. $E_2 = \{B_1, B_2, B_3\}$ The number of blue marbles is 3.

Therefore, $P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{10}$. The probability of selecting a blue marble is $\frac{3}{10}$.

c. $E_3 = \{\,\}$ The number of white marbles is 0.

Therefore, $P(E_3) = \frac{n(E_3)}{n(S)} = \frac{0}{10} = 0$. It is intuitively obvious that a white marble cannot be selected. Mathematically, a probability of 0 means that the event is impossible.

Apply the Skills

1. A sock drawer has 6 blue socks, 2 white socks, and 12 black socks. If one sock is selected at random, find the probability of the event.

   a. E_1: A blue sock is selected.

   b. E_2: A black sock is selected.

   c. E_3: A brown sock is selected.

The number of outcomes in an event is a nonnegative number less than or equal to the number of outcomes in the sample space. Therefore, the value of a probability is a number between 0 and 1, inclusive. That is, $0 \leq P(E) \leq 1$. Furthermore,

- If $P(E) = 0$, then the event E is called an **impossible event**.
- If $P(E) = 1$, then the event E is called a **certain event**.

In Example 1(c), it is impossible to draw a white marble from the box. However, we would be 100% certain of drawing a red, blue, or green marble from the box and the probability would be 1.

Insights

If the probability of an event is close to 0, the event is unlikely to happen. If the probability is close to 1, the event is very likely to happen. If the probability is close to 0.5, the event is roughly as likely to happen as not to happen.

Example 2: Finding the Probability of an Event

An American roulette wheel has 38 slots, numbered 1 through 36, 0, and 00. Eighteen slots are red, 18 are black, and 2 are green. The dealer spins the wheel in one direction and rolls a small ball in the opposite direction until both come to rest. The ball is equally likely to fall in any one of the 38 slots. On a given spin of an American roulette wheel, find the probability of the event.

- **a.** E_1: The ball lands on an odd number.
- **b.** E_2: The ball lands on a green slot.
- **c.** E_3: The ball does not land on a green slot.

Solution:

The sample space has 38 outcomes. That is $n(S) = 38$.

a. There are 18 odd numbers. Therefore, $n(E_1) = 18$ and $P(E_1) = \frac{n(E_1)}{n(S)} = \frac{18}{38} = \frac{9}{19} \approx 0.4737$.

b. There are 2 green slots on the wheel. Therefore, $n(E_2) = 2$ and $P(E_2) = \frac{n(E_2)}{n(S)} = \frac{2}{38} = \frac{1}{19} \approx 0.0526$.

c. There are 36 slots that are not green. Therefore, $n(E_3) = 36$ and $P(E_3) = \frac{n(E_3)}{n(S)} = \frac{36}{38} = \frac{18}{19} \approx 0.9474$.

Insights

We say that the ball is "*equally likely*" to fall on any one of the 38 slots. This means that the probability of the ball falling on the "1" slot is the same as falling on the "2" slot, and the same as falling on the "3" slot, etc.

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Apply the Skills

2. On a given spin of an American roulette wheel, find the probability of the event.

- **a.** E_1: The ball lands on the number 7.
- **b.** E_2: The ball lands on a red slot.
- **c.** E_3: The ball does not land on a red slot.

From Example 2(b) and 2(c), notice that event E_3 consists of all elements in the sample space not in event E_2. Therefore, the union of the two events makes up the entire sample space, and $P(E_2) + P(E_3) = \frac{1}{19} + \frac{18}{19} = 1$. In this case, we say that events E_2 and E_3 are *complementary events.*

Definition of Complementary Events

Let E be an event relative to sample space S. The **complement of *E***, denoted by E (or sometimes by $\sim E$ or E') is the set of outcomes in the sample space but not in event E. It follows that

$$P(E) + P(E) = 1.$$

Example 3: Finding the Probabilities of Complementary Events

Suppose that 2 dice are rolled. Determine the probability that

a. The sum of the numbers showing on the dice is 7.

b. The sum of the numbers showing on the dice is *not* 7.

Solution:

Each individual die has 6 equally likely outcomes. Therefore, by the fundamental principle of counting, the number of ways the pair of dice can fall is $6 \cdot 6 = 36$ (Figure 11-11).

a. Let E represent the event that the numbers showing on the dice have a sum of 7. A sum of 7 will occur if the dice land on one of the following 6 outcomes.

$E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$. Therefore, $n(E) = 6$ and

$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}$. The probability of rolling a sum of 7 is $\frac{1}{6}$.

Second Die

First Die		1	2	3	4	5	6
	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Figure 11-11

b. The event that the numbers on the dice do not present a sum of 7 is the complement of event E. Therefore, we have the option of using the formula $P(E) + P(\overline{E}) = 1$.

$P(E) = 1 - P(E) = 1 - \frac{1}{6} = \frac{5}{6}$ The probability of rolling a sum other than 7 is $\frac{5}{6}$.

Apply the Skills

3. Suppose that two dice are rolled. Determine the probability that

a. The sum of the numbers showing on the dice is 8.

b. The sum of the numbers showing on the dice is *not* 8.

In Example 4, we use the techniques of counting learned in Lesson 11.6 to determine the probability of an event.

Example 4: Finding the Probability of an Event by Using Counting Techniques

Suppose that 5 good batteries and 3 bad batteries are mixed in a drawer and are indistinguishable from one another. If a homeowner selects two batteries at random, find the probability that both batteries are good.

Solution:

Let G represent the event that two good batteries are chosen. The number of elements in G is the number of possible ways to select 2 good batteries from a group of 5 good batteries without regard to order.

$$n(G) = {}_5C_2 = 10$$

The sample space S consists of all possible ways in which 2 batteries (good or bad) can be selected from 8 batteries without regard to order. That is,

$$n(S) = {}_8C_2 = 28$$

The probability that both batteries will be good is

$$P(G) = \frac{n(G)}{n(S)} = \frac{{}_5C_2}{{}_8C_2} = \frac{10}{28} = \frac{5}{14} \approx 0.3571$$

Apply the Skills

4. Suppose that a group of three songs is to be selected from a list of 8 country songs and 6 rock songs. Find the probability that the group of three songs will consist of all country songs.

Learn: Determine Empirical Probabilities

Suppose that the Centers for Disease Control wants to measure the 1-yr survival rates for Americans for specific ages. There is no way that this can be derived theoretically. Instead, the probability can be approximated through observations. A probability computed in this way is called an **empirical probability**.

Computing Empirical Probability

The empirical probability of an event E is given by

$$P(E) = \frac{\text{Number of times the event } E \text{ occurs}}{\text{Number of times the experiment is performed}}$$

Example 5: Computing an Empirical Probability

Data from the Centers for Disease Control for a recent year indicate that there were approximately 4,295,000 individuals of age 20 in the United States. One year later 3887 of these individuals had died and 4,291,113 had lived. Determine the probability that a 20-yr-old will survive to age 21. (*Source*: www.cdc.gov)

Solution:

Let E represent the event that a 20-yr-old lives to age 21.

The experiment involves 4,295,000 individuals of age 20. The number of observed outcomes where an individual lives to the age of 21 is 4,291,113. Therefore, the empirical probability is given by

$$P(E) = \frac{4{,}291{,}113}{4{,}295{,}000} \approx 0.9991$$

Apply the Skills

5. Suppose that of approximately 4,224,100 individuals of age 40 in the United States, 8038 die before the age of 41 and 4,216,062 survive. Determine the probability that a 40-yr-old will live to age 41.

Learn: Find the Probability of the Union of Two Events

We now study the probability of the union of two events. Suppose that one card is drawn from a standard deck of cards. The sample space has the following properties (Figure 11-12).

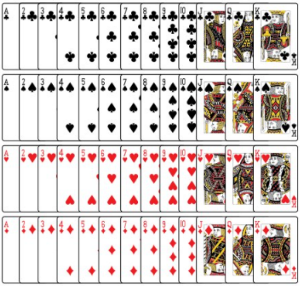


Figure 11-12

- The deck consists of 52 cards.
- The cards are divided into four suits (or categories) called spades (♠), clubs (♣), hearts (♥), and diamonds (♦).

- Each suit consists of 13 cards labeled: Ace (A), 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack (J), Queen (Q), and King (K).
- There are 26 black cards (spades and clubs) and 26 red cards (hearts and diamonds).

Let *A* be the event that an ace is drawn: {A♠, A♣, A♥, A♦}.

Let *K* be the event that a king is drawn: {K♠, K♣, K♥, K♦}.

Events *A* and *K* have no elements in common (a single card cannot be both an ace and a king). Therefore, the intersection of *A* and *K* is the empty set, $A \cap K = \{\ \}$, and we say that events *A* and *K* are **mutually exclusive** (do not overlap).

Now let event *A* be the event that an ace is drawn, and let *S* be the event that a spade is drawn (Figure 11-13).

$A \cap S$: Overlap of *A* and *S*

A = {A♠, A♣, A♥, A♦}

S = {A♠, 2♠, 3♠, 4♠, 5♠, 6♠, 7♠, 8♠, 9♠, 10♠, J♠, Q♠, K♠}

Figure 11-13

Notice that events *A* and *S* share the comment element of the ace of spades, A♠. Therefore, events *A* and *S* are *not* mutually exclusive, meaning that they overlap.

We are now ready to generalize. To find the probability of the event (*A* or *B*), denoted *P*(*A* or *B*), find the probability of the *union* of *A* and *B*.

Probability of *A* or *B*

Given events *A* and *B* in the same sample space, *P*(*A* or *B*) is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note: If *A* and *B* are mutually exclusive, then $P(A \cap B) = 0$ and we have $P(A \cup B) = P(A) + P(B)$.

Example 6: Computing the Probability of *A* or *B*

Suppose that one card is selected at random from a standard deck of cards. Find the probability that

a. The card is an ace or a king.

b. The card is an ace or a spade.

Solution:

Let A be the event that an ace is drawn: {A♠, A♣, A♥, A♦}.

Let K be the event that a king is drawn: {K♠, K♣, K♥, K♦}.

Let S be the event that a spade is drawn:

S = {A♠, 2♠, 3♠, 4♠, 5♠, 6♠, 7♠, 8♠, 9♠, 10♠, J♠, Q♠, K♠}.

a. Events A and K are mutually exclusive. Therefore, to find $P(A \text{ or } K)$, we have

$P(A \cup K) = P(A) + P(K)$

$= \frac{4}{52} + \frac{4}{52}$ There are 4 aces in the deck out of 52 cards. $P(A) = \frac{4}{52}$.

There are 4 kings in the deck out of 52 cards. $P(K) = \frac{4}{52}$.

$= \frac{8}{52}$ or $\frac{2}{13}$

Insights In Example 6(a), since A and K are mutually exclusive, then $P(A \cap K) = 0$.

b. From Figure 11-13, events A and S are *not* mutually exclusive. Therefore, to find $P(A \text{ or } S)$, we have:

$P(A \cup S) = P(A) + P(S) - P(A \cap S)$ There are 4 aces in the deck out of 52 cards, so $P(A) = \frac{4}{52}$. There are 13 spades in the deck out of 52 cards, so $P(S) = \frac{13}{52}$. There is 1 spade that is an ace in the deck. Therefore, $P(A \cap S) = \frac{1}{52}$.

$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$

$= \frac{16}{52}$ or $\frac{4}{13}$

Insights

As an alternative to computing $P(A) + P(S) - P(A \cap S)$, count the number of elements in the event $(A \cup S)$, being careful not to count the events common to A and S twice. That is, count 13 spades plus the 3 aces that are not spades. This gives a total of 16 elements in event $(A \cup S)$. Therfore,

$$P(A \cup S) = \frac{16}{52} = \frac{4}{13}$$

Apply the Skills

6. Suppose that one card is drawn at random from a standard deck. Find the probability that

a. The card is a 2 or a 10.

b. The card is a 2 or a red card.

Example 7: Computing an Empirical Probability of *A* or *B*

The safety and security department at a college asked a sample of 265 students to respond to the following question.

"Do you think that the college has adequate lighting on campus at night?"

The table gives the results of the survey based on whether the student lives off campus or on campus.

	Yes	No	No Opinion	Total
Off campus	92	7	4	103
On campus	36	102	24	162
Total	128	109	28	265

If one student is selected at random from the group, find the probability that

a. The student answered "Yes" or had "No Opinion."

b. The student answered "No" or lived on campus.

Solution:

a. Let Y be the event that the student answered "Yes."
Let O be the event that the student had "No Opinion."

Y and O are mutually exclusive events (do not overlap).

	Yes	No	No Opinion	Total
Off campus	92	7	4	103
On campus	36	102	24	162
Total	128	109	28	265

$$P(Y \text{ or } O) = P(Y) + P(O)$$

$$= \frac{128}{265} + \frac{28}{265} \qquad P(Y) = \frac{128}{265} \text{ and } P(O) = \frac{28}{265}.$$

$$= \frac{156}{265}$$

b. Let N be the event that the student answered "No."
Let C be the event that the student lives on campus.

N and C are *not* mutually exclusive (the events overlap).

	Yes	No	No Opinion	Total
Off campus	92	7	4	103
On campus	36	102	24	162
Total	128	109	28	265

$$P(N \text{ or } C) = P(N) + P(C) - P(N \cap C)$$

$$= \frac{109}{265} + \frac{162}{265} - \frac{102}{265} \qquad P(N) = \frac{109}{265},\ P(C) = \frac{162}{265},\text{ and } P(N \cap C) = \frac{102}{265}$$

$$= \frac{169}{265}$$

The 102 students who live on campus and answered "No" are the elements in the intersection of N and C.

Insights

As an alternative to computing $P(N) + P(C) - P(N \cap C)$, count the number of elements in the event $(N \cup C)$, being careful not to count the elements common to N and C twice; that is, count the 162 students who live on campus plus the 7 students who live off campus and answered "No" for a total of 169 elements in event $(N \cup C)$. So, $P(N \cup C) = \frac{169}{265}$.

Apply the Skills

7. Refer to the data given in Example 7. If one student is selected at random from the group, find the probability that

a. The student answered "Yes" or "No."

b. The student lives off campus or had no opinion.

Learn: Find the Probability of Sequential Independent Events

When a coin is tossed, the probability that it lands heads up is $\frac{1}{2}$. Now suppose that the coin is flipped two times in succession. What is the probability that it lands heads up twice in a row? Our intuition tells us that the probability should be less than $\frac{1}{2}$ because it is more unlikely that the desired outcome will happen twice in a row than one time. We also know that the outcome of the first event (heads up) does not affect the outcome of the second event (also heads up). For this reason, the two events are called **independent events**.

Probability of a Sequence of Independent Events

If events A and B are independent events, then the probability that both A and B will occur is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

For example,

Let A represent the event that a coin lands heads up on the first toss.
Let B represent the event that a coin lands heads up on the second toss.

Then the probability that the coin will land heads up on both tosses is

$$P(A \text{ and } B) = P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Example 8: Finding the Probability of Independent Events

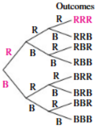
A carnival game is played in which a player selects a card at random from a standard 52-card deck. After each pick, the player returns the card to the deck, the deck is shuffled, and the player selects again. If the player selects three red cards in a row, the player wins a prize. What is the probability that the player will win?

Solution:

First note, that each selection of a card is independent of the previous selection. This is because the cards from previous selections are returned to the deck. In a sense, the player is "starting over" with a fresh deck of 52 cards each time. Furthermore, because the number of black cards equals the number of red cards (26 each), the probability of selecting a red card (R) is:

$$P(R) = \frac{26}{52} = \frac{1}{2}$$

$$\begin{aligned} P(\text{all red}) &= P(\text{red on 1st}) \cdot P(\text{red on 2nd}) \cdot P(\text{red on 3rd}) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \end{aligned}$$



Insights

In Example 8 there are 8 elements in the sample space:

RRR, RRB, RBR, RBB, BRR, BRB, BBR, BBB

The event RRR occurs 1 time. So, the probability of all red cards is $\frac{1}{8}$.

Apply the Skills

8. Suppose that the game from Example 8 is extended so that the player wins a better prize if five black cards are selected in a row. What is the probability that the player will win?

Example 9: Finding the Probability of Independent Events

In baseball, a player's batting "average" is the quotient of the number of hits to the number of "at bats." It can also be interpreted as the probability that the player will get a hit on a given time at bat. If a player had a batting average of 0.279 for a recent season, determine the probability of getting a hit on four consecutive times at bat.

Solution:

Let H represent the event that the player gets a hit on a given time at bat. Furthermore, each time at bat is independent of the time before.

We have that $P(H) = 0.279$. Therefore,

$$\begin{aligned} P(4 \text{ consecutive hits}) &= (0.279)(0.279)(0.279)(0.279) \\ &\approx 0.006 \end{aligned}$$

If the player has four times at bat, there is less than a 1% chance of getting a hit all four times.

Apply the Skills

9. Suppose the probability that a person will catch a winter "cold" is 0.16. What is the probability that four unrelated people will all catch winter "colds"?

Practice 11-7

Practice Exercises

Prerequisite Review

R.1. Given: $P = \{a, b, c, d, e, f, g, h, i\}$ and $Q = \{c, f, h, o, u\}$

List the elements of the following sets.

a. $P \cap Q$ **b.** $P \cup Q$

R.2. Given: $M = \{-4, -2, 1, 3, 5\}$ and $N = \{-5, -4, -3, -2, -1\}$

List the elements of the following sets.

a. $M \cap N$ **b.** $M \cup N$

Concept Connections

1. The ______________ ______________ of an experiment is the set of all possible outcomes.

2. An ______________ is a subset of the sample space of an experiment.

3. If $P(E) = 0$, then E is called an ______________ event. If $P(E) = 1$, then E is called a ______________ event.

4. The notation E represents the ______________ of event E. Furthermore, $P(E) + P(\overline{E}) =$ ______________.

5. Two events in a sample space are ______________ ______________ if they do not share any common elements. That is, the two events do not overlap.

6. If two events A and B are not mutually exclusive, then $P(A \cup B)$ can be computed by the formula $P(A \cup B) =$ ______________.

7. If two events A and B are mutually exclusive, then $P(A \cap B) =$ ______________. As a result, $P(A \cup B)$ can be computed from the formula $P(A \cup B) =$ ______________.

8. For two independent events A and B, $P(A \text{ and } B) =$ ______________.

Learn: Determine Theoretical Probabilities

9. Which of the values can represent the probability of an event?

a. 1.84 **b.** $-\frac{3}{7}$ **c.** 0.00 **d.** 1.00
e. 250% **f.** 6.1 **g.** 0.61 **h.** 6.1%

10. Which of the values can represent the probability of an event?

a. 2.32 **b.** 0.231 **c.** 2.31% **d.** -1
e. $\frac{3}{8}$ **f.** $-\frac{1}{4}$ **g.** $\sqrt{2}$ **h.** 125%

For Exercises 11–16, match the probability with a statement a, b, c, d, e, or f.

a. Event E is certain to happen.
b. Event E cannot happen.
c. Event E is very likely to happen.
d. Event E is very unlikely to happen.
e. Event E is somewhat likely to happen.
f. Event E is as likely to happen as not to happen.

11. $P(E) = 0.994$

12. $P(E) = 1$

13. $P(E) = 0.75$

14. $P(E) = 0.003$

15. $P(E) = 0.5$

16. $P(E) = 0$

For Exercises 17–24, consider an experiment where a single 10-sided die is rolled with the outcomes 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Determine the probability of each event.

17. A number less than 5 is rolled.

18. A number less than 3 is rolled.

19. A number between 4 and 10, inclusive, is rolled.

20. A number between 2 and 7, inclusive, is rolled.

21. A number greater than 10 is rolled.

22. A number less than 1 is rolled.

23. A number greater than or equal to 1 is rolled.

24. A number less than or equal to 10 is rolled.

25. A course in early civilization has 6 freshmen, 8 sophomores, and 16 juniors. If one student is selected at random, find the probability of the following events. (**See Example 1, p. 1252**)

a. A junior is selected.

b. A freshman is selected.

c. A senior is selected.

26. Suppose that a box contains 4 chocolate chip cookies, 8 molasses cookies, and 12 raisin cookies. If one cookie is selected at random, find the probability for each of the following events.

a. A chocolate chip cookie is selected.

b. A molasses cookie is selected.

c. A ginger cookie is selected.

For Exercises 27–28, consider an American roulette wheel. For a given spin of the wheel, find the probability for each of the following events. (See Example 2, p. 1253)

27. a. The ball lands on an even number (do not include 0 and 00).

b. The ball lands on a number that is a multiple of 5 (do not include 0 and 00).

c. The ball does not land on the number 8.

28. a. The ball lands on a black slot.

b. The ball lands on a number that is a multiple of 6 (do not include 0 and 00).

c. The ball does not land on the number 12.

29. If $P(E) = 0.842$, what is the value of $P(E)$?

30. If $P(A) = 0.431$, what is the value of $P(\overline{A})$?

31. According to the Centers for Disease Control, the probability that a live birth will be of twins in the United States is 0.016. What is the probability that a live birth will *not* be of twins?

32. A baseball player with a batting average of 0.291 has a probability of 0.291 of getting a hit for a given time at bat. What is the probability that the player will *not* get a hit for a given time at bat?

For Exercises 33–36, consider the sample space when two fair dice are rolled. Determine the probabilities for the following events. (See Example 3, p. 1254)

33. a. The sum of the numbers on the dice is 4.

b. The sum of the numbers on the dice is *not* 4.

34. a. The sum of the numbers on the dice is 12.

b. The sum of the numbers on the dice is *not* 12.

35. a. The sum of the numbers on the dice is greater than 9.

b. The sum of the numbers on the dice is less than 4.

36. a. The sum of the numbers on the dice is greater than or equal to 8.

b. The sum of the numbers on the dice is less than or equal to 5.

37. After a nationally televised trial, a poll of viewers indicated that 68% thought the defendant was guilty, 22% thought the defendant was not guilty, and the rest were undecided. What is the probability that a person selected from the viewing audience was undecided?

38. A student flips a coin five times, and the probability that "tails" comes up all five times is 0.03125. What is the probability of getting at least one "head?"

39. Suppose that a drug study has a group of participants consisting of 18 people in the placebo group and 16 people in the experimental group. The individuals in the placebo group are administered pills that look like a real drug but have no medicinal value. The individuals in the experimental group are administered the drug that is under study. Suppose that a doctor selects a group of 9 patients at random from all of those in the study.

a. What is the probability that the 9 patients are all in the placebo group?

b. What is the probability that the 9 patients are all in the experimental group?

c. Why do the probabilities from parts (a) and (b) not add up to 1?

40. Suppose that 20 good batteries and 6 defective batteries are in a drawer.

a. If 4 batteries are drawn at random, what is the probability that all four will be defective?

b. What is the probability that all four will be good?

c. Why do the probabilities from parts (a) and (b) not add up to 1?

41. In an Illinois state lottery game, a player wins the grand prize by choosing the same group of five numbers from 1 through 39 as is chosen by the computer. What is the probability that a player will win the grand prize by playing 1 ticket?

42. In a New York state lottery game, a player wins the grand prize by choosing the same group of 6 numbers from 1 through 59 as is chosen by the computer. What is the probability that a player will win the grand prize by playing 5 different tickets?

Scientist Gregor Mendel (1822–1884) is often called the "father of modern genetics" and is famous for his work involving the inheritance of certain traits in pea plants. Suppose that the genes controlling the color of peas are Y for yellow and y for green. Each plant has two genes, one from the female (seed) and one from the male (pollen). The Y gene is dominant, and therefore a plant with genes YY will have yellow peas, a plant with genes Yy or yY will have yellow peas, and a plant with genes yy will have green peas.

If a plant with two yellow genes (YY) is crossed with a plant with two green genes (yy), the result is four hybrid offspring with genotypes Yy. The offspring will be yellow, but will carry the recessive green gene.

		Parent 2	
		y	y
Parent 1	Y	Yy	Yy
	Y	Yy	Yy

43. Suppose that both parent pea plants are hybrids with genotype Yy.

a. Make a chart showing the possible genotypes of the offspring.

		Parent 2	
		Y	y
Parent 1	Y		
	y		

b. What is the probability that a given offspring will have green peas?

c. What is the probability that a given offspring will have yellow peas?

44. Suppose that one parent pea plant has genotype YY and the other has genotype Yy.

a. Make a chart showing the possible genotypes of the offspring.

Parent 2

		Y	y
Parent 1	Y		
	Y		

b. What is the probability that a given offspring will have green peas?

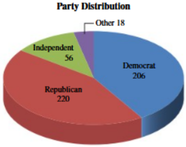
c. What is the probability that a given offspring will have yellow peas?

Learn: Determine Empirical Probabilities

45. At a hospital specializing in treating heart disease, it was found that 222 out of 4624 patients undergoing open heart mitral valve surgery died during surgery or within 30 days after surgery. Determine the probability that a patient will not survive the surgery or 30 days after the surgery. This is called the mortality rate. Round to 3 decimal places. (**See Example 5, p. 1256**)

46. China has the largest population of any country with approximately 1.5 billion people. In a recent year, census results indicated that 199,500,000 Chinese were over the age of 60. If a person is selected at random from the population of China, what is the probability that the person is over 60 years old? Round to 3 decimal places.

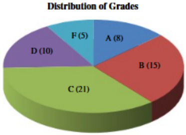
47. For a certain district, a random sample of registered voters results in the distribution by political party given in the graph. Based on these results, what is the probability of selecting a voter at random from the district and getting



a. A Democrat?

b. A voter who is neither Democrat, Republican, nor Independent?

48. The final exam grades for a sample of students in a Freshmen English class at a large university result in the following grade distribution. Based on these results, what is the probability of selecting a student at random taking Freshmen English and getting a student who received



a. An "A."

b. A "C."

Learn: Find the Probability of the Union of Two Events

For Exercises 49–58, consider the sample space for a single card drawn from a standard deck. (See Example 6, p. 1257 and Figure 11-12, p. 1256) Find the probability that the card drawn is

49. A jack or a queen.

50. An ace or a 2.

51. A jack or a diamond.

52. A 5 or a heart.

53. A face card (jack, queen, or king).

54. A card numbered between 5 and 10, inclusive.

55. A face card or a red card.

56. A card numbered between 5 and 10, inclusive, or a black card.

57. A heart, club, or spade.

58. An ace, 2, or 3.

For Exercises 59–66, use the data in the table categorizing cholesterol levels by the ages of the individuals in a study. If one person from the study is chosen at random, find the probability of the given event. (See Example 7, p. 1259)

	Normal Cholesterol	Elevated Cholesterol	Total
30 and under	14	4	18
31–60	52	28	80
61 or older	22	80	102
Total	88	112	200

59. The person has elevated cholesterol.

60. The person is 61 or older.

61. The person is 60 or under.

62. The person is 31 or older.

63. The person has normal cholesterol or is 61 or older.

64. The person has elevated cholesterol or is 30 or under.

65. The person is between 31 and 60, inclusive, or has elevated cholesterol.

66. The person is 61 or older or has elevated cholesterol.

Learn: Find the Probability of Sequential Independent Events

67. Suppose that a die is rolled followed by the flip of a coin. Find the probability that the outcome is a 5 on the die followed by the coin turning up heads. (**See Example 8, p. 1261**)

68. If a code for an alarm is a 4-digit sequence, determine the probability that someone guesses each digit correctly.

69. The 5-yr survival rate for stage I breast cancer is 88%. (*Source*: American Cancer Society, www.cancer.org) If two women with stage I breast cancer are selected at random, what is the probability that they both survive 5 yr? (**See Example 9, p. 1262**)

70. Basketball player Steph Curry makes approximately 91% of free throws. If he plays in a game in which he shoots 6 free throws, what is the probability that he will make all 6?

71. In the 2010 Wimbledon Championships, John Isner from the United States and Nicolas Mahut from France played a first-round tennis match that became the longest match in tennis history. (The match stretched over a 3-day period with Isner winning 70-68 in the fifth set.) In 2011, after a random draw, the two men met again in the first round of Wimbledon. This is highly improbable. If there are 128 players in the draw, estimate the probability that

- **a.** Isner and Mahut would meet in the first round at Wimbledon in any given year. Assume that any player can play any other player in the first round (that is, disregard the fact that seeded players do not play one another in the first round).
- **b.** Isner and Mahut would meet in the first round 2 yr in a row.

72. A traffic light at an intersection has a 120-sec cycle. The light is green for 80 sec, yellow for 5 sec, and red for 35 sec.

a. When a motorist approaches the intersection, find the probability that the light will be red. (Assume that the color of the light is defined as the color when the car is 100 ft from the intersection. This is the approximate distance at which the driver makes a decision to stop or go.)

b. If a motorist approaches the intersection twice during the day, find the probability that the light will be red both times.

73. A test has 10 questions. Five questions are true/false and five questions are multiple-choice. Each multiple-choice question has 4 possible responses of which exactly one is correct. Find the probability that a student guesses on each question and gets a perfect score.

74. A quiz has 6 multiple-choice questions and each question has 5 possible responses, of which exactly one is correct. Find the probability that a student answers each question *incorrectly*.

Mixed Exercises

The blood type of an individual is classified according to the presence of certain antigens, substances that cause the immune system to produce antibodies. These antigens are denoted by A, B, and Rh. If an individual's blood contains either the A or B antigen, these letters are listed in the blood type. If neither A nor B is present, then the letter O is used. If the Rh antigen is present, the blood is said to be Rh positive (Rh^+); otherwise, the blood is Rh negative (Rh^-). Under this system, a person with AB^+ blood has all three antigens, and group O^- is absent all three antigens.

The distribution of blood types for people living in the United States is given in the table. (*Source*: Stanford School of Medicine, http://bloodcenter.stanford.edu) Refer to the table for Exercises 75–82. Round to 3 decimal places when necessary.

Type	Probability
O^+	0.374
O^-	0.066
A^+	0.357
A^-	0.063
B^+	0.085
B^-	0.015
AB^+	0.034
AB^-	0.006

75. a. If an individual is randomly selected from the population, find the probability that the individual will have the Rh factor.

b. If three people are selected at random, find the probability that they all have the Rh factor.

76. a. If a person is selected at random from the population, find the probability that the individual has the A antigen.

b. If four people are randomly selected, find the probability that all four have the A antigen.

77. a. Which blood type is most common?

b. Which blood type is most rare?

78. Across all blood types, is a person more likely to have the Rh factor or less likely to have the Rh factor?

Doctors know that certain restrictions apply when considering the administration of blood to a patient. The antigens of the blood donor and recipient must be compatible. If an antigen is absent from the recipient's blood, then the recipient cannot receive blood from a person who has that antigen. For example, a person with B^+ blood cannot receive AB^+ blood because of the presence of the A antigen from the donor. Use this information and the table with Exercises 75–78 for Exercises 79–82.

79. Suppose that a person is randomly selected from the population. What is the probability that this individual's blood can be used for a transfusion for a person with type B^+ blood?

80. Suppose that a person is randomly selected from the population. What is the probability that this individual's blood can be used for a transfusion for a person with type A^+ blood?

81. a. What is the probability that an individual from the population can donate blood to a person with type O^- blood?

b. Explain why a person with type O^- blood is called a universal donor.

82. a. What is the probability that an individual from the population can donate blood to a person with AB^+ blood?

b. Explain why a person with AB^+ blood can receive blood from anyone.

83. A slot machine in a casino has three wheels that all spin independently. Each wheel has 11 stops, denoted by 0 through 9, and bar. What is the probability that a given outcome is bar-bar-bar?

84. Airlines often overbook flights because a small percentage of passengers do not show up (perhaps due to missed connections). Past history indicates that for a certain route, the probability that an individual passenger will not show up is 0.04. Suppose that 61 people bought tickets for a flight that has 60 seats. Determine the probability that there will not be enough seats. Round to 3 decimal places.

85. South Florida humorist Dave Barry often wrote about his dog, Zippy. Suppose that the Barry home has 3200 ft^2 of living area with tile floor. Further suppose that an expensive 8 ft by 10 ft oriental rug is placed on the floor. If Zippy had an "accident" in the house, what is the probability that it would happen on the expensive rug?

For Exercises 86–88,

a. Shade the area bounded by the given inequalities on a coordinate grid showing $-5 \le x \le 5$ and $-5 \le y \le 5$.

b. Suppose that an enthusiastic mathematics student makes a square dart board out of the portion of the rectangular coordinate system defined by $-5 \le x \le 5$ and $-5 \le y \le 5$. Find the probability that a dart thrown at the target will land in the shaded region.

86. $y \ge |x|$ and $y \le 4$

87. $|y| \le 3$ and $|x| \le 2$

88. $x^2 + y^2 \le 9$

Write About It

89. In a carnival game, players win a prize if they can toss a ring around the neck of a bowling pin. How would you go about estimating the probability of winning the game?

90. Give an example of two events that are mutually exclusive.

91. Give an example of two events that are not mutually exclusive.

92. Explain why a probability of $\frac{5}{4}$ is impossible.

Expanding Your Skills

93. Suppose that a box of DVDs contains 10 action movies and 5 comedies.

a. If two DVDs are selected from the box with replacement, determine the probability that both are comedies.

b. It probably seems more reasonable that someone would select two *different* DVDs from the box. That is, the first DVD would not be replaced before the second DVD is selected. In such a case, are the events of selecting comedies on the first and second picks independent events?

c. If two DVDs are selected from the box *without* replacement, determine the probability that both are comedies.

94. Suppose that 12 students (5 freshmen and 7 sophomores) are being considered for two different scholarships. One scholarship is for $500 and the other is for $250.

a. Two students are selected at random from the group of 12 to receive the scholarships. If a student may receive both scholarships, determine the probability that both students are freshmen.

b. Now suppose that an individual student may not receive both scholarships. Determine the probability that both students chosen are freshmen.

95. If five cards are dealt from a standard deck of 52 cards, find the probability that

a. The cards are all hearts.

b. The cards are all of the same suit.

96. If five cards are dealt from a standard deck of 52 cards, find the probability that

a. The cards consist of four aces.

b. The cards are four of a kind (four cards with the same face value).

97. Three different mathematics books, 5 different science books, and 2 different history books are placed on a shelf. What is the probability that the books of each subject are shelved together? Round to four decimal places.

98. Suppose that a box has 4 white balls and 6 red balls. Three players, A, B, and C will play a game where they each select one ball at random from the box, without replacement. Person A goes first, person B goes second, and person C goes third. The first person to pick a white ball wins the game. Find the probability of a win for each person.

99. Suppose that 5 dice are thrown. Find the probability that a "4" appears on exactly 2 of the 5 dice. Round to four decimal places.

100. Suppose that 6 dice are thrown. Find the probability that an even number appears on 4 of the 6 dice. Round to four decimal places.

101. Suppose a basketball player typically makes 45% of shots taken.

a. On a given shot, what is the probability that the player will not make the basket?

b. Now suppose that the player takes four shots in a game. What is the probability that the player makes none of the shots? Round to four decimal places.

c. What is the probability that the player makes exactly 1 shot? Round to four decimal places.

d. What is the probability that the player makes exactly 2 shots? Round to four decimal places.

e. What is the probability that the player makes exactly 3 shots? Round to four decimal places.

f. What is the probability that the player makes all 4 shots? Round to four decimal places.

g. What is the sum of the probabilities from parts (b)–(f)?

h. What is the probability that the player makes at least 2 shots?

102. A study found that for a particular strain of the influenza virus, a person who had received the flu vaccine had a 0.025 chance of contracting the virus after being exposed to an infectious person. The probability that an unvaccinated person would contract the virus after exposure was 0.4. If two people, one vaccinated and one unvaccinated, are exposed to the flu virus, what is the probability that at least one will contract the flu?

103. A box contains 4 quarters, 5 dimes, and 8 nickels. Find the probability that if three coins are selected, the value will be \$0.40. Round to four decimal places.

104. A box contains 5 white ping pong balls and 1 yellow ping pong ball. A second box has 6 white ping pong balls. Suppose that 4 balls are drawn at random from the first box and placed in the second box. Then, 2 balls are drawn from the second box and placed in the first.

a. Determine the probability that after both moves, the yellow ball is in the second box.

b. Determine the probability that after both moves, the yellow ball is in the first box.

Probability is an important tool used to compute mathematical expectation or expected value. The term *expected value* refers to the return expected for some kind of action. For example, suppose that you take a 24-question multiple choice test in which the possible responses are a, b, c, and d and for which there is exactly one correct answer. The probability of guessing correctly on a given question is $\frac{1}{4}$. So, if you guessed on every question, you would expect to get $\frac{1}{4} \cdot 24 = 6$ questions correct.

Expected value is often used to determine the expected winnings (or loss) in a game of chance. If p is the probability that an individual will win (or lose) m dollars, then the expected value, E, is given by $E = pm$. If an experiment has two possible outcomes, the expected value is the average, long-run payoff (or loss) weighted by the probabilities of each outcome. That is,

$$\textbf{Expected value} = \begin{pmatrix}\textbf{Pobability of}\\ \textbf{outcome 1}\end{pmatrix}\begin{pmatrix}\textbf{Payoff (or loss)}\\ \textbf{of outcome 1}\end{pmatrix} + \begin{pmatrix}\textbf{Probability of}\\ \textbf{outcome 2}\end{pmatrix}\begin{pmatrix}\textbf{Payoff (or loss)}\\ \textbf{of outcome 2}\end{pmatrix}$$

Use this formula for Exercises 105–108.

105. At a carnival, a game is constructed so that a player will win a \$2 prize approximately 5% of the time. If the cost to play the game is \$0.50, find the player's expected value by following these steps. Note that there are two possible outcomes: the player can win the game, or the player can lose.

a. What is the probability that the player wins? What is the net payoff if the player wins?

b. What is the probability that the player loses? What is the net loss if the player loses? (Write the net loss as a negative dollar value.)

c. Compute the player's expected value.

d. Interpret the meaning of the expected value both for the player and for the carnival.

e. If the carnival game is played 10,000 times during a two-week carnival, how much will the carnival make in profit from this game?

106. At a fund-raiser for a charity, a game is set up with a box of 9 red balls and 1 yellow ball. A player pays $1 to play the game, then reaches into the box and takes out a ball, sight-unseen. If the yellow ball is selected, the player wins a $5 bill. However, if a red ball is selected, the player wins nothing.

a. What is the probability that the player will win? What is the net payoff if the player wins?

b. What is the probability that the player will lose? What is the net loss if the player loses? (Write the net loss as a negative dollar value.)

c. Compute the player's expected value.

d. Interpret the meaning of the expected value both for the player and for the charity.

e. If the game is played 500 times, how much will the charity make from this game?

107. An insurance company charges a premium of $850 per year to insure a certain car for a certain driver. Given the driver's age and driving history, the company estimates the probability of an accident within the year to be 0.008. If the driver has an accident, the company estimates an average payout of $20,000 (this includes payment for car repair for all vehicles involved and medical expenses).

a. What is the probability that the insurance company *will not* have to pay out money for a claim during the course of the year (that is, the driver does not have an accident)? If there is no accident, what is the revenue collected by the insurance company for this policy?

b. What is the probability that the insurance company *will* have to pay out money for a claim (that is, the driver has an accident during the year)? In the event of an accident, what is the net loss that the insurance company incurs? (*Hint*: The net loss is the yearly premium minus the average payout.)

c. Compute the insurance company's expected value.

d. Interpret the meaning of the result of part (c).

108. A person buys a car in Florida, and the car dealership offers a warranty on the air-conditioning system. The consumer's cost for the warranty is $150. The dealership knows from long-term data that the probability of an air-conditioner failing during the warranty period is 1.5%. But if the AC unit does fail, the dealership will pay out an average of $1600 to fix it.

a. What is the probability that the dealership *will not* have to pay out money to fix the AC unit (that is, the AC unit does not fail within the warranty period)? If the AC unit does not fail during the warranty period, what is the revenue collected by the dealership?

b. What is the probability that the dealership *will* have to pay out money to fix the AC unit? In the event that the AC unit fails, what is the net loss that the dealership incurs? (*Hint*: The net loss is the $150 in revenue for the warranty minus the average cost to fix the AC unit.)

c. Compute the dealership's expected value.

d. Interpret the meaning of the result of part (c).

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Chapter 11 Review

Chapter 11 Key Concepts

Lesson 11-1 Sequences and Series

Key Concepts	Reference
An **infinite sequence** is a function whose domain is the set of positive integers. A **finite sequence** is a function whose domain is the set of the first n positive integers.	p. 1166
A sequence in which consecutive terms alternate in sign is an **alternating sequence**.	p. 1168
A **recursive formula** defines the nth term of a sequence as a function of one or more terms preceding it.	p. 1169
For a positive integer n, the quantity $\boldsymbol{n!}$ ("n factorial") is defined as $n! = (n)(n-1)(n-2)\cdots(2)(1)$ By definition, $0! = 1$.	p. 1170
The **nth partial sum** of a sequence $\{a_n\}$ is a finite series and is given by $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$	p. 1172
The sum of *all* terms in an infinite sequence is an **infinite series**: $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$	p. 1172
Properties of summation: If $\{a_n\}$ and $\{b_n\}$ are sequences, and c is a real number, then **1.** $\sum_{i=1}^{n} c = cn$ **2.** $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$ **3.** $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$	p. 1175

Lesson 11-2 Arithmetic Sequences and Series

Key Concepts	Reference
An **arithmetic sequence** is a sequence in which each term after the first differs from its predecessor by a common difference d.	p. 1182
The nth term of an arithmetic sequence: $a_n = a_1 + (n-1)d$ where a_1 is the first term of the sequence and d is the common difference.	p. 1182
Sum of the first n terms of an arithmetic sequence (nth partial sum): $S_n = \frac{n}{2}(a_1 + a_n)$ where a_1 is the first term of the sequence and a_n is the nth term of the sequence.	p. 1188

Lesson 11-3 Geometric Sequences and Series

Key Concepts	Reference
A **geometric sequence** is a sequence in which each term after the first is the product of the preceding term and a fixed nonzero real number, called the **common ratio**. If the first term of a geometric sequence is a_1, and the common ratio is r, then the nth term of a geometric sequence is given by $a_n = a_1 r^{n-1}$	p. 1196
***n*th partial sum of a geometric sequence:** The sum S_n of the first n terms of a geometric sequence is given by $S_n = \frac{a_1(1-r^n)}{1-r}$ where a_1 is the first term of the sequence and r is the common ratio, $r \neq 1$.	p. 1201
Sum of an infinite geometric series: Given an infinite geometric series $a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots$ with $\lvert r \rvert < 1$, the sum S of all terms in the series is given by $S = \frac{a_1}{1-r}$ *Note*: If $\lvert r \rvert \geq 1$, then the sum does not exist.	p. 1203
Future value of an ordinary annuity: Suppose that P dollars is invested at the end of each compounding period n times per year at interest rate r. Then the value A (in \$) of the annuity after t years is given by $A = \frac{P\left[\left(1+\frac{r}{n}\right)^{nt} - 1\right]}{\frac{r}{n}}$	p. 1206

Lesson 11-4 Mathematical Induction

Key Concepts	Reference
Mathematical induction is a technique of mathematical proof that is often used to prove that a statement is true for all positive integers.	p. 1215
The principle of mathematical induction: Let P_n be a statement involving the positive integer n, and let k be an arbitrary positive integer. Then P_n is true for all positive integers n if 1. P_1 is true, and 2. The truth of P_k implies the truth of P_{k+1}. The assumption that P_k is true is called the **inductive hypothesis**.	p. 1216 p. 1217
Extended principle of induction: Mathematical induction can be extended to prove statements that might hold true only for integers greater than or equal to some positive integer j. In such a case, • First prove the statement for $n = j$. • Then show that the truth of the statement for an integer greater than or equal to j implies the truth of the statement for the integer that follows.	p. 1220

Lesson 11-5 The Binomial Theorem

Key Concepts	Reference
Binomial theorem: Let n be a positive integer. The expansion of $(a+b)^n$ is given by $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$ $= \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$ The values $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ are called the **binomial coefficients** of the expansion of $(a+b)^n$. The binomial coefficients can also be found by using **Pascal's triangle.** $\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \ldots \end{array}$	p. 1226
Finding the kth term of a binomial expansion: Let n and k be positive integers with $k \le n+1$. The kth term of $(a+b)^n$ is $\binom{n}{k-1}a^{n-(k-1)}b^{k-1}$	p. 1228

Lesson 11-6 Principles of Counting

Key Concepts	Reference
Fundamental principle of counting: If one event can occur in m different ways and a second event can occur in n different ways, then the number of ways that the two events can occur in sequence is $m \cdot n$.	p. 1234
A **permutation** is an ordered arrangement of distinct items. • The number of permutations of n distinct elements is $n!$.	p. 1236
• Consider a set of n elements with r_1 duplicates of one kind, r_2 duplicates of a second kind, ..., r_k duplicates of a kth kind. Then the number of distinguishable permutations of the n elements of the set is: $\dfrac{n!}{r_1! \cdot r_2! \cdot \cdots \cdot r_k!}$	p. 1237
• The number of permutations of n elements taken r at a time is given by ${}_nP_r = \dfrac{n!}{(n-r)!}$ or equivalently, ${}_nP_r = n(n-1)(n-2)\cdots(n-r+1)$	p. 1239
A **combination** is a collection of distinct items taken without regard to order. The number of combinations of n elements taken r at a time is given by ${}_nC_r = \dfrac{n!}{r! \cdot (n-r)!}$.	p. 1240

Lesson 11-7 Introduction to Probability

Key Concepts	Reference
Theoretical probability of an event: Let S represent a sample space with equally likely outcomes, and let E be a subset of S. Then the probability of event E, denoted by $P(E)$, is given by $P(E) = \frac{n(E)}{n(S)}$	p. 1251
Let E be an event relative to sample space S. The **complement of E**, denoted by $\bar{E}$, is the set of outcomes in the sample space but not in event E. It follows that $P(E) + P(\bar{E}) = 1$.	p. 1254
Empirical probability is computed based on observed outcomes of the relative frequency of an event to the number of times an experiment is performed.	p. 1255
Two events are **mutually exclusive** if they share no common elements. **The probability of *A* or *B*:** Given events A and B in a same sample space, $P(A \text{ or } B)$ is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$.	p. 1257
Probability of a sequence of independent events: Two events are **independent** if the probability of one event does not affect the probability of the second event. If events A and B are independent events, then the probability that both A and B will occur is $P(A \text{ and } B) = P(A) \cdot P(B)$.	p. 1260

Go online for a detailed chapter summary.

Chapter 11 Test

1. Write the first six terms of the sequence defined by $a_1 = -2$, $a_2 = 3$, $a_n = a_{n-2} + a_{n-1}$ for $n \geq 3$.

2. Simplify. $\dfrac{(3n+1)!}{(3n-1)!}$

For Exercises 3–5,

a. Determine whether the sequence is arithmetic, geometric, or neither.

b. If the sequence is arithmetic, determine d. If the sequence is geometric, determine r.

c. Write an expression a_n for the nth term of the sequence.

3. 0.139, 0.00139, 0.0000139, ...

4. 0.52, 0.68, 0.84, 1.00, ...

5. $\frac{3}{5}, \frac{6}{25}, \frac{9}{125}, \frac{12}{125}, \ldots$

6. Write the first four terms of the geometric sequence with $a_1 = 4$ and $r = \frac{3}{2}$.

7. Write the first five terms of the arithmetic sequence with $a_1 = 10$ and $a_{20} = 67$.

8. Find the 78th term of an arithmetic sequence with $a_1 = 64$ and $d = -11$.

9. Find the 6th term of a geometric sequence with $a_1 = -3$ and $r = 2$.

10. Find the number of terms in the arithmetic sequence.

$$-15, -19, -23, -27, \ldots, -679$$

11. Find the number of terms in the geometric sequence.

$$16, 8, 4, 2, \ldots, \frac{1}{16}$$

12. Given an arithmetic sequence with $a_{12} = -21$ and $a_{50} = -97$, find a_1 and d.

13. Given a geometric sequence with $a_3 = 20$ and $a_8 = 640$, find a_1 and r.

For Exercises 14–18, evaluate the sum if possible.

14. $\frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \cdots$

15. $\sum_{k=1}^{54} (3k + 7)$

16. $\sum_{i=1}^{6} 4\left(\frac{3}{2}\right)^{i-1}$

17. $\sum_{k=1}^{4} k!$

18. $\sum_{i=1}^{\infty} \frac{1}{2}\left(\frac{6}{5}\right)^{i-1}$

19. Suppose that a county fair has an estimated 50,000 people attend over a 2-week period. Admission to the fair is \$5.00. In addition, suppose that each person spends an average of \$10 on food, drinks, and rides.

a. How much money is initially infused into the local community for admission, food, drinks, and rides?

b. If the money is respent in the local community over and over again at a rate of 65%, determine the total amount spent. Assume that the money is respent an infinite number of times. Round to the nearest dollar.

20. An employee invests \$400 per month in an ordinary annuity. If the interest rate is 5.2%, find the value of the annuity after 25 yr.

21. Lakeisha wants to install new tile. The tile costs \$3.50/ft² and labor is \$4.00/ft². Write the nth term of a sequence representing the cost to tile an n by n square foot area where n is an integer and $n \geq 1$ ft.

22. How many days will it take Johan to read a 920-page book if Johan reads 20 pages the first day, 25 pages the second day, 30 pages the third day, and so on?

For Exercises 23–25, use mathematical induction to prove the given statement for all positive integers *n*.

23. $6 + 10 + 14 + \cdots + (4n + 2) = n(2n + 4)$

24. $1 + 5 + 25 + \cdots + 5^{n-1} = \frac{1}{4}(5^n - 1)$

25. 2 is a factor of $7^n - 5$.

For Exercises 26–27, expand the binomial by using the binomial theorem.

26. $\left(\frac{x}{2} + 3\right)^4$

27. $(4c^2 - t^4)^5$

For Exercises 28–29, find the indicated term of the binomial expansion.

28. $(-2t + v^2)^{10}$; eighth term

29. $(3x + y^2)^7$; Term containing y^6.

30. Evaluate ${}_{13}P_5$ and ${}_{13}C_5$.

31. Explain the difference between a permutation and a combination of n items taken r at a time.

32. A musician plans to perform 9 selections. In how many ways can the selections be arranged?

33. How many outfits can be made from 4 pairs of slacks, 5 shirts, and 3 ties, if one selection from each category is made? Assume that all items fashionably match.

34. In how many ways can the letters in the word HIPPOPOTAMUS be arranged and misspelled?

35. Suppose that a jury pool consists of 30 senior citizens and 26 non-seniors.

a. In how many ways can a jury of 12 people be selected?

b. In how many ways can a jury of 6 seniors and 6 non-seniors be selected?

c. What is the probability of randomly selecting a jury of 6 seniors and 6 non-seniors?

d. What is the probability of randomly selecting a jury of all non-seniors?

36. A review sheet for a history test has 10 essay questions. Suppose that the professor picks 3 questions from the review sheet to put on the test. In how many ways can the professor choose 3 questions from 10 questions?

37. After a service call by a plumber, the company follows up with a survey to rate the service and professionalism of the technician. The survey has 6 yes/no questions and 4 multiple-choice questions each with 3 possible responses. In how many different ways can a customer fill out the survey?

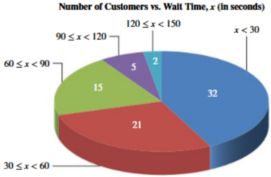
38. Suppose that 50 people buy raffle tickets.

a. In how many ways can 4 people who bought tickets be selected if each is to receive a $20 gift certificate to a restaurant?

b. In how many ways can 4 people who bought tickets be selected if the first person wins a $10 gift certificate, the second person wins a $25 gift certificate, the third person wins a $50 gift certificate, and the fourth person wins a $200 camera?

39. For a recent year, approximately 36,000 people were killed in the United States in motor vehicle accidents. If the population had 300,000,000 people at that time, estimate the probability of being killed in a motor vehicle crash.

40. A cable company advertises short wait times for customer service calls. The graph shows the wait times (in seconds) for a sample of customers.



Based on the data given, if a customer is selected at random, find the probability that

a. The customer will wait less than 30 sec.

b. The customer will wait at least 30 sec.

c. The customer will wait at least 90 sec but less than 120 sec.

d. The customer will wait more than 150 sec.

41. If two fair dice are thrown, find the probability that the sum is between 6 and 8, inclusive.

42. Suppose that two cards are drawn from a standard deck with replacement. What is the probability that an ace is selected, followed by a heart?

43. For a recent year, 31.9% of Americans living below the poverty level were not covered by health insurance. (*Source*: U.S. Census Bureau, www.census.gov) If three people were selected at random from this population, what is the probability that all three would not have coverage?

For Exercises 44–47, use the data in the table categorizing the type of payment used at a grocery store according to the gender of the customer. If one person is chosen at random, find the probability of the given event.

	Cash	Credit Card	Check	Total
Male	19	24	13	56
Female	14	30	20	64
Total	33	54	33	120

44. The customer is female.

45. The customer is male or paid by check.

46. The customer paid by credit card or by cash.

47. The customer paid cash or was female.

Go online for more practice problems.

Chapter 11 Cumulative Review Exercises

For Exercises 1–4, consider sets A and B and determine if the statement is true or false.

$$A = \{x|4 > x\}, B = \{x|x \le 1\}$$

1. $4 \in A$

2. $1 \in B$

3. $A \cap B = B$

4. $A \cup B = B$

5. **a.** Write an expression for the distance between t and 5 on the number line.

b. Simplify the expression from part (a) for $t < 5$.

6. Simplify without using a calculator.

$$\frac{(6.0 \times 10^{13})(9.0 \times 10^{8})}{2.0 \times 10^{-6}}$$

For Exercises 7–10, simplify the expression.

7. $-27^{-4/3}$

8. $(\sqrt{x} + 5\sqrt{2})(3\sqrt{x} - \sqrt{2})$

9. $\frac{4 - x^{-2}}{6 - 3x^{-1}}$

10. i^{127}

For Exercises 11–18, solve the equation or inequality. Write the solution set to the inequalities in interval notation.

11. $\sqrt{2x^2 - x - 14} + 4 = x$

12. $(x^2 + x)^2 - 14(x^2 + x) + 24 = 0$

13. $|3x - 5| = |2x + 1|$

14. $-5x \le 20 \quad \text{and} \quad 3 - [2 - (x - 4)] < 2x + 5$

15. $0 \le |x + 7| - 6$

16. $\frac{2x + 1}{x + 4} \ge 1$

17. $\log_4(2x + 7) = 2 + \log_4 x$

18. $5e^{x+1} - 100 = 0$

19. Given $f(x) = 2x^3 - 5x^2 - 28x + 15$,

a. Find all the zeros of $f(x)$.

b. Identify the x-intercepts of the graph of f.

c. Determine the y-intercept of the graph of f.

d. Graph $y = f(x)$.

e. Solve the inequality $2x^3 - 5x^2 - 28x + 15 < 0$.

20. An object is launched straight upward with an initial speed of 400 ft/sec from a height of 4 ft. The height of the object t seconds after launch can be modeled by $s(t) = -16t^2 + 400t + 4$, where $s(t)$ is the height measured in feet.

a. Determine the time required for the object to reach its maximum height.

b. Determine the maximum height of the object.

c. Determine the time required for the object to hit the ground. Round to 1 decimal place.

21. Given $x^2 + y^2 - 8x - 2y + 1 = 0$,

a. Write the equation of the circle in standard form.

b. Identify the center and radius of the circle.

c. Graph the circle.

For Exercises 22–25, graph the equation.

22. $\frac{y^2}{9} - \frac{x^2}{16} = 1$

23. $\frac{(x + 1)^2}{4} + \frac{(y - 2)^2}{9} = 1$

24. $y = \log_2(x + 3)$

25. $y = \begin{cases} |x| + 1 & \text{for } x \le 1 \\ -x + 1 & \text{for } x > 1 \end{cases}$

26. Explain how the graph of $f(x) = 2\sqrt{x - 1} + 3$ is related to the graph of $y = \sqrt{x}$.

27. Given $f(x) = 4x^3 - 3x$,

- **a.** Find the difference quotient.
- **b.** Find the average rate of change on the interval [1, 3].
- **c.** Determine if the function is even, odd, or neither.

28. Given $g(x) = 5x - 1$,

- **a.** Is g a one-to-one function?
- **b.** Write an equation for $g^{-1}(x)$.

29. Given $f(x) = x^3 - x^2 - 7x + 15$,

- **a.** Is $2 + i$ a zero of $f(x)$?
- **b.** Is $(x + 3)$ a factor of $f(x)$?

30. Given $r(x) = \dfrac{3x^2 + 2}{x^2 - 5x - 14}$,

- **a.** Determine the vertical asymptote(s).
- **b.** Determine the horizontal or slant asymptote if either exists.

31. The population of a city was 320,000 in the year 2000. By 2012, the population had reached 360,800.

- **a.** Write a model of the form $P(t) = P_0e^{kt}$ to represent the population $P(t)$ for a time t years since 2000.
- **b.** Approximate the population in the year 2015 assuming that this trend continued. Round to the nearest 100 people.
- **c.** Determine the amount of time for the population to reach 400,000 if this trend continues. Round to the nearest tenth of a year.

32. Write the expression in terms of $\log x$, $\log y$, and $\log z$.

$$\log\left(\frac{x^3}{y^5\sqrt{z}}\right)$$

33. Simplify. $\log_2 \frac{1}{16}$

34. Approximate the value of $\log_5 417$ to 4 decimal places.

For Exercises 35–37, solve the system of equations.

35.
$$\begin{aligned} 2x - 5y &= 13 \\ -3x + 2y &= -3 \end{aligned}$$

36.
$$\begin{aligned} 4x \qquad - 2z &= -4 \\ 6y + 5z &= 8 \\ 7x - 3y \qquad &= 13 \end{aligned}$$

37.
$$\begin{aligned} 2x - y &= 6 \\ x^2 + y &= 9 \end{aligned}$$

38. Use Gaussian elimination or Gauss-Jordan elimination to solve the system of equations.

$$\begin{aligned} -2x - 9y + 16z &= -15 \\ x + y - z &= 4 \\ x - 2y + 5z &= 1 \end{aligned}$$

39. Use Cramer's rule to solve the system.

$$\begin{aligned} 13x - 2y &= 11 \\ 5x + 3y &= 6 \end{aligned}$$

40. Graph the solution set.

$$\begin{aligned} x^2 + y^2 &\le 16 \\ y &> -x^2 + 4 \end{aligned}$$

For Exercises 41–46, refer to the matrices given and perform the indicated operations, if possible.

$$A = \begin{bmatrix} 4 & -3 \\ 5 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 6 \\ 3 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 4 & 0 \\ 3 & 6 & 1 \\ 2 & -5 & 7 \end{bmatrix} \qquad D = \begin{bmatrix} -5 & 1 & 0 \\ 8 & 4 & 6 \end{bmatrix}$$

41. $-5A + 2B$

42. $3C - A$

43. $|A|$

44. $|C|$

45. CD

46. AD

47. Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$, find A^{-1}.

48. Use the inverse matrix from Exercise 47 to solve the system of equations.

$$\begin{aligned} x + 2y - z &= -5 \\ y + 3z &= 13 \\ x \qquad + 2z &= 5 \end{aligned}$$

49. Given $(y + 2)^2 = -8(x - 1)$,

a. Determine the vertex of the graph of the parabola.

b. Identify the focus.

c. Write an equation for the directrix.

50. Determine whether the sequence is arithmetic or geometric, and find the value of the common difference or the common ratio.

$$-a^2, a^5, -a^8, a^{11}, \ldots$$

51. Find the 500th term of an arithmetic sequence with $a_1 = 6.9$ and $d = 0.3$.

For Exercises 52–53, find the sum.

52. $\sum_{k=1}^{74}(5k + 3)$

53. $\sum_{i=1}^{8} 6(2)^{i-1}$

54. Find the sum if possible. $6 + 2 + \frac{2}{3} + \frac{2}{9} + \cdots$

55. Evaluate the expressions.

a. $8!$

b. $\binom{12}{11}$

c. ${}_{15}P_3$

56. Expand the binomial. $(3x - y^2)^5$

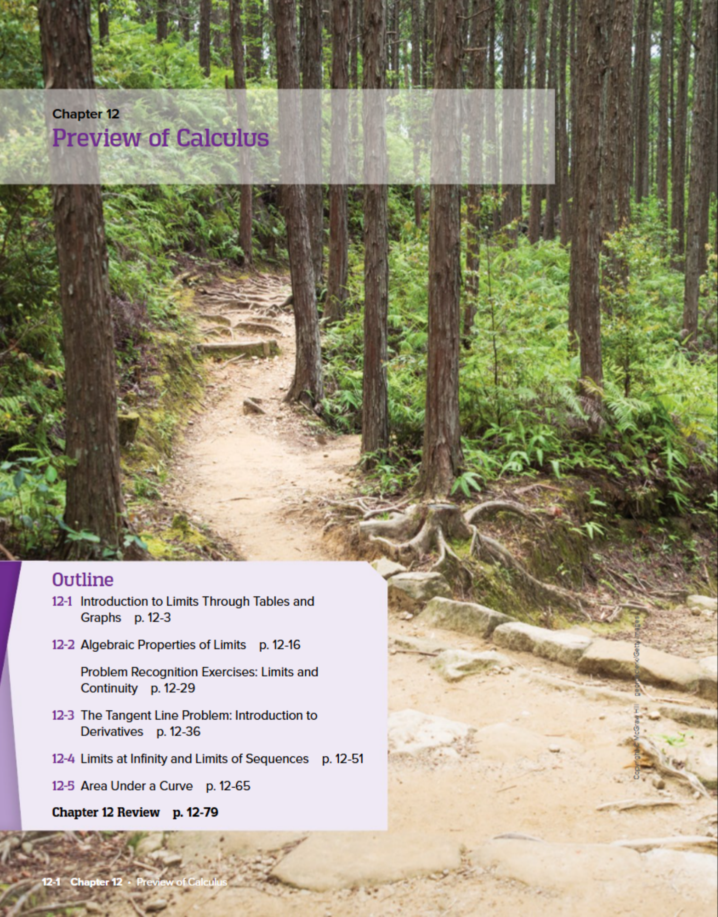
57. Simplify. $\dfrac{(3n - 1)!}{2!(3n + 1)!}$

58. In how many ways can 5 children be arranged in a line for a photograph?

59. Suppose that four students are to be randomly selected from a group of 8 freshmen and 5 sophomores. What is the probability of selecting 2 freshmen and 2 sophomores?

60. If one card is selected from a standard deck of cards, what is the probability that the card selected is a diamond or an ace? (*Hint*: Refer to Figure 11-12 from page 1256.)

GO ONLINE **Go online for more practice problems.**



Chapter 12

Preview of Calculus

Outline

12-1 Introduction to Limits Through Tables and Graphs p. 12-3

12-2 Algebraic Properties of Limits p. 12-16

Problem Recognition Exercises: Limits and Continuity p. 12-29

12-3 The Tangent Line Problem: Introduction to Derivatives p. 12-36

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Launch Activity: Mathematical Modeling

Identifying Quantities

Chris and Mika are both preparing to run the mile for their Cross Country meet. They started exactly two miles apart and ran directly toward each other. They met face-to-face on either side of a chain-link fence. Chris ran a steady 6-minute mile pace. His run started by going directly up a hill for 2½ minutes. He then ran on a flat surface for 1 minute before running down a hill for the remainder of his run until he reached the chain-link fence. Mika ran a 6½ minute mile, first running on a flat surface for 3 minutes then running up a hill for the remaining portion of the mile until he met Chris face-to-face at the chain-link fence.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer this question: How could you represent this situation with a graph?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

GO ONLINE to answer these questions

Lesson 12-1

Introduction to Limits Through Tables and Graphs

What will you learn?

After completing this lesson, you should be able to:

- Estimate Limits Numerically
- Identify Limits that Fail to Exist
- Estimate Limits Graphically

Learn: Estimate Limits Numerically

Suppose that a ball is dropped from a 256-ft building. The height $s(t)$ of the ball above ground level is given by $s(t) = 256 - 16t^2$, where t is the time in seconds after the ball is dropped and $s(t)$ is in feet. In Table 12-1, the height of the ball is given in 1-sec increments during its fall, and a graph of the height versus time is given in Figure 12-1.

TABLE 12-1 Height $s(t)$ for Selected Values of Time t

t (seconds)	$s(t)$ (feet)
0	256
1	240
2	192
3	112
4	0

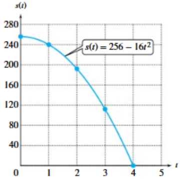


Figure 12-1

Intuitively, we know that the ball picks up speed as it falls. The average velocity of the ball during the first second after release is the *average rate of change* on the interval [0,1]. The negative sign indicates that the velocity is directed downward.

$$\begin{pmatrix}\text{Average velocity} \\ \text{between } t = 0 \text{ and } t = 1\end{pmatrix} = \frac{s(1) - s(0)}{1 - 0} = \frac{240 - 256}{1} = -16 \text{ ft/sec}$$

The average velocity of the ball between 1 and 2 sec after release is the average rate of change on the interval [1,2].

$$\begin{pmatrix}\text{Average velocity} \\ \text{between } t = 1 \text{ and } t = 2\end{pmatrix} = \frac{s(2) - s(1)}{2 - 1} = \frac{192 - 240}{1} = -48 \text{ ft/sec}$$

The average speed of the ball is 16 ft/sec during the first second of its fall. The average speed increases to 48 ft/sec between $t = 1$ and $t = 2$ sec. Now suppose that we want to know how fast the ball is traveling at a time *exactly* 1 sec after release. This value will be between 16 ft/sec and 48 ft/sec, but the exact value can be answered by using the concept of a *limit*.

Informal Definition of the Limit of a Function

Consider an open interval containing *c* and a function *f* defined on the open interval except possibly at *c*.

- The statement $\lim\limits_{x \to c} f(x) = L$ is read as "the limit of *f*(*x*) as *x* approaches *c* equals *L*."
- $\lim\limits_{x \to c} f(x) = L$ means that *f*(*x*) can be made arbitrarily close to *L* provided that *x* is taken sufficiently close to *c* (Figure 12-2).*

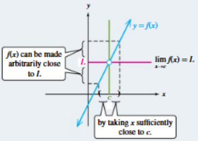

y
y = f(x)
f(x) can be made arbitrarily close to L.
L
lim f(x) = L
x→c
x
c
by taking x sufficiently close to c.

Figure 12-2

*The statements "arbitrarily close" and "sufficiently close" will be quantified in a formal definition of a limit given in a calculus course.

The following points are important to understand when considering the definition of the limit of a function.

- Even if *f*(*c*) is undefined, the limit at *c* may still exist.
- For *x* values taken close to *c* (but not equal to *c*), the corresponding values of *f*(*x*) will be close to *L*.
- The closer *x* is to *c*, the closer *f*(*x*) is to *L*.

Point of Interest

The idea of a limit is preceded by an ancient technique, called the Method of Exhaustion, used by the Greek mathematicians Antiphon and Eudoxus in the 5th century BCE. The modern definition of a limit was originally developed by Bernard Bolzano in 1817.

Example 1: Estimating a Limit Numerically

Use a table to estimate the limit. $\lim\limits_{x \to 2} \dfrac{2x^2 - 3x - 2}{x - 2}$

Solution:

Given $\lim\limits_{x \to 2} \dfrac{2x^2 - 3x - 2}{x - 2}$, we want to determine the behavior of $f(x) = \dfrac{2x^2 - 3x - 2}{x - 2}$ for *x* values close to 2, but not equal to 2 (Table 12-2).

TABLE 12-2 Values of *f*(*x*) as Values of *x* Approach 2

x taken closer to 2 → ← *x* taken closer to 2

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
***f*(*x*)**	4	4.8	4.98	4.998		5.002	5.02	5.2	6

f(*x*) gets closer to 5 → ← *f*(*x*) gets closer to 5

From Table 12-2, we estimate the limiting value to be 5. So, $\lim\limits_{x \to 2} \dfrac{2x^2 - 3x - 2}{x - 2} = 5$.

Apply the Skills

1. Use a table to estimate the limit. $\lim\limits_{x \to 1} \dfrac{x^2 + 2x - 3}{x - 1}$

Figure 12-3 illustrates the behavior of $f(x) = \dfrac{2x^2 - 3x - 2}{x - 2}$ near $x = 2$. Notice that $f(2)$ is undefined, as indicated by the open circle in the graph. However, as x gets closer and closer to 2 from either the right or left, $f(x)$ gets closer and closer to $y = 5$.

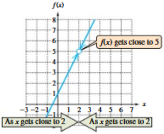


Figure 12-3

TECHNOLOGY CONNECTIONS

Identifying a Limit on a Graphing Calculator

The graph of $f(x) = \dfrac{2x^2 - 3x - 2}{x - 2}$ shown on a calculator may not illustrate the "hole" at $x = 2$ (Figure 12-4). Recall that a calculator renders a graph by plotting points for x values between Xmin and Xmax. If the viewing window is set up such that the exact value $x = 2$ is not one of the input values for x, then the calculator will "connect" the two points immediately to the left and right of $x = 2$. This gives the appearance of a continuous curve without a "hole."

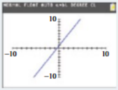


Figure 12-4

However, if we use a "friendly window" designated by the "ZDecimal" option, then the x values seen on the viewing window will include $x = 2$. In such a case, we will see a "gap" in the curve where f is undefined (Figure 12-5).

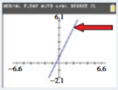


Figure 12-5

In addition to the graph of the function, we can use the table feature on a calculator to estimate the value of the limit as x approaches 2. Notice that at $x = 2$, $f(x)$ is undefined (Figure 12-6).

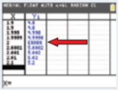


Figure 12-6

In Example 1 we used Table 12-2 to estimate $\lim\limits_{x \to 2} f(x) = 5$ where $f(x) = \dfrac{2x^2 - 3x - 2}{x - 2}$. This means that $f(x)$ can be taken arbitrarily close to 5, provided that we choose x close enough to 2. In Example 2, we illustrate this idea.

Example 2: Applying the Definition of a Limit

Consider $\lim\limits_{x \to 2} f(x) = 5$, where $f(x) = \dfrac{2x^2 - 3x - 2}{x - 2}$. Use Table 12-2 (on page 12-4) to determine an open interval for x that would guarantee that the distance between $f(x)$ and 5 is less than

- **a.** 0.02 units.
- **b.** 0.002 units.

Solution:

The distance between $f(x)$ and 5 is given by $|f(x) - 5|$.

a.

Take x on the interval (1.99, 2.01)

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	4	4.8	4.98	4.998		5.002	5.02	5.2	6

$L = 5$

$|f(x) - 5| < 0.02$ $|f(x) - 5| < 0.02$

Taking x on the interval (1.99, 2.01) guarantees that $|f(x) - 5|$ is within 0.02 units.

b.

Take x on the interval (1.999, 2.001)

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	4	4.8	4.98	4.998		5.002	5.02	5.2	6

$L = 5$

$|f(x) - 5| < 0.002$ $|f(x) - 5| < 0.002$

Taking x on the interval (1.999, 2.001) guarantees that $|f(x) - 5|$ is within 0.002 units.

Apply the Skills

2. Repeat Example 2 by expanding Table 12-2 to determine an interval for x that would guarantee that the distance between $f(x)$ and 5 is less than

- **a.** 0.0002 units
- **b.** 0.00002 units

In Examples 3 and 4, we estimate more limits by using tables.

Example 3: Estimating a Limit by Using a Table

Use a table to estimate the limit. $\lim_{x\to 1}(-3x+1)$

Solution:

Given $\lim_{x\to 1}(-3x+1)$, we want to determine the behavior of $f(x) = -3x + 1$ for x values close to 1, but not equal to 1 (Table 12-3).

TABLE 12-3 Values of $f(x)$ as Values of x Approach 1

x taken closer to 1 → ← x taken closer to 1

x	0.5	0.9	0.99	0.999	1	1.001	1.01	1.1	1.5
$f(x)$	−0.5	−1.7	−1.97	−1.997		−2.003	−2.03	−2.3	−3.5

$f(x)$ gets closer to −2 → ← $f(x)$ gets closer to −2

From Table 12-3, we estimate the limiting value to be −2. So, $\lim_{x\to 1}(-3x+1) = -2$.

Apply the Skills

3. Use a table to estimate the limit. $\lim_{x\to -2}(-4x-3)$

In Example 3, $f(x) = -3x + 1$. Notice that $\lim_{x\to 1}(-3x+1) = -2$ and $f(1) = -2$. In this case, the limit of $f(x)$ as x approaches 1 is the same as the value of f at $x = 1$ (Figure 12-7). That is, $\lim_{x\to 1} f(x) = f(1)$. This will not always be the case. In many applications of limits, a function f will not be defined at c, but will have a limit as x approaches c (see Example 1 on page 12-4).

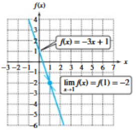


Figure 12-7

Example 4: Estimating a Limit by Using a Table

Use a table to estimate the limit. $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$

Solution:

Given $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$, we want to determine the behavior of $f(x) = \frac{\sqrt{x+4}-2}{x}$ for x values close to 0, but not equal to 0 (Table 12-4).

TABLE 12-4 Values of $f(x)$ as Values of x Approach 0

x taken closer to 0 → ← x taken closer to 0

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$	0.25158	0.25016	0.25002		0.24998	0.24984	0.24846

$f(x)$ gets closer to 0.25 → ← $f(x)$ gets closer to 0.25

From Table 12-4, we estimate the limiting value to be 0.25 or $\frac{1}{4}$. So, $\lim\limits_{x\to 0}\frac{\sqrt{x+4}-2}{x}=\frac{1}{4}$.

Apply the Skills

4. Use a table to estimate the limit. $\lim\limits_{x\to 0}\frac{\sqrt{x+16}-4}{x}$

Learn: Identify Limits that Fail to Exist

In Examples 5–7, we investigate situations in which a limit fails to exist. The first case involves the unbounded behavior of a function as x approaches c.

Example 5: Identifying Unbounded Behavior

Determine whether the limit exists. $\lim\limits_{x\to 1}\frac{1}{x-1}$

Solution:

Given $\lim\limits_{x\to 1}\frac{1}{x-1}$, we want to determine the behavior of $f(x)=\frac{1}{x-1}$ for x values close to 1, but not equal to 1 (Table 12-5).

TABLE 12-5 Values of $f(x)$ as Values of x Approach 1

x taken closer to 1 → ← x taken closer to 1

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
$f(x)$	−10	−100	−1000	−10,000		10,000	1000	100	10

$f(x)$ approaches $-\infty$ → ← $f(x)$ approaches ∞

Notice from Table 12-5 that as x gets closer to 1, the value of $|f(x)|$ becomes increasingly large and does not converge to a unique real number. It appears that as x approaches 1 from the left, $f(x)$ decreases without bound (approaches $-\infty$). Likewise, as x approaches 1 from the right, $f(x)$ increases without bound (approaches ∞). We say that

$$\lim_{x\to 1}\frac{1}{x-1}\text{ does not exist}$$

because the function is unbounded as x approaches 1.

The graph of $f(x) = \frac{1}{x-1}$ has a vertical asymptote at $x = 1$, which supports our result (Figure 12-8).

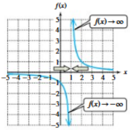


Figure 12-8

Apply the Skills

5. Determine whether the limit exists. $\lim\limits_{x \to -3} \frac{1}{(x+3)^2}$

In Example 6, we illustrate a case in which $\lim\limits_{x \to c} f(x)$ does not exist because the limiting values from the left and right of c are different.

Example 6: Comparing Function Behavior to the Left and Right of c

Determine whether the limit exists. $\lim\limits_{x \to 0} \frac{x}{|x|}$

Solution:

By definition, $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$.

So,

$$\frac{x}{|x|} = \frac{x}{x} = 1 \text{ if } x > 0,$$

and

$$\frac{x}{|x|} = \frac{x}{-x} = -1 \text{ if } x < 0.$$

The graph of $f(x) = \frac{x}{|x|}$ is shown in Figure 12-9.

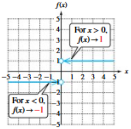


Figure 12-9

Check Your Work

Note that $\frac{x}{|x|}$ is not defined at 0.

For an open interval containing 0, values to the left of 0 show the function approaching −1. Values to the right of 0 show the function approaching 1. Since there is no open interval containing $x = 0$ for which $f(x)$ approaches a single real number, $\lim\limits_{x \to 0} \frac{x}{|x|}$ does not exist.

Apply the Skills

6. Determine whether the limit exists. $\lim\limits_{x \to -4} \frac{x+4}{|x+4|}$

In Example 7, we investigate a situation in which a limit fails to exist because $f(x)$ oscillates between two values without converging to a single real number. It is also important to note that when estimating a limit of a trigonometric function by using a table or graph, make sure that the calculator is in radian mode.

Example 7: Investigating the Limit of a Function that Oscillates Near c

Determine whether the limit exists. $\lim\limits_{x \to 0} \sin\frac{1}{x}$

Solution:

As x approaches 0, the value $\frac{1}{x}$ approaches ∞ if x is positive and $-\infty$ if x is negative. So, the behavior of $\sin\frac{1}{x}$ oscillates between –1 and 1 without converging to a single real number (Figure 12-10).

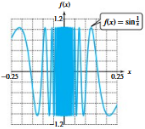


Figure 12-10

So, $\lim\limits_{x \to 0} \sin\frac{1}{x}$ does not exist.

Apply the Skills

7. Determine whether the limit exists. $\lim\limits_{x \to 0} 4\cos\frac{\pi}{x}$

Learn: Estimate Limits Graphically

The graph of a function can also help us to determine whether a limit exists and to estimate the value of a limit.

Example 8: Estimating Limits from a Graph

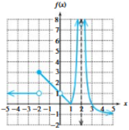
Use the graph to estimate the value of each limit or state that the limit does not exist.

a. $\lim\limits_{x \to -2} f(x)$

b. $\lim\limits_{x \to 0} f(x)$

c. $\lim\limits_{x \to 1} f(x)$

d. $\lim\limits_{x \to 2} f(x)$



Solution:

a. $\lim_{x \to -2} f(x)$ does not exist

For an open interval containing $x = -2$, the value of $f(x)$ does not converge to a single real number. Notice that as x gets close to -2 from the *left*, $f(x)$ approaches 1. In contrast, as x gets close to -2 from the *right*, $f(x)$ approaches 3.

b. $\lim_{x \to 0} f(x) = 1$

The open dot at the point (0, 1) indicates that $f(0)$ is undefined. However, as x gets closer and closer to 0 from both the left and right of 0, the y value approaches 1. So, the limit as x approaches 0 of $f(x)$ is 1.

c. $\lim_{x \to 1} f(x) = 0$

As x gets closer and closer to 1 from both the left and right of 1, the y value approaches 0. So, the limit as x approaches 1 of $f(x)$ is 0.

d. $\lim_{x \to 2} f(x)$ does not exist

$f(x)$ is unbounded for x close to 2.

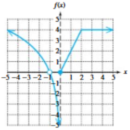
Apply the Skills

8. Use the graph to estimate the value of each limit or state that the limit does not exist.

a. $\lim_{x \to -1} f(x)$

b. $\lim_{x \to 0} f(x)$

c. $\lim_{x \to 2} f(x)$



Practice 12-1

Practice Exercises

Prerequisite Review

R.1. Determine the vertical asymptote(s) of the graph of the function $f(x) = \frac{6}{x-3}$.

R.2. Determine the vertical asymptote(s) of the graph of the function $g(x) = \frac{x-4}{3x^2 - 7x + 2}$.

R.3. Simplify by writing the expression without absolute value bars.

a. $\frac{|x-2|}{x-2}$ for $x > 2$ **b.** $\frac{|x-2|}{x-2}$ for $x < 2$

R.4. Graph the piecewise-defined function.

$$f(x) = \begin{cases} 1 - x & \text{for } x \le -2 \\ x^2 - 1 & \text{for } x > -2 \end{cases}$$

Concept Connections

1. How is the statement $\lim_{x \to c} f(x) = L$ read?

2. $\lim_{x \to c} f(x) = L$ means that $f(x)$ can be taken arbitrarily close to __________ provided that x is taken sufficiently close to __________.

3. Determine if the statement is true or false and explain your answer.

$f(c)$ must be defined in order for $\lim_{x \to c} f(x) = L$ to exist.

4. Given $\lim_{x \to 2} f(x) = 7$, can you conclude anything about $f(2)$?

5. If $f(1) = 5$, can you conclude anything about $\lim_{x \to 1} f(x)$? Explain.

6. Given $f(x) = \frac{1}{x^2 - x - 12}$, write the equation(s) of the vertical asymptote(s). Why does $\lim_{x \to 4} \frac{1}{x^2 - x - 12}$ fail to exist?

Learn: Estimate Limits Numerically

For Exercises 7–12, recreate and complete the table, then estimate the limit. (See Example 1, p. 12-4)

7. $\lim_{x \to 1} (4x - 1)$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$						

8. $\lim_{x \to 2} (12 - 8x)$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

9. $\lim_{x \to 4} \frac{2x^2 - 7x - 4}{x - 4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

10. $\lim_{x \to -2} \frac{x^2 - 2x - 8}{x + 2}$

x	−2.1	−2.01	−2.001	−1.999	−1.99	−1.9
$f(x)$						

11. $\lim_{x \to -3} \frac{x + 3}{x^2 + 2x - 3}$

x	−3.1	−3.01	−3.001	−2.999	−2.99	−2.9
$f(x)$						

12. $\lim_{x \to 1} \frac{x - 1}{x^2 - 4x + 3}$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$						

For Exercises 13–16, refer to the tables in Exercises 7–10. (See Example 2, p. 12-6)

13. Refer to Exercise 7. Determine an interval for x that would guarantee the distance between $f(x)$ and 3 is less than

a. 0.4 units **b.** 0.04 units

14. Refer to Exercise 8. Determine an interval for x that would guarantee the distance between $f(x)$ and −4 is less than

a. 0.08 units **b.** 0.008 units

15. Refer to Exercise 9. Determine an interval for x that would guarantee the distance between $f(x)$ and 9 is less than

a. 0.002 units **b.** 0.2 units

16. Refer to Exercise 10. Determine an interval for x that would guarantee the distance between $f(x)$ and −6 is less than

a. 0.1 units **b.** 0.01 units

For Exercises 17–26, use a table to estimate the limit. (See Examples 3–4, p. 12-7)

17. $\lim_{x \to -5} (2x + 1)$

18. $\lim_{x \to 3} (4x - 8)$

19. $\lim_{x \to 2} \frac{x^2 - 5}{4}$

20. $\lim_{x \to -6} \frac{3x + 4}{28}$

21. $\lim_{x \to 13} \frac{\sqrt{x + 3} - 4}{x - 13}$

22. $\lim_{x \to -3} \frac{\sqrt{1 - x} - 2}{x + 3}$

23. $\lim_{x \to 0} \frac{e^x + e^{-x}}{2}$

24. $\lim_{x \to 0} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

25. $\lim_{x \to 0} \frac{\sin x}{x}$

26. $\lim_{x \to 0} \frac{\cos x - 1}{x}$

Learn: Identify Limits that Fail to Exist

For Exercises 27–34, determine whether the limit exists. If the limit exists, estimate its value. (See Examples 5–7, pp. 12-8–12-10)

27. a. $\lim_{x \to -2} \frac{x}{x + 2}$ **b.** $\lim_{x \to -2} \frac{x + 2}{x}$

28. a. $\lim_{x \to 4} \frac{(x - 4)^2}{x^2}$ **b.** $\lim_{x \to 4} \frac{x^2}{(x - 4)^2}$

29. a. $\lim_{x \to 5} \frac{x - 5}{|x - 5|}$ **b.** $\lim_{x \to 5} \frac{|x - 5|}{x - 5}$

30. a. $\lim_{x \to 0} \frac{3x}{|2x|}$ **b.** $\lim_{x \to 0} \frac{|3x|}{|2x|}$

31. a. $\lim_{x \to \pi/2} \cot x$ **b.** $\lim_{x \to \pi/2} \tan x$

32. a. $\lim_{x \to \pi/2} \frac{1}{\sin x - 1}$ **b.** $\lim_{x \to \pi/2} \frac{1}{\cos x - 1}$

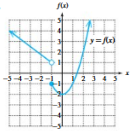
33. a. $\lim_{x \to 0} \left(-2\cos\frac{\pi}{x}\right)$ **b.** $\lim_{x \to 0} \left(-2\cos\frac{x}{\pi}\right)$

34. a. $\lim_{x \to 1} \left(\sin\frac{x - 1}{\pi}\right)$ **b.** $\lim_{x \to 1} \left(\sin\frac{\pi}{x - 1}\right)$

Learn: Estimate Limits Graphically

For Exercises 35–38, use the graph to estimate the value of the limit or state that the limit does not exist. (See Example 8, p. 12-10)

35.

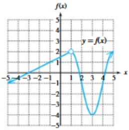


a. $\lim_{x \to 0} f(x)$

b. $\lim_{x \to -5} f(x)$

c. $\lim_{x \to -1} f(x)$

36.

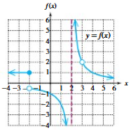


a. $\lim_{x \to 1} f(x)$

b. $\lim_{x \to -3} f(x)$

c. $\lim_{x \to 3} f(x)$

37.

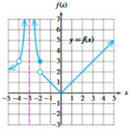


a. $\lim_{x \to 3} f(x)$

b. $\lim_{x \to -2} f(x)$

c. $\lim_{x \to 1} f(x)$

38.

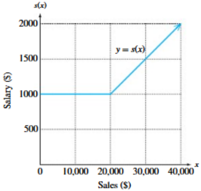


a. $\lim_{x \to -3} f(x)$

b. $\lim_{x \to -4} f(x)$

c. $\lim_{x \to -2} f(x)$

39. A pharmaceutical salesperson earns a weekly salary of \$1000. In addition, she earns a 5% commission on sales over \$20,000 during the week. Let $s(x)$ represent her weekly salary (in dollars), and let x be the number of dollars in sales during the week. Find the limits.

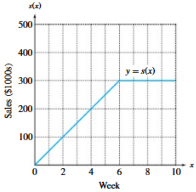


a. $\lim\limits_{x \to 10{,}000} s(x)$

b. $\lim\limits_{x \to 20{,}000} s(x)$

c. $\lim\limits_{x \to 30{,}000} s(x)$

40. A company has a rapid increase in sales for 6 consecutive weeks, followed by steady sales for the next 4 weeks. The graph shows the amount in sales $s(x)$ (in \$1000s) for week x. Find the limits.



a. $\lim\limits_{x \to 2} s(x)$

b. $\lim\limits_{x \to 6} s(x)$

c. $\lim\limits_{x \to 9} s(x)$

For Exercises 41–44, graph the function and estimate the given limits or state that a limit does not exist.

41. $f(x) = \begin{cases} 4 & \text{if } x \leq 1 \\ -1 & \text{if } x > 1 \end{cases}$

a. Graph the function.

b. $\lim\limits_{x \to -3} f(x)$

c. $\lim\limits_{x \to 1} f(x)$

d. $\lim\limits_{x \to 5} f(x)$

42. $f(x) = \begin{cases} 2x & \text{if } x \leq 1 \\ 3 & \text{if } x > 1 \end{cases}$

a. Graph the function.

b. $\lim\limits_{x \to -1} f(x)$

c. $\lim\limits_{x \to 5} f(x)$

d. $\lim\limits_{x \to 1} f(x)$

43. $f(x) = \begin{cases} 5 & \text{if } x < -2 \\ x^2 & \text{if } -2 \leq x \leq 2 \\ 4 & \text{if } x > 2 \end{cases}$

a. Graph the function.

b. $\lim\limits_{x \to -2} f(x)$

c. $\lim\limits_{x \to -1} f(x)$

d. $\lim\limits_{x \to 2} f(x)$

44. $f(x) = \begin{cases} -3 & \text{if } x \leq -3 \\ -|x| & \text{if } -3 < x \leq 3 \\ -4 & \text{if } x > 3 \end{cases}$

a. Graph the function.

b. $\lim\limits_{x \to -3} f(x)$

c. $\lim\limits_{x \to 3} f(x)$

d. $\lim\limits_{x \to 5} f(x)$

Mixed Exercises

45. Sketch a function for which $f(2) = 4$ and $\lim\limits_{x \to 2} f(x) = 3$.

46. Sketch a function for which $f(-3) = -2$ and $\lim\limits_{x \to -3} f(x) = 2$.

Write About It

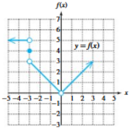
47. Explain why $\lim\limits_{x \to -10} \dfrac{3}{x + 10}$ does not exist.

48. Explain why $\lim\limits_{x \to 5} \dfrac{x - 5}{|x - 5|}$ does not exist.

Expanding Your Skills

For Exercises 49–50, use the graph to find each limit or function value. Otherwise, state that the limit does not exist or the function value is undefined.

49.



a. $\lim_{x\to -3} f(x)$

b. $f(-3)$

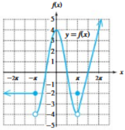
c. $\lim_{x\to 0} f(x)$

d. $f(0)$

e. $\lim_{x\to 4} f(x)$

f. $f(4)$

50.



a. $\lim_{x\to \pi} f(x)$

b. $f(\pi)$

c. $\lim_{x\to -\pi} f(x)$

d. $f(-\pi)$

e. $\lim_{x\to -3} f(x)$

f. $f(0)$

Technology Connections

For Exercises 51–54, use a graphing utility to estimate the value of each limit.

51. $\lim_{x\to 0}\frac{x}{\tan x}$

52. $\lim_{x\to 2}\frac{\ln(3-x)}{x-2}$

53. $\lim_{x\to 0}\frac{e^{2x}-1}{x}$

54. $\lim_{x\to 0}\frac{\sin x}{\tan 2x}$

55. a. Given $\lim_{x\to 0}\sin\frac{\pi}{x}$, recreate and complete the table.

x	$\sin\frac{\pi}{x}$
0.1	
0.01	
0.001	
−0.1	
−0.01	
−0.001	

b. Based on the result in the table, what limiting value might you suspect $\sin\frac{\pi}{x}$ approaches as x approaches 0?

c. Use a graphing utility to graph $f(x) = \sin\frac{\pi}{x}$ on a square viewing window.

d. From the graph, does it appear that $\lim_{x\to 0}\sin\frac{\pi}{x}$ exists?

e. Explain why the table suggests that the limit exists but the graph confirms that the limit does not exist.

56. a. Given $\lim_{x\to 0}\cos\frac{2\pi}{x}$,recreate and complete the table.

x	$\cos\frac{2\pi}{x}$
0.1	
0.01	
0.001	
−0.1	
−0.01	
−0.001	

b. Based on the result in the table, what limiting value might you suspect $\cos\frac{2\pi}{x}$ approaches as x approaches 0?

c. Use a graphing utility to graph $f(x) = \cos\frac{2\pi}{x}$ on a square viewing window.

d. From the graph, does it appear that $\lim_{x\to 0}\cos\frac{2\pi}{x}$ exists?

e. Explain why the table suggests that the limit exists but the graph confirms that the limit does not exist.

GO ONLINE **Go online for more practice problems.**

Lesson 12-2

Algebraic Properties of Limits

What Will You Learn?

After completing this lesson, you should be able to:

- Apply Properties of Limits
- Evaluate Limits Using Direct Substitution
- Apply Strategies to Evaluate Limits in the Case of 0/0
- Evaluate One-Sided Limits

Learn: Apply Properties of Limits

In the previous lesson we estimated the values of limits by using tables and graphs. However, this technique can sometimes lead to erroneous results (See Exercises 55 and 56 on page 12-27). Therefore, we will now study algebraic methods to find the value of a limit. We begin by presenting four basic limits.

Basic Limits

If b and c are real numbers and n is a positive integer, then

1. $\lim\limits_{x \to c} b = b$

2. $\lim\limits_{x \to c} x = c$

3. $\lim\limits_{x \to c} x^n = c^n$

4. $\lim\limits_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, provided that $\sqrt[n]{c}$ is defined.

The formal proofs of the four basic limits given above are left for a first-semester calculus course. However, these limits may be intuitively obvious. The graph of each function has no gaps or unbounded behavior.

For example, to illustrate basic limit (2) the graph of $f(x) = x$ is shown in Figure 12-11. From the graph, $\lim\limits_{x \to c} f(x) = f(c)$.

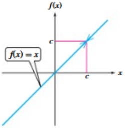


Figure 12-11

Likewise, the values of the other limits given in the *Basic Limits* box can be found by evaluating the function at $x = c$.

Example 1: Finding Limits by Using the Basic Limit Laws

Find the limits.

a. $\lim\limits_{x \to -2} 6$ **b.** $\lim\limits_{x \to \sqrt{3}} x$ **c.** $\lim\limits_{x \to 4} x^3$ **d.** $\lim\limits_{x \to 8} \sqrt[3]{x}$

Solution:

Using direct substitution we have the following results.

a. $\lim\limits_{x \to -2} 6 = 6$ **b.** $\lim\limits_{x \to \sqrt{3}} x = \sqrt{3}$

c. $\lim\limits_{x \to 4} x^3 = (4)^3 = 64$ **d.** $\lim\limits_{x \to 8} \sqrt[3]{x} = \sqrt[3]{8} = 2$

Apply the Skills

1. Find the limits.

a. $\lim\limits_{x \to 5} 2$ **b.** $\lim\limits_{x \to \pi} x$ **c.** $\lim\limits_{x \to -2} x^4$ **d.** $\lim\limits_{x \to 64} \sqrt{x}$

The limits provided in the *Basic Limits* box along with the following properties of limits enable us to evaluate limits of more complicated functions.

Properties of Limits

Suppose that b and c are real numbers and n is a positive integer. If $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$ exist, then

5. $\lim\limits_{x \to c} [b \cdot f(x)] = b \cdot \lim\limits_{x \to c} f(x)$	Limit of a scalar multiple
6. $\lim\limits_{x \to c} [f(x) \pm g(x)] = \lim\limits_{x \to c} f(x) \pm \lim\limits_{x \to c} g(x)$	Limit of a sum or difference
7. $\lim\limits_{x \to c} [f(x) \cdot g(x)] = \lim\limits_{x \to c} f(x) \cdot \lim\limits_{x \to c} g(x)$	Limit of a product
8. $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$ $\left[\text{provided that } \lim\limits_{x \to c} g(x) \neq 0\right]$	Limit of a quotient
9. $\lim\limits_{x \to c} [f(x)]^n = \left[\lim\limits_{x \to c} f(x)\right]^n$	Limit of a power where n is a positive integer
10. $\lim\limits_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to c} f(x)}$ $\left[\text{provided that } \lim\limits_{x \to c} f(x) > 0 \text{ if } n \text{ is even}\right]$	Limit of a root where n is a positive integer

These limit properties are quite intuitive. For example, consider property (6).

$$(6) \quad \lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

If $f(x)$ gets close to L and $g(x)$ gets close to M as x approaches c, then it seems reasonable that $f(x) + g(x)$ will get close to $L + M$.

Example 2: Finding Limits by Using the Properties of Limits

Find the limits.

a. $\lim\limits_{x\to1}(3x^2 - 2x + 4)$ **b.** $\lim\limits_{x\to4}\dfrac{2x-1}{x^3}$

Solution:

a. $\lim\limits_{x\to1}(3x^2 - 2x + 4) = \lim\limits_{x\to1}3x^2 + \lim\limits_{x\to1}(-2x) + \lim\limits_{x\to1}4$	Property (6). Write the original limit as the sum of limits.
$= 3\lim\limits_{x\to1}x^2 - 2\lim\limits_{x\to1}x + \lim\limits_{x\to1}4$	Property (5). Limit of a scalar multiple.
$= \left[3\lim\limits_{x\to1}x\right]^2 - 2\lim\limits_{x\to1}x + \lim\limits_{x\to1}4$ $= 3(1)^2 - 2(1) + 4$	Property (9). Limit of a power.
$= 5$	Simplify.
b. $\lim\limits_{x\to4}\dfrac{2x-1}{x^3} = \dfrac{\lim\limits_{x\to4}(2x-1)}{\lim\limits_{x\to4}x^3}$	Property (8). Limit of a quotient.
$= \dfrac{\lim\limits_{x\to4}2x - \lim\limits_{x\to4}1}{\lim\limits_{x\to4}x^3}$	Property (6). Limit of a difference.
$= \dfrac{2\lim\limits_{x\to4}x - \lim\limits_{x\to4}1}{\left[\lim\limits_{x\to4}x\right]^3}$	Property (5) in the numerator. Limit of a scalar multiple. Property (9) in the denominator. Limit of a power.
$= \dfrac{2(4)-1}{(4)^3} = \dfrac{7}{64}$	Simplify.

Apply the Skills

2. Find the limits.

a. $\lim\limits_{x\to-3}(2x^3 - 5x + 4)$ **b.** $\lim\limits_{x\to1}\dfrac{x^2-4}{2x+7}$

Learn: Evaluate Limits Using Direct Substitution

In Example 2(a) we found that $\lim\limits_{x\to1}(3x^2 - 2x + 4) = 5$. This means that $f(x) = 3x^2 - 2x + 4$ approaches 5 for x values taken close to 1. We might also note that $f(1) = 5$. That is, the limiting value of $f(x)$ as x gets close to 1 is the same as the value of f at 1. So, $\lim\limits_{x\to1}(3x^2 - 2x + 4)$ could have been found by using direct substitution: $3(1)^2 - 2(1) + 4 = 5$. Similarly, in Example 2(b), given $f(x) = \dfrac{2x-1}{x^3}$, the value of $\lim\limits_{x\to4}\dfrac{2x-1}{x^3}$ is the same as f evaluated at 4.

$$\lim_{x\to4}\frac{2x-1}{x^3} = \frac{7}{64} \quad \text{and} \quad f(4) = \frac{2(4)-1}{(4)^3} = \frac{7}{64}$$

The functions in Example 2 are polynomial and rational functions. By using tools in a calculus course, we can prove the following property.

Limit Properties of Polynomial and Rational Functions

If f is a polynomial or rational function, and c is in the domain of f then,

$$\lim_{x \to c} f(x) = f(c)$$

This means that the limit of $f(x)$ as x approaches c is the same as the value of f at c. So, direct substitution can be used to find the value of the limit.

There are other functions for which direct substitution can be used to find the value of a limit. For example:

$$\lim_{x \to c} \sin x = \sin c \quad \text{and} \quad \lim_{x \to c} \cos x = \cos c$$

Intuitively, these statements seem reasonable because the graphs of the sine and cosine functions have neither gaps nor unbounded behavior.

Example 3: Finding Limits by Using Direct Substitution

Find the limits.

a. $\lim_{x \to 3} \frac{x^2 - x - 12}{x + 1}$ **b.** $\lim_{x \to -1} (x^2 + 4x + 4)$ **c.** $\lim_{x \to \pi} x \cos x$

Solution:

a. $\lim_{x \to 3} \frac{x^2 - x - 12}{x + 1} = \frac{(3)^2 - (3) - 12}{(3) + 1} = \frac{-6}{4} = -\frac{3}{2}$ Since 3 is in the domain of the rational function, direct substitution can be used to find the limit.

b. $\lim_{x \to -1} (x^2 + 4x + 4) = (-1)^2 + 4(-1) + 4 = 1$ This is a limit of a polynomial function. Therefore, direct substitution can be used.

c. $\lim_{x \to \pi} x \cos x = \lim_{x \to \pi} x \cdot \lim_{x \to \pi} \cos x$ Use property (7): $\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

$= (\pi) \cdot \cos(\pi)$ Apply direct substitution.

$= \pi \cdot (-1)$ Simplify.

$= -\pi$

Apply the Skills

3. Find the limits.

a. $\lim_{x \to -2} (2x^4 - 5x^2 + 1)$ **b.** $\lim_{x \to 3} \frac{-5x - 6}{2x^2 + 4}$ **c.** $\lim_{x \to \pi/2} \frac{\cos x}{x}$

Learn: Apply Strategies to Evaluate Limits in the Case of 0/0

In Example 3, we evaluated limits by using direct substitution. However, in most applications involving limits, this technique cannot be used. We often encounter limits in which direct substitution would produce a result of $\frac{0}{0}$. This is called an **indeterminate form.** This means that the limit cannot be determined directly. In fact, $\frac{0}{0}$ is not the only type of indeterminate form. Other types are $\frac{\infty}{\infty}$, $0 \cdot \infty$, 1^∞, ∞^0, 0^0, and $\infty - \infty$. In these cases, we use algebraic manipulation and the properties of limits to find the limit.

Example 4: Finding a Limit by Dividing Out a Common Factor

Find the limit. $\lim\limits_{x \to 4} \frac{x^2 - x - 12}{x - 4}$

Solution:

$f(x) = \frac{x^2 - x - 12}{x - 4}$ is not defined at $x = 4$, so property (8) cannot be applied to find $\lim\limits_{x \to 4} \frac{x^2 - x - 12}{x - 4}$. Furthermore, direct substitution would result in the indeterminate form $\frac{0}{0}$. However, notice that the numerator and denominator share a common factor of $x - 4$.

$$\frac{x^2 - x - 12}{x - 4} = \frac{\cancel{(x - 4)}(x + 3)}{\cancel{x - 4}} = x + 3 \quad \text{for} \quad x \neq 4$$

Since we are interested in the behavior of f *near* 4, but not at 4, we know that $x - 4 \neq 0$. So, we can divide out the common factor of $x - 4$ before taking the limit.

$$\lim_{x \to 4} \frac{x^2 - x - 12}{x - 4} = \lim_{x \to 4} \frac{\cancel{(x - 4)}(x + 3)}{\cancel{x - 4}} = \lim_{x \to 4}(x + 3) = (4) + 3 = 7$$

Check Your Work

Be sure to keep the limit notation until the last step when direct substitution is used.

Apply the Skills

4. Find the limit. $\lim\limits_{x \to -5} \frac{x^2 + 3x - 10}{x + 5}$

In Example 4, the functions $f(x) = \frac{x^2 - x - 12}{x - 4}$ and $g(x) = x + 3$ are equal at all values of x *except* at $x = 4$ (Figures 12-12 and 12-13).

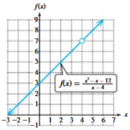


Figure 12-12

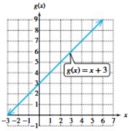


Figure 12-13

Since $\lim\limits_{x\to 4}\frac{x^2-x-12}{x-4}$ defines the behavior of f *close* to 4, but not at 4, we have that $\lim\limits_{x\to 4}\frac{x^2-x-12}{x-4} = \lim\limits_{x\to 4}(x+3)$. The simpler limit, $\lim\limits_{x\to 4}(x+3)$, can be found by using direct substitution. The principle illustrated in Example 4 can be generalized as follows.

Limits of Functions that Agree at All Values Except *c*

If $f(x) = g(x)$ for $x \neq c$ on some open interval containing c, and if $\lim\limits_{x\to c} g(x)$ exists, then

$$\lim_{x\to c} f(x) \text{ exists and } \lim_{x\to c} f(x) = \lim_{x\to c} g(x)$$

Example 5: Finding a Limit by Rationalizing the Numerator

Find the limit. $\lim\limits_{x\to 0}\frac{\sqrt{x+4}-2}{x}$

Solution:

$f(x) = \frac{\sqrt{x+4}-2}{x}$ is not defined at $x = 0$, so property (8) cannot be applied to find $\lim\limits_{x\to 0}\frac{\sqrt{x+4}-2}{x}$. Furthermore, direct substitution would result in the indeterminate form $\frac{0}{0}$. However, we can manipulate the expression $\frac{\sqrt{x+4}-2}{x}$ into a more manageable form by rationalizing the numerator.

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x} = \lim_{x\to 0}\frac{(\sqrt{x+4}-2)}{x}\cdot\frac{(\sqrt{x+4}+2)}{(\sqrt{x+4}+2)} = \lim_{x\to 0}\frac{x+4-4}{x\cdot(\sqrt{x+4}+2)}$$

$= \lim\limits_{x\to 0}\frac{x}{x\cdot(\sqrt{x+4}+2)}$ Simplify the numerator.

$= \lim\limits_{x\to 0}\frac{\overset{1}{x}}{x\cdot(\sqrt{x+4}+2)}$ Since we are interested in the behavior of f *near* 0, but not at 0, $x \neq 0$. Consequently, we can divide out the common factor of x before taking the limit.

$= \lim\limits_{x\to 0}\frac{1}{\sqrt{x+4}+2}$

Check Your Work

Be sure to keep the limit notation until the step where direct substitution is used.

$= \frac{\lim\limits_{x\to 0}1}{\sqrt{\lim\limits_{x\to 0}(x+4)}+\lim\limits_{x\to 0}2}$ Apply the properties of limits.

$= \frac{1}{\sqrt{0+4}+2}$ Simplify.

$= \frac{1}{4}$

Apply the Skills

5. Find the limit. $\lim\limits_{x\to 0}\frac{\sqrt{9-x}-3}{x}$

To reinforce the results of Example 5, we can construct a table of values of $f(x) = \frac{\sqrt{x+4} - 2}{x}$ for x close to zero (Table 12-6). From the table, $f(x)$ appears to get close to 0.25 or equivalently $\frac{1}{4}$ as x approaches 0.

TABLE 12-6 Values of $f(x)$ as Values of x Approach 0

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$	0.25158	0.25016	0.25002		0.24998	0.24984	0.24846

Example 6: Finding a Limit Involving a Complex Fraction

Find the limit. $\lim\limits_{x\to 0} \frac{\frac{1}{x+5} - \frac{1}{5}}{x}$

$f(x) = \frac{\frac{1}{x+5} - \frac{1}{5}}{x}$ is not defined at $x = 0$, so property (8) cannot be applied to find $\lim\limits_{x\to 0} \frac{\frac{1}{x+5} - \frac{1}{5}}{x}$. Furthermore, direct substitution would result in the indeterminate form $\frac{0}{0}$. However, we can manipulate the expression $\frac{\frac{1}{x+5} - \frac{1}{5}}{x}$ into a more manageable form by simplifying the complex fraction.

$$\lim_{x\to 0} \frac{\frac{1}{x+5} - \frac{1}{5}}{x} = \lim_{x\to 0} \frac{\left(\frac{1}{x+5} - \frac{1}{5}\right)}{x} \cdot \frac{5(x+5)}{5(x+5)} = \lim_{x\to 0} \frac{5 - (x+5)}{5x(x+5)}$$

$= \lim\limits_{x\to 0} \frac{-x}{5x(x+5)}$ Simplify the numerator.

$= \lim\limits_{x\to 0} \frac{-\overset{1}{x}}{5x(x+5)}$ Since we are interested in the behavior of f *near* 0, but not at 0, $x \neq 0$. Consequently we can divide out the common factor of x before taking the limit.

$= \lim\limits_{x\to 0} \frac{-1}{5(x+5)}$ Simplify.

Check Your Work

Be sure to keep the limit notation until the step where direct substitution is used.

$= \frac{-1}{5(0+5)}$ The limit of a rational function can be found by using direct substitution.

$= -\frac{1}{25}$ Simplify.

Apply the Skills

6. Find the limit. $\lim\limits_{x\to 0} \frac{\frac{1}{x+2} - \frac{1}{2}}{x}$

Learn: Evaluate One-Sided Limits

One condition in which $\lim_{x \to c} f(x)$ fails to exist is when $f(x)$ approaches one value from the left of c and a different value to the right of c.

For example, consider the graph of $f(x) = \begin{cases} x+1 & \text{for} \quad x < 2 \\ x+2 & \text{for} \quad x > 2 \end{cases}$ (Figure 12-14).

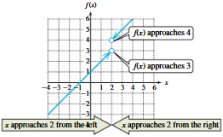


Figure 12-14

From Figure 12-14, as x approaches 2 from the left, $f(x)$ approaches 3. In contrast, as x approaches 2 from the right, $f(x)$ approaches 4.

We can denote these statements by using notation for one-sided limits.

$$\lim_{x \to 2^-} f(x) = 3 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 4$$

Recall that the negative sign to the right of 2 in the notation $x \to 2^-$ indicates that x approaches 2 from the left (that is, from the negative direction). The statement $x \to 2^+$ indicates that x approaches 2 from the right (that is, from the positive direction).

One-Sided Limits

1. $\lim_{x \to c^-} f(x) = L$
   - The statement $\lim_{x \to c^-} f(x) = L$ is read as "the limit of $f(x)$ as x approaches c from the left equals L."
   - The statement $\lim_{x \to c^-} f(x) = L$ means that $f(x)$ can be made arbitrarily close to L provided that x is taken sufficiently close to c but less than c.
2. $\lim_{x \to c^+} f(x) = M$
   - The statement $\lim_{x \to c^+} f(x) = M$ is read as "the limit of $f(x)$ as x approaches c from the right equals M."
   - The statement $\lim_{x \to c^+} f(x) = M$ means that $f(x)$ can be made arbitrarily close to M provided that x is taken sufficiently close to c but greater than c.

In Figure 12-15, we have a function for which the one-sided limits exist for x taken to the left and right of 2. That is, $\lim_{x \to 2^-} f(x) = 3$ and $\lim_{x \to 2^+} f(x) = 4$. However, it is important to note that the two-sided limit $\lim_{x \to 2} f(x)$ does not exist because the limits from the left and right are not equal.

$\lim_{x \to 2} f(x)$ does not exist because $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$.

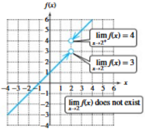


Figure 12-15

Example 7: Finding One-Sided Limits of a Piecewise-Defined Function

Given $f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ -x + 5 & \text{for } x \geq 1 \end{cases}$ find

a. $\lim\limits_{x \to 1^-} f(x)$ **b.** $\lim\limits_{x \to 1^+} f(x)$ **c.** $\lim\limits_{x \to 1} f(x)$

Solution:

a. $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 = (1)^2 = 1$

For $x \to 1^-$ we take $x < 1$. Therefore, use the rule $f(x) = x^2$.

b. $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+}(-x + 5) = -(1) + 5 = 4$

For $x \to 1^+$ we take $x > 1$. Therefore, use the rule $f(x) = -x + 5$.

c. $\lim\limits_{x \to 1} f(x)$ does not exist because $\lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$.

The two-sided limit does not exist because the limit from the left does not equal the limit from the right.

Apply the Skills

7. Given $f(x) = \begin{cases} 5 - x & \text{for } x < 0 \\ \sqrt{x} & \text{for } x \geq 0 \end{cases}$ find

a. $\lim\limits_{x \to 0^-} f(x)$ **b.** $\lim\limits_{x \to 0^+} f(x)$ **c.** $\lim\limits_{x \to 0} f(x)$

The results of Example 7 can be confirmed from the graph of $f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ -x + 5 & \text{for } x \geq 1 \end{cases}$ shown in Figure 12-16.

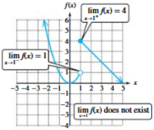


Figure 12-16

Example 8: Evaluating One-Sided Limits

Find the limits.

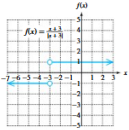
a. $\lim\limits_{x \to -3^-} \frac{x+3}{|x+3|}$ **b.** $\lim\limits_{x \to -3^+} \frac{x+3}{|x+3|}$ **c.** $\lim\limits_{x \to -3} \frac{x+3}{|x+3|}$

Solution:

First note that $f(x) = \frac{x+3}{|x+3|}$ can be written as

$$f(x) = \begin{cases} \frac{x+3}{-(x+3)} & \text{for } x < -3 \\ \frac{x+3}{x+3} & \text{for } x > -3 \end{cases}$$ which simplifies to

$$f(x) = \begin{cases} -1 & \text{for } x < -3 \\ 1 & \text{for } x > -3 \end{cases}$$



a. $\lim\limits_{x \to 3^-} \frac{x+3}{|x+3|} = \lim\limits_{x \to -3^-} (-1) = -1$

b. $\lim\limits_{x \to -3^+} \frac{x+3}{|x+3|} = \lim\limits_{x \to -3^+} 1 = 1$

c. $\lim\limits_{x \to -3} \frac{x+3}{|x+3|}$ does not exist because $\lim\limits_{x \to -3^-} \frac{x+3}{|x+3|} \neq \lim\limits_{x \to -3^+} \frac{x+3}{|x+3|}$.

Apply the Skills

8. Find the limits.

a. $\lim\limits_{x \to 9^-} \frac{|9-x|}{9-x}$ **b.** $\lim\limits_{x \to 9^+} \frac{|9-x|}{9-x}$ **c.** $\lim\limits_{x \to 9} \frac{|9-x|}{9-x}$

Practice 12-2

Practice Exercises

Prerequisite Review

R.1. Factor by grouping. $10x^3 - 15x^2 - 8x + 12$

R.2. Rationalize the denominator. $\dfrac{25}{\sqrt{15} - \sqrt{10}}$

R.3. Evaluate the function for the given values of x.

$$g(x) = \begin{cases} -3x + 7 & \text{for } x < -1 \\ x^2 + 3 & \text{for } -1 \le x < 5 \\ 6 & \text{for } x \ge 5 \end{cases}$$

a. $g(2)$ **b.** $g(-2)$ **c.** $g(-1)$

d. $g(5)$ **e.** $g(6)$

Concept Connections

1. For real numbers b and c, $\lim\limits_{x \to c} b =$ ________ and $\lim\limits_{x \to c} x =$ ________.

2. For a real number c and positive integer n, $\lim\limits_{x \to c} x^n =$ ________ and $\lim\limits_{x \to c} \sqrt[n]{x} =$ ________.

3. The statement $x \to 4^+$ is read as "________."

4. The statement $x \to 3^-$ is read as "________."

5. Suppose $f(x) = g(x)$ for $x \neq c$ on some open interval containing c. If $\lim\limits_{x \to c} g(x) = L$, then $\lim\limits_{x \to c} f(x) =$ ________.

6. The expression $\frac{0}{0}$ is called an ________ form.

7. If $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = M$, what is the value of $\lim\limits_{x \to c} [f(x) + g(x)]$?

8. Use limit notation to represent the statement "the limit of $f(x)$ as x gets close to -4 from the right."

Learn: Apply Properties of Limits

For Exercises 9–18, find the limit. (See Examples 1–2, pp. 12-17–12-18)

9. $\lim\limits_{x \to 2} \pi$ **10.** $\lim\limits_{x \to 2} e$

11. $\lim\limits_{x \to -8} x^2$ **12.** $\lim\limits_{x \to 8} \sqrt{x}$

13. $\lim\limits_{x \to 3} (x^2 - 5x + 6)$ **14.** $\lim\limits_{x \to -2} (x^3 - 27)$

15. $\lim\limits_{x \to 0} \dfrac{2x + 1}{x^2 - 3}$ **16.** $\lim\limits_{x \to 15} \dfrac{3x^2}{x + 10}$

17. $\lim\limits_{x \to 0} \sqrt{\cos x + 15}$ **18.** $\lim\limits_{x \to \pi/2} \sqrt[3]{9 - \sin x}$

Learn: Evaluate Limits Using Direct Substitution

For Exercises 19–28, find the limit. (See Example 3, p. 12-19)

19. $\lim\limits_{x \to -2} (8x^2 - 3x + 1)$ **20.** $\lim\limits_{x \to -5} (2x^2 + 6x - 14)$

21. $\lim\limits_{x \to 6} \dfrac{x^2 + 6}{x + 6}$ **22.** $\lim\limits_{x \to 10} \dfrac{10 - x^2}{20 - x}$

23. $\lim\limits_{x \to -3} \sqrt[3]{8 - x^2}$ **24.** $\lim\limits_{x \to 12} \sqrt{x^2 + 3}$

25. $\lim\limits_{x \to \frac{2\pi}{3}} x \sec x$ **26.** $\lim\limits_{x \to \frac{\pi}{4}} x \tan x$

27. $\lim\limits_{x \to 1} \dfrac{e^{x-1}}{x}$ **28.** $\lim\limits_{x \to 1} \dfrac{\ln x}{x}$

Learn: Apply Strategies to Evaluate Limits in the Case of 0/0

For Exercises 29–42, find the limit. (See Examples 4–6, pp. 12-20–12-22)

29. $\lim\limits_{x \to 5} \dfrac{x^2 + x - 30}{x - 5}$ **30.** $\lim\limits_{x \to 8} \dfrac{x - 8}{3x^2 - 26x + 16}$

31. $\lim\limits_{x \to 2} \dfrac{x - 2}{x^3 - 8}$ **32.** $\lim\limits_{x \to -3} \dfrac{x^3 + 27}{x + 3}$

33. $\lim\limits_{x \to -3} \dfrac{x^3 - 3x^2 - 9x + 27}{x + 3}$

34. $\lim\limits_{x \to 4} \dfrac{x^3 - 4x^2 + 25x - 100}{x - 4}$

35. $\lim\limits_{x \to 14} \dfrac{\sqrt{x + 2} - 4}{x - 14}$ **36.** $\lim\limits_{x \to 9} \dfrac{\sqrt{x - 5} - 2}{x - 9}$

37. $\lim\limits_{x \to 5} \dfrac{x - 5}{\sqrt{x} - \sqrt{5}}$ **38.** $\lim\limits_{x \to 2} \dfrac{\sqrt{x} - \sqrt{2}}{x - 2}$

39. $\lim\limits_{x \to 3} \dfrac{\frac{1}{x} - \frac{1}{3}}{3 - x}$ **40.** $\lim\limits_{x \to 5} \dfrac{\frac{1}{5} - \frac{1}{x}}{x - 5}$

41. $\lim\limits_{x \to 2} \dfrac{\frac{1}{8} - \frac{1}{x + 6}}{x - 2}$ **42.** $\lim\limits_{x \to 0} \dfrac{\frac{1}{x - 4} + \frac{1}{4}}{x}$

Learn: Evaluate One-Sided Limits

For Exercises 43–48, given the piecewise-defined function, find the limits. (See Example 7, p. 12-24)

43. $f(x) = \begin{cases} 2 - x & \text{for } x \le -2 \\ x + 2 & \text{for } x > -2 \end{cases}$

a. $\lim\limits_{x \to -2^-} f(x)$ **b.** $\lim\limits_{x \to -2^+} f(x)$ **c.** $\lim\limits_{x \to -2} f(x)$

44. $f(x) = \begin{cases} x+2 & \text{for } x<1 \\ 2x-1 & \text{for } x \geq 1 \end{cases}$

a. $\lim\limits_{x \to 1^-} f(x)$ **b.** $\lim\limits_{x \to 1^+} f(x)$ **c.** $\lim\limits_{x \to 1} f(x)$

45. $f(x) = \begin{cases} -\frac{1}{2}x - \frac{1}{2} & \text{for } x < -1 \\ x^2 - 2x - 3 & \text{for } x \geq -1 \end{cases}$

a. $\lim\limits_{x \to -1^-} f(x)$ **b.** $\lim\limits_{x \to -1^+} f(x)$ **c.** $\lim\limits_{x \to -1} f(x)$

46. $f(x) = \begin{cases} 2x^2 - 10 & \text{for } x \leq 5 \\ 2x + 30 & \text{for } x > 5 \end{cases}$

a. $\lim\limits_{x \to 5^-} f(x)$ **b.** $\lim\limits_{x \to 5^+} f(x)$ **c.** $\lim\limits_{x \to 5} f(x)$

47. $f(x) = \begin{cases} \sin x & \text{for } x < 0 \\ 2x & \text{for } 0 \leq x \leq 3 \\ 9 + x & \text{for } x > 3 \end{cases}$

a. $\lim\limits_{x \to 0^-} f(x)$ **b.** $\lim\limits_{x \to 0^+} f(x)$ **c.** $\lim\limits_{x \to 0} f(x)$

d. $\lim\limits_{x \to 3^-} f(x)$ **e.** $\lim\limits_{x \to 3^+} f(x)$ **f.** $\lim\limits_{x \to 3} f(x)$

48. $f(x) = \begin{cases} 1 & \text{for } x < -1 \\ -x^2 & \text{for } -1 \leq x < 1 \\ \ln x & \text{for } x \geq 1 \end{cases}$

a. $\lim\limits_{x \to -1^-} f(x)$ **b.** $\lim\limits_{x \to -1^+} f(x)$ **c.** $\lim\limits_{x \to -1} f(x)$

d. $\lim\limits_{x \to 1^-} f(x)$ **e.** $\lim\limits_{x \to 1^+} f(x)$ **f.** $\lim\limits_{x \to 1} f(x)$

For Exercises 49–52, find the limits. (See Example 8, p. 12-25)

49. a. $\lim\limits_{x \to 5^-} \frac{5-x}{|x-5|}$ **b.** $\lim\limits_{x \to 5^+} \frac{5-x}{|x-5|}$

c. $\lim\limits_{x \to 5} \frac{5-x}{|x-5|}$

50. a. $\lim\limits_{x \to 7^-} \frac{x-7}{|7-x|}$ **b.** $\lim\limits_{x \to 7^+} \frac{x-7}{|7-x|}$

c. $\lim\limits_{x \to 7} \frac{x-7}{|7-x|}$

51. a. $\lim\limits_{x \to 1/2^-} \frac{4x-2}{|2x-1|}$ **b.** $\lim\limits_{x \to 1/2^+} \frac{4x-2}{|2x-1|}$

c. $\lim\limits_{x \to 1/2} \frac{4x-2}{|2x-1|}$

52. a. $\lim\limits_{x \to -4^-} \frac{|x+4|}{-3x-12}$ **b.** $\lim\limits_{x \to -4^+} \frac{|x+4|}{-3x-12}$

c. $\lim\limits_{x \to -4} \frac{|x+4|}{-3x-12}$

Mixed Exercises

For Exercises 53–56, find the limit.

53. a. $\lim\limits_{x \to 1} \frac{x^2 + 2x - 3}{x - 1}$ **b.** $\lim\limits_{x \to 0} \frac{e^{2x} + 2e^x - 3}{e^x - 1}$

54. a. $\lim\limits_{x \to 0} \frac{x^2 - x}{x}$ **b.** $\lim\limits_{x \to \pi} \frac{\sin^2 x - \sin x \cos x}{\sin x}$

55. a. $\lim\limits_{x \to 1} \frac{x^2 - x}{x - 1}$ **b.** $\lim\limits_{x \to 0} \frac{\sin x - \sin x \cos x}{1 - \cos x}$

56. a. $\lim\limits_{x \to 1/3} \frac{9x^2 - 1}{3x - 1}$ **b.** $\lim\limits_{x \to 0} \frac{9e^{2x} - 1}{3e^x - 1}$

57. The **signum** function (Latin for *sign*) is defined as equal to –1 for all real numbers less than 0, equal to 0 when x is 0, and equal to 1 for all real numbers greater than 0.

a. Write a piecewise-defined function for the signum function, sgn(x).

Evaluate the limits.

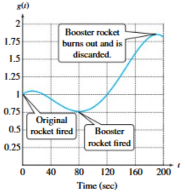
b. $\lim\limits_{x \to 0^-} \operatorname{sgn}(x)$ **c.** $\lim\limits_{x \to 0^+} \operatorname{sgn}(x)$ **d.** $\lim\limits_{x \to 0} \operatorname{sgn}(x)$

58. A salesperson makes a base monthly salary of \$2500. In addition, she makes a 5% commission on the monthly amount in sales, x.

a. Write a function defining the monthly salary $s(x)$ (in dollars) in terms of her monthly sales x (in dollars).

b. Find $\lim\limits_{x \to 10{,}000} s(x)$.

59. A rocket carrying a communications satellite into low Earth orbit is launched from Port Canaveral, Florida. A second booster rocket is fired 80 sec after launch. At approximately 185 sec into the launch, the booster rocket burns all its fuel and is discarded to fall back to Earth over the ocean. The graph shows the number of G-forces, $g(t)$, experienced by the rocket, t sec into the flight. The value $g(t)$ is the acceleration in 9.8-m/sec² increments.



Booster rocket burns out and is discarded.

Original rocket fired

Booster rocket fired

a. Estimate the limiting value of the G-force as the rocket approaches the time at which the booster rocket is fired.

b. Estimate the limiting value of the G-force as the rocket approaches the time at which the booster rocket is burned out and discarded.

Write About It

60. What algebraic technique would be a good first step to rewrite the expression in a more manageable form from which to find the limit? $\lim\limits_{x \to 0} \frac{\sqrt{25 - x} - 5}{x}$

61. If $\lim\limits_{x \to 1^-} f(x) = 5$ and $\lim\limits_{x \to 1^+} f(x) = 5$, what can you conclude about $\lim\limits_{x \to 1} f(x)$? From this information, can you make any conclusions about $f(1)$?

Expanding Your Skills

For Exercises 62–66, find the limit. Use the following limits that will be proved in calculus:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

62. $\lim\limits_{x \to 0} \frac{\cos^2 x - 1}{x}$

63. $\lim\limits_{x \to 0} \frac{\tan x}{x}$

64. $\lim\limits_{x \to 0} \frac{\cos x \sin x + \cos x - \sin x - 1}{x}$

65. $\lim\limits_{x \to 0} \frac{\cos x \tan x}{x}$

66. $\lim\limits_{x \to 0} \frac{\sin x - 2\sin x \cos x}{x}$

67. In the opening of this chapter, we discussed Zeno's paradox that asks whether reducing a distance repeatedly by half results in a distance of zero? Here is another paradox to consider. Suppose that a speedy hare runs 10 times faster than a tortoise. The tortoise and the hare run a race, but the tortoise has a 100-yd head start.

- **a.** The race starts and the hare runs 100 yd. How far has the tortoise run during this time? How far ahead is the tortoise?
- **b.** If the hare runs 10 yd farther, how far has the tortoise run during this time? How far ahead is the tortoise?
- **c.** If the hare runs 1 yd farther, how far has the tortoise run during this time? How far ahead is the tortoise?
- **d.** From parts (a)–(c), the tortoise is always ahead of the hare. Yet with the concept of a limit, we can show that the hare will indeed catch up to the tortoise. Find the total distance that the hare runs when it catches up to the tortoise. (*Hint*: Find the value of the geometric series: $100 + 10 + 1 + 0.1 + \cdots$)

Technology Connections

68. $\lim\limits_{x \to \pi} \frac{\tan^2 x}{\sec x + 1}$

- **a.** Use a graphing utility to graph the function $y = \frac{\tan^2 x}{\sec x + 1}$ in the ZTRIG window. Use the graph to approximate the value of the limit as x approaches π.
- **b.** Multiply numerator and denominator of the expression by the conjugate of the denominator and simplify the result by using a Pythagorean identity. Then use direct substitution to confirm the result of part (a).

69. $\lim\limits_{x \to \pi/2} \frac{\csc x - 1}{\cot^2 x}$

- **a.** Use a graphing utility to graph the function $y = \frac{\csc x - 1}{\cot^2 x}$ in the ZTRIG window. Use the graph to approximate the value of the limit as x approaches $\frac{\pi}{2}$.
- **b.** Multiply numerator and denominator of the expression by the conjugate of the numerator and simplify the result by using a Pythagorean identity. Then use direct substitution to confirm the result of part (a).

Problem Recognition Exercises

In this section, we present exercises that connect the concept of a limit to the concept of the *continuity* of a function. First we need to define what continuity means.

Limits and Continuity: Determine If a Function Is Continuous at *c*

We have already learned that the value of $f(c)$ has no effect on the existence or value of $\lim_{x \to c} f(x)$. However, in some cases, such as with a polynomial function, $\lim_{x \to c} f(x) = f(c)$. In such a case, we say that the function is *continuous* at c.

Continuity of a Function at *c*

A function f is **continuous** at c if the following conditions are met.

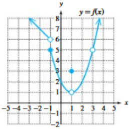
1. f is defined at c. This means that c is in the domain of f and that $f(c)$ is a real number.
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

If f fails any of these conditions, then we say that f is **discontinuous** at c.

Example 1: Determining If a Function Is Continuous at *c*

From the graph of $y = f(x)$, determine if f is continuous or discontinuous at c.

a. $c = -1$ **b.** $c = 0$ **c.** $c = 1$ **d.** $c = 3$



Solution:

a. f is defined at -1: $f(-1) = 5$. Condition 1 is met.

$\lim_{x \to -1} f(x)$ does not exist. Condition 2 fails.

Since $\lim_{x \to -1} f(x)$ does not exist, f is discontinuous at -1.

b. f is defined at 0: $f(0) = 2$. Condition 1 is met.

$\lim_{x \to 0} f(x) = 2$ Condition 2 is met.

$\lim_{x \to 0} f(x) = f(0)$ Condition 3 is met.

Therefore, f is continuous at 0.

c. f is defined at 1: $f(1) = 3$. Condition 1 is met.

$\lim_{x \to 1} f(x) = 1$ Condition 2 is met.

$\lim_{x \to 1} f(x) \neq f(1)$ Condition 3 fails.

Since $\lim_{x \to 1} f(x) \neq f(1)$, f is discontinuous at 1.

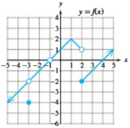
d. f is undefined at 3. Condition 1 fails.

Therefore, f is discontinuous at 3.

Apply the Skills

1. From the graph of $y = f(x)$, determine if f is continuous or discontinuous at c.

a. $c = -3$ **b.** $c = -1$ **c.** $c = 1$ **d.** $c = 2$



Limits and Continuity: Determine Values at Which a Function is Discontinuous

As Example 1 illustrates, a function is discontinuous at c if it has a “break” or “interruption” in the graph. A function is continuous at c if there is no “break” or “interruption” in the graph at c (Figure 12-17).

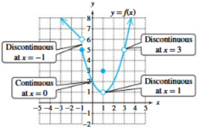


Figure 12-17

Many of the functions we have studied so far are continuous at every value in their domains. When we first introduced functions, we informally described a function as being continuous over an interval if we could draw its graph without lifting the pencil from the paper.

Some examples of continuous functions are polynomial functions, exponential functions, and the sine and cosine functions. Rational functions and the tangent, cotangent, secant, and cosecant functions are continuous at all values except those for which they are not defined (where they have vertical asymptotes). In applications, it is often important to know where a function is discontinuous.

Example 2: Determining Where a Function Is Discontinuous

Determine the value(s) of x (if any) for which the function is discontinuous.

a. $f(x) = \dfrac{4x+1}{x^2 - 2x - 35}$ **b.** $g(x) = \cot x$ **c.** $h(x) = \dfrac{10x}{x^2+4}$

Solution:

a. $f(x) = \dfrac{4x+1}{x^2 - 2x - 35} = \dfrac{4x+1}{(x+5)(x-7)}$ is undefined where the denominator is zero. So, f is undefined at -5 and 7. Since f is a rational function, it is continuous at all real numbers except those where it is undefined.

Therefore, f is discontinuous at $x = 7$ and $x = -5$.

b. $g(x) = \cot x = \dfrac{\cos x}{\sin x}$ is undefined where $\sin x = 0$ which is at all multiples of π.

Therefore, g is discontinuous at $x = n\pi$ for all integers n.

c. Given $h(x) = \dfrac{10x}{x^2+4}$, the denominator is positive for all real numbers x. Since the denominator is never zero, h is defined for all real numbers.

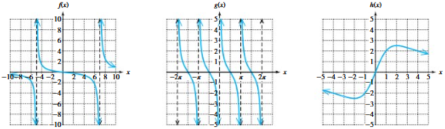
Furthermore, since h is a rational function defined for all real numbers, h is continuous for all real numbers.

Apply the Skills

2. Determine the value(s) of x (if any) for which the function is discontinuous.

a. $f(x) = \dfrac{x-5}{x^2+2x}$ **b.** $g(x) = \sec x$ **c.** $h(x) = \dfrac{x-3}{x^4+1}$

The graphs of the functions from Example 2 are shown in Figure 12-18. Notice that the rational function defined by $f(x) = \dfrac{4x+1}{x^2-2x-35}$ and the cotangent function are not continuous at the values of x at which the graphs have vertical asymptotes.



(a) $f(x) = \dfrac{4x+1}{x^2-2x-35}$

Discontinuous at −5 and 7

(b) $g(x) = \cot x$

Discontinuous at $n\pi$

(c) $h(x) = \dfrac{10x}{x^2+4}$

Continuous for all real numbers

Figure 12-18

The condition for continuity $\lim\limits_{x\to c} f(x) = f(c)$ is equivalent to saying that $\lim\limits_{x\to c^-} f(x) = f(c)$ and $\lim\limits_{x\to c^+} f(x) = f(c)$. We use this fact in Example 3.

Example 3: Determining Where a Function Is Discontinuous

Determine the value(s) of x (if any) for which the function is discontinuous.

$$f(x) = \begin{cases} 4 & \text{if } x < -2 \\ -x & \text{if } -2 \le x \le 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$

Solution:

The functions defined by $y = 4$, $y = -x$, and $y = \sqrt{x}$ are all continuous on their individual domains. However, we must check for continuity at the values of x where the transitions between "pieces" of the function are made. That is, we must check for continuity at $x = -2$ and $x = 0$.

Test for continuity at −2:

1. Determine whether f is defined at −2.	At $x = -2$, use the second rule $f(x) = -x$.
$f(-2) = -(-2) = 2$	Condition 1 is met.
2. Determine whether $\lim_{x \to -2} f(x)$ exists.	Check that the limits from the left and right of −2 exist and are equal.
$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} 4 = 4$	To find the limit from the left of −2 (where x is near −2, but slightly *less than* −2), use the first rule, $f(x) = 4$.
$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (-x) = -(-2) = 2$	To find the limit from the right of −2 (where x is near −2, but slightly *greater than* −2), use the second rule $f(x) = -x$.
Since $\lim_{x \to -2^-} f(x) \neq \lim_{x \to -2^+} f(x)$, $\lim_{x \to -2} f(x)$ does not exist.	Condition 2 is not met.
Therefore, f is discontinuous at $x = -2$.	

Test for continuity at 0:

1. Determine whether f is defined at 0.	At $x = 0$, use the second rule $f(x) = -x$.
$f(0) = -(0) = 0$	Condition 1 is met.
2. Determine whether $\lim_{x \to 0} f(x)$ exists.	Check that the limits from the left and right of 0 exist and are equal.
$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = -(0) = 0$	To find the limit from the left of 0, (where x is near 0, but slightly *less than* 0) use the second rule $f(x) = -x$.
$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0$	To find the limit from the right of 0 (where x is near 0, but slightly *greater than* 0) use the third rule $f(x) = \sqrt{x}$.
Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$, $\lim_{x \to 0} f(x)$ exists and $\lim_{x \to 0} f(x) = 0$.	Condition 2 is met.
3. Determine whether $\lim_{x \to 0} f(x) = f(0)$.	
$\lim_{x \to 0} f(x) = 0 = f(0)$.	Condition 3 is met.

f meets all three conditions for continuity at $x = 0$ and f is continuous at 0.

In summary, f is discontinuous only for $x = -2$.

Apply the Skills

3. Determine the value(s) of x (if any) for which the function is discontinuous.

$$f(x) = \begin{cases} x+4 & \text{if} \quad x \le -2 \\ |x| & \text{if} \; -2 < x < 3 \\ 2 & \text{if} \quad x \ge 3 \end{cases}$$

To confirm the results of Example 3, the graph of

$$f(x) = \begin{cases} 4 & \text{if} \;\; x < -2 \\ -x & \text{if} \; -2 \le x \le 0 \\ \sqrt{x} & \text{if} \;\; x > 0 \end{cases}$$

is shown in Figure 12-19.

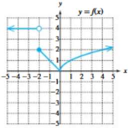


Figure 12-19

Problem Recognition Practice Exercises

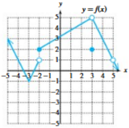
1. What are the criteria for a function *f* to be continuous at *c*?

2. By looking at the graph of a function *f*, how would you determine whether *f* is continuous at *c*?

3. Sketch the graph of a function *f* that is defined at *c* and for which $\lim_{x\to c} f(x)$ exists, but $\lim_{x\to c} f(x) \neq f(c)$.

4. Suppose that $\lim_{x\to c} f(x)$ exists, but $f(c)$ is undefined. Describe what the graph of *f* looks like at *c*.

For Exercises 5–24, use the given graph to evaluate the function, find the limit, or determine whether *f* is continuous at the given value. If a limit does not exist, indicate "Does not exist." If a function is not defined at a given value, indicate "Undefined." (See Example 1, p. 12-29)



5. $\lim_{x\to -3^-} f(x)$

6. $\lim_{x\to -3^+} f(x)$

7. $\lim_{x\to -3} f(x)$

8. $f(-3)$

9. Is *f* continuous at −3?

10. $\lim_{x\to -2^-} f(x)$

11. $\lim_{x\to -2^+} f(x)$

12. $\lim_{x\to -2} f(x)$

13. $f(-2)$

14. Is *f* continuous at −2?

15. $\lim_{x\to 3^-} f(x)$

16. $\lim_{x\to 3^+} f(x)$

17. $\lim_{x\to 3} f(x)$

18. $f(3)$

19. Is *f* continuous at 3?

20. $\lim_{x\to 5^-} f(x)$

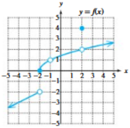
21. $\lim_{x\to 5^+} f(x)$

22. $\lim_{x\to 5} f(x)$

23. $f(5)$

24. Is *f* continuous at 5?

For Exercises 25–44, use the given graph to evaluate the function, find the limit, or determine whether *f* is continuous at the given value. If a limit does not exist, indicate "Does not exist." If a function is not defined at a given value, indicate "Undefined." (See Example 1, p. 12-29)



25. $\lim_{x\to -4^-} f(x)$

26. $\lim_{x\to -4^+} f(x)$

27. $\lim_{x\to -4} f(x)$

28. $f(-4)$

29. Is *f* continuous at −4?

30. $\lim_{x\to -2^-} f(x)$

31. $\lim_{x\to -2^+} f(x)$

32. $\lim_{x\to -2} f(x)$

33. $f(-2)$

34. Is *f* continuous at −2?

35. $\lim_{x\to -1^-} f(x)$

36. $\lim_{x\to -1^+} f(x)$

37. $\lim_{x\to -1} f(x)$

38. $f(-1)$

39. Is *f* continuous at −1?

40. $\lim_{x\to 2^-} f(x)$

41. $\lim_{x\to 2^+} f(x)$

42. $\lim_{x\to 2} f(x)$

43. $f(2)$

44. Is *f* continuous at 2?

For Exercises 45–54, use $f(x) = \dfrac{2}{x^2 + 9x - 22}$ to evaluate the function, find the limit, or determine whether *f* is continuous at the given value. If a limit does not exist, indicate "Does not exist." If a function is not defined at a given value, indicate "Undefined." (See Example 2, p. 12-31)

45. $f(1)$

46. $\lim_{x\to 1} f(x)$

47. Is *f* continuous at 1?

48. $f(-11)$

49. $\lim_{x\to -11} f(x)$

50. Is *f* continuous at −11?

51. $f(2)$

52. $\lim_{x\to 2} f(x)$

53. Is *f* continuous at 2?

54. Is *f* continuous at 0?

For Exercises 55–60, determine the value of x (if any) for which the function is discontinuous. (See Example 2, p. 12-31)

55. $f(x) = \dfrac{x}{x^2 - 9}$

56. $f(x) = \dfrac{1 - x}{x^2 + 8x + 16}$

57. $f(x) = \dfrac{x}{x^2 + 9}$

58. $f(x) = \dfrac{1 - x}{x^2 + 16}$

59. $f(x) = \tan x$

60. $f(x) = \csc x$

For Exercises 61–70, use

$$f(x) = \begin{cases} 4 & \text{for } -5 \le x < -2 \\ 3 & \text{for } -2 \le x < 1 \\ -x + 4 & \text{for } \quad x > 1 \end{cases}$$

to evaluate the function, find the limit, or determine whether f is continuous at the given value. If a limit does not exist, indicate "Does not exist." If a function is not defined at a given value, indicate "Undefined." (See Example 3, p. 12-32)

61. $\lim\limits_{x \to -2^-} f(x)$

62. $\lim\limits_{x \to -2^+} f(x)$

63. $\lim\limits_{x \to -2} f(x)$

64. $f(-2)$

65. Is f continuous at -2?

66. $\lim\limits_{x \to 1^-} f(x)$

67. $\lim\limits_{x \to 1^+} f(x)$

68. $\lim\limits_{x \to 1} f(x)$

69. $f(1)$

70. Is f continuous at 1?

For Exercises 71–80, use

$$f(x) = \begin{cases} -2x + 1 & \text{for } x < -2 \\ x^2 & \text{for } -2 \le x < 0 \\ \sqrt{x} & \text{for } x \ge 0 \end{cases}$$

to evaluate the function, find the limit, or determine whether f is continuous at the given value. If a limit does not exist, indicate "Does not exist." If a function is not defined at a given value, indicate "Undefined." (See Example 3, p. 12-32)

71. $\lim\limits_{x \to 0^-} f(x)$

72. $\lim\limits_{x \to 0^+} f(x)$

73. $\lim\limits_{x \to 0} f(x)$

74. $f(0)$

75. Is f continuous at 0?

76. $\lim\limits_{x \to -2^-} f(x)$

77. $\lim\limits_{x \to -2^+} f(x)$

78. $\lim\limits_{x \to -2} f(x)$

79. $f(-2)$

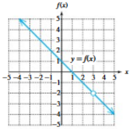
80. Is f continuous at -2?

For Exercises 81–82, find the value of t so that the function is continuous.

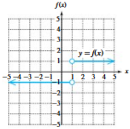
81. $f(x) = \begin{cases} 3 & \text{if } x < 1 \\ tx - 3 & \text{if } x \ge 1 \end{cases}$

82. $f(x) = \begin{cases} -2x + 5 & \text{if } x < 2 \\ 2x + t & \text{if } x \ge 2 \end{cases}$

83. The graph of f is shown in the figure. Is it possible to define f at 3 to make f continuous? Explain.



84. The graph of f is shown in the figure. Is it possible to define f at 1 to make f continuous? Explain.



85. Given $f(x) = \frac{1}{x - 2}$, is it possible to redefine f at 2 to make f continuous at 2? Explain.

Go online for more practice problems.

Lesson 12-3

The Tangent Line Problem: Introduction to Derivatives

What Will You Learn?

After completing this lesson, you should be able to:

- Find the Slope of a Line Tangent to a Graph
- Use Limits to Find the Derivative of a Function
- Apply the Derivative

Learn: Find the Slope of a Line Tangent to a Graph

When we introduced the concept of a limit, we used the model $s(t) = 256 - 16t^2$ to describe the height of a ball dropped from a 256-ft building (Figure 12-20).

The value $s(t)$ represents the height (in ft), t sec after being dropped.

The average velocity (average rate of change of height versus time) between 0 and 1 sec after release is given by the slope of the secant line between the points (0, 256) and (1, 240).

The average velocity between 1 and 2 sec after release is given by the slope of the secant line between the points (1, 240) and (2, 192).

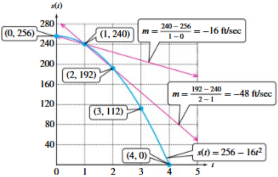


Figure 12-20

Now suppose that we want to find the velocity of the ball at a time *exactly* 1 sec after release. We would need to find the slope of the line *tangent* to the curve at $t = 1$ (Figure 12-21). Intuitively, the **tangent line** at a point $(a, f(a))$ on the graph of a function f is the line whose slope most closely matches the slope of the curve at that point.

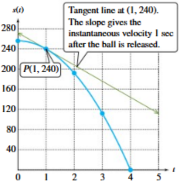


Figure 12-21

The slopes of the secant lines given in Figure 12-20 (on page 12-36) represent the average rate of change in the height of the ball versus time over the intervals [0, 1] and [1, 2]. The slope of the tangent line at point $P(1, 240)$ represents the *instantaneous* rate of change in height versus time exactly 1 sec after release of the ball (Figure 12-21 on page 12-36).

We sometimes refer to the slope of the tangent line at a point on a curve as the slope of the curve at that point. The rationale is that on a small viewing window containing the given point, the graph of a smooth function appears to be almost linear (Figure 12-22).

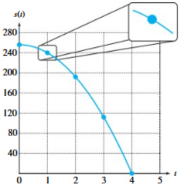
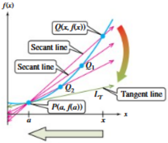


Figure 12-22

The tangent line L_T to the graph of a function f at the point $P(a, f(a))$ must contain point P. We can approximate the slope of the tangent line at P by selecting another point on the curve $Q(x, f(x))$ and then computing the slope of the secant line between the points. Taking points Q_1, Q_2, and so on, progressively closer to P, improves the approximation (Figure 12-23).



As $x \to a$, the secant line approaches the tangent line.

Figure 12-23

- The slope of the secant line m_s is the slope of the line between $Q(x, f(x))$ and $P(a, f(a))$.

  Slope of the secant line: $m_s = \frac{f(x) - f(a)}{x - a}$

- If we take the limit of the slope of the secant line as x approaches a, then we have the slope of the tangent line m_T.

  Slope of the tangent line: $m_T = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Point of Interest

The word secant comes from the verb *secare* in Latin, meaning "to cut," while the word tangent comes from *tangere* meaning "to touch."

Slope of a Tangent Line

The line tangent to the curve $y = f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ with slope

(1) $m_T = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, provided that the limit exists.

Example 1: Finding the Slope and Equation of a Tangent Line

Given $f(x) = \frac{4}{x}$,

- **a.** Find the slope of the tangent line at $x = 1$.
- **b.** Find an equation of the tangent line at $x = 1$. Write the result in slope-intercept form.

Solution:

a. $m_T = \lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$

Use the formula for the slope of the tangent line.

$= \lim\limits_{x \to 1} \frac{\frac{4}{x} - \frac{4}{1}}{x - 1}$

$a = 1$, $f(x) = \frac{4}{x}$, and $f(1) = \frac{4}{1}$.

$= \lim\limits_{x \to 1} \frac{x \cdot \left(\frac{4}{x} - \frac{4}{1}\right)}{x \cdot (x - 1)} = \lim\limits_{x \to 1} \frac{4 - 4x}{x(x - 1)} = \lim\limits_{x \to 1} \frac{4\overset{-1}{\cancel{(1 - x)}}}{x\cancel{(x - 1)}}$

Clear denominators and simplify.

$= \lim\limits_{x \to 1} \frac{-4}{x} = -4$

The slope of the tangent line at $x = 1$ is -4.

b. At $x = 1$, the point on the function is $(1, f(1))$, which is $(1, 4)$.

To find an equation of the tangent line, use the point-slope formula.

$y - y_1 = m_T(x - x_1)$

$y - 4 = -4(x - 1)$

$y - 4 = -4x + 4$

$y = -4x + 8$

The graph of $f(x) = \frac{4}{x}$ and the tangent line $y = -4x + 8$ are shown in Figure 12-24.

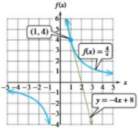


Figure 12-24

Apply the Skills

1. Given $f(x) = -\frac{2}{x}$,
   - **a.** Find the slope of the tangent line at $x = 2$.
   - **b.** Find an equation of the tangent line at $x = 2$. Write the result in slope-intercept form.

Example 2: Finding the Slope and Equation of a Tangent Line

Given $f(x) = x^2 - 6x + 8$,

- **a.** Find the slope of the tangent line at $x = 5$.
- **b.** Find an equation of the tangent line at $x = 5$. Write the result in slope-intercept form.

Solution:

a. $m_T = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

Use the formula for the slope of the tangent line.

$= \lim\limits_{x \to 5} \dfrac{(x^2 - 6x + 8) - (3)}{x - 5}$

$a = 5$, $f(x) = x^2 - 6x + 8$, and $f(5) = (5)^2 - 6(5) + 8 = 3$.

$= \lim\limits_{x \to 5} \dfrac{x^2 - 6x + 5}{x - 5} = \lim\limits_{x \to 5} \dfrac{\overset{1}{\cancel{(x - 5)}}(x - 1)}{\cancel{x - 5}}$

$= \lim\limits_{x \to 5} (x - 1) = 4$

Simplify.

The slope of the tangent line at $x = 5$ is 4.

b. At $x = 5$, the point on the function is $(5, f(5))$, which is $(5, 3)$.

To find an equation of the tangent line, use the point-slope formula.

$y - y_1 = m_T(x - x_1)$

$y - 3 = 4(x - 5)$

$y - 3 = 4x - 20$

$y = 4x - 17$

The graphs of $f(x) = x^2 - 6x + 8$ and the tangent line $y = 4x - 17$ are shown in Figure 12-25.

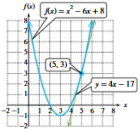


Figure 12-25

Apply the Skills

2. Given $f(x) = x^2 + 4x$,

a. Find the slope of the tangent line at $x = -2$.

b. Find an equation of the tangent line at $x = -2$. Write the result in slope-intercept form.

Learn: Use Limits to Find the Derivative of a Function

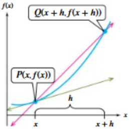
In Example 2, we found the slope of the tangent line to the graph of $f(x) = x^2 - 6x + 8$ at $x = 5$. What if we wanted to find the slope of the tangent line at numerous values of x, such as at $x = 1$, $x = 2$, $x = 3$, $x = 4$, and so on? It would be cumbersome to apply formula (1) numerous times. Instead, we can find the slope of the tangent line at a general point $P(x, f(x))$, where x is any real number in the domain of f.

Suppose we take a second point $Q(x + h, f(x + h))$ on the graph of f (Figure 12-26).

The slope of the secant line between P and Q is given by

$$m_s = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x}$$
$$= \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}$$

This is the familiar *difference quotient* introduced when we studied slope. The numerator is the change in y between P and Q, and the denominator is the change in x.



As $h \to 0$, the secant line approaches the tangent line.

Figure 12-26

As h approaches zero, the secant line approaches the tangent line, which implies that the slope of the secant line approaches the slope of the tangent line. Taking the limit as h approaches zero gives the slope of the tangent line.

$$(2) \quad m_T = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The slope of the tangent line gives us a measure of the slope or "steepness" of the curve at the point $P(x, f(x))$. Formula (2) is also called the *derivative* of f at x and has monumental significance in applications.

Definition of the Derivative of a Function

The **derivative of f at x**, denoted $f'(x)$, is given by

(2) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided that the limit exists.

Notes:

- The notation $f'(x)$ is read "f prime of x."
- The derivative of a function is itself a function.
- The derivative of f evaluated at x is the slope of the tangent line to the graph of f at the point $P(x, f(x))$. The derivative measures the slope or "steepness" of the curve at point P.
- In applications, the derivative represents the instantaneous rate of change of $f(x)$ versus x at $P(x, f(x))$.

Example 3: Finding the Derivative of a Function

Given $f(x) = x^2 - 6x + 8$,

a. Find $f'(x)$.

b. Find the slope of the line tangent to f at $x = 1$, $x = 2$, $x = 3$, and $x = 4$.

Solution:

a. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ — Use the formula for the derivative of a function: Take the limit of the difference quotient.

$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 - 6x - 6h + 8) - (x^2 - 6x + 8)}{h}$

$$\begin{aligned} f(x) &= x^2 - 6x + 8 \\ f(x+h) &= (x+h)^2 - 6(x+h) + 8 \\ &= x^2 + 2xh + h^2 - 6x - 6h + 8 \end{aligned}$$

$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 6x - 6h + 8 - x^2 + 6x - 8}{h}$ — Simplify the difference quotient.

$= \lim_{h \to 0} \frac{2xh + h^2 - 6h}{h}$

$= \lim_{h \to 0} \frac{h(2x + h - 6)}{h}$ — Factor out h.

$= \lim_{h \to 0} (2x + h - 6)$ — Simplify the ratio $\frac{h}{h}$.

$= 2x + (0) - 6$ — Evaluate the limit.

$= 2x - 6$

So $f'(x) = 2x - 6$.

b. The slope of the line tangent to the parabola at $x = 1, x = 2, x = 3$, and $x = 4$ can be found by evaluating the derivative at these values of x (Figure 12-27).

$f'(x) = 2x - 6$

$f'(1) = 2(1) - 6 = -4$

$f'(2) = 2(2) - 6 = -2$

$f'(3) = 2(3) - 6 = 0$

$f'(4) = 2(4) - 6 = 2$

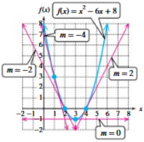


Figure 12-27

Apply the Skills

3. Given $f(x) = x^2 + 4x$,

a. Find $f'(x)$.

b. Find the slope of the line tangent to f at $x = -3, x = -2, x = -1$, and $x = 0$.

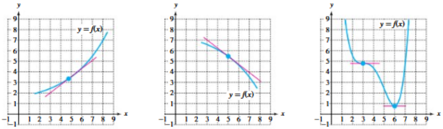
Insights

From Example 3, the derivative of $f(x) = x^2 - 6x + 8$ is $f'(x) = 2x - 6$.

At $x = 5$ the derivative is $f'(5) = 2(5) - 6 = 4$. This is consistent with the result from Example 2(a).

Example 3 supports the following properties of derivatives (Figure 12-28).

- The derivative is positive on intervals over which f is increasing.
- The derivative is negative on intervals over which f is decreasing.
- The derivative is zero where the curve has a horizontal tangent line.



f is increasing.
Slope is positive.
$f'(x) > 0$

f is decreasing.
Slope is negative.
$f'(x) < 0$

Slope is zero (horizontal tangent line).
$f'(x) = 0$

Figure 12-28

Example 4: Finding the Derivative of a Function

Given $f(x) = \sqrt{x+4}$,

a. Find $f'(x)$.

b. Find the slope of the line tangent to f at $x = -3$ and $x = 0$.

Solution:

a. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Use the formula for the derivative of a function: Take the limit of the difference quotient.

$$= \lim_{h \to 0} \frac{\sqrt{x+h+4} - \sqrt{x+4}}{h}$$

$f(x) = \sqrt{x+4}$

$f(x+h) = \sqrt{(x+h)+4}$

$= \sqrt{x+h+4}$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h+4} - \sqrt{x+4})}{h} \cdot \frac{(\sqrt{x+h+4} + \sqrt{x+4})}{(\sqrt{x+h+4} + \sqrt{x+4})}$$

Rationalize the numerator.

$$= \lim_{h \to 0} \frac{x+h+4-(x+4)}{h(\sqrt{x+h+4} + \sqrt{x+4})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h+4} + \sqrt{x+4})}$$

Simplify the numerator.

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h+4} + \sqrt{x+4}}$$

Simplify the ratio $\frac{h}{h}$.

$$= \frac{1}{\sqrt{x+(0)+4} + \sqrt{x+4}}$$

Evaluate the limit.

$$= \frac{1}{\sqrt{x+4} + \sqrt{x+4}}$$

Add like radicals in the denominator.

$$= \frac{1}{2\sqrt{x+4}}$$

So, $f'(x) = \frac{1}{2\sqrt{x+4}}$.

b. The slope of the line tangent to the function at $x = -3$ and $x = 0$ can be found by evaluating the derivative at these values of x (Figure 12-29).

$$f'(x) = \frac{1}{2\sqrt{x+4}}$$

$$f'(-3) = \frac{1}{2\sqrt{(-3)+4}} = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

$$f'(0) = \frac{1}{2\sqrt{(0)+4}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

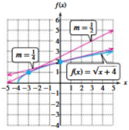


Figure 12-29

Apply the Skills

4. Given $f(x) = 2\sqrt{x}$,

a. Find $f'(x)$.

b. Find the slope of the line tangent to f at $x = 1$ and $x = 9$.

Learn: Apply the Derivative

The derivative of a function can be used to find the slope of the tangent line of a graph at a given point and to find an instantaneous rate of change. In Example 5, we use the derivative in an application involving a rate of change.

Example 5: Applying the Derivative of a Function

The model $s(t) = 256 - 16t^2$ represents the height of a ball dropped from a 256-ft building. The value $s(t)$ represents the height, in ft, t sec after being dropped.

a. Determine the velocity of the ball exactly 1 sec after being released.

b. Determine the velocity of the ball at the instant it hits the ground.

Solution:

The velocity of the ball at any given time t is the rate of change of its position (height) with respect to time. This is given by $s'(t)$.

a. $$s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

Use the formula for the derivative of a function: Take the limit of the difference quotient.

$$= \lim_{h \to 0} \frac{(256 - 16t^2 - 32th - 16h^2) - (256 - 16t^2)}{h}$$

$s(t) = 256 - 16t^2$

$s(t+h) = 256 - 16(t+h)^2$

$= 256 - 16(t^2 + 2th + h^2)$

$= 256 - 16t^2 - 32th - 16h^2$

$$= \lim_{h \to 0} \frac{256 - 16t^2 - 32th - 16h^2 - 256 + 16t^2}{h}$$

Simplify the difference quotient.

$$= \lim_{h \to 0} \frac{-32th - 16h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(-32t - 16h)}{h}$$

Factor out h.

$$= \lim_{h \to 0} (-32t - 16h)$$

Simplify the ratio $\frac{h}{h}$.

$$= -32t - 16(0)$$

Evaluate the limit.

$$= -32t$$

So, $s'(t) = -32t$.

The velocity of the ball after 1 sec is $s'(1) = -32(1) = -32$ ft/sec.

The negative sign indicates that the ball is traveling in the negative direction as it falls to the ground.

b. The ball hits the ground when the height is zero. This is when $s(t) = 0$.

$s(t) = 256 - 16t^2$

$0 = 256 - 16t^2$ Solve $s(t) = 0$.

$0 = 16(16 - t^2)$

$0 = 16(4 - t)(4 + t)$

$t = 4$ or $t = -4$ We reject the negative value for time.

The ball hits the ground 4 sec after release.

$s'(4) = -32(4) = -128$ ft/sec The velocity of the ball 4 sec after release is given by $s'(4)$.

The velocity of the ball at the instant it hits the ground is -128 ft/sec.

Apply the Skills

5. The model $s(t) = -16t^2 + 64t + 80$ represents the height of a model rocket shot upwards from a point 80 ft above the ground. The value $s(t)$ represents the height (in ft), t sec after launch.

a. Determine the velocity of the rocket exactly 1 sec after launch.

b. Determine the velocity of the rocket at the instant it hits the ground.

Practice 12-3

Practice Exercises

Prerequisite Review

R.1. a. Use the slope-intercept form to write an equation of a line that passes through $(-10, -9)$ with slope $m = \frac{5}{3}$.

b. Write the equation using function notation, where $y = f(x)$.

R.2. Find the vertex of the parabola by applying the vertex formula.

$$f(x) = 3x^2 - 24x + 29$$

R.3. For $f(x) = x^2 + 2x - 8$

a. Find the vertex.

b. Find the x-intercepts.

c. Find the y-intercept.

d. Use this information to graph the function.

Concept Connections

1. The ________ line at a point $(a, f(a))$ on the graph of a function f is the line whose slope best matches the slope of the curve at that point.

2. Given a point $P(a, f(a))$ on a function f and another point $Q(x, f(x))$ on f, write an expression for the slope m_s of the secant line between P and Q.

3. Given a point $P(a, f(a))$ on a function f and another point $Q(x, f(x))$ on f, write an expression for the slope m_T of the tangent line to f at P.

4. The expression $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ represents the ________ of f at x, provided that the limit exists.

5. Consider a function f for which the derivative is defined for all x in the domain of f. On the intervals where the derivative is negative, the function is (increasing/decreasing/constant).

6. The tangent line is horizontal. For a function f, if the derivative of f at x equals zero, what do you know about the tangent line at that point?

Learn: Find the Slope of a Line Tangent to a Graph

For Exercises 7–18, find

a. The slope of the tangent line to the graph for the given value of x.

b. An equation of the tangent line to the graph for the given value of x. Write the result in slope-intercept form. **(See Examples 1–2, p. 12-38)**

7. $f(x) = 3x + 8$ for $x = 2$

8. $f(x) = 3 - 2x$ for $x = 2$

9. $f(x) = -\frac{2}{x}$ for $x = 1$

10. $f(x) = \frac{5}{x}$ for $x = 1$

11. $m(x) = x^2 + 4x + 5$ for $x = -3$

12. $n(x) = -x^2 + 4x - 3$ for $x = 0$

13. $s(x) = -4x^2 - 10$ for $x = 0$

14. $z(x) = 2x^2 + 8$ for $x = 0$

15. $p(x) = \sqrt{x + 2}$ for $x = 2$

16. $f(x) = \sqrt{15 - 3x}$ for $x = 2$

17. $t(x) = \frac{9}{x + 4}$ for $x = -2$

18. $r(x) = \frac{3}{x - 1}$ for $x = -1$

Learn: Use Limits to Find the Derivative of a Function

For Exercises 19–28,

a. Find the derivative of each function with respect to x.

b. Find the slope of the tangent line for the given values of x. **(See Examples 3–4, pp. 12-40, 12-42)**

19. a. $f(x) = 15x - 4$

b. $x = 2, x = -3$

20. a. $g(x) = 2 - 8x$

b. $x = 0, x = \frac{1}{3}$

21. a. $m(x) = -2x^2 + 5x - 4$

b. $x = -4, x = 2$

22. a. $h(x) = 3x^2 - 4x + 7$

b. $x = \frac{2}{3}, x = -2$

23. a. $p(x) = x^3 - 1$

b. $x = -3, x = 0$

24. a. $q(x) = 2x^3$

b. $x = 0, x = -3$

25. a. $s(x) = \sqrt{1 - x}$

b. $x = 1, x = -3$

26. a. $r(x) = 5 + \sqrt{x}$

b. $x = 9, x = 0$

27. a. $t(x) = \frac{5}{x^2}$

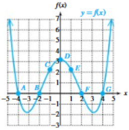
b. $x = 10, x = \frac{1}{2}$

28. a. $w(x) = \frac{3}{x+1}$

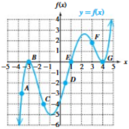
b. $x = -2, x = 2$

For Exercises 29–30, use the graph to determine if the value of $y = f'(x)$ is positive, negative, or zero at the given points.

29.



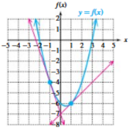
30.



For Exercises 31–32, use the graph to estimate the value of $f'(x)$ for the given values of x.

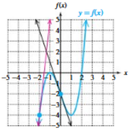
31. a. $x = -1$

b. $x = 1$

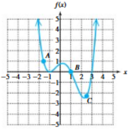


32. a. $x = -2$

b. $x = 0$



For Exercises 33–35, use the graph to select the best estimate for the slope of the tangent line at the point.



33. Point A

a. $m = -4$ **b.** $m = -\frac{1}{4}$

c. $m = 4$ **d.** $m = \frac{1}{4}$

34. Point B

a. $m = 2$ **b.** $m = -2$

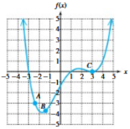
c. $m = 0$ **d.** Undefined

35. Point C

a. $m = -\frac{1}{2}$ **b.** $m = \frac{1}{2}$

c. $m = -2$ **d.** $m = 2$

For Exercises 36–38, use the graph to select the best estimate for the slope of the tangent line at the point.



36. Point *A*

a. $m = 3$
b. $m = -3$
c. $m = \frac{1}{3}$
d. $m = -\frac{1}{3}$

37. Point *B*

a. $m = 4$
b. $m = -4$
c. $m = 1$
d. $m = -1$

38. Point *C*

a. $m = 0$
b. Undefined
c. $m = 3$
d. $m = -3$

Learn: Apply the Derivative

39. The height of an object launched straight upward from ground level follows a path modeled by $s(t) = -16t^2 + 64t$, where t is the time in seconds and $s(t)$ is the height in feet. **(See Example 5, p. 12-43)**

a. Write a function that represents the velocity of the object t sec after launch.

b. What is the exact velocity of the object 0.5 sec after launch? Is the object on the way up or on the way down?

c. What is the exact velocity of the object 3 sec after launch? Is the object on the way up or on the way down?

40. The function defined by $s(t) = 22 - 16t^2$ models the height of a ball dropped from a 22-ft platform. The value $s(t)$ represents the height (in ft), t sec after being dropped.

a. Write a function that represents the velocity of the ball exactly t sec after being released.

b. Determine the velocity of the ball 0.2 sec after its release.

41. Consider a spherical balloon of radius r inches. As the balloon inflates or deflates, both the radius and volume change.

a. Write a function that represents the rate of change in the volume of the balloon as a function of its radius. Recall that the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.

b. How fast is the volume changing relative to the radius when the radius is 4 in.?

42. Consider a spherical balloon of radius r centimeters. As the balloon inflates or deflates, both the radius and surface area change.

a. Write a function that represents the rate of change of the surface area of the balloon as a function of its radius. The surface area of a sphere is given by $A = 4\pi r^2$.

b. How fast is the surface area changing relative to the radius when the radius is 10 cm?

43. An architect designs a rectangular storage container such that the length of one side of the container is twice the length of the other side, and the height of the container is the same as the length of the shorter side. As the container side are increased or decreased, the volume also increases or decreases.

a. Write a function respresenting the volume $V(x)$ of the container as a function of the length of the shortest side x.

b. How fast is the volume changing relative to the shortest side when the shortest side is 5 ft?

44. Consider a right circular cylinder with a fixed height of 12 in. If the radius r is increased or decreased, the volume also increases or decreases.

a. Write a function that represents the rate of change in volume as a function of the radius of the cylinder. Recall that the volume of a right circular cylinder is given by $V = \pi r^2 h$.

b. How fast is the volume changing relative to the radius when the radius is 5 in.?

Mixed Exercises

For Exercises 45–46,

a. Find the equation of the tangent line to the function at the given point.

b. Graph the function and the tangent line.

45. $f(x) = 4 - x^2$, $(-1, 3)$

46. $f(x) = x^2 - 4x$, $(1, -3)$

47. The equation $s(t) = -16t^2 + 96t + 112$ represents the height of an object shot directly upward from a point 112 ft above the ground. The value $s(t)$ represents the height (in ft), t sec after launch.

a. Find $s'(t)$.

b. Determine the value of the derivative at $t = 2$ and interpret the meaning of the derivative as a rate of change.

c. Determine the vertex of the function.

d. Determine the derivative of the function at the vertex and interpret its meaning as a rate of change.

e. Determine the value of the derivative at $t = 5$ and interpret the meaning of the derivative as a rate of change.

48. The equation $s(t) = -4.9t^2 + 196t$ represents the height of an object shot straight upwards from the ground. The value $s(t)$ represents the height (in m), t sec after launch.

a. Find $s'(t)$.

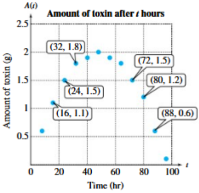
b. Determine the value of the derivative at $t = 25$ and interpret the meaning of the derivative as a rate of change.

c. Determine the vertex of the function.

d. Determine the number of seconds after launch at which the velocity equals zero. Assume this is before the object hits the ground. What is the significance of this time in terms of the height of the object?

e. What is the velocity of the object when it hits the ground?

49. *Tetanus bacillus* bacteria are cultured to produce tetanus toxin used in an inactive form for the tetanus vaccine. The amount of toxin produced per batch increases with time and then decreases as the culture becomes unstable. The variable t is the time in hours after the culture has started and $A(t)$ is the amount of toxin in grams.



a. Find the average rate of change of the amount of toxin on the interval [16, 24].

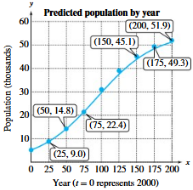
b. Find the average rate of change of the amount of toxin on the interval [24, 32].

c. Next, approximate the instantaneous rate of change of the amount of toxin produced 24 hr after the culture has started. Do this by finding the average of the two average rates of change from parts (a) and (b).

d. Approximate the rate of change of the amount of toxin produced 80 hr after the culture started

e. Interpret the meaning of the signs of the rates of change found in parts (c) and (d).

50. The predicted population of a Jasper, Florida, for 200 yr after the year 2000 is shown in the graph. The predictions are based on population data from earlier years and fitted to a logistic growth pattern.



a. Find the average rate of change in population on the interval [25, 50].

b. Find the average rate of change in population on the interval [50, 75].

c. Next, approximate the rate of change in population in the year 2050 (that is, at time $t = 50$). Do this by finding the average of the two average rates of change from parts (a) and (b).

d. Approximate the rate of change in population in the year 2175.

e. Compare the rates of change found in parts (c) and (d).

51. The table shows the distance $d(t)$ (in ft) run by a sprinter t seconds after the start of a race. Estimate the velocity of the sprinter at the 5-sec mark. (*Hint*: Find the average rate of change on the intervals [4.9, 5.0] and [5.0, 5.1] and average the results.)

Time t(sec)	Distance $d(t)$(ft)
4.5	89.15
4.8	98.24
4.9	101.61
5.0	105.00
5.1	108.41
5.2	112.36
5.5	121.15

52. The distance $d(t)$ (in ft) that a sprinter runs in a short race is given by $d(t) = 2.8t^2 + 9t$, where t is the time in seconds after the start of the race.

a. Determine the speed (in ft/sec) of the sprinter after 5 sec.

b. Determine the speed (in mph) after 5 sec. Round to the nearest tenth of a mph. (*Hint*: 1 mi = 5280 ft.)

Write About It

53. Consider the graph of $f(x) = 2x + 3$. Explain why the derivative of f is a constant at every real number x.

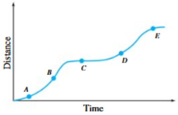
54. Suppose that the graph of $y = f(x)$ is a parabola. Further suppose that $f'(1) = -2$ and $f'(3) = 2$. Is the vertex of the parabola a relative minimum or maximum of the function? Explain.

55. Consider the graph of $f(x) = \ln x$. For all points x in the domain of f, is the derivative positive, negative, or zero? Explain.

56. Consider the graph of $g(x) = -2^x$. For all points x in the domain of g, is the derivative positive, negative, or zero? Explain.

Expanding Your Skills

57. The graph represents the distance a car travels versus the time of travel.



a. At which point A, B, C, D, or E is the car moving the fastest?

b. At which point is the car stopped?

c. At point A, is the car speeding up or slowing down?

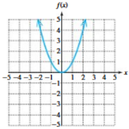
d. At point E, is the car speeding up or slowing down?

58. Verify that the function $f(x) = |x|$ does not have a derivative at $x = 0$ by analyzing the slope of the tangent line at $x = 0$ and using left- and right-hand limits.

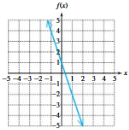
59. Verify that the function $f(x) = x^{1/3}$ does not have a derivative at $x = 0$ by analyzing the slope of the tangent line at $x = 0$.

60. Match each graph a–c with the graph of its derivative i–iii.

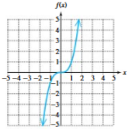
a.



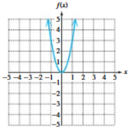
b.



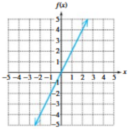
c.



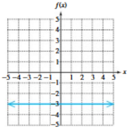
i.



ii.



iii.



Technology Connections

For Exercises 61–64, use the derivative to find the value(s) of *x* where the graph of the function has a horizontal tangent line. Verify the result by graphing the original function.

61. $f(x) = \frac{4}{3}x^3 - 4x$, $f'(x) = 4x^2 - 4$

62. $g(x) = x^4 - 4x^2$, $g'(x) = 4x^3 - 8x$

63. $p(x) = x^2e^{-x}$, $p'(x) = 2xe^{-x} - x^2e^{-x}$

64. $h(x) = x^3e^x$, $h'(x) = 3x^2e^x + x^3e^x$

GO ONLINE Go online for more practice problems.

Lesson 12-4

Limits at Infinity and Limits of Sequences

What Will You Learn?

After completing this lesson, you should be able to:

- Evaluate Limits at Infinity
- Evaluate the Limit of a Sequence

Learn: Evaluate Limits at Infinity

Recall that the horizontal asymptotes of the graph of a function give us the limiting value of $f(x)$ as x approaches infinity or negative infinity.

For example, the graph of $f(x) = \frac{2x+1}{\sqrt{x^2+1}}$ has two horizontal asymptotes (Figure 12-30). As $x \to \infty$, $f(x) \to 2$, and as $x \to -\infty$, $f(x) \to -2$.

From the graph, we see that $f(x)$ gets closer and closer to 2 as x approaches infinity. In fact, we can make $f(x)$ as close to 2 as we want, provided that x is chosen sufficiently large. This is expressed in limit notation as

$$\lim_{x \to \infty} f(x) = 2$$

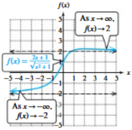


Figure 12-30

Infinity is not actually a number, so we read this statement as "the limit of $f(x)$ as x approaches infinity equals 2." This means that as x increases without bound, $f(x)$ approaches the real number 2.

Likewise, $f(x)$ can be taken arbitrarily close to −2, provided that x is chosen sufficiently "large" in the negative direction. As a limit this is denoted by

$$\lim_{x \to -\infty} f(x) = -2$$

Insights

If we say that x is "large" in the negative direction, we mean that x is negative, but $|x|$ is large.

Informal Definition of a Limit at Infinity or Negative Infinity

1. Let f be a function defined on an interval (a, ∞). Then
$$\lim_{x\to\infty} f(x) = L$$
means that $f(x)$ can be made arbitrarily close to L, provided that x is taken sufficiently large.
2. Let f be a function defined on an interval $(-\infty, b)$. Then
$$\lim_{x\to-\infty} f(x) = M$$
means that $f(x)$ can be made arbitrarily close to M, provided that x is taken sufficiently "large" in the negative direction.

The definition of a horizontal asymptote is given in terms of limits at infinity.

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of $y = f(x)$ if
$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

Example 1: Estimating Limits at Infinity or Negative Infinity Using a Table

Use a table to estimate the limits.

a. $\lim_{x\to\infty} \frac{1}{x}$ **b.** $\lim_{x\to-\infty} \frac{1}{x}$ **c.** $\lim_{x\to-\infty} \frac{1}{x^2}$ **d.** $\lim_{x\to-\infty} \frac{1}{x^2}$

Solution:

a. $\lim_{x\to\infty} \frac{1}{x} = 0$

b. $\lim_{x\to-\infty} \frac{1}{x} = 0$

TABLE 12-7 Values of $f(x) = \frac{1}{x}$ for Select Values of x

x	$f(x) = \frac{1}{x}$
±100	±0.01
±1000	±0.001
±10,000	±0.0001
±100,000	±0.00001

From Table 12-7, $\frac{1}{x}$ gets closer to zero for greater values of $|x|$.

c. $\lim\limits_{x \to \infty} \frac{1}{x^2} = 0$

d. $\lim\limits_{x \to -\infty} \frac{1}{x^2} = 0$

TABLE 12-8 Values of $f(x) = \frac{1}{x^2}$ for Select Values of x

x	$f(x) = \frac{1}{x^2}$
±100	0.0001
±1000	0.000001
±10,000	0.00000001
±100,000	0.0000000001

From Table 12-8, $\frac{1}{x^2}$ gets closer to zero as x approaches ∞ or $-\infty$.

Apply the Skills

1. Use a table to estimate the limits.

   a. $\lim\limits_{x \to \infty} \frac{1}{x^3}$ **b.** $\lim\limits_{x \to -\infty} \frac{1}{x^3}$

The results of Example 1 indicate that the line $y = 0$ is a horizontal asymptote of the graph of $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$ (Figures 12-31 and 12-32).

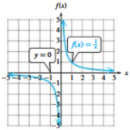


Figure 12-31

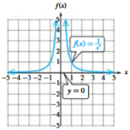


Figure 12-32

Example 1 also illustrates the following important result.

Limits at Infinity for $f(x) = \frac{1}{x^k}$

- If k is a positive real number, then $\lim\limits_{x \to \infty} \frac{1}{x^k} = 0$.
- Furthermore, for a positive real number k, if $\frac{1}{x^k}$ is defined for $x < 0$, then $\lim\limits_{x \to -\infty} \frac{1}{x^k} = 0$.

The properties of limits presented earlier in this chapter also apply to limits at infinity.

Example 2: Evaluating Limits at Infinity

Find the limit or indicate that the limit does not exist.

a. $\lim\limits_{x \to -\infty}\left(10 + \frac{1}{\sqrt[3]{x}}\right)$ **b.** $\lim\limits_{x \to \infty}\frac{2 + \frac{5}{x} - \frac{1}{x^2}}{3 + \frac{1}{x}}$

Solution:

a. $\lim\limits_{x \to -\infty}\left(10 + \frac{1}{\sqrt[3]{x}}\right) = \lim\limits_{x \to -\infty} 10 + \lim\limits_{x \to -\infty}\frac{1}{\sqrt[3]{x}}$ Apply the properties of limits.

$= 10 + 0$ Evaluate each limit. $\frac{1}{\sqrt[3]{x}} = \frac{1}{x^{1/3}}$ is defined for negative values of x, so $\lim\limits_{x \to -\infty}\frac{1}{\sqrt[3]{x}} = 0$.

$= 10$

b. $\lim\limits_{x \to \infty}\frac{2 + \frac{5}{x} - \frac{1}{x^2}}{3 + \frac{1}{x}} = \frac{\lim\limits_{x \to \infty} 2 + 5 \cdot \lim\limits_{x \to \infty}\frac{1}{x} - \lim\limits_{x \to \infty}\frac{1}{x^2}}{\lim\limits_{x \to \infty} 3 + \lim\limits_{x \to \infty}\frac{1}{x}}$ Apply the properties of limits.

$= \frac{2 + 5 \cdot 0 - 0}{3 + 0}$ Evaluate each limit.

$= \frac{2}{3}$ Simplify.

Apply the Skills

2. Find the limits.

a. $\lim\limits_{x \to -\infty}\frac{4}{x^2} - 5$ **b.** $\lim\limits_{x \to \infty}\frac{6 + \frac{1}{x^2} - \frac{1}{x^3}}{2 + \frac{1}{x^3}}$

To evaluate the limit at infinity of a rational function, we divide the numerator and denominator by the greatest power of x that occurs in the denominator. This is possible because as x approaches infinity or negative infinity, $x \neq 0$.

Insights

The rationale for dividing numerator and denominator by the greatest power of x appearing in the denominator is to form fractions of the form $\frac{a}{x^k}$, which approach zero when x approaches infinity or negative infinity.

Example 3: Evaluating Limits at Infinity

Find the limit or indicate that the limit does not exist.

a. $\lim\limits_{x \to \infty}\frac{2x^2 + x - 1}{x^2 - 4}$ **b.** $\lim\limits_{x \to -\infty}\frac{x + 2}{5x^3 - 7x}$

Solution:

a. $\lim\limits_{x\to\infty}\dfrac{2x^2+x-1}{x^2-4}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x^2}\cdot(2x^2+x-1)}{\frac{1}{x^2}\cdot(x^2-4)}$ The greatest power of x in the denominator is x^2. Dividing by x^2 is the same as multiplying by $\frac{1}{x^2}$.

$=\lim\limits_{x\to\infty}\dfrac{2+\frac{1}{x}-\frac{1}{x^2}}{1-\frac{4}{x^2}}$ Apply the distributive property.

$=\dfrac{\lim\limits_{x\to\infty}2+\lim\limits_{x\to\infty}\frac{1}{x}-\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty}1-4\cdot\lim\limits_{x\to\infty}\frac{1}{x^2}}$ Apply the properties of limits.

$=\dfrac{2+0-0}{1-4\cdot 0}$ Evaluate the limits.

$=2$

b. $\lim\limits_{x\to-\infty}\dfrac{x+2}{5x^3-7x}=\lim\limits_{x\to-\infty}\dfrac{\frac{1}{x^3}\cdot(x+2)}{\frac{1}{x^3}\cdot(5x^3-7x)}$ The greatest power of x in the denominator is x^3. Dividing by x^3 is the same as multiplying by $\frac{1}{x^3}$.

$=\lim\limits_{x\to-\infty}\dfrac{\frac{1}{x^2}+\frac{2}{x^3}}{5-\frac{7}{x^2}}$ Apply the distributive property.

$=\dfrac{\lim\limits_{x\to-\infty}\frac{1}{x^2}+2\left(\lim\limits_{x\to-\infty}\frac{1}{x^3}\right)}{\lim\limits_{x\to-\infty}5-7\left(\lim\limits_{x\to-\infty}\frac{1}{x^2}\right)}$ Apply the properties of limits.

$=\dfrac{0+2\cdot 0}{5-7\cdot 0}$ Evaluate the limits.

$=0$

Apply the Skills

3. Find the limits.

a. $\lim\limits_{x\to-\infty}\dfrac{4x^3-2x}{5x^3-x^2+6}$ **b.** $\lim\limits_{x\to\infty}\dfrac{5x^2+2x-3}{2x^4-5x}$

The limits in Example 3 both exist and indicate that the related functions have horizontal asymptotes. From the result of Example 3(a), we know that the graph of $f(x)=\dfrac{2x^2+x-1}{x^2-4}$ has horizontal asymptote $y=2$ (shown in red in Figure 12-33). From Example 3(b), we know that the graph of $f(x)=\dfrac{x+2}{5x^3-7x}$ has horizontal asymptote $y=0$ (Figure 12-34).

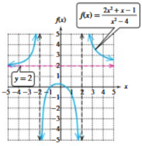


Figure 12-33

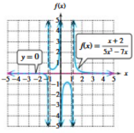


Figure 12-34

The limits found in Example 3 are consistent with our earlier discussion involving the horizontal asymptotes of rational functions. By comparing the degrees of the numerator and denominator, we can identify a horizontal asymptote by inspection. Suppose that a rational function is of the form

$f(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \ldots + b_1 x + b_0}$. The degree of the numerator is n and the degree of the denominator is m. Then,

1. Degree of the numerator equals the degree of the denominator: If $n = m$, then $\lim\limits_{x\to\infty} f(x) = \dfrac{a_n}{b_m}$ and $\lim\limits_{x\to-\infty} f(x) = \dfrac{a_n}{b_m}$. That is, the value of the limit at infinity or negative infinity equals the ratio of the leading coefficients of the numerator and denominator. This in turn implies that the line $y = \dfrac{a_n}{b_m}$ is a horizontal asymptote.
2. Degree of the numerator is less than the degree of the denominator: If $n < m$, then $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to-\infty} f(x) = 0$. This implies that the line $y = 0$ is a horizontal asymptote.
3. Degree of the numerator is greater than the degree of the denominator: If $n > m$, then $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ both do not exist because the function increases or decreases without bound. This is demonstrated in Example 4.

Example 4: Evaluating Limits at Infinity

Find the limit or indicate that the limit does not exist.

a. $\lim\limits_{x\to\infty} \dfrac{x^3 - x + 4}{3x + 1}$ **b.** $\lim\limits_{x\to\infty} \cos x$

Solution:

a. $\lim\limits_{x\to\infty} \dfrac{x^3 - x + 4}{3x + 1} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}\cdot(x^3 - x + 4)}{\frac{1}{x}\cdot(3x + 1)}$ The greatest power of x in the denominator is x. Dividing by x is the same as multiplying by $\frac{1}{x}$.

$= \lim\limits_{x\to\infty} \dfrac{x^2 - 1 + \frac{4}{x}}{3 + \frac{1}{x}}$ Apply the distributive property.

This limit does not exist.

$= \dfrac{\lim\limits_{x\to\infty} x^2 - \lim\limits_{x\to\infty} 1 + 4\cdot\lim\limits_{x\to\infty}\frac{1}{x}}{\lim\limits_{x\to\infty} 3 + \lim\limits_{x\to\infty}\frac{1}{x}}$ Apply the properties of limits.

The first limit in the numerator does not exist because x^2 increases without bound as x approaches infinity. So, since the limit in the numerator is infinite but the limit in the denominator is finite, $\lim\limits_{x\to\infty} \dfrac{x^3 - x + 4}{3x + 1}$ does not exist.

b. $\lim\limits_{x\to\infty} \cos x$

The cosine function is periodic and oscillates between –1 and 1 (Figure 12-35). The value of $\cos x$ does not converge to a single real number as x approaches either ∞ or $-\infty$. So, $\lim\limits_{x\to\infty} \cos x$ does not exist.

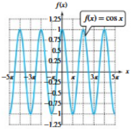


Figure 12-35

Apply the Skills

4. Find the limit or indicate that the limit does not exist.

a. $\lim\limits_{x\to\infty}\dfrac{2x^4+4x^2}{5x^2+3}$ **b.** $\lim\limits_{x\to\infty}\sin 2x$

Insights

In Example 4a, the degree of the numerator is greater than the degree of the denominator. Therefore, the limit as *x* approaches infinity or negative infinity does not exist.

Learn: Evaluate the Limit of a Sequence

In an earlier chapter, we defined a sequence $a_1, a_2, a_3, \ldots$ as a function whose domain is the set of positive integers. In many applications in calculus, we are interested in the behavior of a sequence as $n \to \infty$.

For example, consider the sequence $a_n = \dfrac{3n+5}{n}$ as $n \to \infty$ (Figure 12-36). From the graph it appears that $a_n \to 3$ as $n \to \infty$.

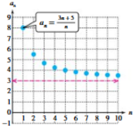


Figure 12-36

We write this using limit notation as

$$\lim_{n\to\infty}\frac{3n+5}{n} = 3.$$

Informal Definition of the Limit of a Sequence

Given $\{a_n\} = a_1, a_2, a_3, \ldots$, we say that the sequence has limit *L*, denoted

$$\lim_{n\to\infty} a_n = L$$

if the *n*th term of the sequence can be made arbitrarily close to *L* by taking *n* sufficiently large.

- If $\lim\limits_{n\to\infty} a_n$ exists, we say that the sequence is **convergent**.
- If $\lim\limits_{n\to\infty} a_n$ does not exist, we say that the sequence is **divergent**.

Notice that the definition of the limit of a sequence as $n \to \infty$ is similar to the definition of a limit of a function as $x \to \infty$. This leads to the following result.

Limit of a Sequence and Limit of a Function

Let f be a real-valued function. If $\{a_n\}$ is a sequence such that $a_n = f(n)$ for every positive integer n, then

$$\text{if } \lim_{x\to\infty} f(x) = L, \text{ then } \lim_{n\to\infty} a_n = L$$

This result tells us that if the limit of a function exists as $x \to \infty$, then the limit of the related sequence also exists as $n \to \infty$, and the two limits are equal (Figure 12-37 and Figure 12-38).

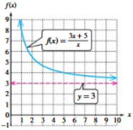


Figure 12-37

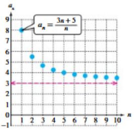


Figure 12-38

The relationship between a function and its related sequence also implies that if $\lim_{x\to\infty} \frac{1}{x^k} = 0$, then $\lim_{n\to\infty} \frac{1}{n^k} = 0$ for positive integers n. Furthermore, the properties of limits of a function also hold for the limits of sequences.

Example 5: Evaluating the Limit of a Sequence

Find the limit or indicate that the limit does not exist. $\lim_{n\to\infty} \frac{3n+5}{n}$

Solution:

$$\lim_{n\to\infty} \frac{3n+5}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}\cdot(3n+5)}{\frac{1}{n}\cdot(n)}$$ The greatest power of n in the denominator is n.

$$= \lim_{n\to\infty} \frac{3+\frac{5}{n}}{1}$$ Apply the distributive property.

$$= \lim_{n\to\infty} 3 + 5 \cdot \lim_{n\to\infty} \frac{1}{n}$$ Apply the properties of limits.

$$= 3 + 5 \cdot 0$$ Evaluate the limits.

$$= 3$$

Apply the Skills

5. Find the limit or indicate that the limit does not exist.

$$\lim_{n\to\infty} \frac{4n^2+7}{3n^3+2n}$$

Insights

In Example 5, the degree of the numerator is the same as the degree of the denominator. Therefore, the limit as *n* approaches infinity or negative infinity equals the ratio of the leading coefficients. So, the limit equals $\frac{3}{1}$, which is 3.

Example 6: Evaluating the Limit of a Sequence

Find the limit or indicate that the limit does not exist. $\lim\limits_{n\to\infty} \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

Solution:

$\lim\limits_{n\to\infty} \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

$= \lim\limits_{n\to\infty} \frac{8}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right)$ Expand the expression in the numerator.

$= \lim\limits_{n\to\infty} \frac{4}{3}\left(\frac{2n^3+3n^2+n}{n^3}\right)$ Simplify the ratio $\frac{8}{6}$ as $\frac{4}{3}$.

$= \frac{4}{3}\lim\limits_{n\to\infty}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$ Simplify.

$= \frac{4}{3}(2)$ Evaluate the limit. Note that $\frac{3}{n}$ and $\frac{1}{n^2}$ both approach 0 as $n \to \infty$.

$= \frac{8}{3}$

Apply the Skills

6. Find the limit or indicate that the limit does not exist.

$$\lim_{n\to\infty} \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

In the next lesson, you will learn that the limit from Example 6 gives the exact area between the graph of $f(x) = x^2$ and the x-axis on the interval [0, 2] (Figure 12-39).

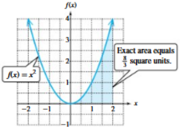


Figure 12-39

Example 7: Evaluating the Limit of a Sequence

Find the limit or indicate that the limit does not exist. $\lim_{n\to\infty}(-2)^n$

Solution:

Writing a few terms of the sequence $a_n = \{(-2)^n\}$ gives the alternating sequence −2, 4, −8, 16, −32, ...

The terms alternate between positive and negative values and their absolute values increase without bound. Therefore, $(-2)^n$ does not converge to a unique real number.

So, $\lim_{n\to\infty}(-2)^n$ does not exist.

Apply the Skills

7. Find the limit or indicate that the limit does not exist.

$$\lim_{n\to\infty}\left(\frac{3}{2}\right)^n$$

Practice 12-4

Practice Exercises

Prerequisite Review

For Exercises R.1–R.3, identify the horizontal asymptotes (if any).

R.1. $f(x) = \frac{3}{x^2 + 8x + 16}$

R.2. $g(x) = \frac{2x^2 + 6x - 4}{x^2 + 1}$

R.3. $t(x) = \frac{x^3 - 3x + 2}{5x + 5}$

R.4. The *n*th term of a sequence is given. Write the first four terms.

$$a_n = 6n^3 + 1$$

R.5. Simplify the expression. $\frac{(n-4)!}{(n-2)!}$

Concept Connections

1. The notation $\lim_{x \to \infty} f(x) = L$ means that $f(x)$ can be taken arbitrarily close to ___________ provided that ___________ is sufficiently large.

2. In terms of limits, give two conditions that imply that the line $y = L$ is a horizontal asymptote of the graph of $y = f(x)$.

3. To evaluate $\lim_{x \to \infty} \frac{x^3 - 4x + 1}{2x^5 - x^3 + x^2}$, by what power of x should the numerator and denominator be divided?

4. To evaluate $\lim_{x \to \infty} \frac{x^4 + x - 3}{-3x^4 - 2x^2}$, by what power of x should the numerator and denominator be divided?

5. Given a sequence $\{a_n\} = a_1, a_2, \ldots$, if $\lim_{n \to \infty} a_n$ exists, then we say that the sequence ___________. Otherwise, we say that the sequence ___________.

6. For $k > 0$ and positive integers n, $\lim_{n \to \infty} \frac{1}{n^k} =$ ___________.

Learn: Evaluate Limits at Infinity

For Exercises 7–10, use a table to estimate the limits. Round table values to 4 decimal places. (See Example 1, p. 12-52)

7. $\lim_{x \to \infty} \frac{2x}{x+1}$

x	$f(x) = \frac{2x}{x+1}$
100	
1000	
10,000	
100,000	

8. $\lim_{x \to -\infty} \frac{2x}{x-1}$

x	$f(x) = \frac{2}{x-1}$
−100	
−1000	
−10,000	
−100,000	

9. $\lim_{x \to \infty} e^{-x}$

x	$f(x) = e^{-x}$
1	
5	
15	
20	

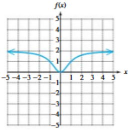
10. $\lim_{x \to \infty} (1 + x)^{1/x}$

x	$f(x) = (1 + x)^{1/x}$
100	
1000	
10,000	
100,000	
1,000,000	

For Exercises 11–16, use the graph to estimate the limits.

a. $\lim_{x\to\infty} f(x)$ **b.** $\lim_{x\to-\infty} f(x)$

11.



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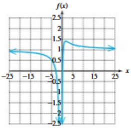

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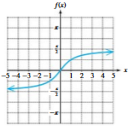

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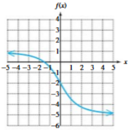

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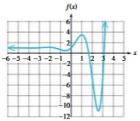

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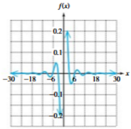


For Exercises 17–36, find the limit or indicate that the limit does not exist. (See Examples 2–4, pp. 12-54–12-56)

17. $\lim_{x\to\infty}\left(\frac{3}{x}+2\right)$

18. $\lim_{x\to-\infty}\left(6-\frac{4}{\sqrt[5]{x}}\right)$

19. $\lim_{x\to-\infty}\frac{3-\frac{3}{x}+\frac{6}{x^2}}{8+\frac{1}{x^3}}$

20. $\lim_{t\to\infty}\frac{4+\frac{3}{t}}{2-\frac{5}{t}+\frac{1}{t^2}}$

21. $\lim_{x\to\infty}\frac{x^2-3x+4}{2x^2+1}$

22. $\lim_{x\to\infty}\frac{6x^3-2x+4}{x^3+5}$

23. $\lim_{x\to-\infty}\frac{4x^3+2x-8}{7-6x^4}$

24. $\lim_{x\to\infty}\frac{x-5}{x^2+25}$

25. $\lim_{x\to\infty}\frac{4x^2-9}{2x+3}$

26. $\lim_{x\to-\infty}\frac{12-4x^3}{3x^2}$

27. $\lim_{x\to-\infty}\cos x^2$

28. $\lim_{x\to-\infty}\tan x$

29. $\lim_{x\to-\infty}\left(8+\frac{2x}{x-3}\right)$

30. $\lim_{x\to-\infty}\left(\frac{3x^2+4}{6x^3}-12\right)$

31. $\lim_{x\to\infty}\sin 4x$

32. $\lim_{x\to\infty}\cos\frac{x}{2}$

33. $\lim_{x\to\infty}\left(\frac{3x-4}{x-5}+\frac{2x^2}{x^2-25}\right)$

34. $\lim_{x\to-\infty}\left(\frac{8x^3-1}{16x^3}-\frac{2x^2}{(x+2)^2}\right)$

35. $\lim_{x\to-\infty}e^x$

36. $\lim_{x\to\infty}\arctan x$

Learn: Evaluate the Limit of a Sequence

For Exercises 37–42, find the limit or indicate that the limit does not exist. (See Examples 5, 7, pp. 12-58, 12-60)

37. $\lim_{n\to\infty}\frac{2n}{n-3}$

38. $\lim_{n\to\infty}\frac{8n^2}{1-2n^2}$

39. $\lim_{n\to\infty}\frac{2n-3}{n^2}$

40. $\lim_{n\to\infty}\left(\frac{n^2}{2n^5-3}+1\right)$

41. $\lim_{n\to\infty}\left(\frac{5}{2}\right)^n$

42. $\lim_{n\to\infty}2^n$

For Exercises 43–48, find the limit or indicate that the limit does not exist. (See Example 6, p. 12-59)

43. $\lim\limits_{n\to\infty}\left\{\frac{9}{n^2}\left[\frac{n(n+1)}{2}\right]+12\right\}$

44. $\lim\limits_{n\to\infty}\left\{24-\frac{16}{n^2}\left[\frac{n(n+1)}{2}\right]\right\}$

45. $\lim\limits_{n\to\infty}\left\{8-\frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\right\}$

46. $\lim\limits_{n\to\infty}\left\{\frac{216}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]+6\right\}$

47. $\lim\limits_{n\to\infty}\left\{128-\frac{256}{n^2}\left[\frac{n(n+1)}{2}\right]+\frac{128}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\right\}$

48. $\lim\limits_{n\to\infty}\left\{32-\frac{64}{n^2}\left[\frac{n(n+1)}{2}\right]+\frac{32}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\right\}$

Mixed Exercises

49. An object that is 45°C is placed in a freezer. The temperature T (in °C) of the object t hours after being placed in the freezer can be approximated by $T=\frac{450}{t^2+3t+10}$. Determine the limiting temperature.

50. The concentration C (in ng/mL) of a drug in the bloodstream t hours after ingestion is approximated by $C=\frac{500t}{t^3+100}$. Determine the limiting concentration.

51. A personal trainer has monthly costs of \$70/month for phone service and \$30/month for her website and advertising. In addition, she pays a \$15 fee to the gym for each session in which she works with a client.

- **a.** Write a cost function to represent the cost $C(x)$ for x training sessions for a 1-month period.
- **b.** Write a function representing the average cost $\overline{C}(x)$ per session if the trainer has x sessions during a 1-month period.
- **c.** The trainer realistically has approximately 240 training sessions per month. However, if the number of sessions were unlimited for a given month, what would the average cost approach?

52. An on-demand printing company has monthly overhead costs of \$1500 in rent, \$200 in electricity, \$100 for phone service, and \$150 for advertising and marketing. The printing cost is \$45 per thousand pages for paper and ink.

- **a.** Write a cost function to represent the cost $C(x)$ to print x thousand pages during a 1-month period.
- **b.** Write a function representing the average cost $\overline{C}(x)$ per thousand pages if x thousand pages are printed during a 1-month period.
- **c.** For a given month, if the printing company could print an unlimited number of pages, what value would the average cost per thousand pages approach?

We know that $\lim\limits_{x\to\infty}\frac{1}{x^k}=0$ for positive real numbers k. Likewise $\lim\limits_{x\to\infty}\frac{1}{e^{kx}}=0$ for positive real numbers k. Use this information for Exercises 53–54.

53. The number of computers $N(t)$ (in millions) infected by a computer virus can be approximated by $N(t)=\frac{4.5}{1+\frac{15}{e^{0.72t}}}$, where t is the time in months after the virus was first detected.

- **a.** Find $\lim\limits_{t\to\infty}N(t)$.
- **b.** Interpret the result of part (a).

54. After a new product is launched, the cumulative sales $S(t)$ (in \$1000) is given by $S(t)=\frac{82}{1+9e^{-0.36t}}$, where t is the number of weeks after launch.

- **a.** Find $\lim\limits_{t\to\infty}S(t)$.
- **b.** Interpret the result of part (a).

For Exercises 55–62, the nth term of the sequence is given. Write several terms of the sequence and then estimate the value of $\lim\limits_{n\to\infty}a_n$ or state that the limit does not exist.

55. $a_n=5+4\left(\frac{1}{2}\right)^{n-1}$

56. $a_n=5\left(\frac{2}{5}\right)^n$

57. $a_n=(-1)^n\left(\frac{1}{n}\right)$

58. $a_n=(-1)^n n^2$

59. $a_n=40-5n$

60. $a_n=12+2n$

61. $a_n=\frac{(n+2)!}{(n+4)!}$

62. $a_n=\frac{(n+4)!}{(n+2)!}$

Write About It

63. If $y = 0$ is a horizontal asymptote of the graph of a function f, can we always conclude that $\lim_{x \to \infty} f(x) = 0$?

64. If $\lim_{x \to \infty} f(x) = 2$, can we always conclude that the line $y = 2$ is a horizontal asymptote of the graph of f? Explain.

Expanding Your Skills

For Exercises 65–68, find the limit or indicate that the limit does not exist. (Hint: Use the technique of dividing by the greatest power of *x* in the denominator.)

65. $\lim_{x \to \infty} \frac{x^{-4} + 6x^{-1}}{3x^{-1} + x^{-3}}$

66. $\lim_{x \to \infty} \frac{3x^{-2} + 12}{2x^{-3} + x^{-1}}$

67. $\lim_{x \to \infty} \frac{x^{4/5} - 2x^{2/5} - 3}{x^{8/5} - x}$

68. $\lim_{x \to \infty} \frac{\sqrt{x} - \sqrt[4]{x}}{\sqrt[3]{x} + \sqrt{x}}$

For Exercises 69–70, use the fact that $\lim_{x \to \infty} \frac{\cos x}{x} = 0$ and the algebraic techniques from this lesson to evaluate the limit.

69. $\lim_{x \to \infty} \frac{x^2}{2x^2 + x\cos x}$

70. $\lim_{x \to \infty} \frac{x^2\cos x - 3x^3}{x^3}$

Technology Connections

For Exercises 71–74,

a. Complete the table and estimate $\lim_{x \to \infty} f(x)$ or state that the limit does not exist. Round table values to 4 decimal places.

b. Graph $y = f(x)$ on a graphing utility to support your answer to part (a).

71. $f(x) = x - \sqrt{x^2 + 4}$

x	$f(x)$
10^2	
10^3	
10^4	
10^5	

72. $f(x) = x - \sqrt{x + 4}$

x	$f(x)$
10^2	
10^3	
10^4	
10^5	

73. $f(x) = x\sin\left(\frac{1}{x}\right)$

x	$f(x)$
1	
10	
10^2	

74. $f(x) = \left(\frac{1}{x}\right)\cos x$

x	$f(x)$
1	
10	
10^2	
10^3	
10^4	
10^5	

GO ONLINE Go online for more practice problems.

Lesson 12-5

Area Under a Curve

What Will You Learn?

After completing this lesson, you should be able to:

- Approximate Area Under a Curve
- Use Limits to Find Area Under a Curve
- Represent an Area with a Definite Integral

Learn: Approximate Area Under a Curve

Earlier in this chapter, we learned that limits can be used to find the slope of a tangent line and to find an instantaneous rate of change. These are two key concepts in the study of differential calculus. In this lesson, we will see that limits can also be used to find the area of a region with a curved boundary. This is one of the fundamental applications in the study of integral calculus. For example, consider the region *S* bounded by the graph of $f(x) = x^2$ and the *x*-axis on the interval [0, 2] (Figure 12-40).

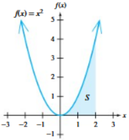


Figure 12-40

We can approximate the area by using a finite number of rectangular regions. This is demonstrated in Example 1.

Example 1: Approximating Area Under a Curve

Use four rectangles to approximate the area bounded by the graph of $f(x) = x^2$ and the *x*-axis on the interval [0, 2] (pictured in Figure 12-41). For the height of the rectangles, use the function value at the

a. Right endpoint of each subinterval.

b. Left endpoint of each subinterval.

Solution:

In Figure 12-41, we show the interval [0, 2] divided into four subintervals of equal width. We denote the width by Δx and the value of Δx is given by

$$\Delta x = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} \text{ or } 0.5$$

a. Four rectangles are constructed with width Δx and height equal to the value of the function at the right endpoint x_R of each subinterval. The area of each rectangle is computed by $f(x_R) \cdot \Delta x$. Since f is increasing, the right endpoint gives the highest point of the function on each subinterval. Therefore, the sum of the rectangular areas using the right endpoint for height *over* approximates the area under the curve. We will denote this area as A_R.

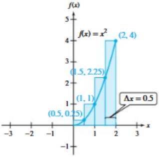


Figure 12-41

$$\begin{aligned} A_R &\approx f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) \\ &\approx 0.5[f(0.5) + f(1) + f(1.5) + f(2)] \qquad \text{Factor out } \Delta x = 0.5. \\ &\approx 0.5[0.25 + 1 + 2.25 + 4] \\ &\approx 3.75 \end{aligned}$$

b. If we use the left endpoint x_L of each subinterval on which to compute the height of the rectangles (Figure 12-42), we would *under* approximate the area under the curve. We will denote this area as A_L.

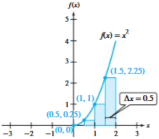


Figure 12-42

$$\begin{aligned} A_L &\approx f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) \\ &\approx 0.5[f(0) + f(0.5) + f(1) + f(1.5)] \\ &\approx 0.5[0 + 0.25 + 1 + 2.25] \\ &\approx 1.75 \end{aligned}$$

The true area A bounded by the graph of $f(x) = x^2$ and the x-axis on the interval [0, 2] is between 1.75 and 3.75. That is, $1.75 < A < 3.75$.

Apply the Skills

1. Use four rectangles to approximate the area bounded by the graph of $f(x) = x^2 + 5$ and the x-axis on the interval [0, 4]. For the height of the rectangles, use the function value at the

- **a.** Right endpoint of each subinterval.
- **b.** Left endpoint of each subinterval.

Insights

For each subinterval, any value of x from that interval can be used to compute the height of the rectangle. In Example 1, we used the left and right endpoints for convenience.

Intuitively, we might expect that using a greater number of rectangles over a greater number of subintervals would result in a better approximation of the true area. This is demonstrated in Example 2.

Example 2: Approximating Area Under a Curve

Use eight rectangles to approximate the area bounded by the graph of $f(x) = x^2$ and the x-axis on the interval $[0, 2]$. For the height of the rectangles, use the function value at the

a. Right endpoint of each subinterval.

b. Left endpoint of each subinterval.

Solution:

In Figure 12-43, we show the interval $[0, 2]$ divided into eight subintervals of equal width. We denote the width by Δx and the value of Δx is given by

$$\Delta x = \frac{2-0}{8} = \frac{2}{8} = \frac{1}{4} \text{ or } 0.25.$$

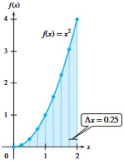


Figure 12-43

The function values at increments of 0.25 are shown in the table.

x	$f(x)$
0	0
0.25	0.0625
0.5	0.25
0.75	0.5625
1	1
1.25	1.5625
1.5	2.25
1.75	3.0625
2	4

a. Using the right endpoint x_R of each subinterval to compute the height of an approximating rectangle, we have

$$\begin{aligned} A_R &\approx f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + \\ &\quad f(1.5)(0.25) + f(1.75)(0.25) + f(2)(0.25) \\ &\approx 0.25[f(0.25) + f(0.5) + f(0.75) + f(1) + f(1.25) + f(1.5) + f(1.75) + f(2)] \\ &\approx 0.25[0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 + 3.0625 + 4] \\ &\approx 3.1875 \end{aligned}$$

b. Using the left endpoint x_L of each subinterval to compute the height of an approximating rectangle, we have

$$\begin{aligned} A_L &\approx f(0)(0.25) + f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + \\ &\quad f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) \\ &\approx 0.25[f(0) + f(0.25) + f(0.5) + f(0.75) + f(1) + f(1.25) + f(1.5) + f(1.75)] \\ &\approx 0.25[0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 + 3.0625] \\ &\approx 2.1875 \end{aligned}$$

The true area A bounded by the graph of $f(x) = x^2$ and the x-axis on the interval $[0, 2]$ is between 2.1875 and 3.1875. That is, $2.1875 < A < 3.1875$ (Figure 12-44).

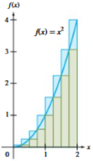


Figure 12-44

Apply the Skills

2. Use eight rectangles to approximate the area bounded by the graph of $f(x) = x^2 + 5$ and the x-axis on the interval $[0, 4]$. For the height of the rectangles, use the function value at the

a. Right endpoint of each subinterval.

b. Left endpoint of each subinterval.

Learn: Use Limits to Find Area Under a Curve

In Example 1, we used four rectangles to approximate the area A under the curve $f(x) = x^2$ on the interval $[0, 2]$ and determined that $1.75 < A < 3.75$. Using eight rectangles, we determined that $2.1875 < A < 3.1875$. Notice that as the number of rectangles increased, the upper approximation of the area decreased and the lower approximation increased. As we further increase the number of rectangles, the upper and lower approximations converge to the true area (Figure 12-45).

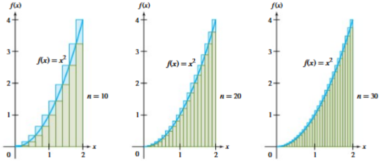


Figure 12-45

Figure 12-45 illustrates the convergence toward the true area under a curve as the number of approximating rectangles increases. This is the basis of the following definition.

Definition of Area Under a Curve

Suppose that *f* is continuous and nonnegative over an interval $[a, b]$. Let $[a, b]$ be partitioned into *n* subintervals of equal width Δx and let $x_1, x_2, \ldots x_n$ represent the right endpoints of the subintervals.* Then the area *A* of the region bounded by the graph of *f* and the *x*-axis on the interval $[a, b]$ is given by

$$A = \lim_{n \to \infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \right]$$

The area can be written more concisely using summation notation.

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{f(x_i)}_{\text{Height of rectangle}} \underbrace{\Delta x}_{\text{Width of rectangle}}, \text{ where } x_i \text{ is the right endpoint of the } i\text{th subinterval.}$$

Notes:

- The total area is the limit of the sum of the areas of the approximating rectangles as the number of rectangles becomes infinitely large.
- The width of the approximating rectangles is given by: $\Delta x = \frac{b-a}{n}$
- The right endpoint of the *i*th subinterval is given by: $x_i = a + i\Delta x$
- The height of the *i*th approximating rectangle is given by: $f(x_i) = f(a + i\Delta x)$

*For each subinterval, any value of *x* from that interval can be used. We choose the right endpoint for convenience.

To make sense of the limit definition of area under a curve, consider Figure 12-46. Notice that function *f* is positive over the interval $[a, b]$ and suppose that the total area between *f* and the *x*-axis is called *S*. The interval $[a, b]$ is partitioned into *n* subintervals corresponding to regions $S_1, S_2, S_3, \ldots, S_n$ that together make up the area of region *S*.

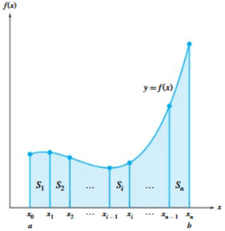


Figure 12-46

- The width of each subinterval is found by dividing the length of interval $[a, b]$ by n. So, $\Delta x = \frac{b-a}{n}$.
- The subintervals are $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]$.
- The right endpoints of regions $S_1, S_2, \ldots S_i, \ldots, S_n$ are given by

  $x_1 = a + 1\Delta x$

  $x_2 = a + 2\Delta x$

  $x_3 = a + 3\Delta x$

  ...

  $x_i = a + i\Delta x \longleftarrow$ Right endpoint of the ith region

  ...

  $x_n = a + n\Delta x$
- The area A of region S is equal to the sum of the areas of $S_1, S_2, S_3, \ldots, S_n$. Using rectangles to approximate the area of each region, we have $A \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$.
- The area A of region S is *exactly* equal to

  $A = \lim\limits_{x\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right]$ or $A = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i)\Delta x$.

In Example 3, we will find the *exact* area between $f(x) = x^2$ and the x-axis on the interval $[0, 2]$. We will also need the properties of summation presented earlier in our discussion of sequences and series.

Properties of Summation

If $\{a_n\}$ and $\{b_n\}$ are sequences and c is a real number, then

1. $\sum\limits_{i=1}^{n} c = cn$

2. $\sum\limits_{i=1}^{n} ca_i = c\sum\limits_{i=1}^{n} a_i$

3. $\sum\limits_{i=1}^{n} (a_i \pm b_i) = \sum\limits_{i=1}^{n} a_i \pm \sum\limits_{i=1}^{n} b_i$

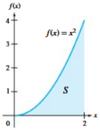
4. $\sum\limits_{i=1}^{n} i = \frac{n(n+1)}{2}$

5. $\sum\limits_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

6. $\sum\limits_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Example 3: Using a Limit to Find the Exact Area Under a Curve

Find the exact area of the region S bounded by $f(x) = x^2$ and the x-axis on the interval $[0, 2]$.



Solution:

In Figure 12-47, region S is approximated by n rectangles with the height of each rectangle given by the function value at the right endpoint.

Width: $\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

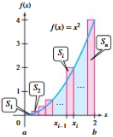


Figure 12-47

Right endpoint of the ith subinterval:

$$x_i = a + i\Delta x$$

$$= 0 + i\left(\frac{2}{n}\right)$$

Substitute 0 for a and substitute $\frac{2}{n}$ for Δx.

$$= \frac{2i}{n}$$

Height of ith rectangle: $f(x_i) = x_i^2$

$$= \left(\frac{2i}{n}\right)^2$$

Substitute $\frac{2i}{n}$ for x_i.

$$= \frac{4i^2}{n^2}$$

Simplify.

The area is given by $A = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\Delta x$.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

Substitute $\frac{4i^2}{n^2}$ for $f(x_i)$ and $\frac{2}{n}$ for Δx.

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8i^2}{n^3}$$

Simplify within the summation.

$$= \lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2$$

The variable i varies within the summation, but the upper limit of summation, n, is constant. Therefore, the expression $\frac{8}{n^3}$ can be removed from the summation.

$$= \lim_{n\to\infty} \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

Replace $\sum\limits_{i=1}^{n} i^2$ by $\frac{n(n+1)(2n+1)}{6}$.

$$= \lim_{n\to\infty} \frac{8}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right)$$

Expand the product in the numerator.

$$= \lim_{n\to\infty} \frac{16n^3+24n^2+8n}{6n^3}$$

Simplify numerator and denominator.

$$= \lim_{n\to\infty} \left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right)$$

Divide each term in the numerator by the denominator.

$$= \lim_{n\to\infty} \frac{8}{3} + \lim_{n\to\infty} \frac{4}{n} + \lim_{n\to\infty} \frac{4}{3n^2}$$

Take the limit of each term.

$$= \frac{8}{3}\lim_{n\to\infty} 1 + 4\lim_{n\to\infty} \frac{1}{n} + \frac{4}{3}\lim_{n\to\infty} \frac{1}{n^2}$$

$4\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and $\frac{4}{3}\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$

$$= \frac{8}{3}$$

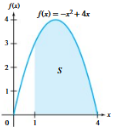
The exact area of the region bounded by $f(x) = x^2$ and the x-axis on the interval $[0, 2]$ is $\frac{8}{3}$.

Apply the Skills

3. Find the exact area of the region S bounded by $f(x) = x^2 + 5$ and the x-axis on the interval $[0, 4]$.

Example 4: Using a Limit to Find the Exact Area Under a Curve

Find the exact area of the region S bounded by $f(x) = -x^2 + 4x$ and the x-axis on the interval $[1, 4]$.



Solution:

In Figure 12-48, region S is approximated by n rectangles with the height of each rectangle given by the function value at the right endpoint.

Width: $\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$

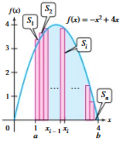


Figure 12-48

Right endpoint of the ith subinterval:

$$\begin{aligned} x_i &= a + i\Delta x \\ &= 1 + i\left(\frac{3}{n}\right) \\ &= 1 + \frac{3i}{n} \end{aligned}$$

Height of ith rectangle:

$$f(x_i) = -x_i^2 + 4x_i$$

$$= -\left(1 + \frac{3i}{n}\right)^2 + 4\left(1 + \frac{3i}{n}\right)$$

Substitute $1 + \frac{3i}{n}$ for x_i.

$$= -\left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right) + 4 + \frac{12i}{n}$$

Square the binomial and apply the distributive property.

$$= -\frac{9i^2}{n^2} + \frac{6i}{n} + 3$$

Simplify.

The area is given by $A = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\Delta x$.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(-\frac{9i^2}{n^2} + \frac{6i}{n} + 3\right)\left(\frac{3}{n}\right)$$

Substitute $-\frac{9i^2}{n^2} + \frac{6i}{n} + 3$ for $f(x_i)$ and $\frac{3}{n}$ for Δx.

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-\frac{27i^2}{n^3} + \frac{18i}{n^2} + \frac{9}{n}\right)$$

Simplify within the summation.

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \left(-\frac{27i^2}{n^3}\right) + \sum_{i=1}^{n} \frac{18i}{n^2} + \sum_{i=1}^{n} \frac{9}{n}\right]$$

Write individual sums of each term.

$$= \lim_{n\to\infty} \left[\left(-\frac{27}{n^3}\right) \sum_{i=1}^{n} i^2 + \frac{18}{n^2} \sum_{i=1}^{n} i + \frac{9}{n} \sum_{i=1}^{n} 1\right]$$

Remove constant factors from each sum.

$= \lim_{n\to\infty}\left\{\left(-\frac{27}{n^3}\right)\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{18}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{9}{n}(n)\right\}$ Use the properties of summation presented on page 12-70.

$= \lim_{n\to\infty}\left[\left(-\frac{27}{n^3}\right)\left(\frac{2n^3+3n^2+n}{6}\right) + \frac{18}{n^2}\left(\frac{n^2+n}{2}\right) + 9\right]$ Expand the products in the numerator.

$= \lim_{n\to\infty}\left[\left(-9 - \frac{27}{2n} - \frac{9}{2n^2}\right) + \left(9 + \frac{9}{n}\right) + 9\right]$ Apply the distributive property.

$= \lim_{n\to\infty}\left(9 - \frac{9}{2n} - \frac{9}{2n^2}\right)$ Combine like terms.

$= \lim_{n\to\infty} 9 - \lim_{n\to\infty}\frac{9}{2n} - \lim_{n\to\infty}\frac{9}{2n^2}$

$= 9\lim_{n\to\infty} 1 - \frac{9}{2}\lim_{n\to\infty}\frac{1}{n} - \frac{9}{2}\lim_{n\to\infty}\frac{1}{n^2}$ $\frac{9}{2}\lim_{n\to\infty}\frac{1}{n} = 0$ and $\frac{9}{2}\lim_{n\to\infty}\frac{1}{n^2} = 0$

$= 9$

The exact area of the region bounded by $f(x) = -x^2 + 4x$ and the x-axis on the interval $[1, 4]$ is 9.

Apply the Skills

4. Find the exact area of the region S bounded by $f(x) = -x^2 + 6x$ and the x-axis on the interval $[1, 6]$.

Learn: Represent an Area with a Definite Integral

In calculus, (thankfully) we will learn a much easier approach to compute the area under a curve. This approach uses a theorem called the Fundamental Theorem of Calculus and a structure called a *definite integral*.

Definite Integral

If f is continuous on an interval $[a, b]$, the "**definite integral** from a to b of $f(x)$" is denoted by $\int_b^a f(x)dx$, where

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

The values of a and b are called the **lower** and **upper limits of integration**, respectively.

If $y = f(x) \geq 0$, then the definite integral represents the area between the graph of f and the x-axis on the interval $[a, b]$.

Example 5: Using a Geometric Approach to Evaluate a Definite Integral

Given $\int_0^3 (2x + 1)dx$,

a. Sketch the region represented by the integral.

b. Use geometry to compute the value of the definite integral.

Solution:

a. The integral $\int_0^3 (2x + 1)dx$ represents the area between the graph of $f(x) = 2x + 1$ and the x-axis on the interval $[0, 3]$.

$f(x) = 2x + 1$ defines a line with a slope of 2 and y-intercept $(0, 1)$ (Figure 12-49). The region represented by the integral is a trapezoid.

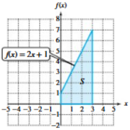


Figure 12-49

b. Since $f(x) = 2x + 1$ is positive on the interval $[0, 3]$, the definite integral represents the area of the shaded region.

The area of a trapezoid is given by $A = \frac{1}{2}(b_1 + b_2)h$.

The values of b_1 and b_2 are the lengths of the parallel sides of the trapezoid.

So, $b_1 = f(0) = 2(0) + 1 = 1$ and $b_2 = f(3) = 2(3) + 1 = 7$.

The height h is the distance between the two parallel sides, which is 3.

The area is given by $A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(1 + 7)(3) = 12$.

Therefore, $\int_0^3 (2x + 1)dx = 12$.

Apply the Skills

5. Given $\int_0^4 (-x + 6)dx$,

a. Sketch the region represented by the integral.

b. Use geometry to compute the value of the definite integral.

Point of Interest

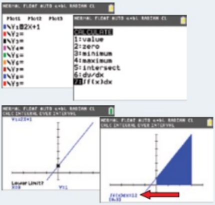
The integral notation is attributed to Gottfried Wilhelm von Leibniz a German mathematician, who alongside Isaac Newton is one of the fathers of calculus.

TECHNOLOGY CONNECTIONS

Approximating a Definite Integral on a Calculator

A definite integral can be approximated using a calculator. First enter the function in the function editor. Then invoke the CALC menu and select $\int f(x)dx$.

The calculator will graph $y = f(x)$ and ask you for the lower and upper limits of integration. After entering the limits and hitting ENTER, the calculator graphs the region represented by the integral and gives the value of the area.


NORMAL FLOAT AUTO a+bi RADIAN CL
Plot1 Plot2 Plot3
\Y1=2X+1
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
Lower Limit?

Practice 12-5

Practice Exercises

Prerequisite Review

R.1. Find the sum. $\sum_{i=1}^{5}(4i + 5)$

R.2. Find the sum. $\sum_{i=1}^{3}(-3i^3)$

R.3. Write the equation in slope-intercept form, then graph using the slope and y-intercept.

$$4x = -1 - y$$

R.4. Identify the center and radius of the circle, then graph the circle.

$$(x - 3)^2 + y^2 = 25$$

Concept Connections

1. Suppose that an interval $[a, b]$ is divided into n subintervals. What is the width Δx of each subinterval?

2. If f is continuous on an interval $[a, b]$ and $y = f(x) \geq 0$, what does $\int_a^b f(x)dx$ represent geometrically?

Learn: Approximate Area Under a Curve

For Exercises 3–8, use n rectangles to approximate the area bounded by the graph of the given function and the x-axis on the indicated interval. For the height of the rectangles, use the function value at the

a. Right endpoint of each subinterval. (**See Example 1, p. 12-65**)

b. Left endpoint of each subinterval.

3. $f(x) = x^2 + 3$ on the interval $[0, 4]$ with $n = 4$

4. $f(x) = 64 - x^2$ on the interval $[0, 8]$ with $n = 4$

5. $f(x) = 4 - x^2$ on the interval $[-2, 2]$ with $n = 8$

6. $f(x) = 4x^2$ on the interval $[-2, 2]$ with $n = 8$

7. $f(x) = x^3 + x$ on the interval $[0, 6]$ with $n = 6$

8. $f(x) = 125 - x^3$ on the interval $[-1, 5]$ with $n = 6$

For Exercises 9–10, use n rectangles to approximate the area bounded by the graph of the given function and the x-axis on the indicated interval. For the height of the rectangles, use the function value at the right endpoint. (See Example 2, p. 12-67)

9. $f(x) = -x^3 + 4x$ on the interval $[0, 2]$

a. With $n = 4$ 3.75

b. With $n = 8$ 3.9375

c. How does the number of rectangles affect the approximation. (*Hint*: The exact area of the region is 4.)

10. $f(x) = -x^4 + 64x$ on the interval $[0, 4]$

a. With $n = 4$

b. With $n = 8$

c. How does the number of rectangles affect the approximation. (*Hint*: The exact area of the region is 307.2.)

Learn: Use Limits to Find Area Under a Curve

For Exercises 11–16, use the properties on page 71 to evaluate each sum.

11. $\sum_{i=1}^{20}(3i - 4)$

12. $\sum_{i=1}^{52}(2i + 3)$

13. $\sum_{i=1}^{60}(i^2 + 5i)$

14. $\sum_{i=1}^{40}(i^2 - 8i)$

15. $\sum_{i=1}^{12}5i^3$

16. $\sum_{i=1}^{16}2i^3$

For Exercises 17–24, find the exact area of the region bounded by the graph of the given function and the x-axis on the given interval. (See Examples 3–4, pp. 12-70, 12-72)

17. $f(x) = x + 4$ on the interval $[0, 3]$

18. $f(x) = -x + 6$ on the interval $[0, 4]$

19. $f(x) = -x^2 + 4$ on the interval $[0, 2]$

20. $f(x) = x^2 + 1$ on the interval $[0, 6]$

21. $f(x) = 2x^2$ on the interval $[-4, 0]$

22. $f(x) = \frac{1}{16}x^2$ on the interval $[-8, 0]$

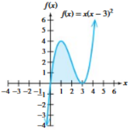
23. $f(x) = -x^2 + 8x$ on the interval $[2, 8]$

24. $f(x) = -x^2 - 10x$ on the interval $[-10, -1]$

Learn: Represent an Area with a Definite Integral

For Exercises 25–28, write a definite integral that represents the shaded area.

25.



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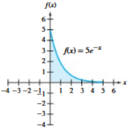

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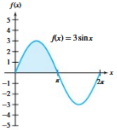

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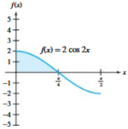


For Exercises 29–34,

a. Sketch the region represented by the integral.

b. Use geometry to compute the value of the definite integral. (**See Example 5, p. 12-74**)

29. $\int_{-3}^{5} 3dx$

30. $\int_{-1}^{3} 2dx$

31. $\int_{3}^{5} (2x - 4)dx$

32. $\int_{-3}^{3} \left(-\frac{1}{3}x + 5\right)dx$

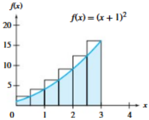
33. $\int_{-2}^{6} \sqrt{16 - (x - 2)^2}dx$

34. $\int_{-3}^{3} \sqrt{9 - x^2}dx$

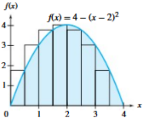
Mixed Exercises

For Exercises 35–36 approximate the area of the shaded region by using the indicated rectangles of equal width.

35.



36.



37. a. Graph the region S bounded by the graph of $f(x) = x + 2$ and the x-axis on the interval $[1, 5]$.

b. Use the limit process to find the exact area of S.

c. Write a definite integral representing the area of S.

d. Confirm the result of part (b) using a geometric formula.

38. a. Graph the region S bounded by the graph of $f(x) = 6 - 2x$ and the x-axis on the interval $[1, 3]$.

b. Use the limit process to find the exact area of S.

c. Write a definite integral representing the area of S.

d. Confirm the result of part (b) using a geometric formula.

Write About It

39. The area between the graph of $f(x) = -x^2 + 9$ and the x-axis is given by $\int_{-3}^{3}(-x^2 + 9)dx$. Graph the region represented by the definite integral and explain why the integral $2\int_{0}^{3}(-x^2 + 9)dx$ gives the same value.

40. The value of the integral $\int_{0}^{4}(-x + 4)dx$ is 8. However, the value of the integral $\int_{0}^{8}(-x + 4)dx$ is 0. Use a graph to explain why this is so.

41. Consider a regular polygon (a polygon with sides of equal length) inscribed inside a circle. Explain how the concept of a limit can be used to define the area of a circle.

Expanding Your Skills

For Exercises 42–43,

a. Sketch the region represented by the integral.

b. Compute the value of the definite integral.

42. $\int_{-4}^{4} f(x)dx$, where

$$f(x) = \begin{cases} x + 5 & \text{for } -4 \le x \le -2 \\ -0.25x + 2.5 & \text{for } -2 < x < 2 \\ 0.5x + 1 & \text{for } 2 \le x \le 4 \end{cases}$$

43. $\int_{-1}^{6} f(x)dx$

$$f(x) = \begin{cases} 1 & \text{for } -1 \le x < 2 \\ 4 - |1.5x - 6| & \text{for } 2 \le x \le 6 \end{cases}$$

Technology Connections

For Exercises 44–45, approximate the value of the definite integral to 3 decimal places by using a calculator that supports definite integration.

44. $\int_{-5}^{5} \ln|x^2 + 2|\,dx$

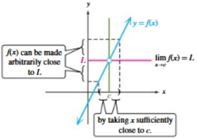
45. $\int_{-\pi}^{\pi} |2\sin x|\,dx$

Go online for more practice problems.

Chapter 12 Review

Chapter 12 Key Concepts

Lesson 12-1 Introduction to Limits Through Tables and Graphs

Key Concepts	Reference
Limit of a Function: Consider an open interval containing c and a function f defined on the open interval except possibly at c. • The statement $\lim_{x\to c} f(x) = L$ is read as "the limit of $f(x)$ as x approaches c equals L." • $\lim_{x\to c} f(x) = L$ means that $f(x)$ can be made arbitrarily close to L provided that x is taken sufficiently close to c.  y y = f(x) f(x) can be made arbitrarily close to L L lim f(x) = L x→c x c by taking x sufficiently close to c.	p. 12-4
$\lim_{x\to c} f(x)$ fails to exist if • The function is unbounded at c. • The function approaches different values from the left and right of c. • The function "oscillates" near c, therefore failing to approach a single number.	pp. 12-8–12-10

Lesson 12-2 Algebraic Properties of Limits

Key Concepts	Reference
Basic Limits: If b and c are real numbers and n is a positive integer, then **1.** $\lim_{x\to c} b = b$ **2.** $\lim_{x\to c} x = c$ **3.** $\lim_{x\to c} x^n = c^n$ **4.** $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$, provided that $\sqrt[n]{c}$ is defined.	p. 12-16
Properties of Limits: Suppose that b and c are real numbers and n is a positive integer. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then **5.** $\lim_{x\to c}[b \cdot f(x)] = b \cdot \lim_{x\to c} f(x)$ Limit of a scalar multiple **6.** $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$ Limit of a sum or difference **7.** $\lim_{x\to c}[f(x) \cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$ Limit of a product	p. 12-17

8. $\lim\limits_{x\to c}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to c}f(x)}{\lim\limits_{x\to c}g(x)}$ Limit of a quotient [provided that $\lim\limits_{x\to c}g(x)\neq 0$] **9.** $\lim\limits_{x\to c}[f(x)]^n=\left[\lim\limits_{x\to c}f(x)\right]^n$ Limit of a power where *n* is a positive integer **10.** $\lim\limits_{x\to c}\sqrt[n]{f(x)}=\sqrt[n]{\lim\limits_{x\to c}f(x)}$ Limit of a root where *n* is a positive integer [provided that $\lim\limits_{x\to c}f(x)>0$ if *n* is even] Example: $\lim\limits_{x\to 1}\frac{2x-1}{x^2+5x}=\frac{\lim\limits_{x\to 1}(2x-1)}{\lim\limits_{x\to 1}(x^2+5x)}=\frac{\lim\limits_{x\to 1}2x-\lim\limits_{x\to 1}1}{\lim\limits_{x\to 1}x^2+\lim\limits_{x\to 1}5x}=\frac{2\lim\limits_{x\to 1}x-\lim\limits_{x\to 1}1}{\left(\lim\limits_{x\to 1}x\right)^2+5\lim\limits_{x\to 1}x}$ $=\frac{2(1)-1}{(1)^2+5(1)}=\frac{1}{6}$	
Limits of Polynomial and Rational Functions: If *f* is a polynomial function or rational function and *c* is in the domain of *f*, then $\lim\limits_{x\to c}f(x)=f(c)$ This means that the limit of *f*(*x*) as *x* approaches *c* is the same as the value of *f* at *c*. So, direct substitution can be used to find the limit. Example: $\lim\limits_{x\to -2}(4x^3-5x+7)=4(-2)^3-5(-2)+7=4(-8)+10+7=-15$	p. 12-19
Strategies to Evaluate Limits in the Case of the Indeterminate Form 0/0: **Divide out by a Common Factor:** $\lim\limits_{x\to -3}\frac{x^2-9}{x+3}=\lim\limits_{x\to -3}\frac{\cancel{(x+3)}(x-3)}{\cancel{x+3}}=\lim\limits_{x\to -3}(x-3)=-3-3=-6$ **Rationalize the Numerator:** $\lim\limits_{x\to 0}\frac{\sqrt{x+64}-8}{x}=\lim\limits_{x\to 0}\frac{(\sqrt{x+64}-8)}{(x)}\cdot\frac{(\sqrt{x+64}+8)}{(\sqrt{x+64}+8)}=\lim\limits_{x\to 0}\frac{x+64-64}{x(\sqrt{x+64}+8)}$ $=\lim\limits_{x\to 0}\frac{\cancel{x}}{\cancel{x}(\sqrt{x+64}+8)}=\lim\limits_{x\to 0}\frac{1}{\sqrt{x+64}+8}=\frac{\lim\limits_{x\to 0}1}{\sqrt{\lim\limits_{x\to 0}(x+64)}+\lim\limits_{x\to 0}8}$ $=\frac{1}{8+8}=\frac{1}{16}$ **Simplify a Complex Fraction:** $\lim\limits_{x\to 0}\frac{\frac{1}{x-1}+1}{x}=\lim\limits_{x\to 0}\frac{\left(\frac{1}{x-1}+1\right)}{(x)}\cdot\frac{(x-1)}{(x-1)}=\lim\limits_{x\to 0}\frac{1+(x-1)}{x(x-1)}=\lim\limits_{x\to 0}\frac{x}{x(x-1)}$ $=\lim\limits_{x\to 0}\frac{1}{x-1}=\frac{1}{(0)-1}=-1$	pp. 12-20–12-22

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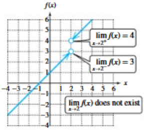
One-Sided Limits:

- The statement $\lim_{x \to c^-} f(x) = L$ means that $f(x)$ can be made arbitrarily close to L provided that x is taken sufficiently close to c but *less* than c.
- The statement $\lim_{x \to c^+} f(x) = M$ means that $f(x)$ can be made arbitrarily close to M provided that x is taken sufficiently close to c but *greater* than c.



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Problem Recognition Exercises: Limits and Continuity

A function f is **continuous** at c if the following condition are met.

1. f is defined at c. **2.** $\lim_{x \to c} f(x)$ exists. **3.** $\lim_{x \to c} f(x) = f(c)$.

</td><td>p. 12-29</td></tr>
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Lesson 12-3 The Tangent Line Problem: Introduction to Derivatives

<table>
<tr><th>Key Concepts</th><th>Reference</th></tr>
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Tangent Line:

The line tangent to the curve $y = f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ with slope $m_T = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, provided that the limit exists.

</td><td>p. 12-37</td></tr>
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Derivative of a Function:

The derivative of f at x, denoted $f'(x)$, is given by

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, provided that the limit exists.

Notes:

- The notation $f'(x)$ is read "f prime of x."
- The derivative of a function is itself a function.
- The derivative of f evaluated at x is the slope of the tangent line to the graph of f at the point $P(x, f(x))$. The derivative measures the slope or "steepness" of the curve at point P.
- In applications, the derivative represents the instantaneous rate of change of $f(x)$ versus x at $P(x, f(x))$.

</td><td>p. 12-40</td></tr>
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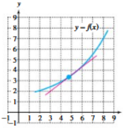
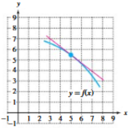
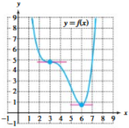
Sign of a Derivative:  - The derivative is positive on the intervals over which *f* is increasing.  - The derivative is negative on the intervals over which *f* is decreasing.  - The derivative is zero where the curve has a horizontal tangent line.	p. 12-41

Lesson 12-4 Limits at Infinity and Limits of Sequences

Key Concepts	Reference
Limits at Infinity or Negative Infinity: **1.** Let *f* be a function defined on an interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that $f(x)$ can be made arbitrarily close to L, provided that x is taken sufficiently large. **2.** Let *f* be a function defined on an interval $(-\infty, b)$. Then $\lim_{x\to-\infty} f(x) = M$ means that $f(x)$ can be made arbitrarily close to M, provided that x is taken sufficiently "large" in the negative direction.	p. 12-52
Horizontal Asymptote: The line $y = L$ is a **horizontal asymptote** of the graph of $y = f(x)$ if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$	p. 12-52
Limit at infinity for $f(x) = \frac{1}{x^k}$: - If k is a positive real number, then $\lim_{x\to\infty}\frac{1}{x^k} = 0$. - Furthermore, for a positive real number k, if $\frac{1}{x^k}$ is defined for $x < 0$, then $\lim_{x\to-\infty}\frac{1}{x^k} = 0$. Example: $\lim_{x\to\infty}\frac{3x^2+5x+2}{4x^2-6} = \lim_{x\to\infty}\frac{\frac{1}{x^2}(3x^2+5x+2)}{\frac{1}{x^2}(4x^2-6)} = \lim_{x\to\infty}\frac{3+\frac{5}{x}+\frac{2}{x^2}}{4-\frac{6}{x^2}} = \frac{\lim_{x\to\infty}3+\lim_{x\to\infty}\frac{5}{x}+\lim_{x\to\infty}\frac{2}{x^2}}{\lim_{x\to\infty}4-\lim_{x\to\infty}\frac{6}{x^2}}$ $= \frac{3\lim_{x\to\infty}1+5\lim_{x\to\infty}\frac{1}{x}+2\lim_{x\to\infty}\frac{1}{x^2}}{4\lim_{x\to\infty}1-6\lim_{x\to\infty}\frac{1}{x^2}} = \frac{3(1)+5(0)+2(0)}{4(1)-6(0)} = \frac{3}{4}$	p. 12-53

Limit of a Sequence:

p. 12-57

Given $\{a_n\} = a_1, a_2, a_3, \ldots$, we say that the sequence has limit L, denoted

$$\lim_{n\to\infty} a_n = L$$

if the nth term of the sequence can be made arbitrarily close to L by taking n sufficiently large.

- If $\lim_{n\to\infty} a_n$ exists, we say that the sequence is **convergent.**
- If $\lim_{n\to\infty} a_n$ does not exist, we say that the sequence is **divergent.**

Example:

$$\lim_{n\to\infty} \frac{5n^2-3}{n^2} = \lim_{n\to\infty} \frac{\frac{1}{n^2}\cdot(5n^2-3)}{\frac{1}{n^2}\cdot(n^2)} = \lim_{n\to\infty} \frac{5-\frac{3}{n^2}}{1} = \lim_{n\to\infty}\left(5-\frac{3}{n^2}\right)$$

$$= \lim_{n\to\infty} 5 - 3\lim_{n\to\infty}\frac{1}{n^2} = 5 - 0 = 5$$

The sequence defined by $a_n = \frac{5n^2-3}{n^2}$ converges.

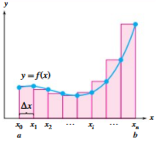
Lesson 12-5 Area Under a Curve

Key Concepts	Reference

Area Under a Curve:

p. 12-69

Suppose that f is continuous and nonnegative over an interval $[a, b]$. Let $[a, b]$ be partitioned into n subintervals of equal width Δx and let $x_1, x_2, \ldots, x_n$ represent the right endpoints of the subintervals. Then the area A of the region bounded by the graph of f and the x-axis on the interval $[a, b]$ is given by



$$A = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

Properties of Summation:

p. 12-70

To evaluate the limit of a sum, the following properties of summation are helpful. If $\{a_n\}$ and $\{b_n\}$ are sequences and c is a real number, then

1. $\sum_{i=1}^{n} c = cn$
2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
3. $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
5. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
6. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Definite Integral:

p. 12-73

If f is continuous on an interval $[a, b]$, the "**definite integral** from a to b of $f(x)$" is denoted by $\int_a^b f(x)dx$, where

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

The values of a and b are called the **lower** and **upper limits of integration**, respectively. If $y = f(x) \geq 0$, then the definite integral represents the area between the graph of f and the x-axis on the interval $[a, b]$.

Go online for a detailed chapter summary.

Chapter 12 Test

For Exercises 1–10, find the limit or indicate that the limit does not exist.

1. $\lim\limits_{x \to 3} \frac{x^2 - 9x}{x + 3}$ **2.** $\lim\limits_{x \to \frac{\pi}{3}} \tan x$

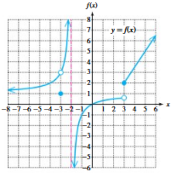
3. $\lim\limits_{x \to 5} \frac{x^2 - 25}{x^2 + 3x - 40}$ **4.** $\lim\limits_{x \to 7} \frac{\sqrt{x + 2} - 3}{x - 7}$

5. $\lim\limits_{x \to -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x}$ **6.** $\lim\limits_{x \to \infty} \frac{x^3}{x^2 - 16}$

7. $\lim\limits_{x \to -\infty} \frac{4x}{x^2 - 8}$ **8.** $\lim\limits_{x \to \infty} \frac{2x^2 + 3x - 4}{6x^2 + 5x}$

9. $\lim\limits_{x \to 3^+} \frac{15 - 5x}{|3 - x|}$ **10.** $\lim\limits_{n \to \infty} \frac{10}{n^2}\left[\frac{n(n + 1)}{2}\right]$

For Exercises 11–14, use the graph to find the limit or indicate that the limit does not exist.



11. a. $\lim\limits_{x \to -\infty} f(x)$ **b.** $\lim\limits_{x \to \infty} f(x)$

12. a. $\lim\limits_{x \to 3^+} f(x)$ **b.** $\lim\limits_{x \to 3} f(x)$

13. a. $\lim\limits_{x \to -3^+} f(x)$ **b.** $\lim\limits_{x \to -3^-} f(x)$

14. a. $\lim\limits_{x \to -2} f(x)$ **b.** $\lim\limits_{x \to 0} f(x)$

For Exercises 15–16, refer to the graph for Exercises 11–14.

15. Is f continuous at $x = -3$? Why or why not?

16. Is f continuous at $x = 3$? Why or why not?

For Exercises 17–18, find the slope of the tangent line at $x = a$ using the formula $m_T = \lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$.

17. $f(x) = 4x^2 - 8$; $a = 3$

18. $f(x) = x^3 - 2x$; $a = -2$

For Exercises 19–20,

a. Find $f'(x)$.

b. Find the slope of the line tangent to f at $x = 4$.

c. Find an equation of the tangent line to the graph at $x = 4$.

19. $f(x) = 2x^2 - 2$ **20.** $f(x) = \frac{5}{x}$

21. A pebble is dropped into a lake. A circular wave then expands outward from the point where the pebble hit the water.

a. Find the rate of change of the area of the wave as a function of the radius of the wave. Recall that the area of a circle is given by $A = \pi r^2$.

b. How fast is the area changing relative to the radius when the radius is 8 ft?

22. Use six rectangles to approximate the area bounded by the graph of $f(x) = 5x - 4$ and the x-axis on the interval [3, 6]. For the height of the rectangles, use the function value at the

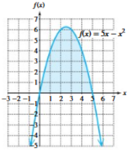
a. Right endpoint of each subinterval.

b. Left endpoint of each subinterval.

c. Use geometry to compute the exact value of $\int_3^6 (5x - 4)dx$.

23. a. Write a definite integral that represents the shaded area.

b. Find the exact area of the shaded region.



Go online for more practice problems.

Appendix: Quick Reference Guide

Properties of Exponents

$b^m \cdot b^n = b^{m+n}$ $\qquad b^0 = 1$

$\frac{b^m}{b^n} = b^{m-n} \quad (b \neq 0)$ $\qquad b^{-n} = \frac{1}{b^n} \quad (b \neq 0)$

$(b^m)^n = b^{m \cdot n}$

$(ab)^m = a^m b^m$ $\qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0)$

Properties of Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real,

$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

$\sqrt[m]{\sqrt[n]{a}} = \sqrt[m \cdot n]{a}$

$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ $\qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad (b \neq 0)$

Factoring and Special Case Products

$a^2 + 2ab + b^2 = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

$a^2 - b^2 = (a + b)(a - b)$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Complex Numbers

$i = \sqrt{-1}$ and $i^2 = -1$

For a real number $b > 0, \sqrt{-b} = i\sqrt{b}$

The complex numbers $a + bi$ and $a - bi$ are conjugates, and $(a + bi)(a - bi) = a^2 + b^2$.

Quadratic Formula

Given $ax^2 + bx + c = 0$, $a \neq 0$, the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Absolute Value Equations and Inequalities

If $k \geq 0$, then

$|u| = k$ is equivalent to $u = k$ or $u = -k$.

$|u| = |w|$ is equivalent to $u = w$ or $u = -w$.

$|u| < k$ is equivalent to $-k < u < k$.

$|u| > k$ is equivalent to $u < -k$ or $u > k$.

Distance Formulas

The distance between two points a and b on a number line is given by $|a - b|$ or $|b - a|$.

The distance between (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

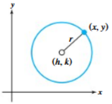
Midpoint Formula

The midpoint of the line segment between (x_1, y_1) and (x_2, y_2) is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Circles

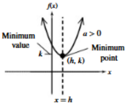
$(x - h)^2 + (y - k)^2 = r^2$ where $r > 0$

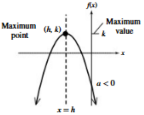


Quadratic Functions

$f(x) = ax^2 + bx + c \ (a \neq 0)$ $\quad$ Vertex: $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

$f(x) = a(x - h)^2 + k \ (a \neq 0)$ $\quad$ Vertex: (h, k)





Slope and Average Rate of Change

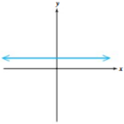
Slope of a line through (x_1, y_1) and (x_2, y_2): $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

Average rate of change of $f(x)$ between (x_1, y_1) and (x_2, y_2): $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$

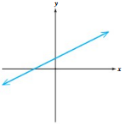
Difference quotient: $\frac{f(x + h) - f(x)}{h}$

Graphs of Basic Functions

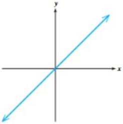
Constant Function
$f(x) = b$



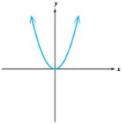
Linear Function
$f(x) = mx + b$



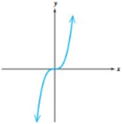
Identity Function
$f(x) = x$



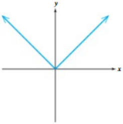
Quadratic Function
$f(x) = x^2$



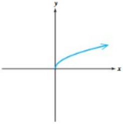
Cubic Function
$f(x) = x^3$



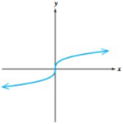
Absolute Value Function
$f(x) = |x|$



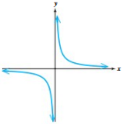
Square Root Function
$f(x) = \sqrt{x}$



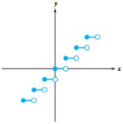
Cube Root Function
$f(x) = \sqrt[3]{x}$



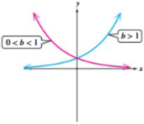
Reciprocal Function
$f(x) = \frac{1}{x}$



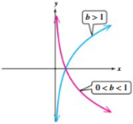
Greatest Integer Function
$f(x) = [\![x]\!]$



Exponential Function
$f(x) = b^x$, where
$b > 0$ and $b \neq 1$



Logarithmic Function
$f(x) = \log_b x$,
where $b > 0$ and $b \neq 1$
$y = \log_b x \Leftrightarrow b^y = x$



Transformations of Graphs

Given $c > 0$ and $h > 0$, the graph of the given function is related to the graph of $y = f(x)$ as follows:

$y = f(x) + c$ Shift the graph of $y = f(x)$ up c units.

$y = f(x) - c$ Shift the graph of $y = f(x)$ down c units.

$y = f(x - h)$ Shift the graph of $y = f(x)$ to the right h units.

$y = f(x + h)$ Shift the graph of $y = f(x)$ to the left h units.

$y = -f(x)$ Reflect the graph of $y = f(x)$ over the x-axis.

$y = f(-x)$ Reflect the graph of $y = f(x)$ over the y-axis.

$y = af(x)$ If $a > 1$, stretch the graph of $y = f(x)$ vertically by a factor of a.

If $0 < a < 1$, shrink the graph of $y = f(x)$ vertically by a factor of a.

$y = f(ax)$ If $a > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $\frac{1}{a}$.

If $0 < a < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $\frac{1}{a}$.

Tests for Symmetry

Consider the graph of an equation in x and y. The graph of the equation is

- Symmetric to the ***y*-axis** if substituting $-x$ for x results in an equivalent equation.
- Symmetric to the ***x*-axis** if substituting $-y$ for y results in an equivalent equation.
- Symmetric to the **origin** if substituting $-x$ for x and $-y$ for y results in an equivalent equation.

Even and Odd Functions

- f is an **even function** if $f(-x) = f(x)$ for all x in the domain of f.
- f is an **odd function** if $f(-x) = -f(x)$ for all x in the domain of f.

Properties of Logarithms

$\log_b 1 = 0$ $\quad \log_b(xy) = \log_b x + \log_b y$

$\log_b b = 1$ $\quad \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

$\log_b b^x = x$ $\quad \log_b x^p = p \log_b x$

$b^{\log_b x} = x$

Change-of-base formula: $\log_b x = \frac{\log_a x}{\log_a b}$

$b^x = b^y$ implies that $x = y$.

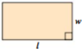
$\log_b x = \log_b y$ implies that $x = y$.

Variation

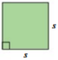
y varies **directly** as x. y is **directly** proportional to x.	$y = kx$
y varies **inversely** as x. y is **inversely** proportional to x.	$y = \frac{k}{x}$
y varies **jointly** as w and x. y is **jointly** proportional to w and x.	$y = kwx$

Perimeter and Circumference

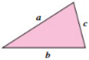
Rectangle
$P = 2l + 2w$



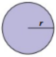
Square
$P = 4s$



Triangle
$P = a + b + c$

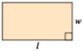


Circle
Circumference: $C = 2\pi r$

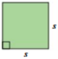


Area

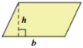
Rectangle
$A = lw$



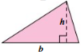
Square
$A = s^2$



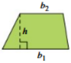
Parallelogram
$A = bh$



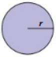
Triangle
$A = \frac{1}{2}bh$



Trapezoid
$A = \frac{1}{2}(b_1 + b_2)h$

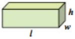


Circle
$A = \pi r^2$

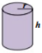


Volume

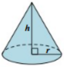
Rectangular Solid
$V = lwh$



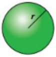
Right Circular Cylinder
$V = \pi r^2 h$



Right Circular Cone
$V = \frac{1}{3}\pi r^2 h$



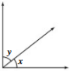
Sphere
$V = \frac{4}{3}\pi r^3$



Angles

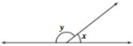
- Two angles are complementary if the sum of their measures is 90°.

  $x + y = 90°$



- Two angles are supplementary if the sum of their measures is 180°.

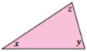
  $x + y = 180°$



Triangles

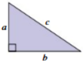
- The sum of the measures of the angles of a triangle is 180°.

  $x + y + z = 180°$



- Given a right triangle with legs of length a and b, and hypotenuse of length c, the Pythagorean theorem indicates that

  $a^2 + b^2 = c^2$



Trigonometric Functions

$\sin t = y$ $\qquad$ $\csc t = \frac{1}{y}(y \neq 0)$

$\cos t = x$ $\qquad$ $\sec t = \frac{1}{x}(x \neq 0)$

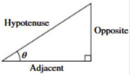
$\tan t = \frac{y}{x}(x \neq 0)$ $\qquad$ $\cot t = \frac{x}{y}(y \neq 0)$

Right Triangle Trigonometry

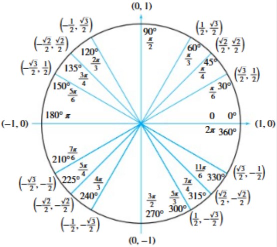
$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\quad$ $\csc\theta = \frac{\text{hyp}}{\text{opp}}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\quad$ $\sec\theta = \frac{\text{hyp}}{\text{adj}}$

$\tan\theta = \frac{\text{opp}}{\text{adj}}$ $\quad$ $\cot\theta = \frac{\text{adj}}{\text{opp}}$


Hypotenuse
Opposite
θ
Adjacent

Unit Circle


(0, 1)
90° π/2
(1/2, √3/2)
60° π/3
(√2/2, √2/2)
45° π/4
(√3/2, 1/2)
30° π/6
(1, 0)
0 0°
2π 360°
(−1/2, √3/2)
120° 2π/3
(−√2/2, √2/2)
135° 3π/4
(−√3/2, 1/2)
150° 5π/6
(−1, 0)
180° π
210° 7π/6
(−√3/2, −1/2)
225° 5π/4
(−√2/2, −√2/2)
240° 4π/3
(−1/2, −√3/2)
3π/2 270°
(0, −1)
5π/3 300°
(1/2, −√3/2)
7π/4 315°
(√2/2, −√2/2)
11π/6 330°
(√3/2, −1/2)

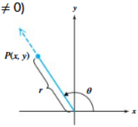
Trigonometric Functions of Any Angle

$\sin\theta = \frac{y}{r}$ $\qquad$ $\csc\theta = \frac{r}{y}(y \neq 0)$

$\cos\theta = \frac{x}{r}$ $\qquad$ $\sec\theta = \frac{r}{x}(x \neq 0)$

$\tan\theta = \frac{y}{x}(x \neq 0)$

$\cot\theta = \frac{x}{y}(y \neq 0)$



Inverse Trigonometric Functions

$y = \sin^{-1}x \quad \Leftrightarrow \quad \sin y = x$

$-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

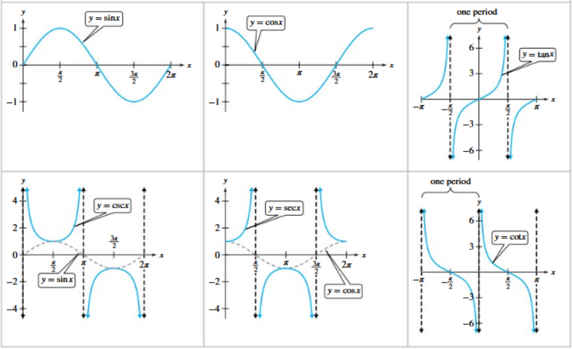
$y = \cos^{-1}x \quad \Leftrightarrow \quad \cos y = x$

$-1 \leq x \leq 1$ and $0 \leq y \leq \pi$

$y = \tan^{-1}x \quad \Leftrightarrow \quad \tan y = x$

$x \in \mathbb{R}$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Graphs of Basic Trigonometric Functions



Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Sum and Difference Formulas

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$

$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Even and Odd Identities

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Double-Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} \qquad = 1 - 2\sin^2\theta$$

$$= 2\cos^2\theta - 1$$

Cofunction Identities

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\quad\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\quad\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\quad\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Power-Reducing Formulas

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$ $\quad\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$

Half-Angle Formulas

$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$ $\quad\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$

Product-to-Sum Formulas

$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$

$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$

$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$

$\cos u \sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$

Sum-to-Product Formulas

$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

Law of Sines

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

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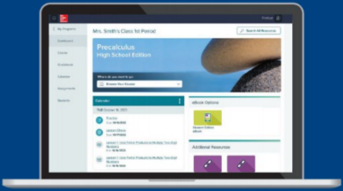
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